

Contents

Preface xi

0 Part 0: Setting the Stage

Prologue: Three Stories 3

Introduction: A Natural System of Units, the Cube of Physics,
Being Overweight, and Hawking Radiation 10

Prelude: Relativity Is an Everyday and Ancient Concept 17

ONE Book One: From Newton to the Gravitational Redshift

I Part I: From Newton to Riemann: Coordinates to Curvature

I.1 Newton's Laws 25

I.2 Conservation Is Good 35

I.3 Rotation: Invariance and Infinitesimal Transformation 38

I.4 Who Is Afraid of Tensors? 52

I.5 From Change of Coordinates to Curved Spaces 62

I.6 Curved Spaces: Gauss and Riemann 82

I.7 Differential Geometry Made Easy, but Not Any Easier! 96

Recap to Part I 110

viii | Contents

II	Part II: Action, Symmetry, and Conservation	
II.1	The Hanging String and Variational Calculus	113
II.2	The Shortest Distance between Two Points	123
II.3	Physics Is Where the Action Is	136
II.4	Symmetry and Conservation	150
	<i>Recap to Part II</i>	155
III	Part III: Space and Time Unified	
III.1	Galileo versus Maxwell	159
III.2	Einstein's Clock and Lorentz's Transformation	166
III.3	Minkowski and the Geometry of Spacetime	174
III.4	Special Relativity Applied	195
III.5	The Worldline Action and the Unification of Material Particles with Light	207
III.6	Completion, Promotion, and the Nature of the Gravitational Field	218
	<i>Recap to Part III</i>	238
IV	Part IV: Electromagnetism and Gravity	
IV.1	You Discover Electromagnetism and Gravity!	241
IV.2	Electromagnetism Goes Live	248
IV.3	Gravity Emerges!	257
	<i>Recap to Part IV</i>	261
TWO	Book Two: From the Happiest Thought to the Universe	
	Prologue to Book Two: The Happiest Thought	265
V	Part V: Equivalence Principle and Curved Spacetime	
V.1	Spacetime Becomes Curved	275
V.2	The Power of the Equivalence Principle	280
V.3	The Universe as a Curved Spacetime	288
V.4	Motion in Curved Spacetime	301
V.5	Tensors in General Relativity	312
V.6	Covariant Differentiation	320
	<i>Recap to Part V</i>	334

VI Part VI: Einstein's Field Equation Derived and Put to Work

VI.1	To Einstein's Field Equation as Quickly as Possible	337
VI.2	To Cosmology as Quickly as Possible	355
VI.3	The Schwarzschild-Droste Metric and Solar System Tests of Einstein Gravity	362
VI.4	Energy Momentum Distribution Tells Spacetime How to Curve	378
VI.5	Gravity Goes Live	388
VI.6	Initial Value Problems and Numerical Relativity	400
	<i>Recap to Part VI</i>	406

VII Part VII: Black Holes

VII.1	Particles and Light around a Black Hole	409
VII.2	Black Holes and the Causal Structure of Spacetime	419
VII.3	Hawking Radiation	436
VII.4	Relativistic Stellar Interiors	451
VII.5	Rotating Black Holes	458
VII.6	Charged Black Holes	477
	<i>Recap to Part VII</i>	485

VIII Part VIII: Introduction to Our Universe

VIII.1	The Dynamic Universe	489
VIII.2	Cosmic Struggle between Dark Matter and Dark Energy	502
VIII.3	The Gamow Principle and a Concise History of the Early Universe	515
VIII.4	Inflationary Cosmology	530
	<i>Recap to Part VIII</i>	537

THREE Book Three: Gravity at Work and at Play

IX Part IX: Aspects of Gravity

IX.1	Parallel Transport	543
IX.2	Precession of Gyroscopes	549
IX.3	Geodesic Deviation	552
IX.4	Linearized Gravity, Gravitational Waves, and the Angular Momentum of Rotating Bodies	563
IX.5	A Road Less Traveled	578
IX.6	Isometry, Killing Vector Fields, and Maximally Symmetric Spaces	585
IX.7	Differential Forms and Vielbein	594

x | Contents

IX.8	Differential Forms Applied	607
IX.9	Conformal Algebra	614
IX.10	De Sitter Spacetime	624
IX.11	Anti de Sitter Spacetime	649
	<i>Recap to Part IX</i>	668

X | Part X: Gravity Past, Present, and Future

X.1	Kaluza, Klein, and the Flowering of Higher Dimensions	671
X.2	Brane Worlds and Large Extra Dimensions	696
X.3	Effective Field Theory Approach to Einstein Gravity	708
X.4	Finite Sized Objects and Tidal Forces in Einstein Gravity	714
X.5	Topological Field Theory	719
X.6	A Brief Introduction to Twistors	729
X.7	The Cosmological Constant Paradox	745
X.8	Heuristic Thoughts about Quantum Gravity	760
	<i>Recap to Part X</i>	775
	<i>Closing Words</i>	777
	<i>Timeline of Some of the People Mentioned</i>	791
	<i>Solutions to Selected Exercises</i>	793
	<i>Bibliography</i>	819
	<i>Index</i>	821
	<i>Collection of Formulas and Conventions</i>	859

1.1 Newton's Laws

The foundational equation of our subject

For in those days I was in the prime of my age for invention and minded Mathematicks & Philosophy more than at any time since.

—Newton describing his youth in his memoirs

Let us start with one of Newton's laws, which curiously enough is spoken as $F = ma$ but written as $ma = F$. For a point particle moving in D -dimensional space with position given by $\vec{x}(t) = (x^1(t), x^2(t), \dots, x^D(t))$, Mr. Newton taught us that

$$m \frac{d^2 x^i}{dt^2} = F^i \tag{1}$$

with the index* $i = 1, \dots, D$. For $D \leq 3$ the coordinates have traditional "names": for example, for $D = 3$, x^1, x^2, x^3 are often called, with some affection, x, y, z , respectively.

Bad notation alert! In teaching physics, I sometimes feel, with only slight exaggeration, that students are confused by bad notation almost as much as by the concepts. I am using the standard notation of x and t here, but the letter x does double duty, as the position of the particle, which more strictly should be denoted by $x^i(t)$ or $\vec{x}(t)$, and as the space coordinates x^i , which are variables ranging from $-\infty$ to ∞ and which certainly are independent of t .

The different status between x and t in say (1) is particularly glaring if $N > 1$ particles are involved, in which case we write $m \frac{d^2 x^i_a}{dt^2} = F^i_a$ or $m \frac{d^2 \vec{x}_a}{dt^2} = \vec{F}_a$ with $x^i_a(t)$ for $a = 1, 2, \dots, N$. But certainly t_a is a meaningless concept in Newtonian physics. In the Newtonian universe, t is the time ticked off by a universal clock, while $\vec{x}_a(t)$ is each particle's private business. We will have plenty more to say about this point. Here $x^i_a(t)$ are $3N$ functions of t , but there are still only 3 x^i .

* See appendix 2.

26 | I. From Newton to Riemann: Coordinates to Curvature

Some readers may feel that I am overly pedantic here, but in fact this fundamental inequality of status between x and t will come to a head when we get to the special theory of relativity. (I now drop the arrow on \vec{x} .) Perhaps Einstein as a student was bothered by this bad notation. One way to remedy the situation is to use q (or q_a) to denote the position of particles, as in more advanced treatments. But here I bow to tradition and continue to use x .

Have differential equation, will solve

After Newton's great insight, we "merely" have to solve some second order differential equations.

To understand Newton's fabulous equation, it's best to work through a few examples. (I need hardly say that if you do not already know Newtonian mechanics, you are unlikely to be able to learn it here.)

A priori, the force F^i could depend on any number of things, but from experience we know that in many simple cases, it depends only on x and not on t or $\frac{dx}{dt}$. As physicists unravel the mysteries of Nature, it becomes increasingly clear that fundamental forces are derived from an underlying quantum field theory and that they have simple forms. Complicated forces often merely result from some approximations we make in particular situations.

Example A

A particle in 1-dimensional space tied to a spring oscillates back and forth.

The force F is a function of space. Newton's equation

$$m \frac{d^2x}{dt^2} = -kx \tag{2}$$

is easily solved in terms of two integration constants: $x(t) = a \cos \omega t + b \sin \omega t$, with $\omega = \sqrt{\frac{k}{m}}$. The two constants a and b are determined by the initial position and initial velocity, or alternatively* by the initial position at $t = 0$ and by the final position at some time $t = T$. Energy, but not momentum, is conserved.

Example B

We kick a particle in 1-dimensional space at $t = 0$.

The force F is a function of time. This example allows me to introduce the highly useful Dirac¹ delta function, or simply delta function.² By the word "kick" we mean that the time scale τ during which the force acts is much less than the other time scales we are

* See part II.

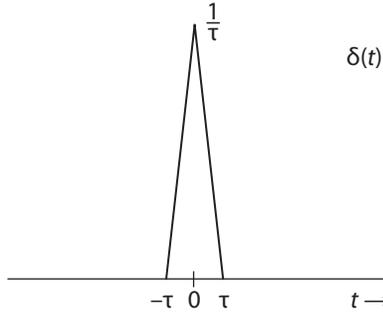


Figure 1 The delta function, which could be thought of as an infinitely sharp spike, is strictly speaking not a function, but the limit of a sequence of functions.

interested in. Thus, take $F(t) = w\delta(t)$, where the function $\delta(t)$ rises sharply just before $t = 0$, rapidly reaches its maximum at $t = 0$, and then sharply drops to 0. Because we included a multiplicative constant w , we could always normalize $\delta(t)$ by

$$\int dt \delta(t) = 1 \tag{3}$$

As we will see presently, the precise form of $\delta(t)$ does not matter. For example, we could take $\delta(t)$ to rise linearly from 0 at $t = -\tau$, reach a peak value of $1/\tau$ at $t = 0$, and then fall linearly to 0 at $t = \tau$. For $t < -\tau$ and for $t > \tau$, the function $\delta(t)$ is defined to be zero. Take the limit $\tau \rightarrow 0$, in which this function is known as the delta function. In other words the delta function is an infinitely sharp spike. See figure 1.

The δ function is somehow treated as an advanced topic in mathematical physics, but in fact, as you will see, it is an extremely useful function that I will use extensively in this book, for example in chapters II.1 and III.6. More properties of the δ function will be introduced as needed.

Integrating

$$\frac{d^2x}{dt^2} = \frac{w}{m}\delta(t) \tag{4}$$

from some time $t_- < 0$ to some time $t_+ > 0$, we obtain the change in velocity $v \equiv \frac{dx}{dt}$:

$$v(t_+) - v(t_-) = \frac{w}{m} \tag{5}$$

Note that in this example, neither energy nor momentum is conserved. The lack of conservation is easy to understand: (4) does not include the agent administering the kick. In general, a time-dependent force indicates that the description is not dynamically complete.

28 | I. From Newton to Riemann: Coordinates to Curvature

Example C

A planet approximately described as a point particle of mass m goes around its sun of mass $M \gg m$.

This is of course the celebrated problem Newton solved to unify celestial and terrestrial mechanics, previously thought to be two different areas of physics. His equation now reads

$$m \frac{d^2 \vec{r}}{dt^2} = -GMm \frac{\vec{r}}{r^3} \quad (6)$$

where we use the notation $\vec{r} = (x, y, z)$ and $r = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x^2 + y^2 + z^2}$.

John Wheeler has emphasized the interesting point that while Newton's law (1) tells us how a particle moves in space as a function of time, we tend to think of the trajectory of a particle as a curve fixed in space. For example, when we think of the motion of a planet around the sun, we think of an ellipse rather than a spiral around the time axis. Even in Newtonian mechanics, it is often illuminating to think in terms of a spacetime picture rather than a picture in space.³

Newton and his two distinct masses

By thinking on it continually.

—Newton (reply given when
asked how he discovered
the law of gravity)

Conceptually, in (6), m represents two distinct physical notions of mass. On the left hand side, the inertial mass measures the reluctance of the object to move. On the right hand side, the gravitational mass measures how strongly the object responds to a gravitational field. The equality of the inertial and the gravitational mass was what Galileo tried to verify in his famous apocryphal experiment dropping different objects from the Leaning Tower of Pisa. Newton himself experimented with a pendulum consisting of a hollow wooden box, which he proceeded to fill with different substances, such as sand and water. In our own times, this equality has been experimentally verified^{4,5} to incredible accuracy.

That the same m appears on both sides of the equation turns out to be one of the greatest mysteries in physics before Einstein came along. His great insight was that this unexplained fact provided the clue to a deeper understanding of gravity. At this point, all we care about this mysterious equality is that m cancels out of (6), so that $\ddot{\vec{r}} = -\kappa \frac{\vec{r}}{r^3}$, with $\kappa \equiv GM$.

Celestial mechanics solved

Since the force is “central,” namely it points in the direction of \vec{r} , a simple symmetry argument shows that the motion is confined to a plane, which we take to be the $(x-y)$ plane. Set $z = 0$ and we are left with

$$\ddot{x} = -\kappa x/r^3 \quad \text{and} \quad \ddot{y} = -\kappa y/r^3 \quad (7)$$

I have already, without warning, switched from Leibniz's notation to Newton's dot notation

$$\dot{x} \equiv \frac{dx}{dt} \quad \text{and} \quad \ddot{x} \equiv \frac{d^2x}{dt^2} \quad (8)$$

Since this is one of the most beautiful problems⁶ in theoretical physics, I cannot resist solving it here in all its glory. Think of this as a warm-up before we do the heavy lifting of learning Einstein gravity. Also, later, we can compare the solution here with Einstein's solution.

Evidently, we should change from Cartesian coordinates (x, y) to polar coordinates (r, θ) . We will do it by brute force to show, in contrast, the elegance of the formalism we will develop later. Differentiate

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta \quad (9)$$

twice to obtain first

$$\dot{x} = \dot{r} \cos \theta - r \sin \theta \dot{\theta} \quad \text{and} \quad \dot{y} = \dot{r} \sin \theta + r \cos \theta \dot{\theta} \quad (10)$$

and then

$$\begin{aligned} \ddot{x} &= \ddot{r} \cos \theta - 2\dot{r} \sin \theta \dot{\theta} - r \cos \theta \dot{\theta}^2 - r \sin \theta \ddot{\theta} \\ \text{and} \quad \ddot{y} &= \ddot{r} \sin \theta + 2\dot{r} \cos \theta \dot{\theta} - r \sin \theta \dot{\theta}^2 + r \cos \theta \ddot{\theta} \end{aligned} \quad (11)$$

(Note that in each pair of these equations, the second could be obtained from the first by the substitution $\theta \rightarrow \theta - \frac{\pi}{2}$, so that $\cos \theta \rightarrow \sin \theta$, and $\sin \theta \rightarrow -\cos \theta$.)

Multiplying the first equation in (7) by $\cos \theta$ and the second by $\sin \theta$ and adding, we obtain, using (11),

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\kappa}{r^2} \quad (12)$$

On the other hand, multiplying the first equation in (7) by $\sin \theta$ and the second by $\cos \theta$ and subtracting, we have

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \quad (13)$$

I remind the reader again that we are doing all this in a clumsy brute force way to show the power of the formalism we are going to develop later.

After staring at (13) we recognize that it is equivalent to

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \quad (14)$$

which implies that

$$\dot{\theta} = \frac{l}{r^2} \quad (15)$$

for some constant l . Inserting this into (12), we have

$$\ddot{r} = \frac{l^2}{r^3} - \frac{\kappa}{r^2} = -\frac{dv(r)}{dr} \quad (16)$$

30 | I. From Newton to Riemann: Coordinates to Curvature

where we have defined

$$v(r) = \frac{l^2}{2r^2} - \frac{\kappa}{r} \quad (17)$$

Multiplying (16) by \dot{r} and integrating over t , we have

$$\int dt \frac{1}{2} \frac{d}{dt} \dot{r}^2 = \int dt \dot{r} \ddot{r} = - \int dt \frac{dr}{dt} \frac{dv(r)}{dr} = - \int dr \frac{dv(r)}{dr}$$

so that finally

$$\frac{1}{2} \dot{r}^2 + v(r) = \epsilon \quad (18)$$

with ϵ an integration constant.

This describes a unit mass particle moving in the potential $v(r)$ with energy ϵ . Plot $v(r)$. Clearly, if ϵ is equal to the minimum of the potential $v_{\min} = -\frac{\kappa^2}{2l^2}$, then $\dot{r} = 0$ and r stays constant. The planet follows a circular orbit of radius l^2/κ . If $\epsilon > v_{\min}$ the orbit is elliptical, with r varying between r_{\min} (perihelion) and r_{\max} (aphelion) defined by the solutions to $\epsilon = v(r)$. For $\epsilon > 0$ the planet is not bound and should not even be called a planet.

We have stumbled across two conserved quantities, the angular momentum l and the energy ϵ per unit mass, seemingly by accident. They emerged as integration constants, but surely there should be a more fundamental and satisfying way of understanding conservation laws. We will see in chapter II.4 that there is.

Orbit closes

One fascinating apparent mystery is that the orbit closes. In other words, as the particle goes from r_{\min} to r_{\max} and then back to r_{\min} , θ changes by precisely 2π . To verify that this is so, solve (18) for \dot{r} and divide by (15) to obtain $\frac{dr}{d\theta} = \pm(r^2/l)\sqrt{2(\epsilon - v(r))}$. Changing variable from r to $u = 1/r$, we see, using (17), that $2(\epsilon - v(r))$ becomes the quadratic polynomial $2\epsilon - l^2u^2 + 2\kappa u$, which we can write in terms of its two roots as $l^2(u_{\max} - u)(u - u_{\min})$. Since u varies between u_{\min} and u_{\max} , we are led to make another change of variable from $u = u_{\min} + (u_{\max} - u_{\min}) \sin^2 \zeta$ to ζ , so that ζ ranges from 0 to $\frac{\pi}{2}$. Thus, as the particle completes one round trip excursion in r , the polar angle changes by (note that $u_{\min} = 1/r_{\max}$ and $u_{\max} = 1/r_{\min}$)

$$\begin{aligned} \Delta\theta &= 2 \int_{r_{\min}}^{r_{\max}} \frac{l dr}{r^2 \sqrt{2(\epsilon - v(r))}} = 2 \int_{u_{\min}}^{u_{\max}} \frac{l du}{\sqrt{2\epsilon - l^2u^2 + 2\kappa u}} \\ &= 2 \int_{u_{\min}}^{u_{\max}} \frac{du}{\sqrt{(u_{\max} - u)(u - u_{\min})}} = 4 \int_0^{\frac{\pi}{2}} d\zeta = 2\pi \end{aligned} \quad (19)$$

That this integral turns out to be exactly 2π is at this stage nothing less than an apparent miracle. Surely, there is something deeper going on, which we will reveal in chapter I.4. Note also that the inverse square law is crucial here. Incidentally, the change of variable

here indicates how the Newtonian orbit* (and also the Einsteinian orbit, as we will see in part VI) could be determined. See exercise 2.

Bad notation alert! In (1), the force on the right hand side should be written as $F^i(x(t))$ (in many cases). In C, the gravitational force exists everywhere, namely $F(x)$ exists as a function, and what appears in Newton's equation is just $F(x)$ evaluated at the position of the particle $x(t)$. In contrast, in A, with a mass pulled by a spring, $F(x)$ does not make sense, only $F(x(t))$ does. The force exerted by the spring does not pervade all of space, and hence is defined only at the position of the particle $x(t)$, not at any old x . I can practically hear the reader chuckling, wondering what kind of person I could be addressing here, but believe me, I have encountered plenty of students who confuse these two basic concepts: spatial coordinates and the location of particles. I may sound awfully pedantic, but when we get to curved spacetime, it is often important to be clear that certain quantities are defined only on so-called geodesic curves, while others are defined everywhere in spacetime.

A historical digression on the so-called Newton's constant

Wouldn't we be better off with the two eyes we now have plus a third that would tell us what is sneaking up behind? . . . With six eyes, we could have precise stereoscopic vision in all directions at once, including straight up. A six-eyed Newton might have dodged that apple and bequeathed us some levity rather than gravity.

—George C. Williams⁷

Physics textbooks by necessity cannot do justice to physics history. As you probably know, in the *Principia*, Newton (1642–1727) converted his calculus-based calculations to geometric arguments,⁸ which most modern readers find rather difficult to follow. Here I want to mention another curious point: Newton never did specifically define what we call his constant G . What he did with $ma = GMm/r^2$ was to compare the moon's acceleration with the apple's acceleration: $a_{\text{moon}} R_{\text{lunar orbit}}^2 = GM_{\text{earth}} = a_{\text{apple}} R_{\text{radius of earth}}^2$. But to write $GM_{\text{earth}} = a_{\text{apple}} R_{\text{radius of earth}}^2$, he had to prove what is sometimes referred to as the first of Newton's two "superb theorems," namely that with the inverse square law the gravitational force exerted by a spherical mass distribution acts as if the entire mass were concentrated in a point at the center of the distribution. (See exercise 4.) Even with his abilities, Newton had to struggle for almost 20 years, the length of which contributed to the bitter priority fight he had with Hooke on the inverse square law, with Newton claiming that he had the law a long time before publication. You should be able to do it faster by a factor of $\sim 10^4$ as an exercise.

* On the old one pound note, a portrait of Newton together with his orbits appears on the back. Amusingly, the artist felt compelled to put the sun at the center, rather than one of the foci, of the ellipse.

32 | I. From Newton to Riemann: Coordinates to Curvature

Knowing the moon's period and $R_{\text{lunar orbit}}$, Newton could calculate a_{moon} . Since $R_{\text{radius of earth}}$ had been known since antiquity, he was thus able to calculate a_{apple} and obtained agreement* with Galileo's measurement of a_{apple} . This of course represents one of the most magnificent advances in physics history, with Newton unifying⁹ the previously disparate subjects of celestial and terrestrial mechanics in one stroke. I don't have space to dwell on this here, but I do want to call your attention to the fact that Newton did not need to know G and M_{earth} to perform his feat.

Indeed, G was not measured until 1798 by Henry Cavendish (1731–1810) using equipment built and designed by his friend John Michell (1724–1793), now of black hole fame, who died before he could carry out the experiment.

Needless to say, what I have presented here should only be regarded as a comic book version of history.

Appendix 1: Where is hell?

You will find it in this appendix, sort of.

Curiously, contrary to what some textbooks and popular books stated, Cavendish's goal was not to measure G , but M_{earth} and hence the earth's density. Why this was of more interest to physicists of the time than G is in itself another interesting tidbit in physics history.

I mentioned that Newton had two superb theorems and that the first triggered his feud with Hooke. His second superb theorem states that there is no gravitational force inside a spherical shell.¹⁰ Are you curious why Newton would even attack such a problem? An erroneous calculation had convinced him that the earth was much less dense than the moon, which led his friend Edmond Halley (1656–1742), who by the way published the *Principia* at his expense, to propose the hollow earth theory.¹¹ Witness the popularity of the idea in science fiction, notably Jules Verne's *Journey to the Center of the Earth* (1864). The idea may seem absurd to us, but at that time, a location for hell had to be found, and leading physicists gave serious thought to this pressing problem. Every epoch in physics has its own top ten problems.

So now we understand Cavendish's interest in M_{earth} and hence in the density of the earth rather than in G . Some textbooks give the impression that people easily obtained M_{earth} by multiplying the density of rock and the volume of the earth. Not so easy if you think that the earth might be hollow! We learn from Newton's second theorem that there is no gravitational force in hell, so the usual portrayal of the leaping flames can't be right!

Appendix 2: Fear of indices

Occasionally, a student or two would profess, unaccountably, a "fear of indices." In fact, there is nothing to fear.¹² At this stage, just stand back and admire how clever the invention of indices is. Instead of giving names to each coordinate axis, such as x , y , and z , we could pass fluidly between different dimensions by writing x^i , with $i = 1, 2, \dots, D$. The length of the alphabet we use does not limit us, and we could easily go beyond 26 dimensions.

When we get to Einstein's theory, there will be a flood of indices, and we will have to distinguish between upper and lower indices. In Newtonian mechanics, there is no significance to whether we write the index as a superscript or a subscript. Have no fear: we will discuss each of these features of indices when the need arises. At this point, we merely note that a quantity can carry more than one index. In the text, we wrote x^i_a , with $i = 1, 2, \dots, D$ labeling the different spatial directions, and $a = 1, 2, \dots, N$ labeling the different particles. We will encounter more examples as we go along.

* Newton's first try did not lead to excellent agreement, because the value for the earth's equatorial radius was off. Just a reminder that physics never progresses as smoothly as textbooks say.

With only slight exaggeration, we could say that the invention of indices represents one of the really clever ideas¹³ in the history of physics and mathematics, almost a “magic trick” that enables us to deal with as many particles in as many spatial dimensions as we like with the mere addition of some subscripts and superscripts.

Exercises

- 1 Show that for some suitably smooth function $f(x)$, the integral $\int_{-\infty}^{\infty} dx \delta(x) f(x) = f(0)$. Then show that $\delta(ax) = \delta(x)/|a|$ by evaluating the integral $\int_{-\infty}^{\infty} dx \delta(ax) f(x)$ for some smooth function $f(x)$.
- 2 Determine the orbit $r(\theta)$ by changing variable from r to $u = 1/r$. We will need the result of this exercise later.
- 3 Newton thought that light consists of “corpuscles.” Calculate the deflection of light by the sun, applying what you learned in the text to the case $\epsilon > 0$. Note that the mass of these minute “particles of light” drops out in Newtonian theory anyway. We will need this result to compare with Einstein’s theory later in chapter VI.3.
- 4 Prove Newton’s first superb theorem: the gravitational force exerted by a spherical mass distribution acts as if the entire mass were concentrated in a point at the center of the distribution.
- 5 Prove Newton’s second superb theorem.
- 6 Suppose engineers can build a straight tunnel connecting two cities on earth. Then we could have a free unpowered “gravity express”¹⁴ by simply dropping a railroad car into the tunnel, allowing it to fall from one city to the other. Use Newton’s two superb theorems to calculate the transit time.

Notes

1. Also introduced by Cauchy, Poisson, Hermite, Kirchoff, Kelvin, Helmholtz, and Heaviside. See J. D. Jackson, *Am. J. Phys.* 76 (2008), pp. 707–709.
2. Rigorous mathematicians go berserk at physicists’ use of the word “function” here; they prefer to call it a distribution, defined as the limit of a function. But working physicists do not give a flying barf about such niceties. In any case, I do not personally know a theoretical physicist suffering any harm by calling $\delta(t)$ a function.
3. Consider a game of tennis. Compare a hard drive down the line and a soft lob high over the net. In both cases, we are to solve Newton’s law $\frac{d^2x}{dt^2} = 0$, $\frac{d^2y}{dt^2} = -g$, with the boundary conditions $x(0) = 0$, $x(T) = L$, and $y(0) = y(T) = 0$. (The problem is so elementary that we won’t bother to explain the notation, that y denotes the vertical direction, that $y = 0$ is the ground, that T is the time of flight before the ball hits the ground, that L is the length of the tennis court, and so on and so forth. You might want to draw your own figure.) The solution is $x = Lt/T$, $y = \frac{1}{2}g(T - t)t$. Note that the two types of tennis shots are governed by the same equation and the same L . Hence we obtain the same solution, but keep in mind that T is small in the case of the hard drive and that T is large in the case of the soft lob. Now eliminate t to obtain y as a function of x , namely $y(x) = \frac{1}{2}gT^2(1 - \frac{x}{L})^2/L$, a parabola in both cases (of course). But compare the curvature of the two parabolas: we have $\frac{d^2y}{dx^2} = -g(T/L)^2$, very small in the case of the hard drive (small T) and very large in the case of the lob (large T). The hard drive down the line barely skimming over the net, and the soft lob climbing lazily high up into the sky, look and feel totally different pictured in space. In contrast, consider y as a function of t . We also have two parabolas (of course), namely $y(t) = \frac{1}{2}g(T - t)t$, as given earlier. Now compare the curvature of the two parabolas: we have $\frac{d^2y}{dt^2} = -g$, the same in both cases. The curvature of the ball’s trajectory in spacetime is universal (universal gravity, get it?). But we tend to see in our mind’s eye the two parabolas $y(x)$ in space, one for the hard drive and one for the lob, which look quite different, rather than the parabolas $y(t)$ in spacetime, which have the same curvature. I learned this long ago from John Wheeler.
4. Currently to one part in 10^{13} . The modern round of experiments started with Loránd Eötvös in 1885 and continues with the Eöt-Wash experiment led by E. Adelberger in our days.

34 | I. From Newton to Riemann: Coordinates to Curvature

5. The equality of the gravitational and inertial mass of the neutron has also been verified to good accuracy using neutron interferometry.
6. For Newton's letter to Halley about Hooke on the inverse square, see P. J. Nahin, *Mrs. Perkins's Electric Quilt*, Princeton University Press, 2009.
7. G. C. Williams, *The Pony Fish's Glow*, Basic Books, 1997, p. 128.
8. S. Chandrasekhar, *Newton's Principia for the Common Reader*, Oxford University Press, 2003.
9. *Fearful*, pp. 74–75.
10. For a popular account, see *Toy/Universe*.
11. N. Kollerstrom, "The Hollow World of Edmond Halley," *J. Hist. Astronomy* 23 (1992), p. 185.
12. Surely most readers are familiar with indices. My son the biologist informs me that even biologists use indices routinely; for example, on p. 20 of *Genetics and Analysis of Quantitative Traits* by M. Lynch and B. Walsh, indices appear without explanation or apology.
13. A colleague told me to mention that indices are crucial in computer programming, something that many readers can relate to.
14. *Toy/Universe*, p. xxix.

1.2 Conservation Is Good

An integrability condition

Conservation has been important to physics from day one.¹ In this chapter, we discuss the origin of various conservation laws in Newtonian mechanics.

The most important case is when the force F^i depends only on x and can be written in the form

$$F^i(x) = -\frac{\partial V(x)}{\partial x^i} \quad (1)$$

for $i = 1, 2, \dots, D$. As we all learned, $V(x)$ is called the potential.

Suppose such a function $V(x)$ exists; then a clever person might have the insight to multiply each of Newton's equations

$$m \frac{d^2 x^i}{dt^2} = F^i = -\frac{\partial V(x)}{\partial x^i} \quad (2)$$

by $\frac{dx^i}{dt}$ to obtain the D equations

$$m \frac{d^2 x^i}{dt^2} \frac{dx^i}{dt} = -\frac{\partial V(x)}{\partial x^i} \frac{dx^i}{dt}, \quad \text{with } i = 1, \dots, D \quad (3)$$

He or she would then recognize that the sum of these D equations could be written as

$$\frac{d}{dt} \left[\frac{1}{2} m \sum_i \left(\frac{dx^i}{dt} \right)^2 + V(x) \right] = 0 \quad (4)$$

which we could verify by explicit differentiation. Lo and behold, the total energy, defined by

$$E = \frac{1}{2} m \sum_i \left(\frac{dx^i}{dt} \right)^2 + V(x) \quad (5)$$

is conserved. It does not change in time.

36 | I. From Newton to Riemann: Coordinates to Curvature

For $D = 1$, (1) holds automatically: $V(x)$ is simply given by $-\int^x dx' F(x')$. For $D > 1$, the D equations in (1), namely $F^i(x) = -\frac{\partial V(x)}{\partial x^i}$, imply the consistency or integrability condition

$$\frac{\partial F^i(x)}{\partial x^j} = \frac{\partial F^j(x)}{\partial x^i} \quad (6)$$

(Since derivatives commute, both sides of (6) are equal to $-\frac{\partial^2 V(x)}{\partial x^i \partial x^j}$.) Thus, given $F^i(x)$, we merely have to check to see whether (6) holds. If not, then V does not exist. If yes, then we could integrate $F^i(x) = -\frac{\partial V(x)}{\partial x^i}$ for each i to determine V .

Apples do not fall down

Suppose $V(r)$ depends only on $r \equiv \left(\sum_{i=1}^D (x^i)^2\right)^{\frac{1}{2}}$. In other words, the potential does not pick out any preferred direction. We take this for granted nowadays, but it represents one of the most astonishing insights of physics.² Newton realized that the apple did not fall down, but toward the center of the earth.

Differentiating $r^2 = \sum_{i=1}^D (x^i)^2$, we obtain $r dr = \sum_i x^i dx^i$ (an “identity,” which we will use again and again in this text) or $\frac{\partial r}{\partial x^j} = \frac{x^j}{r}$, so that

$$F^i = -\frac{x^i}{r} V'(r) \quad \text{and} \quad \frac{\partial F^i(x)}{\partial x^j} = -\frac{1}{r} [\delta^{ij} V'(r) + \frac{x^i x^j}{r^2} (-V'(r) + r V''(r))]$$

which is manifestly symmetric under $i \leftrightarrow j$.

Here we have introduced the Kronecker delta δ^{ij} , defined by

$$\delta^{kj} = 1 \text{ if } k = j, \quad \delta^{kj} = 0 \text{ if } k \neq j \quad (7)$$

(which we can think of as an ancestor of the Dirac delta function³ introduced in chapter I.1).

The important point is not the somewhat involved expression for $\frac{\partial F^i(x)}{\partial x^j}$, but that it is a linear combination of δ^{ij} and $x^i x^j$. We haven’t talked about tensors yet (see chapter I.4), but this result could have been anticipated by a “what else can it be?” type of argument. Not having any preferred direction, we could only construct an object with indices i and j out of δ^{ij} and $x^i x^j$. We could have seen immediately that the integrability condition (6) holds.

Note that this discussion holds for any value of D .

Conservation of angular momentum

Suppose the force in (2) points toward the center, so that it has the form $F^i = f(r)x^i$ (with $f(r) = -V'(r)/r$, as we just saw). Then we obtain angular momentum conservation immediately. To see this, multiply Newton’s equation (2)

$$m \frac{d^2 x^i}{dt^2} = f(r)x^i \quad (8)$$

I.2. Conservation Is Good | 37

by x^j , so that $m \frac{d^2 x^i}{dt^2} x^j = f(r) x^i x^j$. Subtract from this the same equation but with i and j interchanged. Regardless of the function $f(r)$, we find

$$x^j \frac{d^2 x^i}{dt^2} - x^i \frac{d^2 x^j}{dt^2} = 0 \quad (9)$$

But this is the same as

$$\frac{d}{dt} \left(x^j \frac{dx^i}{dt} - x^i \frac{dx^j}{dt} \right) = 0 \quad (10)$$

Clever, eh? I am constantly amazed by how brilliant early physicists were.

The quantity $l^{ij} \equiv \left(x^j \frac{dx^i}{dt} - x^i \frac{dx^j}{dt} \right)$, the angular momentum per unit mass, is conserved. Recall that in the preceding chapter, this fact seemingly fell out when we changed to polar coordinates. Note also that the argument given here holds for any $D \geq 2$.

Exercise

- 1 Let N particles interact according to

$$m_a \frac{d^2 x_a^i}{dt^2} = - \frac{\partial V(x)}{\partial x_a^i} \quad (11)$$

with $a = 1, \dots, N$. Suppose $V(x_1, \dots, x_N)$ depends only on the differences $x_a^i - x_b^i$, with $a, b = 1, \dots, N$. Show that the total momentum $\sum_a m_a \frac{dx_a^i}{dt}$ is conserved.

Notes

1. *Fearful.*
2. I once explained this point to humanists using Einstein's terminology by saying that "The words up and down have no place in the Mind of the Creator." See A. Zee, *New Lit. Hist.* 23 (1992), pp. 815–838. See also web.physics.ucsb.edu/jatila/supplements/zee_lecture.pdf.
3. In the sense that $\delta(x - y)$ is zero for $x \neq y$.

1.3

Rotation: Invariance and Infinitesimal Transformation

Rotation in the plane

My pedagogical strategy for this chapter is to take something you know extremely* well, namely rotations in the plane, present it in a way possibly unfamiliar to you, and go through it slowly in great detail, “beating the subject to death,” so to speak.

I have already mentioned that Monsieur Descartes had the clever idea of reducing geometry to algebra. Put down Cartesian coordinate axes so that a point P is labeled by two real numbers (x, y) . Suppose another observer (call him Mr. Prime) puts down coordinate axes rotated by angle θ with respect to the axes put down by the first observer (call her Ms. Unprime) but sharing the same origin O. Elementary trigonometry tells us that the coordinates (x, y) and (x', y') assigned by the two observers to the same point P are related by[†] (see figure 1)

$$x' = \cos \theta x + \sin \theta y, \quad y' = -\sin \theta x + \cos \theta y \quad (1)$$

The distance from P to the origin O of course has to be the same for the two observers. According to Pythagoras, this requires $\sqrt{x'^2 + y'^2} = \sqrt{x^2 + y^2}$, which you can check using (1).

Introduce the column vectors $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{r}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ and the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (2)$$

so that we can write (1) more compactly as $\vec{r}' = R(\theta)\vec{r}$.

* If you don't know rotations in the plane extremely well, then perhaps you are not ready for this book. A nodding familiarity with matrices and linear algebra is among the prerequisites.

[†] For example, by comparing similar triangles in the figure, we obtain $x' = (x/\cos \theta) + (y - x \tan \theta) \sin \theta$.

I.3. Rotation: Invariance and Infinitesimal Transformation | 39

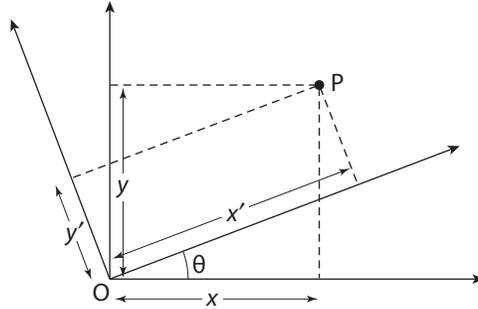


Figure 1 The same point P is labeled by (x, y) and (x', y') , depending on the observer's frame of reference.

As you recall from a course on mechanics, we can either envisage rotating the physical body we are studying or rotating the observer. We will consistently rotate the observer.

We have already used the word “vector.” A vector is a physical quantity (for example the velocity of a particle in the plane) consisting of two real numbers, so that if Ms. Unprime represents it by $\vec{p} = \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}$, then Mr. Prime will represent it by $\vec{p}' = R(\theta)\vec{p}$. In short, a vector is something that transforms like the coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$ under rotation.

Given two vectors $\vec{p} = \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix}$, the scalar or dot product is defined by $\vec{p}^T \cdot \vec{q} = p^1 q^1 + p^2 q^2$. Here T stands for transpose and \vec{p}^T the row vector (p^1, p^2) . By definition, rotations leave $\vec{p}^2 \equiv \vec{p}^T \cdot \vec{p} = (p^1)^2 + (p^2)^2$ invariant. In other words, if $\vec{p}' = R(\theta)\vec{p}$, then $\vec{p}'^2 = \vec{p}^2$. Since this works for any vector \vec{p} , including the case in which \vec{p} happens to be the sum of two vectors $\vec{p} = \vec{u} + \vec{v}$, and since $\vec{p}^2 = (\vec{u} + \vec{v})^2 = \vec{u}^2 + \vec{v}^2 + 2\vec{u}^T \cdot \vec{v}$, rotation also leaves the dot product between two arbitrary vectors invariant: the invariance of \vec{p}^2 implies that $\vec{u}'^T \cdot \vec{v}' = \vec{u}^T \cdot \vec{v}$.

Since $\vec{u}' = R\vec{u}$ (to unclutter things, we often suppress the θ dependence in $R(\theta)$) and so $\vec{u}'^T = \vec{u}^T R^T$, we now have $\vec{u}'^T \cdot \vec{v}' = \vec{u}'^T \cdot (R\vec{v}) = (\vec{u}^T R^T) \cdot (R\vec{v}) = \vec{u}^T \cdot (R^T R)\vec{v}$. (The transpose M^T of a matrix M is of course obtained by interchanging the rows and columns of M .) As this holds for any two vectors \vec{u} and \vec{v} , we must have the matrix equation

$$R^T R = I \tag{3}$$

where, as usual, I denotes the identity or unit matrix: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Indeed, we could verify (3) explicitly:

$$R(\theta)^T R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

Matrices that satisfy (3) are called orthogonal.

Taking the determinant of (3), we obtain $(\det R)^2 = 1$, that is, $\det R = \pm 1$. The determinant of an orthogonal matrix may be -1 as well as $+1$. In other words, orthogonal matrices also include reflection matrices, such as $\mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, a reflection in the y -axis.

40 | I. From Newton to Riemann: Coordinates to Curvature

To focus on rotations, let us exclude reflections by imposing the condition (since $\det \mathcal{P} = -1$)

$$\det R = 1 \tag{5}$$

Matrices with unit determinant are called special.

We define a rotation as a matrix that is both orthogonal and special, that is, a matrix that satisfies both (3) and (5). Thus, the rotation group of the plane consists of the set of all special orthogonal 2 by 2 matrices and is known as $SO(2)$.

Note that matrices of the form $\mathcal{P}R$ for any rotation R are also excluded by (5), since $\det(\mathcal{P}R) = \det \mathcal{P} \det R = (-1)(+1) = -1$. In particular, a reflection in the x -axis $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which is the product of \mathcal{P} and a rotation through 90° , is also excluded.

Act a little bit at a time

The Norwegian physicist Marius Sophus Lie (1842–1899) had the almost childish obvious but brilliant idea that to rotate through, say, 29° , you could just as well rotate through a zillionth of a degree and repeat the process 29 zillion times. To study rotations, it suffices to study rotation through infinitesimal angles. Shades of Newton and Leibniz! A rotation through a finite angle could always be obtained by performing infinitesimal rotations repeatedly. As is typical with many profound statements in physics and mathematics, Lie’s idea is astonishingly simple. Replace the proverb “Never put off until tomorrow what you have to do today” by “Do what you have to do a little bit at a time.”

When the angle is small enough, the rotation is almost the identity, that is, no rotation at all. Thus, we can write

$$R(\theta) \simeq I + A \tag{6}$$

where A denotes some infinitesimal matrix.

Now suppose we have never seen (2). Indeed, suppose we have never even heard of sine and cosine. Instead, let us define rotations as the set of linear transformations on 2-component objects $\vec{u}' = R\vec{u}$ and $\vec{v}' = R\vec{v}$ that leave $\vec{u}' \cdot \vec{v}'$ invariant. Following Lie, we solve this condition on R , namely (3) $R^T R = I$, by considering an infinitesimal transformation $R(\theta) \simeq I + A$. Since by assumption, A^2 can be neglected relative to A , $R^T R \simeq (I + A^T)(I + A) \simeq (I + A^T + A) = I$. We thus obtain $A^T = -A$, namely that A must be antisymmetric. But there is basically only one 2-by-2 antisymmetric matrix:

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{7}$$

In other words, the solution of $A^T = -A$ is $A = \theta \mathcal{J}$ for some real number θ . Thus, rotations close to the identity have the form $R = I + \theta \mathcal{J} + O(\theta^2) = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} + O(\theta^2)$. The antisymmetric matrix \mathcal{J} is known as the generator of the rotation group.

An equivalent way of saying this is that for infinitesimal θ , the transformation $x' \simeq x + \theta y$ and $y' \simeq y - \theta x$ (you could verify that (1) indeed reduces to this to leading order in

I.3. Rotation: Invariance and Infinitesimal Transformation | 41

θ) obviously satisfies the Pythagorean condition $x'^2 + y'^2 = x^2 + y^2$ to first order in θ . Or, write $x' = x + \delta x$, $y' = y + \delta y$ and solve $x\delta x + y\delta y = 0$.

Alternatively, simply draw figure 1 for θ infinitesimal. Since we know the transformation is linear, we could determine the matrix R in (6) by looking at the figure to see what happens to the two points $(x = 1, y = 0)$ and $(x = 0, y = 1)$ under an infinitesimal rotation.

Now recall the identity $e^x = \lim_{N \rightarrow \infty} (1 + \frac{x}{N})^N$ (which you can easily prove by differentiating both sides). Then, for a finite (that is, not infinitesimal) angle θ , we have

$$R(\theta) = \lim_{N \rightarrow \infty} R\left(\frac{\theta}{N}\right)^N = \lim_{N \rightarrow \infty} \left(1 + \frac{\theta \mathcal{J}}{N}\right)^N = e^{\theta \mathcal{J}} \quad (8)$$

The first equality represents Lie's profound idea. For the last equality, we use the identity just mentioned, which amounts to the definition of the exponential.

Some readers may not be familiar with the exponential of a matrix. Given a well-behaved function f with a power series expansion, we can define $f(M)$ for an arbitrary matrix M using that power series. For example, define $e^M \equiv \sum_{n=0}^{\infty} M^n/n!$; since we know how to multiply and add matrices, this series makes perfect sense. (Whether or not any given series converges is of course another issue.) We must be careful, however, in using various identities that may or may not generalize. For example, the identity $e^a e^a = e^{2a}$ for a a real number, which we could prove by applying the binomial theorem to the product of two series (square of a series in this case) generalizes immediately. Thus, $e^M e^M = e^{2M}$. But for two matrices M_1 and M_2 that do not commute with each other, $e^{M_1} e^{M_2} \neq e^{M_1+M_2}$.

This provides an alternative but of course equivalent path to our result. To leading order, we have every right to write $R\left(\frac{\theta}{N}\right) = 1 + \frac{\theta \mathcal{J}}{N} \simeq e^{\frac{\theta \mathcal{J}}{N}}$ and thus $R(\theta) = R\left(\frac{\theta}{N}\right)^N = e^{\theta \mathcal{J}}$.

Finally, we easily check that the formula $R(\theta) = e^{\theta \mathcal{J}}$ reproduces (2) for any value of θ . We simply note that $\mathcal{J}^2 = -I$ and separate the exponential series into even and odd terms. Thus

$$\begin{aligned} e^{\theta \mathcal{J}} &= \sum_{n=0}^{\infty} \theta^n \mathcal{J}^n / n! = \left(\sum_{k=0}^{\infty} (-1)^k \theta^{2k} / (2k)! \right) I + \left(\sum_{k=0}^{\infty} (-1)^k \theta^{2k+1} / (2k+1)! \right) \mathcal{J} \\ &= \cos \theta I + \sin \theta \mathcal{J} = \cos \theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \end{aligned} \quad (9)$$

which is precisely $R(\theta)$ as given in (2). Note this works because \mathcal{J} plays the same role as i in the identity $e^{i\theta} = \cos \theta + i \sin \theta$.

Poor Lie, he never made it into the 20th century.

Two approaches to rotation

Notice that I actually gave you two different approaches to rotation. Let us summarize the two approaches. In the first approach, applying trigonometry to figure 1, we write down (1) and hence (2). In the second approach, we specify what is to be left invariant by rotations and hence define rotations by the condition (3) that rotations must satisfy. Lie then tells us that it suffices to solve (3) for infinitesimal rotations. We could then build up rotations

42 | I. From Newton to Riemann: Coordinates to Curvature

through finite angles by multiplying infinitesimal rotations together, thus also arriving at (2).

It might seem that the first approach is much more direct. One writes down (2) and that is that. The second approach appears more roundabout and involves some “fancy math.” It might even provoke an adherent of the first “more macho” approach to wisecrack, “Why, with a bit of higher education, sine and cosine are not good enough for you any more? You have to go around doing fancy math!” The point is that the second approach generalizes to higher dimensional spaces (and to other situations) much more readily than the first approach does, as we will see presently. Dear reader, in going through life, you would be well advised to always separate fancy but useful math from fancy but useless math.

Before we go on, let us take care of one technical detail. We assumed that Mr. Prime and Ms. Unprime set up their coordinate systems to share the same origin O . We now show that this condition is unnecessary if we consider two points P and Q (rather than one point, as in our discussion above) and study how the vector connecting P to Q transforms.

Let Ms. Unprime assign the coordinates $\vec{r}_P = (x, y)$ and $\vec{r}_Q = (\tilde{x}, \tilde{y})$ to P and Q , respectively. Then Mr. Prime’s coordinates $\vec{r}'_P = (x', y')$ for P and $\vec{r}'_Q = (\tilde{x}', \tilde{y}')$ for Q are then given by $\vec{r}'_P = R(\theta)\vec{r}_P$ and $\vec{r}'_Q = R(\theta)\vec{r}_Q$. Subtracting the first equation from the second and defining $\Delta x = \tilde{x} - x$, $\Delta y = \tilde{y} - y$, and the corresponding primed quantities, we obtain

$$\begin{pmatrix} \Delta x' \\ \Delta y' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \quad (10)$$

Rotations leave the distance between the points P and Q unchanged: $(\Delta x')^2 + (\Delta y')^2 = (\Delta x)^2 + (\Delta y)^2$. You recognize of course that this is a lot of tedious verbiage stating the perfectly obvious, but I want to be precise here. Of course, the distance between any two points is left unchanged by rotations. (This also means that the distance between P and the origin is left unchanged by rotations; ditto for the distance between Q and the origin.)

Invariance and geometry

There is no royal road to geometry.
—Euclid’s advice to a prince

Let no one unversed in geometry enter here.
—Plato’s motto, carved over the
entrance to his academy

Let us take the two points P and Q to be infinitesimally close to each other and replace the differences $\Delta x'$, Δx , and so forth by differentials dx' , dx , and so forth. Indeed, 2-dimensional Euclidean space is defined by the distance squared between two nearby points:

$$ds^2 = dx^2 + dy^2 \quad (11)$$

I.3. Rotation: Invariance and Infinitesimal Transformation | 43

Rotations are defined as linear transformations* $(x, y) \rightarrow (x', y')$, such that

$$dx^2 + dy^2 = dx'^2 + dy'^2 \quad (12)$$

The whole point is that this now makes no reference to the origin O (and whether Mr. Prime and Ms. Unprime even share the same origin).

The column $d\vec{x} = \begin{pmatrix} dx^1 \\ dx^2 \end{pmatrix} \equiv \begin{pmatrix} dx \\ dy \end{pmatrix}$ is defined as the basic or ur-vector, the template for all other vectors. To repeat, a vector is defined as something that transforms like $d\vec{x}$ under rotations.

So, a vector is defined by how it transforms. An array of two numbers $\vec{p} = \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}$ is a vector if it transforms according to $\vec{p}' = R(\theta)\vec{p}$.

Sometimes it is very helpful, in order to understand what something is, to be given an example of something that is not. As a simple example, given a \vec{p} , then $\begin{pmatrix} ap^1 \\ bp^2 \end{pmatrix}$ is definitely not a vector if $a \neq b$. (You could easily write down more outrageous examples, such as $\begin{pmatrix} (p^1)^2 p^2 \\ (p^1)^3 + (p^2)^3 \end{pmatrix}$. That ain't no vector!) You will work out further examples in exercise 1. An array of numbers is not a vector unless it transforms in the right way.¹

Oh, about the advice Euclid gave to the prince who wanted to know a quick way of mastering geometry. Mr. E is also telling you that, to master the material covered in this book, there is no way other than to cogitate over the material until you get it and to work through as many exercises as possible.

From the plane to higher dimensional space

The reader who has wrestled with Euler angles in a mechanics course knows that the analog of (2) for 3-dimensional space is already quite a mess. In contrast, Lie's approach allows us, as mentioned above, to immediately jump to D -dimensional Euclidean space, defined by specifying the distance squared between two nearby points (compare this with (11)), as given by the obvious generalization of Pythagoras' theorem:

$$ds^2 = \sum_{i=1}^D (dx^i)^2 = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^D)^2 \quad (13)$$

This is as good a place as any to say a word about indices. As I said in chapter I.1, in my experience teaching, there are always a couple of students confounded by indices. Dear reader, if you are not, you could simply laugh and skip to the next paragraph. Indices provide a marvelous notational device to save us from having to give names to individual elements belonging to a set. (For example, consider all humans h^i now alive, with $i = 1, 2, \dots, P$ where P denotes the population size.) Take a look at the 19th century physics literature, before the use of indices became widespread. I am always amazed by

* Indeed, most, but not all, of the readers² of this book are constantly rotating between two coordinate systems.

44 | I. From Newton to Riemann: Coordinates to Curvature

the fact that, for example, Maxwell could see through the morass of the electromagnetic equations written out component by component.

Rotations are defined as linear transformations $d\vec{x}' = R d\vec{x}$ that leave ds unchanged. The preceding discussion allows us to write this condition as $R^T R = I$. As before, we want to focus on rotations by imposing the additional condition $\det R = 1$. The set of D -by- D matrices R that satisfy these two conditions forms the simple orthogonal group $SO(D)$, which is just a fancy way of saying the rotation group in D -dimensional space.

Lie in higher dimensions

The power of Lie now shines through when we want to work out rotations in higher dimensional spaces. All we have to do is satisfy the two conditions $R^T R = I$ and $\det R = 1$.

So let us follow Lie and write $R \simeq I + A$. Then $R^T R = I$ is solved by requiring $A = -A^T$, namely that A must be antisymmetric. But it is very easy to write down all possible antisymmetric D -by- D matrices! For $D = 2$, there is basically only one: the \mathcal{J} introduced earlier. For $D = 3$, there are basically three of them:

$$\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{J}_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (14)$$

Any 3-by-3 antisymmetric matrix can be written as $A = \theta_x \mathcal{J}_x + \theta_y \mathcal{J}_y + \theta_z \mathcal{J}_z$, with three real numbers θ_x , θ_y , and θ_z . At this point, you can verify that $R \simeq I + A$, with A as given here, satisfies the condition $\det R = 1$.

The three matrices \mathcal{J}_x , \mathcal{J}_y , \mathcal{J}_z are known as the generators of the 3-dimensional rotation group $SO(3)$. They generate rotations, but are of course not to be confused with rotations, which are by definition 3-by-3 orthogonal matrices with determinant equal to 1.

The upshot of this whole discussion is that any 3-dimensional rotation (not necessarily infinitesimal) can be written as $R(\theta) = e^A$ and is thus characterized by three real numbers. As I said, those readers who have suffered through the rotation of a rigid body in a course on mechanics must appreciate the simplicity of studying the generators of infinitesimal rotations and then simply exponentiating them.

Index notation and rotations

Some readers will find this obvious, but others might find it helpful if we derive the condition $R^T R = I$ explicitly once again using the index notation. I prefer to go slow here, since we will need some of the same formalism later when we get to special relativity. Once the reader feels sure-footed, we could then dispense with indices.

Let me start by reminding the reader that a D -by- D matrix M carries two indices and has entries M^{ij} , with the standard convention that the first index labels the rows, the second the column (for $i, j = 1, 2, \dots, D$). For example, for $D = 2$, $M = \begin{pmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{pmatrix}$, and M^{12} is

I.3. Rotation: Invariance and Infinitesimal Transformation | 45

the entry in the first row and the second column, whereas M^{21} is the entry in the second row and the first column. Note that the transpose of a matrix M is given by $(M^T)^{ji} \equiv M^{ij}$. Thus, if \vec{v} is a column vector with entries v^j , then the entries of the column vector $\vec{u} = M\vec{v}$ are given by $u^i = \sum_j M^{ij}v^j$. For A and B two D -by- D matrices, the product AB is defined as the matrix with the entries $(AB)^{ij} = \sum_k A^{ik}B^{kj}$. (If everything here is news to you, see the first footnote in this chapter.)

Under a rotation,

$$dx^i = \sum_j R^{ij} dx^j = R^{i1} dx^1 + R^{i2} dx^2 + \cdots + R^{iD} dx^D \quad (15)$$

(I have written the sum out explicitly for the benefit of the rare reader afflicted by fear of indices.) Also, as was mentioned in chapter I.1, at this stage it is completely arbitrary whether we write upper or lower indices.

Let us pause and recall the Kronecker delta symbol δ^{ij} introduced in (I.2.7), defined to be equal to +1 if $i = j$ and 0 otherwise, and which we can also think of as a D -by- D unit matrix. We will be encountering the highly useful Kronecker delta often in this book. For example, $\sum_j A^j B^j = \sum_k \sum_j \delta^{kj} A^k B^j$. Since δ^{kj} vanishes unless k is equal to j , the double sum on the right hand side collapses to the single sum on the left hand side. In other words, the Kronecker delta allows us to write a single sum as a double sum. It seems like a really silly thing to do, but as we will see presently, it is an extremely useful trick that we use quite often in this book.

We now determine how the matrix R must be restricted for it to be a rotation. The statement that $ds^2 = \sum_{i=1}^D (dx^i)^2$ as defined in (13) is left unchanged by the rotation implies that (with all indices running over $1, \dots, D$)

$$\sum_i (dx^i)^2 = \sum_i \sum_k \sum_j R^{ik} dx^k R^{ij} dx^j = \sum_j (dx^j)^2 = \sum_k \sum_j \delta^{kj} dx^k dx^j \quad (16)$$

In the last step, we used what we just learned.

Since the infinitesimals dx^i can take on arbitrary values, to have the second term equal to the last term in (16), we must equate the coefficients of $dx^k dx^j$ and demand that

$$\sum_i R^{ik} R^{ij} = \delta^{kj} = \sum_i (R^T)^{ki} R^{ij} = (R^T R)^{kj} \quad (17)$$

Indeed, we obtain $R^T R = I$ just as in (3), but now in D -dimensional space for any D .

We end this section with a trivial remark. So far in this chapter, we have written the column vectors as columns. But columns take up so much space, and so for typographical convenience (editors must be placated!) we will henceforth write the entries of a column vector as $d\vec{x} = (dx^1, dx^2, \dots, dx^D)$, a practice we will indulge in throughout this book. (If we want to be insufferably pedantic, we could put in a T for transpose: the column ur-vector $d\vec{x} = (dx^1, dx^2, \dots, dx^D)^T$.)

46 | I. From Newton to Riemann: Coordinates to Curvature

Einstein's repeated index summation

Observe that in all those sums in (16) the indices to be summed over always appear twice, that is, they are repeated. For example, in the second term in (16), $\sum_i \sum_k \sum_j R^{ik} dx^k R^{ij} dx^j$, the indices i , k , and j all appear repeated. Thus, we could adopt the so-called repeated index summation convention proposed by Albert Einstein himself: omit the pesky summation symbol and agree that if an index is repeated, then it is to be summed over. For example, $dx^i = \sum_j R^{ij} dx^j$ can now be written as $dx^i = R^{ij} dx^j$: in the expression on the right hand side, the index j appears twice and is thus to be summed over.* In contrast, i is a “free” index and does not appear twice in the same expression. Notice that free indices must match on opposite sides of any equation. It is rightly said that one of Einstein's greatest contribution to physics is the repeated index summation convention.† When we get to Einstein gravity, we will meet lots of indices to be summed over, and it would be silly to keep on writing the summation symbol.

Vector fields

The vectors we encounter may well vary in space. For example, the flow velocity in a fluid in general would depend on where we are. We are then dealing with a vector field $\vec{V}(\vec{x})$. Again, consider two observers studying the same vector field. Mr. Prime would see

$$\vec{V}'(\vec{x}') = R\vec{V}(\vec{x}) \quad (18)$$

with $\vec{x}' = R\vec{x}$ of course. In other words, the two observers are studying the same vector field at the same point P. See figure 2. As another example, the familiar electric $\vec{E}(\vec{x})$ and magnetic fields $\vec{B}(\vec{x})$ are both vector fields.

Physics should not depend on the observer

Let me stress again why physicists constantly talk about vectors. The laws of physics often involve the statement that one vector is equal to another, for example, Newton's law states $m\vec{a} = \vec{F}$. Applying a rotation matrix $R(\theta)$, we obtain $mR(\theta)\vec{a} = R(\theta)\vec{F}$. If \vec{F} transforms like a vector, then $m\vec{a}' = \vec{F}'$. Ms. Unprime and Mr. Prime see the same Newton's law, and more generally, the same laws of physics!

This statement, while self-evident, is profound, and in some sense, it is what makes physics possible. Physics should not depend on the physicist. Ms. Unprime and Mr. Prime

* When a pair of repeated indices, such as j here, is summed over, they are often said to be contracted with each other. In a tiny abuse of terminology, people also say that R^{ij} is contracted with dx^j .

† It appeared only in his later work. In 1905, Einstein did not even use vector notation! In one system, the coordinates were denoted by x, y, z , in the other, by ξ, η, ζ ; the components of the force acting on the electron were called X, Y, Z . To modern eyes, his notation was a horrific mess.

I.3. Rotation: Invariance and Infinitesimal Transformation | 47

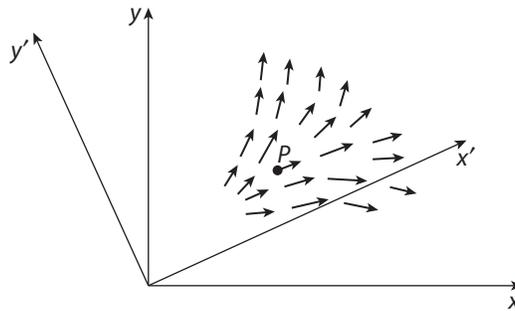


Figure 2 Two observers studying the same vector field.

see different accelerations \vec{a} and \vec{a}' , and different forces \vec{F} and \vec{F}' , but the same Newton's law. We say that Newton's law is invariant—that is, it does not change—under rotation.*

We should also remind ourselves that mass is an example of a scalar: a physical quantity that does not change under rotation. If it does change, Newton's law would not be invariant under rotation and one observer would be preferred over another, which is unacceptable. Physics rests on the democratic ideal.

Let me remind you that the gravitational force in the planetary problem studied in chapter I.1 is derived from what is sometimes called a central potential, namely one without a preferred direction: $F^i(x) = -\frac{\partial}{\partial x^i} V(r) = -\frac{x^i}{r} V'(r)$. Hence, \vec{F} is proportional to \vec{x} and so a fortiori transforms like a vector.

At this point, it may be worthwhile to be a bit more pedantic and professorial. Some authors give long-winded speeches about covariance versus invariance, and take great pain to distinguish the two. We should too. The equation $m\vec{a} = \vec{F}$ is covariant, that is, the two sides transform the same way under rotations. The physics expressed by Newton's second law is, however, invariant, that is, independent of observers related by a rotation. If physics depends on how you tilt your head, we are in trouble. Physics does not, but the way physics is expressed, in terms of equations, does.

Here is the profound and trivial statement. Under a certain set of transformations, a purportedly fundamental equation is said to be covariant if the two sides of the equation transform in the same way. If so, then that transformation is known as a symmetry of physics.³ Physics is said to be invariant under that transformation. As we will see, both sides of Einstein's field equation transform in the same way, as tensors, under what are known as general coordinate transformations. I will explain what a tensor is in the next chapter. I will allow myself the luxury of using the words invariance and covariance interchangeably and simply trust you to be discerning.

Since we can always move the quantity on the right hand side of an equation to the left hand side, we can rewrite a physical law of the form $\vec{u} = \vec{v}$ in the form $\vec{w} \equiv \vec{u} - \vec{v} = 0$. Physics students sometimes joke that they could already write down the ultimate

* The reader who has already been exposed to the special theory of relativity knows that this notion of invariance represents the essence of Einstein's insight. We will of course have a great deal more to say about that!

48 | I. From Newton to Riemann: Coordinates to Curvature

equation of physics, namely $\mathcal{X} = 0$, whatever \mathcal{X} is. Thus, the statement of invariance merely expresses the mathematically obvious fact that if $\vec{w} = 0$, then $R(\theta)\vec{w} = 0$. (Strictly speaking, the 0 on the right hand side should be written as $\vec{0}$, but we don't want to be that pedantic!)

Descartes versus Euclid

I remember how excited I was when I learned about analytic geometry. Surely you were excited too. What a genius, that Descartes! Henceforth, we could prove geometric theorems by doing algebra. After Descartes,⁴ physics can no longer live without the concept of coordinates,* but he also managed to obscure what was once obvious to Euclid. We now must also insist on invariance. Indeed, the notion of invariance is at the heart of what we mean by geometry.

For example, suppose somebody hands you a formula for the area of a triangle with vertices at (a_1, b_1) , (a_2, b_2) , (a_3, b_3) . You better insist that the formula is invariant under rotation. In fact, this requirement, plus the requirement that the area should scale as the square of the separation between the three vertices, suffices to determine the formula. This simple example rings in the central motif of this book.

Appendix 1: Differential operators rather than matrices

Here I have to divide readers into the haves and the have-nots, but only temporarily. What I will say may sound difficult, but really, it amounts to not much more than a notational triviality.

If you have studied quantum mechanics, you would know that the generators \mathcal{J} of rotation studied here are related to angular momentum operators. You would also know that in quantum mechanics, observables are represented by hermitean operators. However, in our discussion, the \mathcal{J} s come out naturally as antisymmetric matrices and are thus antihermitean. To make them hermitean, we multiply them by some multiples of i .

If you have not studied quantum mechanics, then the preceding would sound like gibberish to you, but do not worry. Simply take the attitude that, hey, it is a free country, and we can always invite ourselves to define a new set of physical quantities by multiplying an existing set of physical quantities by some constant. Heck, we could multiply by $\sqrt{17}i$ if we want.

Even though here we are nowhere near quantum mechanics, we will bow to customary usage and define $J_x \equiv -i\mathcal{J}_x$ and so forth. From (14) we see that, for example, J_z acting on the column vector (x, y, z) gives $i(y, -x, 0)$. Thus, instead of using matrices, we could also represent J_z by $i(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})$, since $J_z x = i(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})x = iy$, $J_z y = i(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})y = -ix$, and $J_z z = i(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})z = 0$. Note that J_z is precisely the z -component of the angular momentum operators in quantum mechanics. We can naturally pass back and forth between matrices and differential operators. We will not make use of this differential representation until a later chapter.

* Regarding the argument (which I mentioned in a footnote in the preface) between those who live with coordinates and those who live coordinate free, I would say that the proof of angular momentum conservation, which I already gave, not once, but twice in the two preceding chapters using coordinates, provides an example in favor of the latter group: $\frac{d}{dt}\vec{l} = \frac{d}{dt}(\vec{r} \times \vec{p}) = m\frac{d}{dt}(\vec{r} \times \frac{d\vec{r}}{dt}) = m\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + m\vec{r} \times \frac{d^2\vec{r}}{dt^2} = 0$ for rotationally symmetric potentials. While this indeed looks simpler than the two previous discussions, the former group could also say that this requires learning “considerable formal math,” such as the cross product and its various properties.

Appendix 2: Rotations in higher dimensional space

Here we discuss rotations in D -dimensional Euclidean space. As you have no doubt heard, Einstein combined space and time into a 4-dimensional spacetime. Thus, what you will learn here about $SO(4)$ will be put to good use.* If you prefer, you could skip this discussion and come back to it later.

Start with a D -by- D matrix with 0 everywhere. Generalize (14). Stick a 1 into the m th row and n th column, and a (-1) into the n th row and m th column. Call this matrix $J_{(mn)}$. We put the subscripts (mn) in parentheses to emphasize that (mn) labels the matrix. They are not indices to tell us which element of the matrix we are talking about. As explained before, we define $J_{(mn)} = -i\mathcal{J}_{(mn)}$ so that explicitly

$$J_{(mn)}^{ij} = -i(\delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni}) \quad (19)$$

To repeat, in the symbol $J_{(mn)}^{ij}$, the indices i and j indicate respectively the row and column of the entry $J_{(mn)}^{ij}$ of the matrix $J_{(mn)}$, while the indices m and n , which I put in parentheses for pedagogical clarity, indicate which matrix we are talking about. The first index m on $J_{(mn)}$ can take on D values, and then the second index n can take on only $(D - 1)$ values since, obviously, $J_{(mm)} = 0$. Also, since $J_{(mn)} = -J_{(nm)}$, we require $m > n$ to avoid double counting. Thus, there are only $\frac{1}{2}D(D - 1)$ real antisymmetric D -by- D matrices $J_{(mn)}$, and A could be written as a linear combination of them: $A = i \sum_{m>n} \theta_{mn} J_{(mn)}$, where θ_{mn} denote $\frac{1}{2}D(D - 1)$ real numbers. (As a check, for $D = 2$ and 3, $\frac{1}{2}D(D - 1)$ equals 1 and 3, respectively.) The matrices $J_{(mn)}$ are known as the generators of the group $SO(D)$.

Notice a notational peculiarity: for $SO(3)$, the J s could be labeled with one index rather than two indices. The reason is simple. In this case, the indices m, n take on 3 values, and so we could write $J_x = J_{23}$, $J_y = J_{31}$, and $J_z = J_{12}$. We will, as we do here, often pass freely between the index sets (123) and (xyz) . In general, rotations are labeled by the plane they occur in, say the $(m-n)$ plane spanned by the m th and n th axes. In 3-dimensional space, and only in 3-dimensional space, a plane is uniquely specified by the vector perpendicular to it. Thus, a rotation commonly spoken of as a rotation around the z -axis is better thought of as a rotation in the $(1-2)$ plane, that is, the $(x-y)$ plane. (In this connection, note that the \mathcal{J} in (7) appears as the upper left 2-by-2 block in \mathcal{J}_z in (14).) In contrast, for $SO(4)$ it makes no sense to speak of a rotation around, say, the third axis.

The reader who has studied some group theory knows that the essence of the group is captured by the extent to which the multiplication of two group elements does not commute. For rotations, everyday observations show that $R(\theta)R(\theta')$ is in general quite different from $R(\theta')R(\theta)$. See figure 3.

Following Lie, we could try to capture this essence by focusing on infinitesimal rotations. Let $R_1 \simeq I + A$ and $R_2 \simeq I + B$. Then $R_1R_2 \simeq (I + A)(I + B) \simeq I + A + B + AB + O(A^2, B^2)$ (where rather pedantically we have indicated that to the desired order if we keep AB , we should also keep terms of order $O(A^2, B^2)$, but we will see immediately that they are irrelevant). If we multiply in the other order, we simply interchange A and B , thus $R_2R_1 \simeq (I + A)(I + B) \simeq I + B + A + BA + O(A^2, B^2)$. Hence, R_1R_2 and R_2R_1 differ by the amount $[A, B] \equiv AB - BA$, a quantity known as the commutator between A and B .

More formally, given two matrices X and Y , to measure how they differ from each other, we could ask how $X^{-1}Y$ differs from the identity. If $X = Y$, then this product is equal to the identity. Now, the inverse of a matrix $I + A$ infinitesimally close to the identity is easy to determine: it is just $I - A$, since $(I - A)(I + A) = I + O(A^2)$. Thus, let us calculate $(R_2R_1)^{-1}R_1R_2$:

$$\begin{aligned} (R_2R_1)^{-1}R_1R_2 &= [I - (B + A + BA + O(A^2, B^2))][I + A + B + AB + O(A^2, B^2)] \\ &= I + [A, B] + \dots \end{aligned} \quad (20)$$

For $SO(3)$, for example, A is a linear combination of the J_i s, known as the generators of the Lie algebra. Thus, we could write $A = i \sum_i \theta_i J_i$ and similarly $B = i \sum_j \theta'_j J_j$. Hence $[A, B] = i^2 \sum_{ij} \theta_i \theta'_j [J_i, J_j]$, and so it suffices to calculate the commutators $[J_i, J_j]$.

Recall that for two matrices M_1 and M_2 , $(M_1M_2)^T = M_2^T M_1^T$. Transpose reverses the order. Thus $([J_i, J_j])^T = -[J_i, J_j]$. In other words, the commutator $[J_i, J_j]$ is itself an antisymmetric 3-by-3 matrix and thus could be written as a linear combination of the J_k s:

$$[J_i, J_j] = i c_{ijk} J_k \quad (21)$$

* Higher dimensional rotation groups often pop up in the most unlikely places in theoretical physics. For example, $SO(4)$ is relevant for a deeper understanding of the spectrum of the hydrogen atom.⁵

50 | I. From Newton to Riemann: Coordinates to Curvature

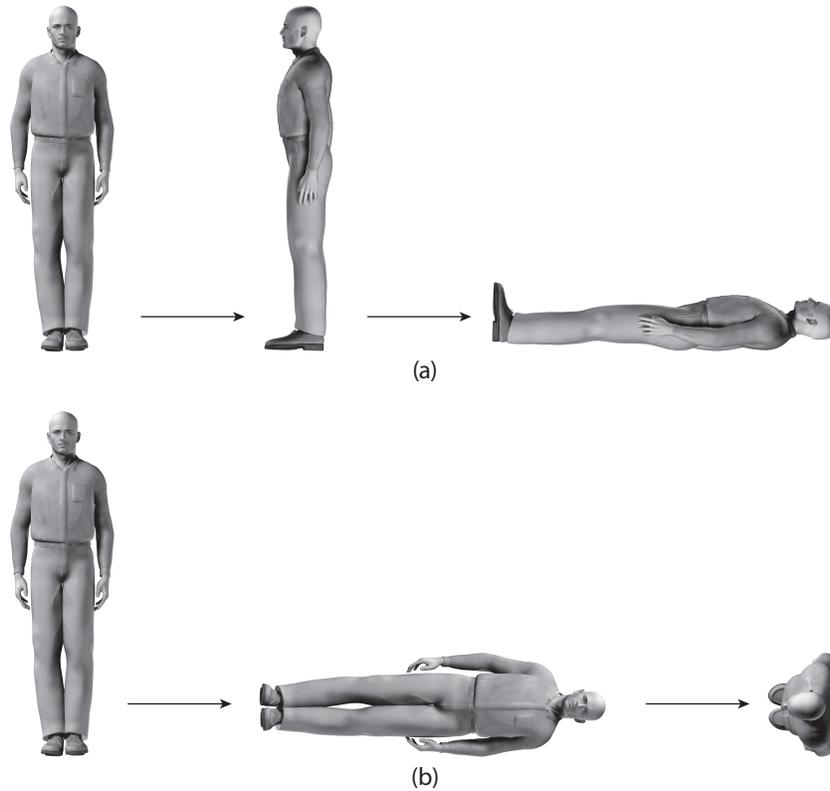


Figure 3 A marine recruit in a boot camp is standing and facing north. When the drill sergeant shouts, “Rotate by 90° eastward around the vertical axis” our recruit turns to face east. Suppose the sergeant next shouts, “Rotate by 90° westward around the north-south axis.” Our recruit ends up lying down on his back with his head pointing west, his feet pointing east. But what would happen if the sergeant reverses his two commands? You could easily verify that our recruit now ends up lying down on his left elbow, with his head pointing north. The order matters. For this reason, the study of rotations has been a *bête noire* for generations of physics students.

for a set of real (convince yourself of this!) numbers c_{ijk} . The summation over k is implied by the repeated index summation convention.

By explicit computation using (14), we find

$$[J_x, J_y] = iJ_z \tag{22}$$

You should work out the other commutators or argue by cyclic substitution $x \rightarrow y \rightarrow z \rightarrow x$. The three commutation relations may be summarized by

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{23}$$

We define the totally antisymmetric symbol ϵ_{ijk} by saying that it changes sign upon the interchange of any pair of indices (and hence it vanishes when any two indices are equal) and by specifying that $\epsilon_{123} = 1$. In other words, we found that $c_{ijk} = \epsilon_{ijk}$.

Lie’s great insight is that the preceding discussion holds for any group whose elements are labeled by a set of continuous parameters (such as θ_i , $i = 1, 2, 3$ in the case of $SO(3)$), groups now known as Lie groups. Expanding the group elements around the origin, we arrive at (20) and hence the structure (21) for any continuous group. The set of all commutation relations of the form (21) is said to define a Lie algebra, with c_{ijk} referred to as the

I.3. Rotation: Invariance and Infinitesimal Transformation | 51

structure constants of the algebra. The matrices J_i are called the generators of the Lie algebra. The idea is that by studying the Lie algebra, we go a long way toward understanding the group.

You should now work out (exercise 4), starting from (19), the Lie algebra for $SO(D)$:

$$[J_{(mn)}, J_{(pq)}] = i(\delta_{mp}J_{(nq)} + \delta_{nq}J_{(mp)} - \delta_{np}J_{(mq)} - \delta_{mq}J_{(np)}) \quad (24)$$

This may look rather involved to the uninitiated, but in fact it is quite simple. First, the right hand side, a linear combination of the J s, as required by the general argument above, is completely fixed by the first term by noting that the left hand side is antisymmetric under three separate interchanges: $m \leftrightarrow n$, $p \leftrightarrow q$, and $(mn) \leftrightarrow (pq)$. Next, all those Kronecker deltas just say that if the two sets (mn) and (pq) have no integer in common, then the commutator vanishes. If they do have an integer in common, you simply “cross off” that integer. This is best explained by using $SO(4)$ as an example. We have $[J_{(12)}, J_{(34)}] = 0$, $[J_{(12)}, J_{(14)}] = iJ_{(24)}$, $[J_{(23)}, J_{(31)}] = -iJ_{(21)} = iJ_{(12)}$, and so forth. The first of these relations says that rotations in the (1-2) plane and in the (3-4) plane commute, as you might expect. Do write down a few more and you will get it.

Exercises

- Suppose we are given two vectors \vec{p} and \vec{q} in ordinary 3-dimensional space. Consider this array of three numbers: $\begin{pmatrix} p^2q^3 \\ p^3q^1 \\ p^1q^2 \end{pmatrix}$. Prove that it is not a vector, even though it looks like a vector. (Check how it transforms under rotation!) In contrast, $\begin{pmatrix} p^2q^3 - p^3q^2 \\ p^3q^1 - p^1q^3 \\ p^1q^2 - p^2q^1 \end{pmatrix}$ does transform like a vector. It is in fact the vector cross product $\vec{p} \times \vec{q}$.

- Show that the product of two delta functions $\delta(x)\delta(y)$ is invariant under rotation around the origin.

- Using (14) show that a rotation around the x -axis through angle θ_x is given by

$$R_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_x & \sin \theta_x \\ 0 & -\sin \theta_x & \cos \theta_x \end{pmatrix}$$

Write down $R_y(\theta_y)$. Show explicitly that $R_x(\theta_x)R_y(\theta_y) \neq R_y(\theta_y)R_x(\theta_x)$.

- Calculate $[J_{(mn)}, J_{(pq)}]$.

- Given a 3-vector \vec{p} , show that the quantity $\vec{p}^i \vec{p}^j$ when averaged over the direction of \vec{p} is given by $\frac{1}{4\pi} \int d\theta d\varphi \cos \theta \vec{p}^i \vec{p}^j = \frac{1}{3} \vec{p}^2 \delta^{ij}$.

Notes

- Outside of physics, people often erroneously call any array of numbers a vector. Of course, people are free to call anything anything, so let’s not quibble about the word “erroneously.”
- I say “most, but not all,” because it is conceivable that you are a native speaker of Guugu Yimithirr. See G. Deutscher, *Through the Language Glass*, H. Holt and Co., 2010, p. 161.
- The intellectual precision of our definition of symmetry is necessary lest we make the same mistake as the ancient Greeks. See *Fearful*, pp. 11–12 and figure 2.2.
- According to one story, take it or leave it, Descartes was lying in bed when he noticed a fly buzzing around the room. He then realized that he could fix the fly’s position given how far the fly was from two intersecting walls and the ceiling.
- For example, J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, pp. 265–268.

1.4 Who Is Afraid of Tensors?

A tensor is something that transforms like a tensor

Long ago, an undergrad who later became a distinguished condensed matter physicist came to me after a class on group theory and asked me, “What exactly is a tensor?” I told him that a tensor is something that transforms like a tensor. When I ran into him many years later, he regaled me with the following story. At his graduation, his father, perhaps still smarting from the hefty sum he had paid to the prestigious private university his son attended, asked him what was the most memorable piece of knowledge he acquired during his four years in college. He replied, “A tensor is something that transforms like a tensor.”

But this should not perplex us. A duck is something that quacks like a duck. Mathematical objects could also be defined by their behavior. We already saw in the preceding chapter that a vector is defined by how it transforms: $V^i = R^{ij} V^j$. Consider a collection of “mathematical entities” T^{ij} with $i, j = 1, 2, \dots, D$ in D -dimensional space. If they transform under rotations according to

$$T^{ij} \rightarrow T'^{ij} = R^{ik} R^{jl} T^{kl} \quad (1)$$

then we say that T transforms like a tensor, and hence is a tensor. (Here we are using the Einstein summation convention introduced in the previous chapter: The right hand side actually means $\sum_{k=1}^D \sum_{l=1}^D R^{ik} R^{jl} T^{kl}$ and is a sum of D^2 terms.) Indeed, we see that we are just generalizing the transformation law of a vector.

Fear of tensors

In my experience teaching, a couple of students are invariably confused by the notion of tensors. The very word “tensor” apparently make them tense. Dear reader, if you are not one of these unfortunates, so much the better for you! You could zip through this chapter. But to allay the nameless fear of the tensorphobe, I will go slow and be specific.

I.4. Who Is Afraid of Tensors? | 53

Think of the tensor T^{ij} as a collection of D^2 mathematical entities that transform into linear combinations of one another. To help the reader focus, I will often specialize to $D = 3$. Compounded and intertwined with their fear of tensors, the unfortunates mentioned above are also unaccountably afraid of indices, as mentioned in chapter I.1. For them, let us list T^{ij} explicitly for $D = 3$. There are $3^2 = 9$ of them: $T^{11}, T^{12}, T^{13}, T^{21}, T^{22}, T^{23}, T^{31}, T^{32}, T^{33}$. That's it, 9 objects that transform into linear combinations of one another. For example, (1) says that $T'^{21} = R^{2k} R^{1l} T^{kl} = R^{21} R^{11} T^{11} + R^{21} R^{12} T^{12} + R^{21} R^{13} T^{13} + R^{22} R^{11} T^{21} + R^{22} R^{12} T^{22} + R^{22} R^{13} T^{23} + R^{23} R^{11} T^{31} + R^{23} R^{12} T^{32} + R^{23} R^{13} T^{33}$. This shows explicitly, as if there were any doubt to begin with, that T'^{21} is given by a particular linear combination of the 9 objects. That's all: the tensor T^{ij} consists of 9 objects that transform into linear combinations of themselves under rotations.

We could generalize further and define* 3-indexed tensors, 4-indexed tensors, and so forth by such transformation laws as $W'^{ijn} = R^{ik} R^{jl} R^{nm} W^{klm}$. Here we will focus on 2-indexed tensors, and if we say tensor without any qualifier, we often, but not always, mean a 2-indexed tensor. With this definition, we might say that a vector is a 1-indexed tensor and a scalar is a 0-indexed tensor, but this usage is not common. A scalar transforms as a tensor with no index at all, namely $S' = S$; in other words, a scalar does not transform.

Tensor field

In the preceding chapter, we introduced the notion of a vector field $V^i(\vec{x})$, nothing more or less than a vector function of position. That it is a vector means that it transforms according to $V'^i(\vec{x}') = R^{ij} V^j(\vec{x})$. Now consider the derivative of this vector field $\frac{\partial V^i(\vec{x})}{\partial x^k}$, which we will call $W^{kj}(\vec{x})$.

Use the fact that $\vec{x}' = R\vec{x}$ implies $\vec{x} = R^{-1}\vec{x}' = R^T\vec{x}'$ and thus $\frac{\partial x^k}{\partial x'^h} = (R^T)^{kh} = R^{hk}$. (The O in the rotation group $SO(D)$ is crucial: the inverse of a rotation is its transpose.) Then

$$\frac{\partial}{\partial x'^h} = \frac{\partial x^k}{\partial x'^h} \frac{\partial}{\partial x^k} = R^{hk} \frac{\partial}{\partial x^k} \quad (2)$$

Thus

$$W'^{hi}(\vec{x}') \equiv \frac{\partial V^i(\vec{x}')}{\partial x'^h} = R^{hk} \frac{\partial}{\partial x^k} (R^{ij} V^j(\vec{x})) = R^{hk} R^{ij} \frac{\partial V^j(\vec{x})}{\partial x^k} = R^{hk} R^{ij} W^{kj}(\vec{x}) \quad (3)$$

Comparing with (1) we see that $W^{kj}(\vec{x})$ transforms like a tensor and, hence, is a tensor. Indeed, it is a tensor field.

Notice that a tensor T^{ij} transforms as if it were composed of two vectors $v^i w^j$, that is, T^{ij} and $v^i w^j$ transform in the same way. (Compare $v^i w^j \rightarrow v'^i w'^j = R^{ik} v^k R^{jl} w^l = R^{ik} R^{jl} v^k w^l$ with (1).) It is important to recognize that only in exceptional cases does a tensor T^{ij} happen to be equal to $v^i w^j$ for some v and w . In general, a tensor cannot be

* Our friend the Jargon Guy tells us that the number of indices carried by a tensor is known as its rank. (The Jargon Guy is a new friend of the author; he did not appear in *QFT Nut*.)

54 | I. From Newton to Riemann: Coordinates to Curvature

written in the form $v^i w^j$. Our tensor field $W^{kj}(\vec{x})$ offers a ready example: in general, it is not equal to some vector U^k multiplied by $V^j(\vec{x})$.

Also, note in our example that the differential operator $\frac{\partial}{\partial x^k}$ transforms (2) like a vector. For example, if $\phi'(x') = \phi(x)$ transforms like a scalar, then $\frac{\partial \phi}{\partial x^k}$ transforms like a vector. Indeed, that's why you have encountered the notation $\vec{\nabla}$ for the gradient in an elementary physics course. This remark will be important later when we revisit Newton's inverse square law in chapter II.3. Do exercise 1 now.

Representation theory

Go back to the 9 objects T^{ij} that form a tensor. Mentally arrange them in a column

$$\begin{pmatrix} T^{11} \\ T^{12} \\ \vdots \\ T^{33} \end{pmatrix}$$

The linear transformation on the 9 objects can then be represented by a 9-by-9 matrix $\mathcal{D}(R)$ acting on this column. (Here we are going painfully slowly because of common confusion on this point. Some authors refer to this column as a 9-component "vector," which is a horrible abuse of terminology. We reserve the word "vector" for something that transforms like a vector $V^i = R^{ij} V^j$. It is not true that any old collection of stuff arranged in a column is a vector. Don't call anything with feathers a duck!)

For every rotation, specified by a 3-by-3 matrix R , we could thus associate a 9-by-9 matrix $\mathcal{D}(R)$ transforming the 9 objects T^{ij} linearly among themselves. We say that the 9-by-9 matrix $\mathcal{D}(R)$ represents the rotation matrix R in the sense that

$$\mathcal{D}(R_1)\mathcal{D}(R_2) = \mathcal{D}(R_1 R_2) \tag{4}$$

Multiplication of $\mathcal{D}(R_1)$ and $\mathcal{D}(R_2)$ mirrors the multiplication of R_1 and R_2 , as it were. The tensor T is said to furnish a 9-dimensional representation of the rotation group $SO(3)$. The 9-by-9 matrices $\mathcal{D}(R)$ represent R . Notice that with this jargon, the vector furnishes a 3-dimensional representation of the rotation group, known as the defining or fundamental representation.

Reducible versus irreducible

Let us now pose the central question of representation theory. Given these 9 entities T^{ij} that transform into each other, consider the 9 independent linear combinations that we can form out of them. Is there a subset among them that only transform into each other? A secret in-club, as it were.

A moment's thought reveals that there is indeed an in-club. Consider $A^{ij} \equiv T^{ij} - T^{ji}$. Under a rotation,

I.4. Who Is Afraid of Tensors? | 55

$$\begin{aligned} A^{ij} \rightarrow A'^{ij} &= T'^{ij} - T'^{ji} = R^{ik} R^{jl} T^{kl} - R^{jk} R^{il} T^{kl} \\ &= R^{ik} R^{jl} T^{kl} - R^{jl} R^{ik} T^{lk} = R^{ik} R^{jl} (T^{kl} - T^{lk}) = R^{ik} R^{jl} A^{kl} \end{aligned} \quad (5)$$

I have again gone painfully slow here, but it is obvious, isn't it? We just verified in (5) that A^{ij} transforms like a tensor and is thus a tensor. Furthermore, this tensor changes sign upon interchange of its two indices ($A^{ij} = -A^{ji}$) and is said to be antisymmetric. The transformation law (1) treats the two indices democratically, without favoring one over the other, and thus preserves the antisymmetric character of a tensor: if $A^{ij} = -A^{ji}$, then $A'^{ij} = -A'^{ji}$ also.

Let us count. The index i in A^{ij} could take on D values; for each of these values, the index j could take on only $D - 1$ values (since the D diagonal elements $A^{ii} = 0$ for $i = 1, 2, \dots, D$, no Einstein repeated index summation here); but to avoid double counting (since $A^{ij} = -A^{ji}$) we should divide by 2. Hence, the number of independent components in A is equal to $\frac{1}{2}D(D - 1)$. For example, for $D = 3$, we have the 3 objects: A^{12} , A^{23} , and A^{31} . The attentive reader would recall that we did the same counting in the previous chapter.

Obviously, the same goes for the symmetric combination $S^{ij} \equiv T^{ij} + T^{ji}$. You could verify as a trivial exercise that $S^{ij} = R^{ik} R^{jl} S^{kl}$. A tensor S^{ij} that does not change sign upon interchange of its two indices ($S^{ij} = S^{ji}$) is said to be symmetric. Evidently, the symmetric tensor S has more components than the antisymmetric tensor A . In addition to the components S^{ij} with $i \neq j$, S also has D diagonal components, namely $S^{11}, S^{22}, \dots, S^{DD}$. Thus, the number of independent components in S is equal to $\frac{1}{2}D(D - 1) + D = \frac{1}{2}D(D + 1)$.

For $D = 3$, the number of components in A and S are $\frac{1}{2} \cdot 3 \cdot 2 = 3$ and $\frac{1}{2} \cdot 3 \cdot 4 = 6$, respectively. (For $D = 4$, the number of components in A and S are 6 and 10, respectively.) Thus, in a suitable basis, the 9-by-9 matrix referred to above actually breaks up into a 3-by-3 block and a 6-by-6 block. We say that the 9-dimensional representation is reducible: it could be reduced to smaller representations.

But we are not done yet. The 6-dimensional representation is also reducible. To see this, note

$$S'^{ii} = R^{ik} R^{il} S^{kl} = (R^T)^{ki} R^{il} S^{kl} = (R^{-1})^{ki} R^{il} S^{kl} = \delta^{kl} S^{kl} = S^{kk} \quad (6)$$

where we have used the O in $SO(D)$. (Here we are using repeated index summation: the indices i and k are both summed over.) In other words, the linear combination $S^{11} + S^{22} + \dots + S^{DD}$, the trace of S , transforms into itself, that is, does not transform at all. It is a loner forming an in-club of one. The 6-by-6 matrix describing the linear transformation of the 6 objects S^{ij} breaks up into a 1-by-1 block and a 5-by-5 block. See figure 1.

Again, for the sake of the beginning student, let us work out explicitly the 5 objects that furnish the representation 5 of $SO(3)$. First define a traceless symmetric tensor \tilde{S} by

$$\tilde{S}^{ij} = S^{ij} - \delta^{ij} (S^{kk} / D) \quad (7)$$

(The repeated index k is summed over.) Explicitly, $\tilde{S}^{ii} = S^{ii} - D(S^{kk} / D) = 0$, and \tilde{S} is traceless. Specialize to $D = 3$. Now we have only 5 objects, namely $\tilde{S}^{11}, \tilde{S}^{22}, \tilde{S}^{12}, \tilde{S}^{13}, \tilde{S}^{23}$.

56 | I. From Newton to Riemann: Coordinates to Curvature

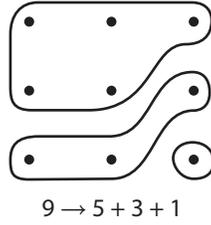


Figure 1 How the collection of 9 objects T^{ij} splits up. The figure is meant to be schematic: the dots do not represent the original 9 objects, but linear combinations of them, and the positions of the dots are not meaningful.

We do not count \tilde{S}^{33} separately, since it is equal to $-(\tilde{S}^{11} + \tilde{S}^{22})$. Under an $SO(3)$ rotation, these 5 objects transform into linear combinations of one another, as we just explained.

Let us be specific: the object \tilde{S}^{13} , for example, transforms into $\tilde{S}'^{13} = R^{1k}R^{3l}\tilde{S}^{kl} = R^{11}R^{31}\tilde{S}^{11} + R^{11}R^{32}\tilde{S}^{12} + R^{11}R^{33}\tilde{S}^{13} + R^{12}R^{31}\tilde{S}^{21} + R^{12}R^{32}\tilde{S}^{22} + R^{12}R^{33}\tilde{S}^{23} + R^{13}R^{31}\tilde{S}^{31} + R^{13}R^{32}\tilde{S}^{32} + R^{13}R^{33}\tilde{S}^{33} = (R^{11}R^{31} - R^{13}R^{33})\tilde{S}^{11} + (R^{11}R^{32} + R^{12}R^{31})\tilde{S}^{12} + (R^{11}R^{33} + R^{13}R^{31})\tilde{S}^{13} + (R^{12}R^{32} - R^{13}R^{33})\tilde{S}^{22} + (R^{12}R^{33} + R^{13}R^{32})\tilde{S}^{23}$, where in the last equality, we used $\tilde{S}^{ij} = \tilde{S}^{ji}$ and $\tilde{S}^{33} = -(\tilde{S}^{11} + \tilde{S}^{22})$. Indeed, \tilde{S}^{13} transforms into a linear combination of \tilde{S}^{11} , \tilde{S}^{22} , \tilde{S}^{12} , \tilde{S}^{13} , \tilde{S}^{23} .

To summarize, what we found is that if, instead of the basis consisting of the 9 entities T^{ij} , we use the basis consisting of the 3 entities A^{ij} , the single entity S^{kk} (remember repeated index summation!), and the 5 entities \tilde{S}^{ij} , the 9-by-9 matrix $\mathcal{D}(R)$ (that represents rotation in the sense of (4)) breaks up into a 3-by-3 matrix, a 1-by-1 matrix, and a 5-by-5 matrix “stacked on top of each other.” This is represented schematically as

$$\mathcal{D}(R) = (9\text{-by-}9 \text{ matrix}) \rightarrow \left[\begin{array}{c|c|c} (3\text{-by-}3 \text{ block}) & 0 & 0 \\ \hline 0 & (1\text{-by-}1 \text{ block}) & 0 \\ \hline 0 & 0 & (5\text{-by-}5 \text{ block}) \end{array} \right] \quad (8)$$

Note that once we chose the new basis, this decomposition holds true for all rotations. (For the readers who know their linear algebra, the technical statement is that there exists a similarity transformation that block-diagonalizes $\mathcal{D}(R)$ for all R . Incidentally, we will encounter plenty of similarity transformations later.)

More generally, the D^2 representation furnished by a general 2-indexed tensor decomposes into a $\frac{1}{2}D(D - 1)$ -dimensional representation, a $(\frac{1}{2}D(D + 1) - 1)$ -dimensional representation, and a 1-dimensional representation. We say that in $SO(3)$, $9 = 5 + 3 + 1$. (In $SO(4)$, $16 = 9 + 6 + 1$.)

You might have noticed that in this entire discussion we never had to write out R explicitly in terms of the 3 rotation angles and how the 5 objects $\tilde{S}^{11}, \dots, \tilde{S}^{23}$ transform into one another in terms of these angles. It is only the counting that matters. You might regard that as the difference between mathematics and arithmetic.

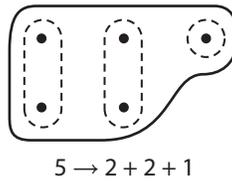


Figure 2 Under $SO(3)$, the 5 objects inside the solid line transform into linear combinations of each other, but under the smaller group of transformations $SO(2)$, the objects inside each of the 3 dashed lines transform into linear combinations of each other. The 5 breaks up as $5 \rightarrow 2 + 2 + 1$. As in figure 1, this figure is meant to be schematic.

Restriction to a subgroup

You definitely do not have to master group theory¹ to read this book, but it would be useful for you to learn a few basic concepts and to be able to count. For instance, the notion of a subgroup. Consider the group $SO(2)$ that we studied to exhaustion, consisting of rotations around the z -axis, say. Evidently, $SO(2)$ is a subgroup of $SO(3)$ in that its elements are all elements of $SO(3)$ and form a group all by themselves. The components of the 3-vector V^i could be split into two sets: (V^1, V^2) and V^3 . Under a rotation around the z -axis, (V^1, V^2) transform as a 2-vector and V^3 as a scalar. We say that upon restriction to the subgroup $SO(2)$, the irreducible representation 3 breaks up into the representations 2 and 1 of the subgroup, a decomposition we write as $3 \rightarrow 2 + 1$. All the group theoretic results we need in this book could be obtained by explicit listing and simple counting.

Look at the 5 objects, $S^{11}, S^{22}, S^{12}, S^{13}, S^{23}$, that furnish the representation 5 of $SO(3)$. Now consider a restriction to the subgroup $SO(2)$. In other words, we restrict ourselves to rotations around the z -axis, that is, rotations under which $V^3 \rightarrow V^3 = V^3$, namely rotations with $R^{33} = 1$ and $R^{13}, R^{23}, R^{31}, R^{32}$ all vanishing. Since $SO(2)$ does not touch the index 3, we conclude immediately that the combination $S^{11} + S^{22} = -S^{33}$ does not transform, or in other words, it transforms as a singlet under $SO(2)$. Similarly, the pair (S^{13}, S^{23}) transforms as a doublet, since the index 3 is “invisible” to $SO(2)$: the group transforms the indices 1 and 2 into each other, while leaving the index 3 alone. Indeed, we see that our earlier expression for S'^{13} collapses to $S'^{13} = R^{11}S^{13} + R^{12}S^{23}$, as expected. Finally, you can verify that the remaining combinations $(S^{12}, S^{11} - S^{22})$ transform like a doublet. These results could be summarized by saying that, upon restriction to the subgroup $SO(2)$, the irreducible representation 5 of the group $SO(3)$ breaks up as $5 \rightarrow 2 + 2 + 1$. See figure 2.

Tensors in Newtonian mechanics

Let us give another example, particularly apt for a book on gravity, of a Newtonian tensor. Consider two nearby particles moving in a potential. Denote their trajectories by $\vec{x}(t)$

58 | I. From Newton to Riemann: Coordinates to Curvature

and $\vec{y}(t)$, respectively, determined by $\frac{d^2x^i}{dt^2} = -\partial^i V(\vec{x})$ and $\frac{d^2y^i}{dt^2} = -\partial^i V(\vec{y})$. (I am also testing whether there are any readers who do not understand thoroughly the concept of notational freedom.) We want to know how the separation vector $\vec{s} \equiv \vec{y} - \vec{x}$ changes with time, keeping terms to leading order in \vec{s} :

$$\frac{d^2s^i}{dt^2} = \frac{d^2y^i}{dt^2} - \frac{d^2x^i}{dt^2} = -\partial^i[V(\vec{y}) - V(\vec{x})] = -\partial^i[V(\vec{x} + \vec{s}) - V(\vec{x})] \simeq -\partial^i\partial^j V(\vec{x})s^j$$

The object $\mathcal{R}^{ij}(\vec{x}) \equiv \partial^i\partial^j V(\vec{x})$ is manifestly a tensor if $V(\vec{x})$ is a scalar. For example, verify that $\mathcal{R}^{ij} = GM(\delta^{ij}r^2 - 3x^ix^j)/r^5$ for the gravitational potential $V(\vec{x}) = -GM/r$. Note that \mathcal{R}^{ij} is a symmetric traceless tensor. Since $\mathcal{R}^{ii} = \partial^i\partial^i V(\vec{x}) = \vec{\nabla}^2 V$, the tracelessness merely reaffirms the fact that the $1/r$ potential satisfies Laplace's equation $\vec{\nabla}^2 V = 0$. Also, \mathcal{R}^{ij} is manifestly not the product of two vectors, but it transforms as if it were.

Let us see how rotational covariance works in the equation

$$\frac{d^2s^i}{dt^2} = -\mathcal{R}^{ij}s^j \tag{9}$$

The right hand side has to be linear in the vector \vec{s} . Since the left hand side transforms like a vector, the right hand side must also: indeed, it is given by a tensor \mathcal{R} contracted* with a vector \vec{s} . A tensor is needed on the right hand side.

Imagine yourself falling toward a spherical planet or star. With no loss of generality, let your location at some instant be $(0, 0, r)$ along the z -axis. The tensor \mathcal{R} written out as a matrix is then diagonal and is given by (for example, $\mathcal{R}^{33} = GM(\delta^{33}r^2 - 3x^3x^3)/r^5 = GM(1 - 3)/r^3$)

$$\mathcal{R} = \frac{GM}{r^3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{10}$$

Thus, the sign of $\frac{d^2s}{dt^2}$ depends on the orientation of \vec{s} .

To see why this is so and to understand what tensors are all about, imagine surrounding yourself with a circular arrangement of balls lying in the $(x-z)$ plane (see figure 3a) and initially at rest in your frame. Using (9) and (10), we can now write down how the separation between two balls along different directions changes.

Since we are going to specify the direction, we will denote the separation simply by s . Along the z -axis, s grows according to (see (9)) $\frac{d^2s}{dt^2} = -\mathcal{R}^{33}s = +2\frac{GM}{r^3}s$. The plus sign indicates that the two balls move away from each other. In contrast, along the x -axis, s decreases according to $\frac{d^2s}{dt^2} = -\mathcal{R}^{11}s = -\frac{GM}{r^3}s$. The two balls approach each other. (Similarly for two balls aligned along the y -axis.) (Note that acting on \vec{s} on the right hand side of (9) by a tensor makes it possible for $\frac{d^2s}{dt^2}$ to change sign depending on the orientation of \vec{s} .)

Inspecting figure 3a, you see why. Look at it as an observer on the planet. In the first case, one of the two balls, being closer to the planet, is falling faster than the other. Thus, they

* When a pair of repeated indices, such as j in (9), is summed over, they are often said to be contracted with each other (as mentioned in a footnote in the preceding chapter) in the sense that this index no longer appears in the result, as shown by the left hand side of (9).

I.4. Who Is Afraid of Tensors? | 59

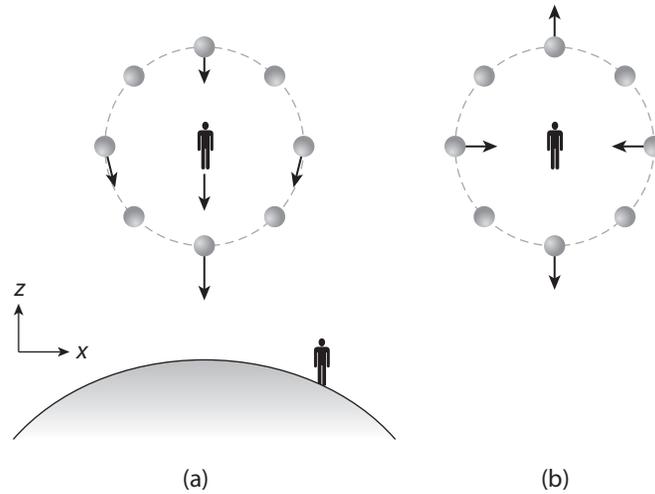


Figure 3 A falling ring of balls as seen by an observer on the planet (a), and as seen by an observer falling with the balls (b).

are moving away from each other. In the second case, the two balls are coming closer due to spherical symmetry: they are both heading toward the center of the planet. As Newton pointed out, objects do not fall down to earth, but toward the center of the earth.

In your rest frame (figure 3b) as you fall along with the balls, however, you see a tidal force acting on the circular ring (or a spherical shell if you prefer) of balls. The force appears to stretch the ring in the z -direction and to squeeze it in the orthogonal direction. When we come to Einstein's prediction of gravitational waves in chapter IX.4, we will see that gravitational waves act on the detector according to equations analogous to (9) and (10). Note also for future reference that the tidal force $\mathcal{R}^{ij}(\vec{x}) \equiv \partial^i \partial^j V(\vec{x})$ involves two derivatives acting on the gravitational potential $V(\vec{x})$.

Invariant tensors

In D -dimensional space, define the antisymmetric symbol $\varepsilon^{ijk\dots n}$ carrying D indices to have the following properties:

$$\varepsilon^{\dots l\dots m\dots} = -\varepsilon^{\dots m\dots l\dots} \quad \text{and} \quad \varepsilon^{12\dots D} = 1 \quad (11)$$

In other words, the antisymmetric symbol ε flips sign upon the interchange of any pair of indices. It follows that ε vanishes when two indices are equal. (Note that the second property listed is just normalization.) Since each index can take on only values $1, 2, \dots, D$, the antisymmetric symbol for D -dimensional space must carry D indices as already noted. For example, for $D = 2$, $\varepsilon^{12} = -\varepsilon^{21} = 1$, with all other components vanishing. For $D = 3$, $\varepsilon^{123} = \varepsilon^{231} = \varepsilon^{312} = -\varepsilon^{213} = -\varepsilon^{132} = -\varepsilon^{321} = 1$, with all other components vanishing (as was already noted in the preceding chapter).

60 | I. From Newton to Riemann: Coordinates to Curvature

Using the Kronecker delta and the antisymmetric symbol, we can write the defining properties of rotations $R^T R = I$ and $\det R = 1$ as

$$\delta^{ij} R^{ik} R^{jl} = \delta^{kl} \quad (12)$$

and

$$\varepsilon^{ijk\dots n} R^{ip} R^{jq} R^{kr} \dots R^{ns} = \varepsilon^{pqr\dots s} \det R = \varepsilon^{pqr\dots s} \quad (13)$$

respectively. In (13) we used the definition of $\det R$. (Verify this for $D = 2$ and 3 .)

Referring to (1), we see that we can describe δ^{ij} and $\varepsilon^{ijk\dots n}$ as invariant tensors: they transform into themselves. For the rest of this text, we will often use, implicitly or explicitly, the notion of invariant tensors.

For example, for $SO(3)$, using (13) you can show that $\varepsilon^{ijk} A^i B^j \equiv C^k$ defines a vector $\vec{C} = \vec{A} \times \vec{B}$, the familiar cross product. Various identities follow. Consider, for example,

$$\varepsilon^{ijk} \varepsilon^{lnk} = \delta^{il} \delta^{jn} - \delta^{in} \delta^{jl} \quad (14)$$

To prove this, simply note that both sides transform as invariant tensors with four indices, and the symmetry properties (such as under $i \leftrightarrow j$) of the two sides match. Contracting with A^j , B^l , and C^n , we obtain an identity you might recognize: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$.

Closing of Newtonian orbits once again

We can now go back to the apparent mystery in chapter I.1, that the Newtonian orbits in a $1/r$ potential close. Out of the conserved angular momentum vector $\vec{l} = \vec{r} \times \vec{p} = \vec{r} \times \dot{\vec{r}}$ (we are using the notation of chapter I.1; we have effectively set the mass to unity and hence the second equality) we can form the Laplace-Runge-Lenz vector $\vec{L} \equiv \vec{l} \times \dot{\vec{r}} + \frac{\kappa}{r} \vec{r}$. Computing the time derivative $\dot{\vec{L}}$, you can verify (see exercise 4) that \vec{L} is conserved for an inverse square central force. When $\dot{\vec{r}}$ is perpendicular to \vec{r} , which occurs at perihelion and aphelion, the vector \vec{L} points in the direction of \vec{r} . We could take the constant vector \vec{L} to point toward the perihelion, and thus the position of the perihelion does not change. Hence the orbit closes.

This result does not hold in Einstein gravity. The precession of the perihelion of Mercury, which we will discuss in chapter VI.3, is of course one of the classic tests of general relativity.

Appendix: Two lemmas for future use

There is a lot more we could say about tensors, but let me mention two simple lemmas that we will happen to need later.

Let S^{ij} and A^{ij} be two arbitrary and unrelated tensors, symmetric and antisymmetric, respectively. Then $S^{ij} A^{ij} = 0$. (See exercise 5.)

I.4. Who Is Afraid of Tensors? | 61

Tensors can have all kinds of symmetry properties, which you can explore on your own and in the exercises. For example, a totally antisymmetric 3-indexed tensor T^{ijk} is such that T flips sign under the interchange of any pair of indices (for example, $T^{ijk} = -T^{jik} = +T^{kji}$). A multi-indexed tensor can also have symmetry properties under the interchange of a specific pair, or may have no symmetry at all. Consider, for example, a tensor G^{kij} symmetric under the interchange of the first pair of indices only, that is, $G^{kij} = G^{ikj}$. To be pedantic and absolutely clear, sometimes I like to put a space or a dot between the indices, thus $G^{ki\cdot j}$ or $G^{ki\cdot j}$ to separate the “special” pair from the other indices. For example, our tensor could happen to be $G^{ki\cdot j} = \partial^k \partial^i W^j(\vec{x})$ for some vector field W^j .

Given $G^{ki\cdot j}$, define $H^{k\cdot ij} \equiv G^{ki\cdot j} + G^{kj\cdot i}$. (Note that $H^{k\cdot ij} = H^{k\cdot ji}$ by definition, but $H^{i\cdot kj}$ is in general not equal to $H^{k\cdot ij}$.) Then we can solve for G in terms of H :

$$G^{ki\cdot j} = \frac{1}{2}(H^{k\cdot ij} + H^{i\cdot jk} - H^{j\cdot ki}) \quad (15)$$

(See exercise 8.)

Exercises

- 1 Define $\vec{\nabla} \equiv \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^D} \right)$. Show that if ϕ is a scalar, then $(\vec{\nabla}\phi)^2 = \vec{\nabla}\phi \cdot \vec{\nabla}\phi = \sum_k \left(\frac{\partial\phi}{\partial x^k} \right)^2$ and $\nabla^2\phi$ transform like a scalar. The Laplacian is defined by

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial(x^1)^2} + \frac{\partial^2}{\partial(x^2)^2} + \dots + \frac{\partial^2}{\partial(x^D)^2}$$

- 2 Show that the symmetric tensor S^{ij} is indeed a tensor.
- 3 Show that the infinitesimal volume element d^3x is a scalar.
- 4 Show that the Laplace-Runge-Lenz vector is conserved.
- 5 Show that $S^{ij}A^{ij} = 0$ if S^{ij} is a symmetric tensor and A^{ij} an antisymmetric tensor.
- 6 Let T^{ijk} be a totally antisymmetric 3-indexed tensor. Show that T has $\frac{1}{3!}D(D-1)(D-2)$ components. Identify the one component for $D = 3$.
- 7 Consider for $SO(3)$ the tensor T^{ijk} from exercise 6. Show that it transforms as a scalar.
- 8 Prove the lemma in (15).
- 9 Verify (13) for $D = 2$ and 3.

Note

1. For a concise introduction to some of the group theory needed in theoretical physics, see *QFT Nut*, appendix B.

(CHAPTER 1 CONTINUES...)

Index

Page numbers followed by letters e, f, and n refer to exercises, figures, and notes, respectively.

- $\frac{1}{2}$ -factor, Einstein's field equation, and metric tensor formalism, 76
- "1–2" test, 326
- 1-forms, 599–600; Hodge star operation on, 724
- (2+1)-dimensional spacetime, Chern-Simons term in, 721
- 2-D metric, 77
- 2-dimensional solid state structures, gauge potential of, 721
- 2-forms, 601
- 2-indexed tensors, definition of, 727
- 2-manifold, without boundary, 727
- 3-D metric, and black holes, 77
- 3-dimensional spaces, embedding into Minkowskian spacetime, 634
- 3-spaces, maximally symmetric, 610
- 3-spheres: cosmological principle, 491; metric tensor of, 296
- 3-vectors, transformation into 4-vectors, 218
- 4-current, 251
- 4-dimensional electromagnetism, 720–721
- 4-dimensional spacetime, 386; divergence theorem generalized to, 386
- 4-gluon scattering, 738, 744e
- 4-momenta: in electromagnetism, from special relativity, 245; lightlike, 782; of particles in box, 227
- 4-vectors: from 3-vectors, 218; length of, 182; relativistic curl of, 252; spacetime metrics, 181
- 4-velocity: around black holes, 414; of finite sized object, 716
- 5-dimensional Einstein field equations, for 2-brane model, 700
- 5-dimensional scalar curvature, 684–685
- 5-dimensional spacetime. *See* Kaluza-Klein theory
- Abbott, E. A., 671
- abelian gauge theory, 681n
- Abraham, Max, on Newton gravity and Lorentz invariance, 580
- acausality, of universe, 754, 783
- "accelerated" thought experiment, 280–283, 286
- acceleration: and curvature, 554; Galilean transformation, 276–277; Galileo's law of, 140; and general relativity, 189; and gravity, 269, 271; in Minkowski spacetime, 190; relativistic particles, 277
- accretion disks, 414–415; around Kerr black holes, 474
- acoustic peak, microwave background, 523–525, 788n
- action: for 2-brane model, 700; constraints in varying, 755–756; containing two powers of time derivative, search for, 338–339; different sectors of matter action, 382–383; dimensions of, 346; at a distance, of Newton's gravity, 145; Einstein-Hilbert (*see* Einstein-Hilbert action); for elastic medium, 771; electromagnetic, 244, 250–251, 333;

822 | Index

- action (*continued*)
for everything else, 347; for fields in spacetime described by a metric, 770; in flat spacetime, 379; formulation by metric or vielbein, 785; of free particle, 162; gravitational time dilation, 284; for gravity, 339, 344, 346; as infinite series of terms, 766; Kaluza-Klein, in Jordan frame, 686; length or energy scale dependence of, 710; local, 246; Lorentz, in Kaluza-Klein theory, 678; of matter (*see* matter action); Maxwell, 325, 332, 675–676; for motion of finite sized objects, 714–715, 716; Newton-Einstein-Hilbert, quantum gravity limit of, 444; Newton’s law of action and reaction, 470; nondependence on metric, 723; nonlocality in time, 754; nonrelativistic, 241–242, 356; offshell information carried by, 782; reasons for emphasis on, 396; relativistic, 284–285, 308; relativistic string, 210n; scalar field, 332; specification of dynamical variables, 395; terms of, behavior at long distances, 722; topological, 720–721; total, Newtonian world, 145; of universe, 346, 356; as usually formulated, 783; of world, and energy momentum tensor, 378; Yang-Mills, 681
- action functional, 138
- action principle, 155; basics of physics, 136–149; different notions of, 138; as fundamental principle of theoretical physics, 783; globality of, compared to equation of motion, 141; kinetic term in, 140; and least time principle, 139, 144; metaphor for life, 140; mystery of, 141, 155; of particles and fields, 145; theories based on, 383; variational calculus, 113
- action variation, holding dynamical variables fixed, 380
- active diffeomorphism, 397
- actual biological time, elapsed between event A and B, 179
- addition of velocities. *See* velocities
- ADM (Arnowitt-Deser-Misner) formulation, of gravitational dynamics, 693
- AdS. *See* anti de Sitter spacetime
- affine parameter, 308
- Aharonov-Bohm effect, 789n
- air resistance, and free fall, 268
- airline example, for proving curvature of earth, 66
- al-jabr, calculation method, 208n
- Al-Khwarizmi, calculation of square roots, 207n
- algebra: conformal, 614–623; de Sitter, and cosmological constant, 755; extensions of, 667; Lie (*see* Lie algebra); Lorentz (*see* Lorentz algebra); matrix, quick review of, 742–743; Poincaré (*see* Poincaré algebra)
- algorithm, etymology of the word, 207n
- ambitwistors: power of, 738; representation of, 736
- American football, relativity of, 171, 172f
- “analog Newtonian” equation, 367
- analytic continuation: de Sitter to anti de Sitter spacetime, 664; hyperbolic coordinates, 661; of stereographic projection, 641
- analytic geometry, role of coordinates, 48
- Anderson, Phil, on particle physicists, 713n
- angles: defined by physicists, 170; hyperbolic, 628; importance of, 620
- angular coordinates: on de Sitter spacetime, 627; suppressed, 422, 426
- angular correlation, cosmic microwave background fluctuations, 523f
- angular deficits: as “measure” of curvature, 727; of polyhedra, 726–727
- angular momentum: around black holes, 412–413, 459; conservation of, 30, 36–37, 48n, 126, 152, 310; Kerr black hole, slow rotation limit, 571; loss, Penrose process, 471–472; of particle on sphere, 148; of rotating black holes, 442, 465, 576; of rotating bodies, 563–577; symmetry of, 150
- angular velocity: around black holes, 414, 460; defined by time coordinate, 550; for Kerr black hole, 462f; slowly rotating gravitational sources, 570; inside stationary limit surface, 471
- annihilated spacetime, 785
- annihilation operator, 447–448
- annus mirabilis, Albert Einstein’s, 265
- ant and honey analogy, 5–6, 5f
- ant movement, as example of variational calculus, 128
- anthropic principle: and cosmological constant paradox, 751–752, 757; and ultimate theory, 789n
- anti de Sitter / conformal field theories (AdS/CFT): AdS/CFT correspondence, 649, 787; conformal coordinates of, 654, 654f; and Poincaré half plane, 68
- anti de Sitter spacetime (AdS), 606e, 612, 649–666; for 2-brane model, 702; AdS^2 boundaries, 664; boundary of, 655; d-dimensional, 650, 650f; different forms of, 660; in hyperbolic coordinates, 661; isometry group of, 650; motion of light, 659; motion of massive particles, 659–660; Poincaré coordinates, 656; slice of, 658f; stereographic projection for, 661; table for, 662
- anti-gravity, discussion of, 392
- anticommutation: of differential forms, 597; Jordan’s manuscript on, 789n
- antimatter: and charge conjugation in Kaluza-Klein theory, 678; creation of, 205, 206; in early universe, 528; Feynman diagram of, 206f; in higher dimensional theories, 683; in quantum field theory, 476
- antiparticles, 26, 437–438
- antipodal condition, space of spheres, 646
- antisymmetric matrices, introduction of, 40
- antisymmetric symbol: in curved spacetime, 723–725; as invariant tensor, 60; role as metric, 734; used to contract indices, 719

- antisymmetric tensors: character of, 55;
decomposition of, 236e
- antisymmetry: useful relations based on, 608. *See also* symmetry
- apparent singularities, at Schwarzschild radius, 409
- apparent violation of causality, in brane models, 703, 705
- apple: falling, 36, 137f, 268; floor rushing up to meet, 270f
- area: approximation by small rectangles, 546;
infinitesimal, enclosed by closed curves, 547;
Planck, and entropy of black holes, 442
- area and volume, concept of, and coordinate transformations, 75–76
- area theorem, Penrose process, 472
- area transformations, in differential forms, 598
- Aristotle, comparison to Newton, 140–141
- arithmetic, difference from mathematics, in terms of rotations, 56
- arithmetic laws, of working in general relativity, 665
- Arkani-Hamed, modified Einstein’s field equation, 754
- Arnowitz-Deser-Misner (ADM) formulation, of gravitational dynamics, 693
- arrays, and vectors, 51n
- astronomy, with gravitational waves, 563
- astrophysical objects: mass and energy for
gravitational waves, 569; Schwarzschild radius to actual radius relation, 366
- asymptotic safety, as approach to quantum gravity, 760
- atomic clock, 287
- atomic physics, in early universe, 518
- atoms, action of, 714–715
- attractor, stable, in cosmic diagram, 511f
- auxiliary fields, 217n
- auxiliary quantities, calculus, 129
- averaging, for many particles, 231
- baby string theory, 215; and Lorentz transformation, 147
- Babylonian tablet, 214, 214f
- background radiation. *See* cosmic microwave background
- bad notation alert: confusion in time dilation, 198;
confusion in relativistic action, 211; geodesic equation, 555
- balls: circularly arranged, falling toward spherical planet, 58–59; separation between falling, 554; in train, 160–161, 161f
- baryogenesis, 526–528
- baryonic matter, 502–503, 506
- basic vector: spacetime metrics, 181; (ur-), definition of, 43
- basis vectors: change by moving on surface, 99–100;
for surface, in Euclidean space, 98; variation of, 100
- Beer, Gillian, on Lewis Carroll, 173n
- Bekenstein-Hawking entropy, 441–442, 444; second law of black hole thermodynamics, 472
- Beltrami, Eugenio, and discovery of Poincaré’s half plane, 67n
- bending of light: “accelerated/dropped” gedanken experiments, 281–282. *See also* deflection of light
- Bentley, Richard, on existence of God, 520
- Bering Strait, “attractive force” of, 275
- Berlin Wall, construction of, 476
- Bernoulli, Jacob and Johann, brachistochrone problem, 120
- Bessel, Friedrich, Bessel functions, 376n
- Besso, Michele, letter of Einstein to son of, 177
- Bethe, Hans, and Peierls’ comments on thinking and calculating, 133
- Bianchi identity, 452; constraints on curvature tensor, 592; contracted, 393, 394; derivation of, 392, 393; and Maxwell’s equations, 724; similarity to differential forms, 599
- Big Bang, 785; analyzed with cosmic potential, 508–509; in cosmic diagram, 502–503; and cosmic microwave background, 517; in cosmic potential diagram, 508f; as creation of space, 498–499, 708; as point of infinite temperature, 496
- Big Crunch, 508–509, 508f, 514
- billiard balls, elastic collision of, 165e
- binary pulsar, emission of gravitational waves, 563
- binary systems, gravitational waves from, 714
- binding energy, gravitational, 455–456
- biological time, actual, elapsed between event A and B, 179
- Birkhoff, George: Newton-Jebsen-Birkhoff theorem, 453; time dependent spherically symmetric mass distribution, 373
- black body radiation, of black holes, 436
- black hole hypothesis, historical, 13
- black holes: and 3-D metrics, 77; binary systems of, 714; charged, 477–484; “dangers of extremes,” 484; in de Sitter spacetime, 635; definition of, 410; distance around extremal, 469; dust ball collapsing into, 422f; entropy of, 15, 436, 441, 448, 766, 788n; estimation of “electric” and “magnetic” components for, 717; eternal, 421–422, 426–427, 479; extremal, 467–468, 478, 481; first and second law of thermodynamics, 472–473; formation of, 373, 421–423, 422f, 423f, 429–431; gravitational potential around, 410–411, 411f; and Hawking radiation, 14–15; horizon of, 416–417, 784; information paradox, 439; internal world of, 781; just sitting there, 482–483; Kerr black hole, 462, 464–468; Kruskal-Szekeres diagram, 426; as limit for measuring device, 763–764; local gravitational field in great distance of, 574; mass determination, 570; mass of, given by Michell and Laplace, 366; mystery of, 410, 441; orbits for light moving around, 416f; orbits with substantial

824 | Index

- black holes (*continued*)
angular momentum, 412–413; particles and light around, 409–418; perihelion shift around, 413; Reissner-Nordström black hole, 479, 483; rotating, 576; Schwarzschild black hole, 429f, 436; stellar collapse into, 455–456; strangeness of, 764–765; sub-/transextremal, 478, 483; tilting light cones, 421; and unitarization of graviton scattering, 765.
See also finite sized objects; rotating black holes
- blue shift, relativistic, of frequency, 186
- blue sky effect: reason for, 715; squared in gravity, 717
- bodies, rotating: angular momentum of, 563–577; slowly, 570; spacetime deformation by, 460
- Bogoliubov transformation, 448
- boiling vacuum, 437–438
- Boltzmann constant, and temperature concept, 16n
- boosts: invariance of, 188; Lorentz transformation for, 169, 187; and rotations, commutation relations of, 191–192
- Born, Max, on Einstein’s gravity, 777
- Bose-Einstein condensate, 332
- bosons: bound to magnetic monopoles, 789; as open strings, 696
- Boulware, David, curved spacetime, 580
- bounce theory, 536n
- boundaries: of AdS^2 , 664; of anti de Sitter spacetime, 655; divergence of metric tensor, 663; numbers of, 664
- boundary: of Euclidean anti de Sitter space, 662; incoming light beam, with Poincaré coordinates, 659; spatial, in anti de Sitter spacetime, 649
- boundary conditions, of energy functional, 116
- bowl: curvature at bottom of, 85; potential energy of moving marble, 113–115
- box: accelerating, laser light, 281f, 283f; Lorentz contraction of, 23; particles in, 223f, 227; for studying physical systems, 649
- Boyer-Lindquist coordinates, 476; description of flat space, 78
- brachistochrone problem, 121f; formulated by Bernoulli, 120
- Brahe and Kepler, work of, importance for Newton, 369n
- brane worlds, 696–707
- branes: 1-brane model, 702; 2-brane model, 700–702; initially static, 707; Poincaré invariant, 707; waves from bulk, 703f. *See also* membranes
- breathing circles, 679–680
- Bright, Ms. (limerick character), 294
- Broglie, Louis de, 773n; particle-wave dualism, 762
- Broglie wavelength, particles at Schwarzschild radius, 442
- Bronstein, Matvei, reconstruction of theory of gravity, 764–765
- Buchdahl’s theorem, 454
- bulk waves, to brane, 703f
- Calabi-Yau manifolds, 695
- calculus: simplification of, auxiliary quantities, 129; of variation (*see* variational calculus)
- caloric, historical concept of, 786
- Calvino, Italo, *Cosmicomics*, 554
- candles: falling, 268, 271; standard, 359
- Carl Friedrich Gauss, and differential geometry, 90–91
- Carroll, Lewis: constant notion of time, 173n; on times, 166
- Cartan, Élie, and Lie algebra, 586
- Cartan’s equations: anti de Sitter spacetime, 612; first, index transformations, 603; for maximally symmetric 3-spaces, 610; in spherically symmetric static spacetimes, 611; structural, 607, 684
- Cartan formalism: calculation of curvature, 602; curvature and covariant derivative, 605
- Carter-Penrose diagrams, 435. *See also* Penrose diagram
- Cartesian coordinates: change to polar coordinates, 29, 62, 71; change to spherical coordinates, 63
- Casimir effect, 748–749, 758n
- Cauchy horizon, 404
- Cauchy problem, in Einstein gravity, 400
- Cauchy surface, initial data on, 402
- Cauchy’s theorem, for analytically continuing integrands into complex plane, 732
- causal structure, of de Sitter spacetime, 638, 639f
- causal structure of spacetime: domains, 530, 531f; Hawking radiation, 438; Penrose diagrams, 427, 431
- causality, 178; apparent violation of, in brane models, 703, 705; as fundamental principle of theoretical physics, 783; at Schwarzschild radius, 421; in special relativity, 204
- Cavendish, Henry, measurement of Newton’s constant, 32
- Cavendish experiment, and non-quantized gravity, 771
- celestial mechanics, Newton’s solution of, 28–30
- ensorship, cosmic, 479–480
- center-of-mass energy, graviton scattering, 761
- central forces, in celestial mechanics, 28
- central potential, and invariance, 47
- centrifugal force, 278; around black holes, 411; and curvature of curve, 97
- “centrifugal” potential, 126
- CFT (conformal field theories), 649n
- chain rule, transformation of Christoffel symbols in, 132
- Chandrasekhar, Subrahmanyan, Kerr solutions, 481
- Chandrasekhar limit, 455
- Chang Heng, and concept of coordinates, 62n
- charge: conjugation, 678; conservation of, during antimatter creation, 205; coupling to electromagnetic field, 250; density of, in Maxwell’s equations, 252; Lorentz force law, 404; and

- momentum in fifth dimension, 677; notion of, 246–247; quantization in Kaluza-Klein theory, 677
- charged black holes, 477–484; Penrose diagram, 480f
- charged particles, individual, worldlines of, 715
- charged scalar fields, in 5-dimensional theories, 687
- Chern-Simons term: in (2+1)-dimensional spacetime, 721; powers of derivatives, 722
- Chinese, and concept of coordinates, 62n
- Christoffel 1-form, definition of, 604
- Christoffel symbol, 129; brute force transformation of, 329; and comoving coordinates, 290; and covariant differentiation, 321; and curved spacetime, 278; definition range in parallel transport, 544; in Fermi normal coordinates, 560; indices, number of, 131; introduction of, 99; schematic form of, 342; around spherically symmetric mass distribution, 310–311; transformation of, 132, 389; use of symmetry properties in Fermi normal coordinates, 561; variation of, 347
- circles: breathing, 679–680; of constant latitude/longitude, on sphere, 105; mistaken for points, 674f
- circular orbit: around black holes, 413–414, 413f; innermost stable, 414, 474; around massive object, 549
- “classical” differential geometry, 96–109
- classical field theory, 119; harmonic oscillator in, 361
- classical gravity, puzzling, 784
- classical mechanics, without Newton’s equation, 145
- classical physics, profound difference from quantum physics, 360–361
- classical relativity, not consistent with quantum field theory, 773n
- classicalization of gravity, 766
- clock paradox, 194n
- clocks: cosmic, 504; invented by Einstein, 166; observed in different frames, 196f; and rulers, role in physics, 719–720; slow running, in special relativity, 197
- closed curved space, 681
- closed forms, 604
- closed orbits, verification of, 30
- closed strings, 696
- closed timelike curves, 484
- closed universe, 296–297, 491; critical density, 497–498; as de Sitter spacetime, 630; Einstein’s field equations, 493–494; with positive cosmological constant, 633
- clothed singularities, 479
- Cohen, I. Bernard, visit to Einstein, 267
- coincidence problem, 499, 778
- collapse: dissipative, 520–521; stellar, 455–456
- collisions: elastic, of billiard balls, 165e; of particles, 219–220, 438; of photons and electrons, 222f
- column vectors, notation of, 45
- common sense, to be abandoned for development of physics, 784
- “common to all the things contained in it,” 18n
- communication, in expanding universe, 293–294
- commutation: and group theory, 49; of matrices, 41
- commutation relations, between boosts and rotations, 191–192
- commutators: between A and B, definition, 49; computation of, cyclic substitution, 50; index-free representation of vector fields, 319; introduction of, 340; and Lie derivative, 328; of two covariant derivatives, 325, 341
- comoving coordinates, 290, 298; preferred flow direction in, 230
- comoving observers: and perfect fluids, 229; spacetime distance of, 174; in universe filled with perfect fluid, 492–493
- compact source approximation, 568
- compactification, of extra dimensions, 683
- completion: and promotion, of gravitational fields, 218; relativistic, 242–243
- complex matrices, and twistors, 730
- complex parameters, rescaling of, 733
- complexification, of variables, 732
- Compton mass, of universe, 747–748
- Compton scattering, 222f; inverse, 235e
- computational effort, by using action principle, 141
- condensed matter physics: dynamical critical exponent, 657n, 754, 758n; gauge potential of solid state structures, 721; scale and conformal invariances, 621
- conformal algebra, 614–623; flat spacetime, 615; generators of, 617; identification of, 618
- conformal coordinates: for anti de Sitter spacetime, 654, 654f; for de Sitter spacetime, 638
- conformal equivalence, of anti de Sitter spacetime, 654
- conformal field theories (CFT), 649n
- conformal flatness: of anti de Sitter spacetime, 662; of de Sitter spacetime, 641–642
- conformal generators, 615
- conformal groups, equality with isometry groups, 656
- conformal Killing condition, 614
- conformal time, and cosmic time, 632
- conformal transformations, 614; generators of, 644; preservation of angles, 620; as solutions of Laplace equation, 616
- conformally equivalent spacetime, 311
- conformally flat metrics, 352e–353e; definition of, 94
- conformally flat space, 80–81e; as bad terminology, 94
- conformally related spacetimes, 622e
- conjugation, charge, 678
- connection 1-form, 599–600; indices of, 607
- conservation: angular momentum (*see* angular momentum); charge, 205; covariant, of energy

826 | Index

- conservation (*continued*)
momentum tensor, 384; current, 226, 253; energy (*see* energy conservation); energy momentum (*see* energy momentum conservation); momentum (*see* momentum conservation); and relativistic fluid dynamics, 233; and symmetry, 150–155
- conservation laws, 155; from action principle, 141; and Killing vectors, 589; for motion in curved spacetime, 310; in Newtonian mechanics, 35–37
- conserved quantities: in Newtonian mechanics, 30; Noether’s theorem, 152
- consistency condition, and determination of potential, 36
- constant latitude/longitude, circles of, on sphere, 105
- constant vector fields, covariant derivative of, 331
- constants, fundamental, 12
- constraints on metric, 403
- container: of anti de Sitter spacetime, 649; rectilinear, infinitesimal volume of, 80e
- continuity equation, for current conservation, 225
- continuum mechanics, notations of coordinates, 117
- contracted Bianchi identity, 393; derived from Einstein-Hilbert action, 394
- contraction: of indices, 46n, 345; of repeated indices, 58n; of spacetime indices, metric for, 719; tensors, 316
- contravariant indices, 72, 315
- contravariant vectors, 183
- coordinate differentials, 312
- coordinate invariance: general, 305–306, 672; local, in higher dimensional theories, 682
- coordinate patches, to cover entire space, 76
- coordinate scalars, to form a metric, 708–709
- coordinate singularities: compared to physical singularities, 91–92; and Einstein-Rosen bridge, 92f; Kerr black holes, 467; Schwarzschild metric, 365–366
- coordinate systems: failure of, 76–77; natural, 134
- coordinate transformations: 5-dimensional, gauge transformations as, 673; accelerated frames, 285; change of metric under, 70–71; Christoffel symbols, by brute force, 329; in curved space and curved spacetime, 317; in differential forms, 597; freedom of, 62; Galilean, of acceleration, 276–277; general, 68–71, 312, 314, 318, 384; for gravitational waves, similarity to electromagnetic gauge transformations, 564; and indices (upper and lower), 73–74; and Jacobian, 75; and Mercator map, 79e; nonlinear, 69; as passive diffeomorphism, 398
- coordinates: angular, 422, 426, 627; Boyer-Lindquist, 476; change of, 64–65, 641; choose of appropriate, 631; comoving, 290, 298; concept of, by Descartes, 48; dimensionless, 665; Eddington-Finkelstein, 431; effect of motion on, 160; geometric significance of, 68; hyperbolic, 661; hyperbolic radial, 653–654; internal, 675; Kruskal-Szekeres, 424–425, 432–433; Kruskal-Szekeres-like, 635; light cone, 146–147, 170–171, 427–429, 704; locally flat, 130, 132, 278, 288, 552, 557; notation, 62n, 117; Painlevé-Gullstrand, 417; poor choice of, 590; primed and unprimed, 18, 38, 39f, 71–73; pseudo-time, 657; relations between different, 159; Rindler, 446; role exchange of, at horizon, 419; of specific point, Fermi normal coordinates, 559; static, 634, 636, 652; time, 652; traditional “names” of, 25; warped polar, 613e
- coordinates, “crazy,” 94e
- coordinatization, of de Sitter spacetime, 634
- Copernican principle, 491
- corotation/counterrotation: light rays, 461, 469; particles, 474
- correlation: angular, cosmic microwave background fluctuations, 474; of quantum fluctuations, 447
- coset manifolds, 590; and classification of space and spacetime, 666; group theory of universe, 644; and maximal symmetry, 625; and spontaneous symmetry breaking, 593
- cosmic censorship, 479–480
- cosmic clock, universe’s ambient temperature as, 504
- cosmic coincidence problem, and cosmological constant paradox, 751
- “cosmic conspiracy,” 297
- cosmic diagram, 496, 502, 503f; flow in, 510–512; phase boundaries in, 513–514; stable attractor and fixed points, 511f
- cosmic expansion. *See* expanding universe
- cosmic microwave background, 236e, 517; angular correlation of fluctuations of, 523f; density fluctuations in early universe, 521–522; first acoustic peak, 523–525; fluctuations, effect of curvature on, 525–526; temperature, 515
- cosmic potential, 508f; Big Bang analyzed with, 508–509
- cosmic ray particle, lifetime of, 198
- cosmic time, 295; and conformal time, 632; and horizon problem, 530
- Cosmicomics* (Calvino), 554
- cosmological action: derivation of energy momentum conservation, 387e; variation of, 391
- cosmological constant: for 2-brane model, 701; added by Einstein, 360; in cosmic diagram, 502–503, 513; in de Sitter spacetime, 456; decaying, 756; and deceleration of cosmic expansion, 507; deletion of, Feynman diagrams for, 756–757; dependence of equation of state parameter on, 359; different spatial curvature, 634; Einstein’s field equation in presence of, 357; and Einstein-Hilbert curvature term, 754; and energy conditions, 557; essential role played by, 512; and expanding universe, 392; in inflationary cosmology, 534; introduction of, 356, 393; as Lagrange multiplier for volume of spacetime, 756; mass scale of, 700; mystery of,

- 356, 711, 751, 782; positive, 633; in quantum world, forbidding removal of, 361; and scale factor of universe, 496f; scaling of, 753–754; universe’s equation of state, 497
- cosmological constant paradox, 745–759; algebraic solution of, 754–755; analogies of, 786; anthropic principle, 751–752, 757; breaking free of local field theory, 756; as challenge for physics, 712; cosmic coincidence problem, 751; deeper understanding of physics, 753; deletion of Feynman diagrams, 756–757; effective action for gravity, 711; and effective field theory, 782; extreme ultra infrared regime, 750–751; inflation, 751; largest and smallest masses, 747–748; linkage between infrared and ultraviolet regime, 752; naturalness doctrine, 749–750; omniscience of gravity, 745; Planck mass, 746–747; potential solution of, 778; quantum fluctuations, 745–746; question for explanation of vacuum energy, 752–753; unimodular gravity, 755–756; vacuum energy density, 749
- cosmological distances, 750; scaling at, 753–754
- cosmological equation, 501; appropriate units, 633; and history of universe, 503–504
- cosmological principle, 289, 491–492; Einstein’s field equations, 494; Newtonian mechanical analogies from, 507, 513
- cosmological redshift, 295
- cosmological time, in outgoing brane wave model, 706
- cosmology: curvature of universe, 490; gases for, 230; golden age of, 491; inflationary, 530–536; nonlocal, 712; observational, 505; physical history of early universe, 515–529; proper distances, 296–297; trans-Planckian, 518; use of scalar fields in, 759n
- couch potato problem: action principle, 143; particles at rest, 142
- Coulomb potential, dilation invariance, 620
- counting: for characterizing intrinsic curvature, 110; and group theory, 56–57; of matrix elements, 87–90
- coupled Einstein and Maxwell equations, static solutions, 482–483
- coupled ordinary differential equations, relativistic stellar interiors, 452
- coupled partial differential equations, transfer of spacetimes, 664
- covariance: difference from invariance, 47; general, 285
- covariant curl, 325; derivation of energy momentum tensor, 381
- covariant curvature, 339
- covariant derivatives: along geodesic, 553; in Cartan formalism, 605; concept of, and differential geometry, 100–101; constructed by parallel transport of vectors, 103; Newton-Leibniz rule, failure for, 342; and objects carrying vectorial arrow, 109; in parallel transport, 543–544; of vectors, 340
- covariant differentiation, 320–333; and Christoffel symbol, 321; along curves, 327
- covariant divergence, 326; of tensors, 332
- covariant indices, 72, 315
- covariant vectors, 183
- CP (charge conjugation followed by parity) violation, 528; in higher dimensional theories, 683
- “crazy” coordinates, 94
- creation of space, 498
- creation operator, 447–448
- critical density, 497–498; and Hubble length, 514; ratio of energy density to, 505
- critical phenomena, theory of, 621
- Crommelin, Andrew, and Royal Society expedition, 367
- cross-product notation, angular momentum conservation, 48n
- cross section scattering, 715
- cube, topology of, 725
- cube of physics, 12–13
- cubic vertex, 739
- curl: covariant, 325; exterior derivative, 599; relativistic, 252
- curled up space, 673–674
- current: conservation of, 225, 253; in relativistic physics, 223; in string theory, 235
- curvature, 667; 5-dimensional scalar, 684–685; and acceleration, 554; angular deficits as “measure” of, 727; calculated on basis of given metric, 66; calculated with Cartan formalism, 602, 605; calculated with differential forms, 607; connection with field strength by differential forms, 602n; constant of scalars, 589; of curve, 89, 97; from curves to surfaces, 106; of cylinders and spheres, 6; of earth, airline example for proving, 66; expressing failure of Newton-Leibniz rule covariant derivatives, 342; extrinsic (*see* extrinsic curvature); of “fixed latitude” circle, 80e; intrinsic (*see* intrinsic curvature); invariant or covariant description of, 339; and least path principle, 5–6; measurement of, 89, 547, 548e; negative, definition of, 85; Riemann (*see* Riemann curvature); scalar (*see* scalar curvature); of space, 65–66; of spacetime (*see* curved spacetime); spatial, effect on CMB fluctuations, 525–526; of surface and curves, 104–105; of universe, 490–491, 495, 748; vanishing, 85
- curvature 2-form, 600
- curvature density, 504, 512; and flatness problem, 531
- curvature tensor: alternative derivation of, 547–548; anti de Sitter spacetime, 651; computation of, with symbolic manipulation software, 607; constraints on, 591; for de Sitter spacetime, 626; directly

828 | Index

- curvature tensor (*continued*)
from 2-form, 611; Fermi normal coordinates, 560; fixed by maximal symmetry, 592; and Hawking radiation, 438; and Kerr metric, 476; in locally flat coordinates, 553; in maximally symmetric spaces, 589; use of symmetry properties, 561. *See also* Riemann curvature tensor
- curved rectangle, displacement of vector, 341f
- curved space: and change of coordinates, 64–65; closed, 681; compared to curved spacetime, 91; and coordinate transformations, 68, 317; determination of curvature, 65–66; embedded in higher dimensional flat spaces, 85–86; sphere as example for, 83
- curved spacetime: antisymmetric symbol in, 723–725; and change of coordinates, 64–65; compared to curved space, 91; coordinate transformations in, 317; determination of Lagrangian in, based on Einstein’s equivalence principle, 712; electromagnetism in, 325–326; energy momentum tensor in, 380; Euclid’s axiom, 552; general, spatial distance in, 290–292; geodesic equation for, 277–278; governed by energy distribution, 390; governed by gravity, 344–346; and gravity, mystery of, 276; independence of mass, 258–259; in lab, 332; Maxwell’s equations, 333; most appreciated, 624; motion in, 289–290, 301–311, 307–309; Newtonian limit, 302–303; quantum field theory in, 780; Raychaudhuri equation, 449; spacelike 3-dimensional hypersurface, 693f; around spherical mass distribution, metric for, 364; spinors in, 604–605; universality of gravity, 275–276; universe as, 288–300; visualizations, 296
- curved surface: parallel transport of vectors on, 102; and tangent plane, 83f
- curves: of constant latitude, 92, 93; curvature of, compared to surface, 89n; decomposition of, 545; in Euclidean space, 96–97; fear of, 82; in geodesic problem, definition of, 123; length in spherical coordinates, 127; minimal length of, 128; in Minkowskian spacetime, 175; parametrized, and parallel transport, 543; reparametrization invariance of, 124; on surface, determination of curvature, 104–105
- cutoff: concept of, in quantum field theory, 758n; instead of infinities in physics, 770
- cyclic substitution, computation of commutators, 50
- cyclic symmetry, of Riemann curvature tensor, 351e
- cylinder: curvature of, 6, 84–85, 107; tangent plane of, 98; topological, 654
- D-branes, Bekenstein-Hawking entropy, 444
- d-dimensional Euclidean space, rotations in, 49–51
- d-dimensional sphere, definition of, 624
- d-dimensional anti de Sitter spacetime, definition of, 650
- Damour, Thibault, on Einstein’s application of Lorentz transformation to physics, 190
- “dangers of extremes,” 484
- Dao, of many-worlds interpretation of quantum, 788n
- dark energy, 359, 627n, 642; coincidence problem in dark energy–dominated universe, 499; and cosmological constant paradox, 711, 747, 778; in de Sitter spacetime, 456; and energy conditions, 557; and Hubble parameter, 391; mystery of, 356; and Nobel Prize in Physics (2011), 361n; observational evidence for, 503; phase boundaries in cosmic diagram, 514; and scalar fields, 788n; struggle with dark matter, 502–503
- dark energy density: negative pressure as consequence of, 360; in spacetime, 356, 359
- dark matter: gravitational lensing, 370–371; observational evidence, 503, 503f, 506; structure formation in early universe, 522–523; struggle with dark energy, 502–503
- de Broglie, Louis, 773n; particle-wave dualism, 762
- de Donder gauge condition, gravitational waves, 564
- de Sitter algebra, and cosmological constant, 755
- de Sitter horizon, 293, 636f; thermal radiation from, 637
- de Sitter-Lanczos-Weyl-Lemaître spacetime, 642
- de Sitter length, inverse of, Hubble constant, 632
- de Sitter metric, history of, 642
- de Sitter precession, 549
- de Sitter spacetime, 456, 624–648, 625f; angular coordinates, 627; to anti de Sitter spacetimes, 664; causal structure of, 638, 639f; containing black holes, 635; d-dimensional, 625f; different forms of, 629; isometry group of, 625; iterative relationships of, 640; Kruskal-Szekeres-like coordinates for, 636f; Lemaître-de Sitter metric, generalized, 489; maximal symmetry of, 626; preview of calculation of, 148; Riemann curvature tensor of, 626; and space of spheres, 646; stereographic projection for, 641; table for, 643
- decomposition, of groups, 56f; definition of, 56–57
- decoupling: of internal and external geometries, 691–692; of matter and radiation in early universe, 516–517
- defining representation, of rotation group, 54
- deflection of light, 368f; by astrophysical objects, Soldner’s calculation of, 366–367
- degree, subdivision of, proposal by Ptolemy, 368n
- degrees of freedom, gravitational waves, 564
- degrees of polarizations, gravitational waves, 564
- delay, and radar echo experiments, 372
- delta function. *See* Dirac delta function
- Denken, before Integration, 133
- density fluctuations: in early universe, 521, 523–525; in inflationary cosmology, 533
- density waves, in static relativistic fluid, 234
- derivatives: covariant (*see* covariant derivatives);

- covariant differentiation); exterior, differential forms, 598; as fractions, 207; of functional, definition of, 116–117; Lie, 327–328, 331–332, 555; order of taking, 340; taken with respect to functions, 113; two, in Newton’s force law, 110
- Descartes, René: approach to questions in physics, 583n; concept of analytical geometry, 18; and concept of coordinates, 62n; versus Euclid, 48; and Euler characteristics, 726; theory of vortices, 578n; watching a fly, and concept of coordinates, 51n
- Deser, Stanley: ADM formulation, 693; curved spacetime, 580
- determinants: antisymmetric, 236; definition of, 60, 719; and intrinsic curvature, 84; introduction of, 39–40; of metric, 215–216; variation of, 381
- DeWitt, Cecile, on “vielbein,” 606n
- Dialogue Concerning the Two Chief World Systems* (Galileo), 17–18
- diffeomorphism, 398
- differences, infinitesimal, and Galileo transformations, 160
- differential equations: coupled ordinary, relativistic stellar interiors, 452; solving problems of motion, 26–27; in variational calculus, 126
- differential forms: applications of, 607–613; calculation of 5-dimensional scalar curvature, 684–685; Cartan’s structural equations, 607; Hodge star operation on, 723–725; jargon of, 604; in Kasner universe, 613e; language of, 596; and magnetic flux, 728; and vielbein, 594–606. *See also* topological entries
- differential geometry: classical, 96–109, 130; and concept of covariant derivatives, 100–101; logic of, 66; pioneering work of Gauss and Riemann, 90–91; of Riemannian manifolds, Cartan’s formulation of, 599–600
- differential operation, definition of, 598
- differential operators, 48; Killing vectors as, 587; shorthand notation for, 72; vector fields as, 319
- differentials: coordinate, 312; manipulations of, 161
- differentiation: along curves, 327; covariant, 320–333; dot notation, 96; of functionals, 114; of scalars and vectors, 318
- dilation: conformal generators, 615; generators of, 644
- dilation invariance, Coulomb potential, 620
- dilaton field, 680; and calculation of 5-dimensional scalar curvature, 686; in outgoing brane wave model, 704
- dimensional analysis, 120; of action, 346; to determine power of scattering amplitude, 717; for effective action of gravity, 711; of graviton scattering amplitude, 770; Hawking temperature, 14–15; scattering amplitude, 761
- dimensions: higher, Poincaré half plane in, 656; invisible, 672–673; large extra, 696–707
- Dirac, P.A.M., and quantization of magnetic flux, 728
- Dirac action, in Minkowskian spacetime, 605
- Dirac delta function: 3-dimensional generalization, 119; continuous variables in functional variations, 122; in electromagnetism, 251; in higher dimensions, 698, 701; introduction of, 26–27; and Kronecker delta, 36; as limit of a sequence of functions, 27f; momentum conservation, 740; and smooth functions, 33e; time, 229
- Dirac equation, commutation relations, 192
- Dirac-Feynman formulation. *See* path integral (Dirac-Feynman) formulation
- Dirac large number hypothesis, 778
- Dirac spinors, 604–605
- directional derivative, covariant differentiation and, 331
- discretization, of functional variation, 121
- disks. *See* accretion disks
- dissipative collapse, 520–521
- distance: of cities, and non-flatness of world, 66f; in Euclidean space, 174; in generally curved spaces, 181; Hubble units, 293; less important than angles, 620; luminosity, 298; measurement of, in spacetime, 180; minimal between points, 123–135; in Minkowskian geometry, 175; operational definition of, 291, 291f; of points, 174–175; proper, 296–297; shortest, 175, 176f, 545; spatial, in general curved spacetime, 290–292; of spheres in 3-spaces, 610; traversed, during lifetime of particles, 198. *See also* length; path
- distribution, compared to functions, 33
- divergence: covariant, 326, 332; notation in various coordinate systems, 78–79; in spherical coordinates, 81e; transformation, 321
- divergence theorem, generalized to 4-dimensional spacetime, 386
- dome, curvature at top of, 85
- dominant energy condition, 557
- Donder, Théophile Ernest de, application of action principle to gravity, 397
- Donoghue, J. F., treating general relativity as effective field theory, 773n
- Doppler effect: in accelerated frames, 282; relativistic, 185–186, 222
- dot notation, Newton’s, 29, 96
- dot product: of four vectors, 182; of vectors, definition of, 39
- dots, as symmetry symbol, 129
- “dropped” thought experiment, 280–283, 286
- Droste effect (for pictures), 375
- Droste’s solution, of Einstein’s field equation, 375
- dS/CFT (de Sitter / conformal field theories) correspondence, 787
- duality, electromagnetic, 255, 483
- dueling thinkers experiment, 7–9
- dust: cosmological, 387e, 495, 514; technical term, 421

830 | Index

- dust ball, collapsing, forming black hole, 422f, 423f
dynamic universes, 489–501
dynamical critical exponent, in condensed matter physics, 657n, 754, 758n
dynamical variables, 249; in continuum mechanics, 117; holding fixed in variation of action, 380; independence of, 133; specification of, 395
Dyson, Freeman: on Einstein's ideas of a field theory of gravity, 119; on Einstein's saying about vanity, 777; on loneliness of Einstein, 388; on non-quantization of gravity, 768–769; on quantization of gravity, 780, 788n
- early universe, 496–497; curvature term, 495; density fluctuations in, 521; history of, 515–529; structure formation in, 520, 522–523
earth: center of, and falling apple, 36; density of, 32; surface of, shortest path on, 275; theory of hollow sphere, 32
earth-moon distance, accurate measurements of as test of Einstein gravity, 366n
Eddington, Sir Arthur: and Chandrasekhar limit, 455; on costs of light, 369; and geometry of universe, 6; making Einstein a worldwide celebrity, 369–370; Royal Society expedition, 367
Eddington-Finkelstein coordinates, 431
edges, in topology, 725–727
“effect of inertia,” 276
effective action: for gravity, 711; Weyl's ansatz, 374
effective field theory, 782; and concept of action, 710; Einstein-Hilbert action, 709; general relativity treated as, 773n; and graviton scattering amplitude, 770; of gravity, 766; low energy, 711–712
Ehrenfest, Paul, letters from Einstein: on fears of going insane, 355; on perihelion motion of Mercury, 368
eigentime, 179f
eigenvalues, of matrix, usual determination of, 106
eigenzeit, 179f
Einstein, Albert, 150; and action for gravity, 339; anger at nostrification of his theory of general relativity, 396; *annus mirabilis*, explanation of light, 213; as classical physicist, 360; confusion concerning the metric, 404; $E = mc^2$, 209, 220–221, 232; 233f; early work, lack of vector notation, 46n; equivalence principle, 271; ether detection, experimental set-up for, 163; factor-of-2 error, 367, 370; field equation, 348f; *gedanken* experiments, on simultaneity, 7–9; on going beyond space and time, 787; greatest blunder, 393; and Grossmann, paper on variational principle for gravity, 396; happiest thought of life, 265, 278, 302; “hole argument,” 404; on influence of philosophers, 159; invention of Palatini formalism, 397; legacy to physics, 253–255; letter from Schwarzschild, 362; letters to Ehrenfest, 355, 368; letters to Kakuza, 693–694; letters to Sommerfeld, 344, 366, 580; longing, 337; and Lorentz, 168; on magic of relativity theory, 195; mathematical elegance of his theory, 777; on mysteries, 778; old man's toy, 267; penance, 500; on pure thought, 172; repeated index summation (*see* summation convention); “second greatest blunder,” 509–510; separation from his wife, 399; and Soldner's calculation, 366–367; stars made of nothing, 456; static universe, 509–510, 514; summation convention (*see* summation convention); understanding of gravity, equality of inertial and gravitational mass, 28; unfinished symphony, ripples in spacetime, 563
Einstein's clock, 166–173; in different frames, 167f
Einstein convention, in general relativity, 314
Einstein's equivalence principle, determination of Lagrangian in curved spacetime, 712
Einstein's field equation: $\frac{1}{2}$ -factor, and metric tensor formalism, 76; 5-dimensional, for 2-brane model, 700; acceleration or deceleration of cosmic expansion, 506–507; anti de Sitter spacetime, 651; for charged black holes, 477; for closed/open/flat universes, 493–494; coupled to Maxwell's equations, static solutions, 482–483; de Sitter spacetime, 627; derived by Palatini formalism, 395; determination of, 347–349; Droste's solution of, 375; easy solutions to, 557; Einstein's search for, 341–342; for empty spacetime, 347–348; flipping between spacetimes, 664–665; Kerr solutions on, 464; in Minkowski metrics, 563; modified by Arkani-Hamed, 754; non-determinism of, 403; nonlinearity of, 400; in post-Newtonian approximation, 577; in presence of cosmological constant, 357; for relativistic stellar interiors, 451; result of derivation of, 390; role of two powers of spacetime derivative, 402; solving, 358; and spacetime thermodynamics, 448–449; time-time component of, 498; traceless part of, 755; vacuum, 647e; variation of, 350
Einstein gravity: from ambitwistor representation, 739; connection to Yang-Mills theory, 782; cube of physics, 13f; discord with quantum physics, 768–769; features of, 777; replaced by something more fundamental, 785; roads leading to, 578–584
Einstein-Hilbert action: alternative form of, 397; cosmological action added to, 356; and cosmological constant, 712, 754; derivation of contracted Bianchi identity, 394; and differential forms, 725; and effective field theory, 782; effective field theory approach, 709; finding of, 344–346; general invariance of, compared to Maxwell action, 394; graviton coupling, 582; higher dimensional, 681; in Kakuza-Klein theory, 675; low dimensional terms, 782; quantum gravity limit, 444; things unknown to, 789n; and twistors, 739; variation of,

- 388; weak field action without, 572; and Weyl's ansatz to Schwarzschild solution, 374
- Einstein-Hilbert Lagrangian, determination of Einstein's gravity, 581
- Einstein potential, compared to Newtonian potential, planetary orbits, 371
- Einstein-Rosen bridge, 433; and coordinate singularities, 91–92, 92f
- Einstein tensor, 388; as result of variation of Einstein-Hilbert action with respect to metric, 394
- Einsteinian mechanics, and cube of physics, 13f
- elastic string: as example of variational calculus, 113; hanging under force of gravity, 114f
- electric charge: role for photon, 383. *See also* charge electric dipole moment, of atom, and action, 715
- electric field, 245; relativistic unification, 247
- electrodynamics: initial value problem in, 404. *See also* Maxwellian electromagnetism
- electromagnetic action, 244, 250–251; in expanding universe, 333; local, 246. *See also* Maxwell action
- electromagnetic coupling constant, 767
- electromagnetic current: conservation of, 225; as flat space analogous to energy momentum tensor, 379; relativistic form of, 226
- electromagnetic duality, 255, 483
- electromagnetic field: as collection of infinite number of harmonic oscillators, 382; coupling to charged particles, 250; derivation of equation of motion, 385; determination of, 338, 342n; energy density of, 255; energy momentum tensor in, 381; in Kaluza-Klein theory, 691; and Lorentz vector potential, 244; Maxwell action in Minkowskian spacetime, 381; and Maxwell's equations, 252; motion in curved spacetime, 301; and mystery of light, 162; at particle position, 246; treated as superposition of modes, 746
- electromagnetic field tensor, 244; dual, 255; gauge invariance, 249; relativistic curl of a 4-vector, 252
- electromagnetic gauge transformations, similarity to coordinate transformations, 564
- electromagnetic interaction, compared to gravitational interaction, 768
- electromagnetic potential, in fifth dimension, 677
- electromagnetic waves: cross section for scattering on atom or molecule, 715; momentum of, derivation of Einstein's formula, 232
- electromagnetism: 4-dimensional, 720–721; in curved spacetime, 325–326; described by differential forms, 598; finite sized objects in, 714–715; fixed gauges, 564; in flat spacetime, 382; gauge invariant derivative in, 342, 353n; Maxwell's laws of, and Galilean transformation, 20; restrictions imposed on by Lorentz symmetry, 339; role of signs in, 382; similarities to gravitational waves, 568; from special relativity, 244–246; theory of, development of, 253; unification with gravity, 674–676
- electromagnetism analogy, Einstein's search for the metric, 404
- electrons: collisions with photons, 222f; degenerate, 455; delayed recombination in early universe, 516–517
- electrostatics, mathematical treatment leading to Maxwell, 582
- electroweak interaction, 527, 765; in M versus R plot, 14f
- elementary particles, masses of, 16n
- elementary physics, definition of mass, 213
- elementary scalar fields, 759n
- embedding: of curved spaces in higher dimensional flat spaces, 85–86; of surface, determination of curvature, 90
- embedding space, geodesics in, 645
- Emerson, Ralph Waldo, dictum of, 235n
- empty spacetime: Einstein's field equation for, 347–348; gravity in, 362
- energy: dark (*see* dark energy); elastic, of hanging string, 113–114; exact meaning of, 383; extraction from Kerr black holes, 470; of membrane, rotational invariance, 118; not conserved, 27; search for minimization function, 114; spatial density of, 228. *See also* gravitational energy
- energy conditions, 557
- energy conservation, 26, 153; historical considerations, 387n; around rotating black holes, 459; in static isotropic spacetime, 310
- energy density: constant, filling universe, 356; electromagnetic field, 255; in flat spacetime, 382; ratio to critical density, 505; replacing mass density, in Newtonian gravitational potential, 379n; as rotational scalar, 226–227; and scale factor of universe, 496f; of universe, 359, 504; vacuum, 749. *See also* dark energy density
- energy distribution, governing curvature of spacetime, 390
- energy functional: boundary conditions, 116; of a membrane, 118; minimization for Newtonian gravity, 119
- energy level splitting, inverse of, at cosmological time scale, 768
- energy momentum, role for graviton, 383
- energy momentum conservation, 227; and Bianchi identity, 393; derivation by using cosmological action, 387e; and general invariance of matter action, 383–384; in gravitational field, 386
- energy momentum pseudotensor, 386
- energy momentum tensor: assumptions about, 557; called stress energy tensor, 228; covariant conservation of, 384; curved spacetime generalization of, 380; in electromagnetic field, 381; of electromagnetic field, tracelessness, 381; under Lorentz transformation, 226–227; “new

832 | Index

- energy momentum tensor (*continued*)
and improved," 712; of perfect fluids, 230, 492;
for scalar action, 387e; sign considerations,
380; slowly rotating bodies, 570; as source for
gravitational field, 379; total, disappearance of,
394; and variation of Maxwell action with respect
to metric, 394. *See also* stress energy tensor
- energy per unit mass, as conserved quantity, 30
- energy scale: grand unified theory, 767; introduced
by gravity, 770
- energy variations, calculated, 115
- entanglement: and mysteries of quantum mechanics,
789n; and quantum gravity, 771
- entropy: Bekenstein-Hawking, 441–442, 444; of
black holes, 15, 436, 441, 448, 788n; lack of
knowledge behind horizon, 648n; Penrose process
area theorem, 472; per particle, relativistic fluid
dynamics, 234; in spacetime, mystery of, 234; of
universe, 527
- Eöt-Wash group, 260
- equation of motion, 155; by action principle, 146;
near boundary, 665; electromagnetism, 245–246,
385; energy conservation of, 153; generalized
for particles under external force, 190; particles
in potential of, 135; for universe, 357; from
variational principle, 137
- equation of motion approach, to Einstein gravity, 396
- equation of state: of ideal gas, 231; of universe, 359,
494, 496–497
- equations: $E = mc^2$ (*see* Einstein); as expression of
physics, 47; versus identities, 403. *See also specific
equations*
- equator, length of, squashed sphere, 80e
- equilibrium, hydrostatic, relativistic stellar interiors,
453–454
- equilibrium macrostates, Bekenstein-Hawking
entropy, 441, 444
- equipartition theorem, Planck and, 789n
- equivalence principle, 271; and definition of
energy momentum, 386; falling living room as
example, 265–266; and general covariance, 286;
motion in curved spacetime described by, 302;
nonimpossibility of deleting Feynman diagrams,
756–757; old man's toy, 267; predictions of,
280; and relativistic stellar interiors, 451; and
symmetry, 317–318
- ergoregion, 467, 469–471
- escape: from black hole, 427, 483; from gravity,
nonimpossibility of, 717n
- escape problem: in Kaluza-Klein theory, 673–674;
with large extra dimensions, 696–697
- eternal black holes, 421–422; Kruskal-Szekeres
diagram, 426–427; Reissner-Nordström, 479
- ether: detection experiments, 163; as dynamical
variable, 783
- Euclid: and curves, 189; versus Descartes, 48; famous
axiom, and curvature of spacetime, 552; on the
non-existence of royal road to geometry, 42;
shortest path between two points, 4
- Euclidean anti de Sitter space, boundary of, 662
- Euclidean ball, 663; boundary of, 664
- Euclidean geometry: flat, 6; rotation invariance of,
190; specification of, 175
- Euclidean group, as symmetry group of physics, 755
- Euclidean metric: in hyperbolic spaces, 93; inducing
curved space metric, 86; locally flat, second order
corrections to, 88; for spaces of any dimension, 87
- Euclidean plane, conformal Killing vector fields,
623e
- Euclidean space: 2-dimensional, definition of, 41;
curves in, 96–97; d-dimensional, 42, 49–51;
described with different coordinates, 62–63;
distance in, 174; as example for Killing vector
fields, 587; object analogs in twistor space, 742;
paths lengths in, 190; surfaces in, 98–109
- Euclidean thinking, trap of, 180
- Euler, Leonhard, variational calculus, 120
- Euler characteristic, 725–726
- Euler equation: in fluid dynamics, 164; relativistic
and fluid dynamics, 234
- Euler-Lagrange action, for material particles, 207
- Euler-Lagrange equation, 116; action principle, 138;
fields, 119; multiple unknown functions, 123;
simplification of, Poincaré half-plane, 133
- events: coordinates of, in discussion of simultaneity,
200; definition of in spacetime, 177; horizon
of, 293, 536; of pole in the barn problem, 203;
separation of, 160, 166; spacetime locations, 195;
and worldlines, in special relativity, 195
- expanding universe: acceleration or deceleration of
expansion, 499–500, 506–507; closed/open/flat,
494, 497–498; communication in, 293–294;
curved, 489; de Sitter spacetime, 456–457, 627,
630; with differential forms, 608; distances
in, 292–293; earth-moon distance not growing
because of, 289; electromagnetic action in, 333;
expansion rate, discovery of, 359; exponentially
expanding, 293–294, 357–358, 631, 642–643; and
Hubble, 500; light cones in, 294, 294f; metric
tensor of, off-diagonal components, 292; and
positive cosmological constant, 392; without
Einstein's field equation, 645. *See also* universe
- expansion parameter, determination of, 556
- exponential, of matrix, 41
- exponential function, and rotations, 41
- extended objects, motion of, 714
- extensive quantities, 441
- exterior derivative, 599; differential forms, 598
- external forces, influencing motion in curved
spacetime, 301–302
- external potential, translation invariance of, 242
- extra dimensions, large, 696–707
- extraction of energy, from Kerr black holes, 470
- extremal black holes, 467–468; charged, 478, 481;

- “dangers of extremes,” 484; distance around, 469;
first and second law of thermodynamics, 473; just
sitting there, 482–483
- extreme ultra infrared regime, 786; and cosmological
constant paradox, 750–751
- extremizing a function, with constraint, 109
- extremum, determination of type, 117
- extrinsic curvature, 5–6; defined by Gauss, 107;
and matrix eigenvalues, 84–85. *See also* intrinsic
curvature
- faces, in topology, 725–727
- fall: through event horizon, 412; into rotating black
holes, 463–464, 470, 472
- falling apple: from hanging string to, 137f; and Isaac
Newton, 268
- falling living room analogy, 265–266
- families, of quarks and leptons, 786
- family problem, mystery of, 7
- far field, of gravitating system, 576
- Faraday, Michael: conception of scientists, 9n; flux
picture, 697; and magnetic flux lines, 728
- fate of universe, 507–509
- Fermat, Pierre, controversy over birth year, 136n
- Fermat’s least time principle, 4; as analog to
Einstein-Hilbert action, 789n; teleological flavor
of, 136
- Fermi, Enrico, theory of weak interaction, 765
- Fermi normal coordinates: locally flat, 558f; metric
in, 559, 561; motivation of, 557
- Fermi pressure, and Chandrasekhar limit, 455
- Fermi-Walker transport, 193e
- fermions: fundamental, 683; as mystery of physics,
781–782; as open strings, 696
- Feynman, Richard P., 145; curved spacetime, 580;
“shut up and calculate,” 445
- Feynman diagrams: of antimatter, 206f; for
cosmological constant, deletion of, 756–757;
for gluon scattering, 735–736, 738; for graviton
scattering, 738; and worldlines, 237n
- Feynman’s path integral formalism. *See* path integral
(Dirac-Feynman) formulation
- Feynman’s path to rescue a drowning girl, 3–4, 4f
- Feynman propagator, for graviton, 573
- fictitious forces, 278–279
- field equations: in Minkowski metrics, 563;
Nordström’s theory, 579. *See also* Einstein’s field
equation
- field strength: connection with curvature, 602n;
relation to electric and magnetic fields, 382
- field theory: classical, 119, 361; quantum (*see*
quantum field theory); topological, 719–728
- fields: conceptual jump from many particle case,
400; to describe universe, 384; notion of, 119; and
particles, 145–146; understanding of, 783
- fine structure constant, 767
- “finger of God” problem, 703, 705
- finite sized objects: in electromagnetism, 714–
715; in gravitational field, 716–715; scattering
amplitude for gravitational wave, 717; sensitivity
to variations, 716; and tidal forces, 716–717. *See
also* black holes
- Finkelstein, David, Eddington-Finkelstein
coordinates, 431
- first acoustic peak, 523–525; effect of curvature on,
525f
- first law of thermodynamics: black holes, 472–473;
and pressure of universe, 360n
- first order formalism for gravity, Palatini formalism,
395
- first stars, 519
- “fixed latitude” circle, curvature of, 80e
- fixed points, in cosmic diagram, 511f
- flame, of falling candle, 268, 271
- flat coordinates, locally, 130, 132; as trick in variation,
389
- flat plane, curvature of, 105
- flat space, 65; conformally, 80–81e; description by
Boyer-Lindquist coordinates, 78; and everyday life,
82–83; metric, 77
- flat spacetime: with conformal algebra, 615;
electromagnetism in, 382; Minkowskian,
governing action of, 581; twistors in, 729–745
- flat universes, 296–297, 491; age of, 513; critical
density of, 497–498; curvature effect on CMB
fluctuations, 526; Einstein’s field equations, 493–
494; observational evidence for, 505; stability of,
512
- flat world, 88
- Flatland* (Abbott), 671
- flatness, local. *See* local flatness
- flatness problem, 531
- floor, rushing up to meet apple, 270f
- flow: in cosmic diagram, 510–512; described by
geodesics, 556; going with the, 328
- fluctuations: of density in early universe, 521, 523–
525; in inflationary cosmology, 533; quantum,
436, 446–447, 533
- fluid dynamics: Euler equation, 164; Galilean
invariance of, 164; symmetry approach to, 164
- fluids: incompressible, 454; motion of, 230,
556; perfect, 229, 451, 492–493; 704–705; as
visualizations of vector fields, 327
- flux picture, Faraday’s, 697
- fly in car, velocity of, 162–163
- foamlike structure, of universe, 754, 758n
- foliation: Kaluza-Klein theory as, 689–690; spherically
symmetric mass distribution, 305–306
- force: central, 28, 36; external, influencing motion in
curved spacetime, 301–302; fictitious, 278–279; as
function of space, 26; as function of time, 26–27;
per unit area, stress as, 228
- forms, closed, 604
- Fourier analysis, and inverse square law, 697–698

834 | Index

- Fourier space, gravitational field in, 758n
Fourier transformation, of scattering amplitude, 736, 740
fractional quantum Hall effect, fluids, 789n
frame dragging, 460–461, 465–466; deformed by rotating body, 460f; etymology, 476n; with Lense-Thirring precession, 550
frame field, 606n
frames. *See* reference frames
free fall, into rotating black holes, and first and second law of thermodynamics, 472. *See also* fall
free Maxwell's equations, 251
free particles, 302; action of, 143, 162; motion of, 180; noninteracting, 221
“free” variables, in variational calculus, 116
freely falling observers, metric for, 561
Frenet-Serret equations, 97
frequency, seen by different observers, 185
frequency dependence, of scattering of gravitational wave (or graviton), 717
frequency shift: relativistic, 186; in relativistic Doppler effect, 222
Freundlich, Erwin, solar eclipse expedition to Crimea, 370
Friedmann, Aleksandr, 501
Friedmann-Robertson-Walker universes, 296, 491; in outgoing brane wave model, 704
Frost, Robert, and mass density transformation, 579
functional derivative, definition of, 116–117
functional variation, 114–115; alternative approach, 121–122
functionals: energy, of a membrane, 118; general, of multiple functions, 123; notation of, 114
functions: and distributions, 33; variational calculus, 115. *See also specific functions*
fundamental constants, three needed, 12
fundamental equations, on glass windows, 138
fundamental fermions, 683
fundamental interactions: action principle description of, 141; unification of (*see grand unified theory*)
fundamental principles, 12
fundamental representation, of rotation group, 54
funnel analogy, misleading for black holes, 432
fusion, nuclear, compared to accretion disk radiation, 415
future light cone, 177–178; particle movement in, 178f
galaxies: formation in early universe, 519–520; forming of, and anthropic principle, 757; as masspoints on geodesics, 554. *See also* universe
“galaxy far far away,” 241–246
Galilean invariance, and fluid dynamics, 164
Galilean limit, of past light cone, 179f
Galilean transformation, 18–20, 19f, 159–160; accelerated frames, 276–277; modifications of, independence from observer, 168; necessary modification of, 166; observed velocities, 161
Galileo: brachistochrone problem, 120; and free fall, 268; law of acceleration, 140; versus Maxwell, 159; relativity principle, 17–19, 159; vision on flying of butterflies, 19f
Galison, Peter, and special relativity theory, 18n
Gamow, George, 177; and Einstein's great blunder, 393n; stars made of nothing, 456
Gamow principle, 515–529; understanding of cosmology, 778
gases: for cosmology, 230; nonrelativistic, 231, 454; relativistic, derivation of speed of sound, 235
gauge: harmonic, 564; transverse-traceless, 565
gauge condition, harmonic, 573
gauge connection, 602
gauge fields, emerging from lattice Hamiltonians, 787
gauge freedom, and initial value: in Einstein gravity, 402; in Maxwell electromagnetism, 401–402
gauge/gravity duality, 649
gauge invariance, 248–250; in Kaluza-Klein theory, 672
gauge invariant derivative, in electromagnetism, 342, 353n
gauge potential: of 2-dimensional solid state structures, 721; as dynamical variable, and energy momentum tensor, 381; and spinor fields, 789n; Yang-Mills, 682, 688
gauge symmetry, local, in higher dimensional theories, 682
gauge theories: and anthropic principle, 757; nonabelian, decoupling of geometries, 692; (non)abelian, 681; topological terms in, 720–721
gauge transformations: as 5-dimensional coordinate transformation, 673; similarity to coordinate transformations, 564; strong gravitational sources, 575
Gauss, Carl Friedrich: determination of curvature of space, 65, 104–105; Theorema Egregium, 90–91
Gauss-Bonnet theorem, 727
Gauss's equation, definition of, 99
Gauss's law: and evolving time, 402; and gauge theory, 401
Gaussian normal coordinates, 298
gears, function of, 109n
gedanken experiments: “accelerated”/“dropped,” 280–283; by Einstein, 166; by Galileo Galilei, 269
Gell-Mann, Murray: on quantization of gravity, 583n; what is not taboo is a commandment, 361n
general coordinate invariance, 305–306; determination of action for gravity, 344, 346; in Kaluza-Klein theory, 672
general coordinate transformations, 312, 318; invariance of physics under, 403
general covariance, 285

- general curved spacetime, spatial distance in, 290–292
- general invariance: of Einstein-Hilbert action, compared to Maxwell action, 394; of matter action, and conservation of energy momentum, 383–384
- general relativity: abstract of, 20; as effective field theory, 773n; and Hamilton’s principle, Lorentz’s paper on, 397; modifications with respect to horizons, 784; solar system tests, 309; tensors in, 312–319. *See also* gravity
- generalized uncertainty principle, 769
- generators: breakup into subgroups, 663; of conformal algebra, 615, 617; of Lie algebra, 49, 51; of rotation, 192; of rotation group, 40; of $SL(4, R)$ group, 737, 739; of $SO(3)$ group, 44
- genus, in Euler characteristics, 726
- GEO600, gravitational wave detector, 577n
- geodesic deviation, 552–561, 554; and Lie derivative, 555
- geodesic equation, 128; alternative derivation of, 130; Christoffel symbols of, 129; comoving coordinates, 298; curved spacetime, 277–278; invariance on rescaling, 559; motion in curved spacetime, 289; and parallel transport, 545; presence of external forces, 301; rotating black holes, 459; transformation of Christoffel symbol, 330–331
- geodesic problem: free parameters, 124f; variational calculus, 123
- geodesics: at black holes, 426–427; collections of, 554; congruence of, 554; covariant derivative, 553; determination on Poincaré half plane, 133; distance of two nearby, 552, 553f; in embedding space, 645, 665; family of, 134; geometric construction of, 133n; intersection of, 134; lightlike, 292, 646; on Poincaré half plane, 134f; separation of, 552; on spheres, 127; timelike, 645
- geodetic precession, 549
- geometrical entities, in topology, 725–727
- geometrical view: of Kaluza-Klein theory, 691–693; of special relativity, 582
- geometrostatics, in higher dimensional theories, 693
- geometry: analytic, role of coordinates, 48; conversion factor between physics and, 211; covariant derivative from, 323; dynamics of, in higher dimensional theories, 693, 693f; and invariance, 42–43, 48; of Minkowski spacetime, 174–193, 238; non-existence of royal road to, Euclid’s remarks, 42; of points, isometry, 585; of relativistic point particle action, 210; of rotation groups, 191; and significance of coordinates, 68; of a world, 6. *See also* differential geometry; Riemannian geometry
- Ghostwriter, The* (Roth), 254
- Gibbons, Gary, discovery of Hawking radiation, 449n
- Gibbons-Hawking radiation, 638; mystery of, 637
- Gibbons-Hawking-York boundary term, 399n
- global character of space, versus local, 76–77
- global positioning system (GPS), 287, 291
- globe, curves of constant latitude on, 92
- gluon scattering: amplitudes for, 785; Feynman diagrams for, 735–736; in terms of abinitwistors, 738; in terms of helicity spinors, 735–736
- gluons: in brane models, 696; in early universe, 526
- GMT (Greenwich Mean Time), 133n
- God: existence of, 520; “What is greater than God?” puzzle, 789n
- Goldberger, Murph, on his aunt, 321n
- golden age of cosmology, 491
- “golden” guiding principle, in theoretical physics, 338
- Gordon, Walter, Klein-Gordon equation, 694
- Grace, Louis, constructor of old man’s toy and of war chariot, 267
- graceful exit problem, 534
- gradient: definition of, 61; notation of, 54; transformation of, 320
- grand unification, mystery of, 527n
- grand unified theory, 527; and charge, 786; in early universe, 518; energy scale, 767; and higher dimensional theories, 681; and Kaluza-Klein theory, 672; in M versus R plot, 14f; magnetic (anti)monopoles, 532
- Grassmann variables, 606n; and supertwistors, 739n
- gravitating system, far field of, 576
- gravitation, field equation for, Einstein’s search for, 341–342
- gravitation law, Einstein’s belief of inconsistency with principle of causation, 404
- gravitational collapse, of spherically symmetric dust cloud, 373
- gravitational constant, 11; time dependent in brane models, 707
- gravitational coupling, 768
- gravitational energy, 580–581; binding, 455–456; of hanging string, 114
- gravitational field: classical, quantum particles in, 771; completion and promotion of, 218; conservation of energy momentum in, 386; determination of, 338; dynamics of, 146; and equivalence principle, 271; finite sized objects in, 716–715; in Fourier space, 758n; in great distance of black hole, 574; momentum of, 580–581; nature of, 218–237, 231; quantization of, 582; strong stationary source, 574; as tensor field, 231
- gravitational field limit, Newtonian gravity as, 391
- gravitational interaction, 581; compared to electromagnetic interaction, 768
- gravitational Lagrangian, 339
- gravitational lensing, 370–371
- gravitational mass, equality to inertial mass, 28, 257, 268–269
- gravitational potential: action principle, 145; around black holes, 410–411, 411f; connection to mass

836 | Index

- gravitational potential (*continued*)
distribution of, 578; Newton's, 119; satisfying Poisson's equation, 708
"gravitational radius," of massive objects, 764
gravitational redshift, 259; "accelerated"/"dropped" gedanken experiments, 282–283; at black hole horizon, 412; measurement of, 284, 287; motion in curved spacetime, 303–304; and time dilation, 284
gravitational sources: approximations for, 570; strong: gauge transformations, 574, 575; weak field approximation, 569–570
gravitational waves, 563–577; astronomy with, 563; from binary systems, 714; degrees of polarizations, 564; detection of, 566, 577n; deviating Minkowskian spacetime, 571–572; emission of, 567; frequency dependence of scattering of, 717; localized packets of, 577n; propagation of, 566, 568; removal by coordinate transformation, 577; similarities to electromagnetism, 568; speed of, 579; time and gravity, 579. *See also* gravitons
gravitomagnetic field, 571
graviton coupling: Einstein-Hilbert action of, 582; to electron line, 756
graviton mass, 785
graviton scattering, 782; frequency dependence, 717; off each other, 731; scattering amplitude of, 739, 761, 770; unitarization by formation of black hole, 765
graviton spin, quadrupole radiation, 571
gravitons, 566; Feynman propagator for, 573; fluctuating, 712; from gravitational waves, 780; Hawking radiation of, 439, 450; interaction among, 738–739; in large extra dimensions, 696; from lattice system, 787; massless, 718; as non-bound states, 785; versus photons, 768; propagator, momentum of, 786; role of energy momentum for, 383; self interaction, 582; and spatial direction, 785; of spin 2, 697. *See also* gravitational waves
gravity: action for (*see* Einstein-Hilbert action); as classical probe, 771; classicalization of, 766; completely altering causal structure of spacetime, 780; connection with time, 579; container for, 649; cubic vertex for, 744; Dysonian view on quantization of, 780, 788n; effective action for, 711; effective field theory of, 766; without Einstein-Hilbert action, 771; in empty spacetime, 362; as fictitious force, 279; first order formalism for, 395; high energy behavior of, 767–768, 782; indifferent to the universe, 778; induced, 770; inherent instability of, 520; introducing an energy scale, 770; introducing natural quantities, 764; linearized, 563–577, 758n; mystery of, 778–779; "naked" singularities, 480; nonlinearity of, 571; non-quantization of, 768–769; omniscience of, and cosmological constant paradox, 745; as part of larger structure, 786; quantization of, 780; quantum, 439, 443–444; and spacetime, origins of, 787; and spacetime curvature, mystery of, 276; speculative thoughts about, 788; surface, 473; symmetry imposed on, 254; theory of, as analog to theory of light, 789n; time and, 257–258; time dilation caused by, 258–259, 284, 304, 412; true scale of, 698–700, 702; unification, 674–676, 767–768, 780; unimodular, 755–756; universality of, 258, 269–270, 275–276. *See also* general relativity; Newtonian gravity
gravity attraction, and strong energy condition, 562n
gravity express, 33
gravity potential, particles moving in, tensor notation of, 57–59
Gravity Probe B, launch of, 551n
great circles, 127; on earth's surface, 275; movement of, particle on sphere, 148; on sphere, determination of curvature, 105
Greek symbols, in tensor notation, 63
Green's function: different determinations of, 573; for gravitational waves, 567–568; for quantum fluctuations, 447
Greenwich Mean Time (GMT), 133n
Grimm stories, and quantum gravity, 773n
Grossmann, Marcel, and Einstein: paper on variational principle for gravity, 396; search for field equation for gravitation, 353
ground states: degeneracy of, 723; in string theory, 757
group theory: and commutation, 49; and counting, 56–57; of exponentially expanding universe, 642–643; metric for expanding universe, 645; of universe, coset manifold, 644
groups: 2-by-2 matrix as generator of, 663; decomposition of, 56–57, 56f; Eöt-Wash, 260; Euclidean, as symmetry group of physics, 755; isometry (*see* isometry group); Lie, characteristic of, 50; Lorentz (*see* Lorentz group); Poincaré, transformations and translations of, 666; renormalization, and scaling, 754; representation of, 225; requirements of, 193; rotation (*see* rotation groups); subgroups, 57, 663. *See also specific groups*
Gullstrand, Allvar, and Painlevé-Gullstrand coordinates, 417
Gupta, Suraj, and curved spacetime, 580
gyroscopes: gravitational precession of, 465; launched with satellite, 549; precession of, 549–551
 \hbar , explanation of symbol, 773n
Hale, George, proposition of solar eclipse observation to test Einstein's theory, 367n
half plane, Poincaré: with differential forms, 608; and metric, 67–68

- Hamilton's principle, and general theory of relativity, Lorentz's paper on, 397
- Hamiltonian: derived from Lagrangian, 144; leading to gauge fields, 787; of zero value, in quantum systems, 723
- handle, in Euler characteristics, 726
- hanging membrane, 118f; as generalization of hanging string, 118
- hanging string: transition to falling apple, 137f; and variational calculus, 113–123
- harmonic gauge condition: in quantum field theory, 573; strong gravitational sources, 575; weak field, 564
- harmonic oscillator: actual path of, 148e; annihilation and creation operators, 447; energy of, 758n; in field theory (classical and quantum), 361; in quantum mechanics, 746; symmetry and invariance, 242
- Harrison, E. R., on masks of the universe, 779
- Hawking, Stephen, 14
- Hawking radiation, 436–450; derived from quantum field theory, 780; fundamental paper on, 14–15; of gravitons, 450; history of discovery, 449; as one key for understanding quantum gravity, 748. *See also* Gibbons *entries*
- Hawking temperature: determination of, 444–445; dimensional analysis, 14–15; and entropy, 441; of Schwarzschild black hole, 436
- heat, understanding of, 786
- Heaviside, Oliver, and Maxwell's equations, 405n
- Heisenberg picture, of quantum physics, consequences for gravity, 771
- Heisenberg's uncertainty principle. *See* uncertainty principle
- helicity, of gravitational waves, 734
- helicity spinors, 731; Lorentz invariance, 734; power of, 735; scattering amplitude expressed in terms of, 734–735
- helicity states, of graviton in QFT, 566
- helium: liquidity at zero temperature, 748; primeval nucleosynthesis of, 518
- hell, and hollow earth theory, 32
- Heron of Alexandria, 149n
- hierarchy problem, 699
- Higgs mass term, 712
- Higgs mechanism, 679
- high energy behavior, of gravity, 767–768, 782
- high-energy physicists, particle physicists renaming themselves, 713n
- high energy physics: linkage to low energy physics, 752; naturalness doctrine in, 749–750
- high temperature superconductivity, 789n
- higher dimensional Einstein-Hilbert action, 681, 782
- higher dimensional metric, 682
- higher dimensional spaces: definition of, 43–44; embedding curved spaces, 85–86; rotation as freedom left, 88; rotations in, 44–45, 49–51
- higher dimensional spheres, metric of, 80e
- higher dimensional theories: dynamics of geometry, 693f; Kaluza-Klein / Yang-Mills, 680–682; string theory, 695
- higher energies, larger structure of energy, 786
- Hilbert, David, and Einstein-Hilbert action. *See* Einstein-Hilbert action
- Hilbert-Einstein priority dispute, on field equations, 396
- historical digressions, Newton's constant, 31–32
- Hodge star operation, 602; on differential forms, 723–725
- hole argument, Einstein's, 404
- “holes,” number of, 726
- hollow earth theory, 32
- holographic principle, 441; black hole entropy, 15; mapping of spacetime, 649
- homogeneity problem, 531
- homogeneous space, 289, 292, 491; definition of with Killing vectors, 588; in outgoing brane wave model, 704
- horizon: crossing in static coordinates, 635; de Sitter, 293, 636f; detection by indirect local measurements, 789n; event vs. particle, 536; inner, Kerr black holes, 469; outer, Kerr black holes, 468–469; at Schwarzschild radius, 412, 419, 431–432; sound, 524; as source of confusion, in Schwarzschild solution, 376n; as switch of Killing vector, 631
- horizon problem, 530–533
- “How do you do?” 333
- Hoyle, Fred, and Big Bang, 498
- Hubble constant, 504, 632; and communication in expanding universe, 293; determination of, 391; discovered by Lemaître, 501; in inflationary cosmology, 535
- Hubble length, and critical density, 514
- Hubble parameter. *See* Hubble constant
- Hubble radius, of universe, 711; and photon mean free path, 517
- Hulse, Russell A., detection of binary pulsar, 563
- humans: distance between head and toe in spacetime, 658f; existence of, and anthropic principle, 757
- hydrogen atom, and $SO(4)$ group, 49n
- hydrostatic equilibrium, relativistic stellar interiors, 453–454
- hyperbolic coordinates: angle, 628; anti de Sitter spacetime in, 661
- hyperbolic radial coordinate, 653–654
- hyperbolic shell, momentum restricted to, 220
- hyperbolic spaces, 92–93, 296, 627; as coset manifolds, 590; cosmological principle, 491; line element of, 628
- hyperboloid, of rotation, de Sitter spacetime, 625
- hypersurface, spacelike 3-dimensional, 693f
- ideal gas, equation of state of, 231

838 | Index

- identities, versus equations, 403
- identity matrix, definition of, 39, 63
- illusion, of time, 177
- images, method of, 620; conformal algebra for, 620
- imaginary time, in derivation of Hawking temperature, 445–446
- impact parameter, light deflection: around black hole, 416; and gravitational lensing, 371; in spacetime, 309, 309f
- incompressible fluids, 454
- index-free representation of vector fields, 319
- index notation: under coordinate transformations, 71–73; fear of, 32, 53; of quantities (in general), 32; and rotations, 44–45; $SO(D)$ group, 49; use of, 43–44
- index summation. *See* summation convention
- indexed objects, handling by human mind, 607
- indices: changes in general relativity, 547; contraction of, 46n, 345; conversion with vielbein, 596, 603; different, 595; explosion of, Einstein’s gravity, 131; of four-vectors, 182; magic of, 140; naming conventions, 608; object without indices not transforming as scalar, 719–720; order of, memorization, 132; repeated, contraction of, 58n; in Riemann curvature tensor, 351; sea of, and differential forms, 599; summation over (*see* summation convention); upper and lower, 64, 314–316; and vectors, scalars and tensors, 73–74. *See also* index notation
- induced gravity, 770
- inertia: “effect” of, 276; law of, action principle, 143; Sylvester’s law of, 193e
- inertial force, 278
- inertial frames, and locally flat coordinates, 278
- inertial mass: equality to gravitational mass, 28, 34, 257, 268–269; Galilean transformation of accelerated frames, 276
- infinitesimal area, enclosed by closed curves, 547
- infinitesimal boosts, of Lorentz transformation, 187
- infinitesimal differences, 160
- infinitesimal rotations, 40
- infinitesimal segments, space and time experience of, 180
- infinitesimal transformations: as generators of conformal algebra for Minkowski spacetime, 615; in Lorentz algebra, 187
- infinitesimal volume element, and metric tensor formalism, 75–76
- infinity, and human mind, 779
- inflation of universe, 534–535; and cosmological constant paradox, 751; and scalar fields, 788n
- inflationary cosmology, 530–536; cosmological constant in, 534; Hubble parameter in, 535
- inflaton field, 534
- inflaton potential, 535f
- information paradox, of black holes, 439
- infrared regime: extreme ultra, and cosmological constant paradox, 750–751; linkage to ultraviolet regime, 752
- inherent instability in dynamics with higher powers of time derivative, discovery of, 338
- initial value formulation, in numerical relativity, 693
- initial value problems: in electrodynamics, 404; and numerical relativity, 400–405
- initial values: on Cauchy surface, 402; evolving in time, basic scheme of, 400–401; and gauge freedom, 401–402
- initially static branes, 707
- inner horizon, of Kerr black holes, 469
- innermost stable circular orbit (ISCO), 414, 474
- instability, inherent, of gravity, 520
- integrability condition, and determination of potential, 36
- integrands, analytically continued into complex plane, 732
- integration: by parts, 115, 326; variational calculus, 116; over volume, at specific time, 226
- integration measure, covariant differentiation, 326
- interaction: with classical fields, 221n; contained in matter action, 384; contribution to energy momentum tensor, 383; as part of matter action, 383
- interaction potential, particle movement in, 162
- interferometry, detection of gravitational waves, 567
- internal coordinates, 675; for points in spacetime, 689f
- internal space: emergence of Yang-Mills theory, 688; spacetime perpendicular to, 689
- intersection, of geodesics, 134
- intrinsic curvature, 5–6; counting for characterizing of, 110; as defined by Gauss, 107; determination without knowledge of embedding of surface, 90; versus extrinsic, 107–108; and matrix eigenvalues, 84–85; metric as prerequisite to calculate, 90–91; of spacetime, compared to extrinsic, 85
- intrinsic lifetime, of particles, 198
- Introduction to the Theory of Relativity* (Bergmann), 376n
- invariance: coordinate, general, 305–306, 672, 682; CP, violation of, 528, 683; difference from covariance, 47; Galilean, of Newtonian mechanics, 161; gauge, 248–250, 672; and geometry, 42–43, 48; local coordinate, in higher dimensional theories, 682; Lorentz, 242, 253; Noether’s theorem, 310; of physical laws, 46–48; of physics under general coordinate transformation, 403; Poincaré invariant brane, 707; rotational, 118, 697; scale and conformal, 621; of separation, 623e; of string action, 216e; and symmetry, 242–243; time reversal, 416–417, 500; under transformations, of Poincaré coordinates, 657; translation, 242, 303–304
- invariance group of physics, rotation group as, 755
- invariant curvature, 339

- invariant scalar products, in parallel transport, 544
invariant tensors, definition of, 59–60
invariants, topological, 725–727
inverse Compton scattering, 235e
inverse length, dimension of, 120
inverse light speed, analogy to cosmological constant paradox, 754
inverse metric, 315
inverse square law, 120; spatial dimensions, 122
inverse temperature, 445
inversion, of spacetime, 743–744
invisible dimensions, 672–673
irreducible representations, 54–57
ISCO (innermost stable circular orbit), 414, 474
isometric conditions, for metric, 586
isometric spacetime, around rotating black holes, 459
isometry, 585–593; conformal transformations, 614; hidden underlying, 631; intuitive account of, 589; light cone coordinates, 631
isometry group: of AdS^3 , 663; for anti de Sitter spacetime, 650; of de Sitter spacetime, 625; equality with conformal groups, 656; and higher dimensional theories, 682; identical, for de Sitter spacetimes, 664
isomorphism: between AdS^3 and $SL(2, R)$, 663; of Lie algebra, and conformal algebra, 618
isoperimetrical problem, 149e; Lagrange, solution of, 144
isotropic fluids, seen by comoving observer, 229
isotropic space, 289, 292, 491; definition of with Killing vectors, 588; in outgoing brane wave model, 704; spherically symmetric mass distribution, 305
isotropic spacetime, static, motion in, 306–307
isotropy problem, 531
- Jacobi identity, Bianchi identity derived as special case of, 393
Jacobian, 216; changes of, 235; and coordinate transformations, 75; differential forms of, 598
Jacobian determinant, for Lorentz transformations, 188
Jeans, James, structure formation in early universe, 520
Jebsen-Birkhoff theorem: with gravitational waves, 568; Newton-Jebsen-Birkhoff theorem, 453; and time dependent spherically symmetric mass distribution, 373–374
Jordan, Pascual: anticommutation manuscript, 789n; stars made of nothing, 456
Jordan frame, 686
- Kaluza, Theodor: letter to Einstein, 671; letters from Einstein, 693–694
Kaluza-Klein action, 686; in Jordan frame, 686
Kaluza-Klein metric, 676, 680; in vielbein formalism, 690–691
Kaluza-Klein theory, 671–695; charge conjugation and antimatter in, 678; charge quantization in, 677; coordinate invariance in, 672; Einstein-Hilbert action in, 675; electromagnetic field in, 691; escape problem in, 673–674; as foliation, 689–690; gauge invariance in, 672; geometrical view of, 691–693; and grand unified theory, 672; higher dimensional, 680–682; linking of internal and external geometries, 691; Lorentz action in, 678; Maxwell action in, 675–676; motion of point particles, 676; phase angle of wave function in, 678; Planck length and charge quantization in, 677; Planck mass in, 675; transformations in, 672; and uncertainty principle, 674; visibility problem in, 673–674; and Weyl, 693–694. *See also* quantum gravity; string theory
Kaluza-Klein towers, 679
Kasner universe: as solution of Einstein’s field equation, 361e; with differential forms, 613e
Kepler’s third law: orbits around black holes, 413–414, 417; precession of gyroscopes, 549
Kerr, Roy, and rotating black hole solution of Einstein’s field equation, 458, 461
Kerr black holes, 462, 464–467; angular momentum, 465, 571; angular velocity for, 462f; mass determination, 570; no-hair theorems, 481–482; and Schwarzschild black holes, 468; Weyl approach, 473. *See also* rotating black holes
Kerr metric, 465–466, 475
Kerr-Newman solution, 477
Kerr-Schild form, 476
Kerr spacetime: Killing vectors, 470–471; radiation from rotating black holes, 473
Killing, Wilhelm, and Lie algebra, 586
Killing condition, conformal, 614
Killing vector fields, 332, 585–593; conformal, 614, 623e; definition of, 586
Killing vectors: admitted by spacetime, 636; derivation of curvature tensor, 591; emergence of Yang-Mills theory, 688; great circles, 127n; and higher dimensional theories, 682; for Kerr spacetime, 470–471; and Lie algebra, 591; linear combinations of, 587; in Riemannian manifold, 588; for spacetime around rotating black holes, 459; for spherically symmetric mass distribution, 305; for static isotropic spacetime, 310; timelike and spacelike, 637; from timelike to spacelike, 631
kinematics, relativistic, 221
Klein, Oskar: Klein-Gordon equation, 694. *See also* Kaluza-Klein entries
Kraichnan, Robert: curved spacetime, 580; “particle physics” approach, 583n
Kretschmann scalar, 365n
Kronecker delta: definition of, 36, 70; discrete variables in functional variations, 121; indices of, 183; as invariant tensor, 60; use of, 45

840 | Index

- Krupp (munitions manufacturer), financing of solar eclipse expedition to the Crimea, 370
- Kruskal, Joseph B., paper on spherical singularity, 376n
- Kruskal coordinates, for elimination of singularity of Schwarzschild solution, 365
- Kruskal-Szekeres coordinates, 424–425; wormholes in, 432–433
- Kruskal-Szekeres diagram, 425–427; of Schwarzschild black hole, 426f; Unruh effect, 447
- Kruskal-Szekeres-like coordinates, for de Sitter spacetime, 635, 636f
- kung fu stories, 470n
- Lagrange, Joseph-Louis: tautochrone problem, 144; variational calculus, 120
- Lagrange multiplier, 148; introduction of, 106; notion of, 109; for volume of spacetime, cosmological constant as, 756
- Lagrangian: for 2-brane model, 700; in action principle, 138; in curved spacetime, determination of, 712; gravitational, 339; infinitesimal transformation, 151; Maxwellian, 249, 255; of motion in static isotropic spacetime, 306; in nonrelativistic mechanics, 138–139; of relativistic point particle action, 211; Schwarzschild spacetime, time reversal invariance, 417; terms added for determination of gravitational with respect to electromagnetic field, 338; without time dependence, energy conservation, 153
- Lanczos, Kornel, corrections to de Sitter metric, 289, 642
- Landau, L. D., Green’s function approach, 577n
- Laplace, Pierre-Simon: black hole hypothesis, 13; Michell-Laplace argument, 409
- Laplace’s equation: for strong gravitational sources, 574; and tensor notation, 58
- Laplace-Runge-Lenz vector, definition of, 60
- Laplacian: definition of, 61; in membrane shape determination, 118; notation in various coordinate systems, 78–79
- “lapse,” 691, 693
- large extra dimensions, 696–707
- Large Hadron Collider, 699
- Larmor, J., Lorentz transformation, 169n
- Laser Interferometer Gravitational Wave Observatory (LIGO), 577n
- Laser Interferometer Space Antenna (LISA), 577n
- laser interferometry, detection of gravitational waves, 567
- laser light, box hit by, 281f, 283f
- Latin symbols, change to Greek symbols, in tensor notation, 63
- lattice gravity, 726n; as approach to quantum gravity, 760
- lattice Hamiltonians, leading to gauge fields, 787
- laws. *See specific laws*
- Le Verrier, Urbain, prediction of Neptune, 368
- Leaning Tower of Pisa, 270
- least path principle: and curvature, 5–6. *See also* path
- least time principle, 4, 136; connection with action principle, 139, 144; Feynman’s path, 3–4. *See also* time
- Legendre polynomials, 523
- legs. *See* reference frames
- Leibniz, Gottfried: brachistochrone problem, 120; discovery of calculus, 113; notation of action principle, 138
- Lemaître, Georges: closed and open universes, 296–297; Hubble constant, 501; as triple winner, 500
- Lemaître–de Sitter cosmology, 712
- Lemaître–de Sitter metric, 357; generalized, 489
- Lemaître–de Sitter spacetime, 642
- length: inverse, dimension of, 120; minimization of, 125; parametrization in general metric, 128; parametrization independence of, 130; of rulers, in special relativity, 199; units for, 10, 633
- length contraction, 199–200
- length element, on unit circle, 80e
- length scales: cosmological, physics on, 750; and cosmological constant, 748; and deviation from Newtonian gravity, 709; leading to cosmological constant paradox, 711
- Lense-Thirring precession, 550; alternative derivation, 551
- leptogenesis, 526–528
- leptons, families of, 786
- Levi-Civita symbol, 252; used to contract indices, 719
- l’Hospital, Marquis de, brachistochrone problem, 120
- Lie, Marius Sophus: infinitesimal rotations, 40; infinitesimal transformations, 154; method for derivation of Lorentz transformation, 187–188
- Lie algebra: definition of, 50–51; discovered by Killing, 586; generators of, 49; isomorphism of, 618; and Killing vectors, 591; of rotation groups, 191
- Lie derivative, 327–328, 331–332; and geodesic deviation, 555
- Lie’s equation, and emergence of Yang-Mills theory, 688
- Lie groups, characteristic of, 50
- Lifschitz, E., structure formation in early universe, 520
- light: “accelerated”/“dropped” gedanken experiments, 281–282; least time principle, 4; Maxwell’s explanation, 162; motion around black holes, 409–418; motion of, 307–309, 416f, 659; propagation of, in medium, 163; theory of, as analog to theory of gravity, 789n; unification with material particles, 207–217, 212–213. *See also* deflection of light; photons

- light cones: closing up, 420f; coordinates of, 146–147, 170–171, 427–429, 619, 631, 704; in expanding universe, 294, 294f; past, 177–178, 179f; spanned in Minkowski space, 177; tilting, at Schwarzschild radius, 420–421, 421f
- light deflection. *See* deflection of light
- light flashes, in trains, 166
- light paths: anti de Sitter spacetime, 656; depending on geodesics, 665
- light pulses, dueling thinkers experiment, 7
- light rays: corotating/counterrotating, 461, 469; more fundamental than spacetime events, 741; moving at 45°, 423; surfaces generated of, 185
- light signal trajectories, in static spacetime, 304f
- light speed, 162; constancy of, effect on notion of simultaneity, 8; determined by Maxwell's theory, 162–163; in expanding universe, 294; inverse, 754; in metals, ratio to sound speed, 749; as velocity of massless particles, 213
- lightfoot, not a unit of time, 773n
- lightlike 4-momenta, 782
- lightlike distance, 175
- lightlike geodesics, 292, 646
- lightlike lines, in general spacetime, 730
- lightlike momentum, complex, 733
- lightlike vectors, 731
- lightsecond, natural unit of distance, 168
- lightyear, as length unit, 10
- LIGO (Laser Interferometer Gravitational Wave Observatory), 577n
- limit surface, stationary, angular velocity inside, 471
- line: of constant time, de Sitter spacetime, 637; in twistor space, 742. *See also* straight line; worldlines
- line element: 5-dimensional, 676; of hyperbolic space, 628; square of, and metric, 64
- linear combinations, and tensors, 53
- linear transformations, rotations as, 68
- linearity of transformation matrix, 313
- linearity requirement, Galilean transformation, 18
- linearized gravity, 563–577, 758n
- LISA (Laser Interferometer Space Antenna), 577n
- lithium, primeval nucleosynthesis, 519
- local action, electromagnetic, 246
- local coordinate invariance, in higher dimensional theories, 682
- local curvature, measurement of, 547
- local field theory: and cosmological constant paradox, 756; invariance of physics, 621
- local flatness: of curved surface, 83; for spaces of any dimension, 86–87
- local gauge symmetry, in higher dimensional theories, 682
- local Lagrangian, in action, 783
- local measurements, indirect, detecting horizons, 789n
- local observables, 765; absence of, in quantum gravity, 772
- locality: as fundamental principle of theoretical physics, 783; of physics, 757
- locally exact forms, 604
- locally flat coordinates, 557; determination of, 132; and inertial frames, 278; for investigations of symmetry relations, 343–344; Minkowskian, 288; nearby geodesics, 552; transformation of polar coordinates into, 89
- locally flat Euclidean metric, second order corrections to, 88
- locations: of events in spacetime, 195; and spatial coordinates of particles, difference between, 31
- long distance behavior, of action terms, 722
- long distance expansion, deviation from Newtonian gravity, 708–709
- “long distance physicists,” 713n
- loop quantum gravity, 772
- Lorentz, Hendrik: and Droste's solution of Einstein's field equation, 375; paper on Hamilton's principle and general theory of relativity, 397; paper on variation of Lagrangian, 396; understanding of waves, 783–784
- Lorentz action, in Kaluza-Klein theory, 678
- Lorentz algebra, 187; extension to Poincaré algebra, 192, 617
- Lorentz boost: of 4-vector, 230; of mass density, 579; $SL(2, C)$ group, 730
- Lorentz contraction: of box with particles, 23; and number density, 223f
- Lorentz-Fitzgerald length contraction, 199–200; pole in the barn problem, 202
- Lorentz force law, 245, 247; movement of charges, 404
- Lorentz group, 188, 218; connection to rotation group, 192; covered, 729–730; $SO(3, 1)$, 730
- Lorentz indices, 594, 608; conversion with vielbein, 603; versus world indices, 595
- Lorentz invariance: beyond cosmological length scale, 754; helicity spinors, 734; Maxwell's equations, 253; Newtonian action, 242; of physics, 218; of spacetime, 666
- Lorentz scalar: definition of, 218; and density distribution, 579
- Lorentz symmetry, restrictions imposed on electromagnetism, 339
- Lorentz tensors, 188, 243
- Lorentz transformation, 166–173; alternative route, 172; within cars, 205; and curved spacetime, 317; definition in Minkowski spacetime, 181–182; invariance of, 186; low velocity limit of, 169; sneak preview of, 147; tensors under, 193e
- Lorentz vector, Pauli spinors as “square root” of, 731
- Lorentz vector potential, 243, 248
- Lorentzian Lagrangian, 249
- Lorenz gauge, in electromagnetism, 564

842 | Index

- low energy effects, of quantum gravity, 767
low energy physics: linkage to high energy physics, 752; understanding of, 750
low energy world, neglect of quantum gravity, 766
lower indices, 314–316; introduction of, 64; transformations in change of coordinates, 71–73
luminosity distance, 297
- macrostates, Bekenstein-Hawking entropy, 441, 444
magnetic field, 245; relativistic unification, 247; Schrödinger equation for (nonrelativistic) charged particle in, 354n
magnetic flux lines, Faraday's picture of, 728
magnetic moment, of atom, and action, 715
magnetic monopoles, 81; bosons bound to, 789; Newtonian approximation of Einstein's field equation, 577; relic problem, 532; topological field theory, 728
Maldacena, Juan, and quantum gravity, 649
manifolds: Calabi-Yau, 695; coset (*see* coset manifolds); Riemannian, 599–600; rotations determined in, 590; topology of, and ground states, 723; without boundary, 727
many particle case, and fields, 400
many particle systems. *See* fluids
many worlds interpretation, of quantum, 780
map. *See* Mercator map
mapping: of heaven and earth, subdivision of degree (proposal by Ptolemy), 368n; of twistor space to spacetime, 742
marble, positional variations in bowl, 114
marine recruit in boot camp, following rotation commands, 50f
mass: changes of, 221; as conversion factor between geometry and physics, 211; definition of, in elementary physics, 213; of elementary particles, 16n; gravitational and inertial, 257, 268–269; Planck (*see* Planck mass); role of, in action principle, 142; spherically symmetric distribution, 304–307, 310–311, 409; of universe, 747–748
mass density transformation, under Lorentz boost, 579
mass dimensions: and dimensions of scalar curvature, 711; role in quantum field theory, 711–712
mass distribution: and gravitational potential, 578; from point masses, 119; rotating, gravitational sources, 569; spherically symmetric, 373–374, 569, 571
mass loss, of radiating atoms, 232
mass scale: of cosmological constant, 700; as limit of understanding of quantum field theory, 746
mass shell condition, 220, 464
massive objects: “gravitational radius” of, 764; motion of, 659–660, 659f; worldlines of, 175
massless particles, 307–309; accelerated relativistic, 277; gravitons, 718; motion around black holes, 415–416; mystery of, 213; natural parametrization, 308; preferred parameter choice for, 215; relativistic action principle for, 213; worldlines of, 175
material particles: Euler-Lagrange action of, 207; unification with light, 207–217, 212–213
mathematical entities, as tensors, 52
mathematical universes, 634
mathematics: difference from arithmetic, in terms of rotations, 56; as poetry of logical ideas, 150
matrices: antisymmetric, introduction of, 40; commutation of, 41; exponential of, 41; as group generator, 663; introduction of, 39–40; and operators, 48; rotation matrix, definition of, 38; of spacetime metrics, 183; transpose of, 45
matrix algebra, quick review of, 742–743
matrix differentiation, 322
matrix elements, counting of, 87–90
matrix theory, for relativistic action, 210
matter: baryonic, 502–503, 506; dark (*see* dark matter); observational evidence, 503f; spherical shell of, 423f
matter action: contribution of Maxwell action to, 378; fields contained in, 384; general invariance of, and conservation of energy momentum, 383–384; generic, 386; interaction as part of, 382–383; as part of action of world, 378; variation of, 378–379
matter-antimatter asymmetry, 528; in higher dimensional theories, 683
matter density, and scale factor of universe, 496f
matter dominance: and coincidence problem, 499; and photon decoupling, 788n
matter equation of motion, and matter action, 386
Matthew principle, 520; Birkhoff theorem as example for, 376n; Lorentz transformation as example for, 169n; the rich inheriting from the wimps, 523
maximal symmetry: anti de Sitter spacetime, 650; and coset manifold, 625
maximally symmetric spaces, 585–593; negatively curved, 610
Maxwell, James C., versus Galileo, 159
Maxwell action, 325, 332; Chern-Simons term added to, 721; contribution to matter action, 378; and differential forms, 724–725; general invariance, 384, 394; in Kaluza-Klein theory, 675–676; long-distance behavior, 722; in Minkowskian spacetime, 381; scale and conformal invariance, 621; vanishing by variation, 384; from weak field action, 572. *See also* electromagnetic action
Maxwell's equations, 252–253; and Bianchi identity, 724; for charged black holes, 477; coupled to Einstein's field equations, static solutions, 482–483; in curved spacetime, 333; free, 251; and initial value problem, 404
Maxwell field, in terms of Yang-Mills field, 789n
Maxwell Lagrangian, 249, 255, 382

- Maxwell's laws of electromagnetism, and Galilean transformation, 20
- Maxwellian electromagnetism: differences to Newtonian gravity, 338; gauge freedom and initial value, 401–402; speed of light, determined by, 163
- Mead, C. Alden, generalized uncertainty principle and quantum gravity, 769
- mean free path of photons, 517
- measuring device, collapsing into black hole, 763–764
- mechanics: immediate formulation of, 142; least action formulation of, 139
- medium, for propagation of light, 163
- membranes: hanging, as generalization of hanging string, 118; from null surfaces, 185
- Mercator, Gerardus, importance of angles, 620
- Mercator map: and coordinate transformations, 79e, 94; singularity at poles, 365; of the world, 620
- Mercury, perihelion shift, 368–369, 369f
- messages, paths through spacetime, 638
- metals, ratio of sound speed to light speed, 749
- metric: in case of isometry, 586; change under coordinate transformations, 70–71, 110; chosen in Riemannian manifold, 88; conformally flat, 80–81e, 94, 352e–353e; constraints on, 403; for contracting spacetime indices, 719; in cosmological action, 357; definition range in parallel transport, 544; determinant of, 215–216; and different indices, 595; differentiation of, 131; as dot product of vielbein, 603; for expanding universe, group theory, 645; in Fermi normal coordinates, 559, 561; flat space, 77; formed by coordinate scalars, 708–709; induced by ambient Euclidean metric, 86; integral over, 770; intrinsic curvature calculation, 90–91; invariance under scaling, Poincaré coordinates, 657; Lemaitre–de Sitter, 357; and line element, 64; for lowering or raising indices, 74; not related by coordinate, 81e; restriction by isometric condition, 586; role in differential geometry, 66; Schwarzschild, discovery of, 364; second order deviation of, 343; of space, 128; in spacetime, 181, 716; on sphere, determination of, 65; in spherical coordinates, 108; for spinor indices, 742; on surface, in Euclidean space, 99; of surface of sphere, 83–84; time-independence of, 636; transformation in terms of matrices, 72–73; two powers of derivatives acting on, 349; unfamiliar of spheres, 585; on unit spheres, 80e; variations in spacetime, 716
- metric formalism, derivation of divergence and Laplacian, 78–79
- metric tensor: of 3-sphere, 296; covariant derivative, 325; divergence near boundary, 663; and general coordinate transformations, 314; general static and isotropic, 306; generalized Lemaitre–de Sitter, 489; higher dimensional, 682; inverse, 315; Kaluza-Klein, 676, 680, 690–691; Kerr, 465–466, 475; Minkowskian (*see* Minkowski metric); near-horizon Schwarzschild, 445–446; off-diagonal components, 292, 459, 466, 474; Rindler, 445–446; for space measurements, 63–64; spatial, cosmic expansion, 491; time dependent, 455; time translation invariance of, 304
- Michell, John, black hole hypothesis, 13
- Michell and Laplace, mass of black hole, 366, 409
- Michelson-Morley experiment, 163; explained by length contraction, 200
- microscopic physics, and topological action, 721
- microstates: Bekenstein-Hawking entropy, 441, 444; in de Sitter spacetime, 638
- microwave background, cosmic. *See* cosmic microwave background
- Mie, Gustav, Newton gravity and Lorentz invariance, 580
- Mills, Robert L. *See* Yang-Mills theory
- minimum, as solution of variational calculus, 117
- minimum length measurement, limited by special and general relativity, 763–764
- Minkowski, Hermann: “mystical” substitution, 640; on physical laws between worldlines, 176; on space, time, and spacetime, 174
- Minkowski metric, 317, 391; and Einstein's field equations, 563; folded into indices, 182; Rindler coordinates, 446
- Minkowski spacetime: (1+1)-dimensional in light cone coordinates, 619; accelerated relativistic particles, 277; acceleration in, 190; coordinate changes, 192e; curves, in, 175; deviations due to gravitational waves, 571–572; Dirac action in, 605; distance in relativistic action, 210; flat, governing action of, 581; generators of conformal algebra for, 615; geometry of, 175, 191; locally flat coordinates, 288; maximal extension of, 434; Maxwell action for electromagnetic field in, 381; Penrose diagram, 428f, 434; spherical shell of photons in, 430f; surfaces in, 184
- Minkowskian sphere, including time, 631
- Minkowskian time, compared to Newtonian time, 372
- minus sign, role of, in energy functional, 139
- Misner, Charles W., ADM formulation of gravitational dynamics, 693
- mites: flat space analogy, 6; geometer measuring curvature, 545
- MLT system, reduction to nothing, 11
- modes, electromagnetic field treated as superposition of, 746
- molecules, appearance in early universe, 519
- momentum: angular (*see* angular momentum); energy (*see* energy momentum *entries*); exact meaning of, 383; of gravitational field, 580–581; of graviton's propagator, 786; Hamiltonian, 144; not conserved, 26, 27; physical, and twistors, 731;

844 | Index

- momentum (*continued*)
restricted to hyperbolic shell, 220; spatial density of, 228; total, conservation of, 37
- momentum conservation, 219; delta function, 740; derivation of Einstein's formula, 232; in terms of helicity spinors, 736
- momentum-twistor space, scattering amplitudes as volumes of polytopes in, 742
- monopoles, magnetic. *See* magnetic monopoles
- Morley, Edward. *See* Michelson-Morley experiment
- Mössbauer effect, measurement of gravitational redshift, 284
- mother: of all headaches, plaguing fundamental physics, 699; of all vectors, 312–313
- motion: around black holes, 412–416; in curved spacetime, 289–290, 301–311; effect on coordinates of, 160; in fifth dimension, 676; of free particles, 180; law of, 25; relative, of observers, 168, 181; in static isotropic spacetime, 306–307
- movement: at constant speed, of objects under special relativity, 189; along curve, through vector field, 544; fuel-economizing, 127
- moving observer, fluids, 230
- moving trihedron, of smooth curve, 97f
- multipole expansion approximations, 568–569
- “Must it be? It must be.”: discovery of action for gravity, 346, 346f
- My World Line* (Gamow), 177
- mysteries: action principle, 141, 155; Bekenstein-Hawking entropy, 444; Bering Strait, 275; black holes, 410, 441; caloric, 786; closing orbits, 30, 60; correspondence between quantum statistical mechanics and quantum field theory, 445; cosmological constant, 356, 711, 751, 782; cosmos, 778; “crazy” coordinates, 94; dark energy, 356, 711; Einstein's field equation, 358; entropy in spacetime, 234; equality of inertial and gravitational mass, 28; family problem, 7; fermions, 781–782; Gibbons-Hawking radiation, 637; grand unification, 527n; gravity, 276, 778–779; holographic principle, 15; light and electromagnetic field, 162; massless particle, 213; neutrino mass, 359; quantum, 780, 789n; quantum gravity, 748, 781; as source of beautiful experience, 778; temperature, 15; three copies of world, 7; time, 787; universe, 779
- “naked” charged black holes, 478
- Nambu-Goto action, 216e
- naming conventions, for indices, 608
- Nash, John, and embedded spaces, 95
- Nasty and Vicious, dueling thinkers experiment, 7–9
- “natural” coordinate systems, 134
- natural parametrization, 308
- natural quantities: introduced by gravity, 764; and unnatural quantities, 218
- natural system of units, 10–12
- naturalness doctrine, 579; in high energy physics, 749–750; and inverse light speed, 754–755
- Navier-Stokes equation, 234; in fluid dynamics, 164
- near-horizon Schwarzschild metric, 445–446
- negative curvature, definition of, 85
- negative pressure, as consequence of constant dark energy density, 360
- negatively curved space, maximally symmetric, 610
- neutral objects, impossibility of under gravity, 716
- neutrino masses, as mystery, 359n
- neutrino oscillations, and cosmological constant paradox, 747
- neutrinos: (non)relativistic, 501; scattering of, 765; “typical” mass scale of, 700
- neutron interferometry, and equality of inertial and gravitational mass, 34
- neutrons: mass of, and anthropic principle, 757; primeval nucleosynthesis of, 517–518
- “new and improved” energy momentum tensor, 712
- Newman, Ezra T., Kerr-Newman solution, 477
- Newton, Isaac: action principle, 144; apple falling on, 268; comparison to Aristotle, 140–141; discovery of calculus, 113; existence of God, 520; on his youth, 25; inherent instability of gravity, 520; miraculous year, 194n; role of second derivative in time, 401; shown with orbits on one pound note, 31; unification of celestial and terrestrial mechanics, 28
- Newton's constant: Cavendish's first measurement of, 32; dimension of, 346; historical digression on, 31–32; and quantum gravity, 761
- Newton's dot notation, 29, 96
- Newton-Einstein-Hilbert action, quantum gravity limit, 444
- Newton-Jebsen-Birkhoff theorem, 453
- Newton's laws, 25–34; law of action and reaction, 470; law of gravity, 11, 28; as result of variation principle, 137; second law, 46–48, 110, 140
- Newton-Leibniz rule: breaking of, 340–341; failure for covariant derivatives, 342
- Newton's superb theorems, 32–33
- Newtonian action, 241–242
- Newtonian approximation, Einstein's field equation in, 577
- Newtonian equation, “analog,” 367
- Newtonian gravitational potential: around black holes, 410–411, 411f; compared to Einstein potential, planetary orbits, 371; dynamical origin of, 578n; fields, 119; quantum gravity corrections to, 767; replacement of mass density by energy density, 379n
- Newtonian gravity: cube of physics, 13f; deviation from, and powers of derivatives, 708–709; differences from Maxwellian electrodynamics, 338; restriction imposed by symmetry, 339; as weak gravitational field limit, 391
- Newtonian Lagrangian, 249

- Newtonian limit, 302–303
- Newtonian mechanical analogies, from cosmological principle, 507, 513
- Newtonian mechanics: and black holes, 13; conservation laws in, 35–37; cube of physics, 13f; Galilean invariance of, 161; initial value problem in, 400; invariance of, 161; invariance under Galilean transformation, 19; reproduction by relativistic particle action, 209; role of differential equations, 26–27; role of signs in, 382; standard notation of coordinates, 25; tensors in, 57–59
- Newtonian orbits: closing of, and tensor notation, 60; determination of, 31
- Newtonian time, compared to Minkowskian time, 372
- Newtonian universe, role of time in, 7
- no-hair theorems, and Kerr black holes, 481–482
- Nobel prize in physics (2011), and dark energy, 361n
- Noether, Emmy, 150; spacetime hidden in scattering amplitude, 739–740
- Noether’s theorem: application of, 152; generality of, 153; motion in static isotropic spacetime, 310; promotion of physical laws, 221; proof of, 151
- non-determinism, of Einstein’s field equations, 403
- non-quantization, of gravity, 768–769
- nonabelian gauge theory, 681; decoupling of geometries, 692
- noninteracting free particles, 221
- nonlinear coordinate transformations, 69
- nonlinear gravity, 571
- nonlocal cosmology, 712
- nonlocal phenomena, removal, 784
- nonlocal terms: in action, 751; and cosmological constant paradox, 751
- “nonphysical” degrees of freedom, 783
- nonrelativistic action, 241–242
- nonrelativistic gases, 454
- nonrelativistic matter. *See* dust
- nonrelativistic mechanics, Lagrangian in, 138–139
- nonrelativistic particles, in potential, action of, 356
- nonrelativistic physics, completion and promotion of quantities in, 218
- nonrelativistic quantum mechanics, 438; in presence of gravitational field, 12–13
- nonrenormalizable interactions, 711–712
- Nordström, Gunnar: derivation of Einstein’s gravity, 579. *See also* Reissner-Nordström entries
- Nordström’s theory, road to higher dimensional theories, 682–683
- normal, to surface: at certain point, 99f; as timelike vector, 184
- normal coordinates, Fermi, 557
- normal vector, tangent plane rotating around, 100
- north pole, and its longitude, 76
- notation: of action principle (Leibniz), 138; of column vectors, 45; confusion in variational calculus, 117; convenient for vectors, 182; of coordinates, 25, 62n; cross-product, angular momentum, 48n; for differential operator, 72; dot: as symbol for symmetry, 29, 96, 129; erroneous, in parallel transport, 543; of functionals, 114; of gradient, 54; group theory of universe, 644; index (*see* index notation); Laplacian, 78–79; of quantities (in general), 32; spacetime metric, 183; tensor (*see* tensor notation)
- notation alert, bad: confusion in time dilation, 198; confusion in relativistic action, 211; geodesic equation, 555
- nothing, waving of, 783
- nuclear force, generated by pions, 205
- nuclear fusion, compared to accretion disk radiation, 415
- nuclear physics, in early universe, 518
- nucleons, formulation of strong interaction, 785
- nucleosynthesis: primeval, 517–518; stellar, 518–519, 758
- null infinities, in Penrose diagrams, 428, 428f
- null lines: in general spacetime, 730; in spacetime, 741f
- null surfaces, 184; acting as membrane, 185; black hole horizons as, 422, 468
- number current: inside 3–volume, 226f; as 4–vector, 225f
- number density: as component of Lorentz-vector, 224; of particles in box, 223f; relativistic completion of, 223; in relativistic form, 224
- numerical relativity: initial value formulation, 693; and initial value problems, 400–405; setting up, 403
- obesity index of universe, Schwarzschild radius and, 443
- observables: appearance of antimatter, 205; Heisenberg picture, 771; local, 765, 772, 781; quantum mechanics, 48
- observational cosmology, 491, 505
- observers: accelerated, 193, 446–447; different, 185; freely falling, metric for, 561; moving and resting, 166–168; relative motion of in spacetime, 181; role in physics, 46–48; studying vector field, 47f; uniform relative motion of, 168. *See also* reference frames
- odd-dimensional space, space reflection in, 721n
- offshell information, carried by action, 782
- old man’s toy, 267f
- Once and Future King, The* (White), 361n
- one pound note, showing Newton with orbits, 31
- open strings, 696
- open universes, 296–297, 491, 629; critical density, 497–498; Einstein’s field equations, 493–494; with positive cosmological constant, 633
- operational definition of distance, 291, 291f
- operators: annihilation and creation, 447–448; differential, 48, 72, 319, 588; quantum, 771, 772
- orbifolds, 700

846 | Index

- orbits: circular, 413–414, 413f, 549; closed, verification of, 30; for light moving around black hole, 416f; properties of, precession of gyroscopes, 550
- ordinary differential equations, coupled, relativistic stellar interiors, 452
- orthogonal matrices, definition of, 39
- orthonormal frames, 594; erecting, 595f
- oscillator, harmonic, 447; symmetry and invariance, 242
- osculating plane, of smooth curve, 97f
- Ostrogradsky, M. V., discoverer of inherent instability, 338
- outer horizon, of Kerr black holes, 468–469
- outgoing brane wave model, 704–707
- p*-form, definition of, 597
- Page, Don, Hawking radiation, 449
- Painlevé, Paul, 417
- Painlevé-Gullstrand coordinates, 417
- pair production, 438
- Palatini formalism: action for Einstein gravity, 395; derivation of Einstein's gravity, 583; invention by Einstein, 397
- Palatini identity, 389–390; mixing up with Palatini formalism, 397
- parabolas, bending in opposite directions, and negative curvature, 85
- paradoxes: pedagogical aspects of special relativity, 203–204. *See also* cosmological constant paradox
- parallel transport, 543–548; precession of gyroscopes, 549; of vectors, 101–102, 102f, 545f
- parameter choice for massless particles, 215
- parametrization: invariance of: current, 133, 235; natural, 125, 308; of surface, 98; ultrarelativistic particle motion, 308
- parametrized post-Newtonian (PPN) approximation, 309–310, 311
- parity: strong gravitational sources, 574; and space reflections, 721n
- partial differential equations, solving, 708
- particle-antiparticle pairs, thermal radiation from horizon, 637
- particle cloud, motion of, described by geodesic, 556
- particle collisions, 438; momentum, 219–220
- particle decay, conservation, 237n
- particle horizon, 536
- particle location, versus spacetime, 224
- particle mass, as proportionality factor in relativistic action, 211
- particle motion, 198; free, 180; in future light cone, 178f; in interaction potential, 162; law of inertia of, 143; multiple coordinates, generalization of, 140; in potential, 57–59, 135, 137; simplest case of, 142
- particle physicists, renaming themselves high-energy physicists, 713n
- particle physics: approach to gravity, 583n; baryogenesis and leptogenesis, 526–528; in early universe, 518; evolving of, 753n; scale and conformal invariances, 621; standard model of, 683
- particle theory, use of scalar fields in, 759n
- particles: accelerated: and general relativity, 189, 193e; anti- (*see* antimatter); birth and death of, 198; around black holes, 409–418; in box, number density of, 223f; corotating/countertorotating, 474; de Broglie wavelength at Schwarzschild radius, 442; electromagnetic field acting on, 246, 250; under external force, 190; and fields, 145–146, 384; of finite size, motion of, 714; free, 302; and gravitational waves, 566; intrinsic lifetime of, 198; massive, 659–660, 659f; massless (*see* massless particles); near barrier, path integral formalism for, 781; noninteracting (*see* dust); notation of position, 117; point (*see* point particles); relativistic action, 208–209; at rest, Newton's laws, 142; ring of, responding to gravitational wave, 567f; scattering of, Lorentz invariance, 236e; separation of, for different polarizations in gravitational waves, 566; on a sphere, 148, 645; spin 1, 256; teleological behavior of, 139; test, 302, 309; wimps, 522; worldlines of, 177f, 211f, 380
- partition function of quantum systems, 445
- passive diffeomorphism, coordinate transformation as, 398
- path: in 2-dimensional Cartesian space, 123; actual, extreme value of action, 141; choosing, as metaphor for life, 139–140; of falling apple, determination of, 137; Feynman's, to rescue a drowning girl, 3–4, 4f; harmonic oscillator, 148e; least path principle, 3, 5–6; length of, 189, 190; of light, 175, 656, 665; mean free path of photons, 517; shortest (*see* shortest path); straight and narrow, deviation from, 143; through spacetime, 638. *See also* distance; length
- path integral (Dirac-Feynman) formulation: determining Hawking radiation, 445; and local observables, 772; quantum gravity, 781, 783; quantum physics, 770; understanding of quantum mechanics via, 141
- Pauli matrices, in Lorentz algebra, 187
- Pauli spinors, as “square root” of Lorentz vector, 731
- Peierls, Rudolf, on thinking and calculating, 133
- Penrose, Roger, and twistors, 730–731
- Penrose diagram, 427–429; black hole formation, 430; for causal structure of de Sitter spacetime, 639f; charged black holes, 480f; de Sitter spacetime, 638; of Minkowskian spacetime, 428f; Schwarzschild black hole, 429f; time translation, 620
- Penrose process, 449, 469–471; angular momentum loss, 471–472; area theorem, 472
- Penrose's vision, on role of light rays, 741
- Penzias, Arno, cosmic microwave background, 517

- perfect fluids: assumptions during discussion, 237n; and comoving observers, 229; definition of, 230; in outgoing brane wave model, 704–705; relativistic stellar interiors, 451; universe filled with, 492–493
- perihelion shift: around black holes, 413; around Mercury, 368–369, 369f; in Schwarzschild metric, 371–372
- perturbation, relevant, in cosmic diagrams, 512
- perturbative correction, to electromagnetic scattering of point charges, 766
- perturbative expansion, failure of, 770
- perturbed spacetime metric, gravitational sources of, 569
- Petrov notation, of Riemann curvature tensor, 352e
- phase angle, of wave function, in Kaluza-Klein theory, 678
- phase boundaries, in cosmic diagrams, 513–514
- philosophic arguments, power of, 779
- photons: collisions with electrons, 222f; compared to gravitons, 768; decoupling of, and matter dominance, 788n; depending on geodesics, 665; frequency shift, in scattering, 222; momentum of, 232; movement along time axis, 665; movement in dueling thinkers experiment, 7–9; parameter choice for propagating, 215; in primeval universe, 516f; relativistic action, 212; role of electric charge for, 383; spherical shell of, 429, 430f; temperature of gas of, 495. *See also* light; massless particles
- physical momentum, and twistors, 731
- physical reasonability, 557
- physical singularities: compared to coordinate singularities, 91–92; and coordinate singularities, 365–366; Kerr black holes, 467, 467f; Schwarzschild black holes, 418, 425; timelike, 479
- physicists: good versus great ones, 167; particle, renaming themselves high-energy physicists, 713n; physics being independent of, 219
- physics: on cosmological distance, 750; cube of, 12–13; Descartes approach to questions in, 583n; effectiveness in understanding the universe, 779; and expression of physics in terms of equations, difference of, 47; fundamental, Mother of All Headaches, 699; goal of, 757–758; independence of physicists, 219; internal consistency of, 780; linkage between high energy and low energy physics, 752; most famous equation of, 220–221; need to be local, 757; present understanding of, 712; quantum (*see* quantum physics); relevance of topology to, 728; role of clocks and rulers, 719–720; role of observer, 46–48; sensitive to topology of spacetime, 720; start of, 143n; teleological discussions in, 136; theoretical (*see* theoretical physics); translation invariance of, in static spacetime, 304f; ultimate equation of, 47–48
- physics terms, least appropriate, 516
- Pioneer anomaly, 311
- pions: formulation of strong interaction, 785; mass prediction of, 205; negatively charged, 206
- Pisa, Leaning Tower of, 270
- planar coordinates, of expanding universe, 630
- Planck, Max: Einstein's appraisal of his understanding of general theory of relativity, 370; personal life, 10; and ultraviolet catastrophe, 789n
- Planck area, and entropy of black holes, 442
- Planck brane, 700
- Planck constant, 11; dependence on mass-energy scale, 781
- Planck length, 11–12; charge quantization in Kaluza-Klein theory, 677; in effective field theory approach, 709; and large extra dimensions, 699; as minimum length to probe quantum effects, 762; as smallest distance experimentalists can measure, 764
- Planck mass, 11–12; amount of, 583; and cosmological constant paradox, 746–747; in higher dimensional theories, 681; in Kaluza-Klein theory, 675; as largest mass fundamental physics, 748; quantum gravity limit, 444
- Planck scale, in early universe, 518
- Planck time, 11–12
- Planck units: and entropy of black holes, 441; and quantum gravity, 761
- plane: flat, curvature of, 105; osculating, of smooth curve, 97f
- planetary orbits, in Schwarzschild metric, 371–372
- planets, celestial mechanics, 28
- Poincaré, Henri: and Lorentz transformation, 169n; and special relativity, 190; understanding of waves, 783–784
- Poincaré algebra: extension to conformal algebra, 617; generators of, 192
- Poincaré coordinates: in anti de Sitter spacetime, 656; numbers of boundaries, 664
- Poincaré group, transformations and translations of, 666
- Poincaré half plane, 67f; and anti de Sitter / conformal field theories (AdS/CFT), 68; determination of geodesics, 127; with differential forms, 608; finding geodesics of, 133; geodesics on, 134f; in higher dimensions, 656; and metric, 67–68; and temporal boundary, 632
- Poincaré invariant brane, 707
- point charges, electromagnetic scattering of, 766
- point of view, local versus global, 141
- point particles: action for, relativistic, 208–209, 210; associated current of, 235; energy and momentum of, 379–380; motion of, 714, 676; nonrelativistic action, 241
- point-to-line map, from twistor space to spacetime, 742
- pointlike particles, worldline length of, 215
- points: circles mistaken for, 674f; distance of in space

848 | Index

- points (*continued*)
and time, 174–175; isometric geometry of, 585; in spacetime, 177, 689f, 742; in twistor space, 741f
- Poisson's equation: gravitational potential satisfying, 708; membrane shape, 118; for Newton's gravity, 231
- polar coordinates: change from Cartesian coordinates, 29, 62, 71; Christoffel symbols of, 129; on flat plane, 125; to solve celestial mechanics, 29; transformation into locally flat coordinates, 89; warped, 613e
- polar-like coordinates, comoving, 298
- polarization tensor, of gravitational waves, 565
- polarization vectors, written in terms of helicity spinors, 735n
- polarizations: degrees of, in gravitational waves, 564; different, in gravitational waves, 566; of gravitational waves, 734; as helicity states of graviton, 566
- Polchinski, J., deletion of Feynman diagrams, 756
- pole in the barn problem: spacetime view of, 202f; of special relativity, 201, 201f
- poles, and their longitudes, 76
- polyhedra, angular deficits of, 726–727
- polytopes in momentum-twistor space, and scattering amplitudes, 742
- position, of particles: as general space coordinate, 26; notation, 117
- position determination, and position of measuring device, 763–764
- positive cosmological constant, and expanding universe, 392
- post-Newtonian approximation: Einstein's field equation in, 577; parametrized, 309–310, 311
- potential: central, 36; and consistency or integrability condition, 36; cosmic, 508–509, 508f; definition of, 35; electromagnetic, in fifth dimension, 677; external, translation invariance, 242; gauge, emergence of Yang-Mills theory, 688; gravitational, 578; inflaton, 535f; introduced into relativistic action, 209; linear, 139; Newtonian, around black holes, 410–411; particles moving in, tensor notation of, 57–59; rotationally invariant, 150; translation invariant, particle movement in, 151; vector, 243, 248; Yang-Mills gauge, 682
- potential energy, of a marble in a bowl, 113
- potential energy functional, action principle, 146
- power series: expansion of functional, 115–116; introduction of, 41
- powers of derivatives, deviation from Newtonian gravity, 708–709
- Poynting vector, emergence of, 382
- PPN (parametrized post-Newtonian) approximation, 309–310, 311
- precession: of gyroscopes, 465, 549–551; Lense-Thirring, 550; in Schwarzschild spacetime, 549
- precession angle, 550
- predictions, verified for Einstein's theory, 777
- pressure: Fermi, Chandrasekhar limit, 455; relativistic energy contribution of, 230; of universe, relation to energy density, 359
- pressure gradient: of relativistic stellar interiors, 453; in universe filled with perfect fluid, 493
- primed coordinates, 18, 38; metric with, 71–73
- primed frames, in algebra, 196
- primeval nucleosynthesis, 517–518
- primeval universe, 516f
- “primeval” vectors, and coordinate transformations, 73
- Princeton University, fundamental physical equations on glass windows, 138
- principles: action (*see* action principle); anthropic (*see* anthropic principle); of causation, and gravitation law, 404; Copernican, 491; cosmological (*see* cosmological principle); equivalence (*see* equivalence principle); fundamental, 12; Galileo's relativity principle, 17–19, 159; “golden” guiding, 338; holographic (*see* holographic principle); least path (*see* least path principle); least time (*see* least time principle); locality (*see* locality); of presumed innocence, 299; uncertainty (*see* uncertainty principle)
- problem: of not enough time, 521–522, 531; prototype of solutions, 222
- Professor Flat: discusses Christoffel symbols, 132–133; on local flat coordinates, 130
- projection: stereographic, 80–81e, 81f, 641; of vectors on tangent plane, 102
- projective space, integrating over, 740
- promotion, law of, 219
- propagator, for graviton, 573
- proper distances, 296–297
- proper time, 181; definition of and motion of light, 659; for different observers, twin paradox, 189; in electromagnetism, from special relativity, 244; in Minkowskian spacetime, 179; parameter choice for massless particles, 215
- proper time duration, of particle, 210
- proper time interval, invariance of, 199
- proton decay, 527; analogy to cosmological constant paradox, 753–754; and anthropic principle, 757
- protons: delayed recombination of, 516–517; primeval nucleosynthesis of, 517–518
- Proust, Marcel, on time, 205
- pseudo-Euclidean spaces, 653
- pseudo-time coordinate, 657
- pseudospheres. *See* hyperbolic spaces
- pseudotensor, energy momentum, 386
- psychological time, 175n
- Ptolemy: and concept of coordinates, 62n; and the term “second” used in measuring angles, 368n
- pulsars, emission of gravitational waves, 563
- pulsating mass distribution, 571

- pulsating stars, 304
punctured surfaces, 726
puzzle, “What is greater than God?” 789n
Pythagoras: calculation of length of hanging string, 113; motion in static isotropic spacetime, 305, 309
Pythagoras theorem, for space and time, 167
Pythagorean time, and radar echo delay, 372
- QFT. *See* quantum field theory
quadratic derivatives, added to Lagrangian, 338
quadrupole formula, derivation of, 576e
quadrupole radiation, gravitational, 571
quantities: auxiliary, calculus, 129; conserved: and Noether’s theorem, 30, 152; definition of conceptually natural, 219; extensive, 441; index notation of, 32; natural, introducing to gravity, 764; transformation of, 218; without qualities, scalar fields as, 788n
quantization: of electromagnetic field, 764; of gravitational field, 582; of gravity, 780, 788n
quantum: dependence of action, 783; mystery of, 780
quantum chromodynamics, 526, 785
quantum electrodynamics, difficulties in, 764
quantum field theory (QFT), 247; antimatter in, 476; calculating vacuum energy by, 752–753; commutation relations, 192; correspondence with quantum statistical mechanics, mystery of, 445; cube of physics, 13f; in curved spacetime, 780; cutoff in, 758n; in de Sitter spacetime, 648; harmonic oscillator in, 361; as low energy effective theory, 711–712; motivation for development of, 384; motivation for studying twistors, 731; not consistent with classical relativity, 773n; questions on, 781; restless vacuum in, 436–438; understanding of, 746; use of scalar fields in, 759n
quantum fields, appearance in action, 213n
quantum fluctuations: contributing to vacuum energy density, 746; of fields, 784; Hawking radiation originating from, 436; in inflationary cosmology, 533; thermal radiation from horizon, 637; Unruh effect, 446; vacuum as boiling sea of, 745–746
quantum gravity: anti de Sitter spacetime, container for, 649; cube of physics, 13f; divergent behavior of, 766; fundamental scales, appearance of, 760–761; governed by attractive ultraviolet fixed point, 773n; handwaving arguments for, 769; Hawking radiation, 439, 443–444; heuristic thoughts about, 760–774; as Holy Grail of physics, 12; impossibility as a quantum field theory, 765; as local field theory, 781; local observables, absence of, 772, 781; loop, 772; mystery of, 748, 781; “naked” singularities, 480; Newtonian potential, corrections to, 767; nonperturbative treatment of, 770; path integrals, 781; Planck length as minimum length to probe, 762; and problem of knowing the position of measuring device, 763–764; and Schrödinger’s cat experiment, 771; and “strangeness” of black holes, 764–765; taming of, 731; thought to follow from quantum electrodynamics, 764; trouble by Planck mass, 761; and ultraviolet completion, 765; from world described by “matter fields” and a metric, 770. *See also* Kaluza-Klein theory; string theory
quantum Hall effect, fractional, 789n
quantum Hall fluid, and ground state degeneracy, 723
quantum hydrodynamics, analogy to quantum gravity, 759n
quantum mechanics: cube of physics, 13f; derivation of Hawking temperature, 445; special relativity and, 437; spin 1 particles, 256; use of operators in, 48
quantum of gravity. *See* gravitons
quantum of light. *See* photons
quantum operators, 771, 772
quantum particles, in classical gravitational field, 771
quantum physics: difference from classical physics, 360–361; discord with Einstein gravity, 768–769; equivalent formulations for, 770; observables in, 772. *See also* physics
quantum statistical mechanics, correspondence with quantum field theory, mystery of, 445
quantum systems: on torus, 723n; with zero Hamiltonian, 723
quantum tunneling, and Hawking radiation, 449
quarks: baryogenesis, 526; families of, 786; masses of, and anthropic principle, 757–758
quotient theorem, 316–317
- r*, use of letter in different situations, 95
radar echo delay experiments, 373f; as test of Einstein gravity, 372–373
radar ranging, 291
radial coordinates, hyperbolic, 653–654
radiation: accretion disk, compared to nuclear fusion, 415; background (*see* cosmic microwave background); black body, of black holes, 436; Gibbons-Hawking, 449, 638; Hawking (*see* Hawking radiation); quadrupole, graviton spin, 571; role in dissipative collapse, 521; from rotating black holes, 473–475; thermal, from de Sitter horizon, 637; universe dominated by, 495–497
radiation density, and scale factor of universe, 496f
radion field, 680; calculation of 5-dimensional scalar curvature, 686
radius, role of, in Schwarzschild metric, 364–365
rapidity, of boosts, 188
Raychaudhuri equation, 449, 555–556
A la recherche du temps perdu (Proust), 205
recombination, delayed, 516–517
rectilinear container, infinitesimal, 80e
redshift: cosmological, 295; gravitational, 259, 282–283, 303–304, 412; infinite, outside Kerr black holes, 462, 466, 469; relativistic, of frequency, 186

850 | Index

- redshift factor, 295
redshift formula, 299, 490; as cosmic clock, 504
reducible representations, 54–57
Reed, Ishmael, stellar nucleosynthesis, 518
reference frames: change of, and covariant derivative, 103; comoving, preferred flow direction, 230; different, for Einstein's clocks, 167f; dueling thinkers experiment, 7–8; and falling ring of balls, 59f; nearby, connected by 1-forms, 600; orthonormal, 594, 595f. *See also* observers
reflections, space, 721n. *See also* rotations
refraction, as principle phenomenon, 4
Regge calculus, 726n
Reissner-Nordström black holes, subextremal, 483
Reissner-Nordström spacetime, 477–478
relativistic action: accelerated frames, 285; gravitational time dilation, 284; matrix theory, 210
relativistic completion, 218, 242–243; of current, 223
relativistic curl, 4–vector, 252
relativistic Doppler shift, 185–186, 222
relativistic fluid dynamics, 233
relativistic kinematics, 221
relativistic matter. *See also* radiation
relativistic particles. *See* massless particles
relativistic stellar interiors, 451–457
relativistic strings, generalization of action for, 210n, 215
relativistic unification, 247
relativistic wave equation, standard, 565
relativity: in American football, 172f; concept of, 17–20; definition of, 17; Galileo's principle of, 17–19, 159; general (*see* general relativity); numerical, 400–405, 693; special (*see* special relativity)
relativity principle, Galilean, 159
relevant events, time dilation, 197
relevant perturbation, in cosmic diagrams, 512
relic photons, 517
relic problem, 532
renormalizable interactions, 711–712
renormalization group flow, 511
renormalization group ideas, and scaling, 754
reparametrization invariance, variational calculus, 123
repeated index summation. *See* summation convention
representation: ambitwistor, 736, 739; defining, of rotation group, 54; fundamental, of rotation group, 54; of groups with subgroups, 225; index-free, of vector fields, 319; reducible versus irreducible, 54–57
representation theory, 54
repulsion, between like electric charges, 707
rescaling: of complex parameters, 733; invariance on, 559
rest frame: of gyroscope in parallel transport, 549; length contraction, 199; with proper time, 179
restless vacuum, in quantum field theory, 436–438
restrictions: of groups to subgroups, 57; by Lorentz symmetry, 339; of metric, by isometric condition, 586; of momentum, to hyperbolic shell, 220
Ricci-Curbastro, Gregorio, 345
Ricci tensor, 449; for 2-brane model, 701; in anti de Sitter spacetime, 612; calculation of 5-dimensional scalar curvature, 685; for charged black holes, 478; combined with scalar tensors, 388; computation of, 357–358, 362; cosmic expansion, 490–491; derivation of Raychaudhuri equation, 556; introduction of, 345; proportional to metric, 492; for relativistic stellar interiors, 451–452; in Schwarzschild solution, 363–364; for spherically symmetric static spacetimes, 611; vanishing of, 348; variation of, 390, 395
Riemann, Bernhard: determination of curvature of space, 65; pioneering work in extending differential geometry, 91; quest for curvature, 339
Riemann curvature: components of, in Einstein gravity, 89; as found by Riemann, 90–91; and parallel transport, 545. *See also* curved spacetime
Riemann curvature tensor, 546; alternative derivation of, 547–548; anti de Sitter spacetime, 651; computation of, 349–350, 362, 607; constraints on, 591; cyclic symmetry of, 351e; for de Sitter spacetime, 626; derivation of variation of, 389; determination of, 341–343; directly from 2-form, 611; form of, 90; formation of scalar curvature from, 345–346; on geodesic, in Fermi normal coordinates, 560; Hawking Radiation, 438; indices, number of, 131; of Kerr metric, 476; in locally flat coordinates, 553; in maximally symmetric spaces, 589; structure of, 351; symmetry properties of, 343, 561; vanishing of, 348; variation of, 347; and variations of metric in spacetime, 716
Riemann normal coordinates. *See* locally flat coordinates
Riemannian manifolds, 599–600
Riemannian geometry, 280; determination of weak field action, 572; fear of, 82
Riemannian manifolds: Cartan formulation of, 601; choice of metric, 88; definition of, 95; Killing vectors of, 588; nearby geodesics on, 552; specification of curvature of, 89
Riemannian spacetime: fundamental scalars in, 365; generalization of parallel transport to, 543
Rindler coordinates, 193f, 660
Rindler metric, 446
Rindler transformation, in Minkowski spacetime, 192e
ripples in spacetime, 563; propagation of, 667
RNA folding, and punctured surfaces, 728
Robertson, Howard P.: rejecting Einstein's article, 564. *See also* Friedmann-Robertson-Walker universes
Rogers, Eric, neighbor of Einstein, 267

- Rosen, Nathan: Einstein-Rosen bridge, 433;
gravitational waves, 563n
- Rosenfeld, L., nonconsistency of quantum field theory and classical relativity, 773n
- rotating black holes, 414, 458–476; angular momentum of, 576; and Boyer-Lindquist coordinates, 78; frame dragging, 460f; offdiagonal metric component, 459; Penrose process, 449; as sources of radiation, 473–475; 't Hooft's bound, 442. *See also* Kerr black holes
- rotating bodies: angular momentum of, 563–577; slow velocity of, 570; spacetime deformation by, 460
- rotating mass distributions, 569
- rotation groups, 317; generalized, 191; generators of, 40, 192; as invariance group of physics, 755; Lie algebra of, 191; representation of, 54; subgroup of Lorentz group, 192. *See also specific groups*
- rotation matrix: and covariant differentiation, 321; definition of, 38
- rotational invariance, 118; inverse square law, 120, 697; in Newton's second law, 140
- rotations: approach generalizing to higher dimensional spaces, 42; under coordinate transformations, 72–73; definition of, in matrix form, 40; determination in manifolds, 590; and exponential function, 41; as freedom left in higher dimensional space, 88; in higher dimensional spaces, 44–45, 49–51; hyperboloid of, 625; and index notation, 44–45; as invariant transformation, 186; as linear transformations, 68; order of, 50f; in plane, 38–40; similarity to metrics, 181; in spacetime, 174
- Roth, Philip, *The Ghostwriter*, 254
- Royal Society, expeditions to test Einstein's theory, 367
- rubber sheet analogy, misleading for black holes, 432
- rulers, observed in different frames, 199f
- Rumford, Count (Benjamin Thomson), energy conservation, 387n
- saddle point, determination of surface curvature, 105f
- Sakharov, Andrei D., grand unified theory, 529
- Sandage, Allan, closed and open universes, 296–297
- satellites: onboard gyroscope measurements, 549; radar echo delay experiments, 373
- scalar action, energy momentum tensor for, 387e
- scalar check, of Schwarzschild metric, 365
- scalar curvature: for 2-brane model, 700; 5-dimensional, 684–686; constant, of maximally symmetric spaces, 589; of expanding universe, 609; formation from Riemann curvature tensor, 345–346; and mass dimensions, 711; and other coordinate scalars to form a metric, 708–709
- scalar fields: action, 332; in AdS/CFT correspondence, 665; charged, in 5-dimensional theories, 687; Lagrangian in, 712; as quantities without qualities, 788n
- scalar product: of four vectors under Lorentz transformation, 182; invariant in parallel transport, 544; of vectors, definition of, 39
- scalar tensors, combined with Ricci tensor, 388
- scalars: and coordinate transformations, 73; differentiation, 318; in general relativity, 315; and invariance, 47; objects without indices not transforming as, 719–720; rotational, 225
- scale and conformal invariances: and naturalness doctrine, 750; in particle physics, 621
- scale factor of universe, 289, 293, 489; and Big Bang, 499f; cosmological equation, 633; and energy density, 496f; in inflationary cosmology, 534; primeval density fluctuations, 524; redshift formula, 299
- scales, 750; physics on different length, 750
- scaling: at cosmological distances, 753–754; metric invariant under, 657
- scaling dimensions, of terms in action, 713n
- scattering: 4-gluon, 744e; Compton, 222f, 235e; electromagnetic, of point charges, 766; of electromagnetic wave on atom or molecule, 715; gluons, Feynman diagrams for, 735–736; of gravitons (*see* graviton scattering); impact parameter for, 309, 309f, 416; of neutrinos, 765; particle, Lorentz invariance, 236e; photons, frequency shift in, 222
- scattering amplitudes: 4-gluons, 738; ambitwistor representation for, 737; dimensional analysis, 717, 761, 770; and effective field theory, 770; expressed in terms of helicity spinors, 734–735; Fourier transformation of, 736; gluons, 785; for gravitational wave on finite sized object, 717; gravitons (*see* graviton scattering); spacetime hidden in, 739–740; in terms of helicity spinors, 735–736; as volume of polytopes in momentum-twistor space, 742
- scattering cross section, electromagnetic wave on atom or molecule, 715
- Schild (Kerr-Schild form), 476
- Schrödinger's cat experiment, quantum gravity, 771
- Schrödinger equation, for (nonrelativistic) charged particle in magnetic field, 354n
- Schwarzschild, Karl: letter to Einstein, 362; meaning of name, 363
- Schwarzschild black holes: escape from, 427; Hawking temperature of, 436; and Kerr black holes, 468; Kruskal-Szekeres coordinates, 635; Kruskal-Szekeres diagram of, 426f; mass determination, 570; Penrose diagrams, 429f
- Schwarzschild–de Sitter spacetime, 375e, 635
- Schwarzschild-Droste metric, and solar system tests of Einstein gravity, 362–371
- Schwarzschild metric: derivation of, 347; discovery

852 | Index

- Schwarzschild metric (*continued*)
of, 364; Kruskal-Szekeres diagram for, 425–427; near-horizon, 445–446; Painlevé-Gullstrand coordinates, 417; perihelion shift in, 371–372; planetary orbits in, 371–372
- Schwarzschild radius, 409; in Kerr solutions, 461, 465; relation to actual radius, 366; role in metric, 364–365; of universe, 514; and universe's obesity index, 443
- Schwarzschild singularity: coordinate, 365–366; impossibility of, in toy model of spherical cluster of noninteracting particles, 376n
- Schwarzschild solution: as limit of Kerr solution, 466; with charged central mass (*see* charged black holes); Kruskal coordinates as extension to, 433; time-dependent mass distribution, 374; Weyl's way to, 374
- Schwarzschild spacetime, 292; precession in, 549; spherical shell of photons in, 430f
- “second,” meaning of term used in measuring angles, 368n
- second derivative in time, role for dynamics, Newton's insight, 401
- second law of black hole thermodynamics, 472
- second order corrections, to locally flat Euclidean metric, 88
- segments, infinitesimal, space and time experience of, 180
- self-interacting scalar field, 387e
- self-tuning, 706
- semi-circles, as geodesics, 133
- Shapiro, Irwin I., radar echo delay experiments, 372–373
- sheets, swept out by strings, 216f
- “shift,” 691, 693
- shortest path: in curved spacetime, 276; determination of, 155; on earth's surface, 275; and parallel transport, 545; in spacetime, 176f. *See also* geodesics; path
- “shut up and calculate,” 445
- sign: most significant in physics, 176; role in electromagnetism, 382
- sign error, in action variation, 380
- sign function, in Green's function, 573
- signature, of spacetime, changing of, 732–733
- Silberstein, Ludwik, understanding of Einstein's theory, 369–370
- similarity transformations, definition of, 56
- simultaneity: dependence on observer, 8; Einstein's gedanken experiments, 7–9; failing of, 166; fall of, 200
- single particles, ignoring gravitational waves, 566
- singularities: at Big Bang, 498; clothed, 479; coordinate, 91–92, 365–366, 467, 467f; physical, 418, 425, 467, 479; at poles of Mercator map, 365; Schwarzschild, impossibility of, 376n; at Schwarzschild radius, 409; of Schwarzschild solution, 365; spacetime, at Big Bang, 498; spherical, paper by Kruskal, 376n; with trapped surfaces, 484
- sink, in cosmic diagram, 511
- sky, reason for being blue, 715
- $SL(2, C)$ group, 730
- $SL(4, R)$ group: explanation of, 739; and twistors, 737
- slow roll scenario, 535–536
- slow rotation limit, Kerr black hole, 571
- smooth functions, and delta function, 33e
- Snell's law, 9e
- $SO(3, 1)$ group, 730
- $SO(3)$ group, generators of, 44
- $SO(3)$ transformations, 57f
- $SO(6)$ group, 619
- $SO(D)$ group: index notation of, 49; Lie algebra for, 51; Minkowski spacetime, 191
- soft photon theorems, 217n
- solar eclipse expeditions, 367; praise by J. J. Thomson, 369
- solar system, tests of Einstein gravity, and Schwarzschild-Droste metric, 309, 362–371
- Soldner, Johann, calculation of deflection of light by astrophysical objects, 366–367
- solid state structures: gauge potential of, 721. *See also* condensed matter physics
- solitons, included in quantum field theory, 781
- $SO(m, n)$ groups, and complexification, 732
- Sommerfeld, Arnold: introduction of fine structure constant, 767; letters from Einstein, 344, 366, 580
- sound horizon, 524
- sound speed: in metals, ratio to light speed, 749; in static relativistic fluid, 234
- south-pointing carriage: function of, 109; modern version of, 104f
- south pole, and its longitude, 76
- space: closed curved, 681; conformally flat, 80–81e; creation of, 498, 787; curled up, 673–674, 674f; determination of curvature, 65–66; dimensionality and inverse square law, 697; homogeneous, 289, 292, 491, 588, 704; hyperbolic, 296, 491, 590, 627, 633; internal, 688, 689; isotropic, 289, 292, 305, 491, 588, 704; local versus global character of, 76–77; maximally symmetric, 585–593, 588; metric in geodesic equation, 128; negatively curved, maximally symmetric, 610; replacing time, 137; and spacetime, classification of, 666; of spheres, and de Sitter spacetime, 646; spherical, of closed universe, 633; and time, lyrical confounding of, 174n
- space coordinates: as dynamical variable, and energy momentum tensor, 381; notation, 25
- space measurements, metric tensor for, 63–64
- space reflections, in odd-dimensional space, 721n
- spacelike 3-dimensional hypersurface, 693f
- spacelike curves, 175
- spacelike distance, 175
- spacelike events, temporal ordering of, 204

- spacelike geodesics, tentacles of, 558
- spacelike hypersurfaces. *See* Cauchy surface
- spacelike infinity, 428, 428f
- spacelike Killing vector, 637
- spacelike surfaces, 184
- spaceship: ball of whiskey in, 270; in orbit around earth, 266
- spacetime(s): 4-dimensional, divergence theorem generalized to, 386; 5-dimensional (*see* Kaluza-Klein theory); annihilated, 785; anti de Sitter, 612, 702; boundary of, 399n; causal structure of, 427, 431, 438, 530, 531f, 780; changing signature of, 732–733; circles mistaken for points, 674f; conformally equivalent, 311; conformally related, 622e; constancy of dark energy density in, 359; constructed by piling sheets, 689f; curved (*see* curved spacetime); dark energy density in, 356; de Sitter, 456, 624–648; deformation by rotating bodies, 460; discretization of, 773n; disguises of anti de Sitter, 654; distance measurements in, 180; distance of comoving observers, 174; divided into regions, 635; Einstein’s equivalence principle, 271; empty, 347–348, 362; event, definition of, 177; flat, with conformal algebra, 615; four dimensional, 174; geometry of, 174–193; and gravity, origins of, 787; how to generate, 338; human in, 658f; inside stars, 453; inversion of, 743–744; isometric, around rotating black holes, 459; Kerr, 470–471, 473; Lagrange multiplier for volume of, 756; and large extra dimensions, 697; mapping of, holographic principle, 649; Minkowskian, 277, 434; Minkowskian metric for, 181; next steps of understanding of, 784; null lines in, 741f; number current as 4–vector in, 225f; paths lengths, 189; perpendicular to internal space, 689; propagation of ripples, 667; pulsation communicated to outside, 571; Pythagoras theorem of, 167; regions of, 635; ripples in, 563; Schwarzschild, 292; Schwarzschild–de Sitter, 375e; separation of events in, 160; “sewing together” of two distinct, 429–431; shortest path in, 176f; singularity at Big Bang, 498; small enough region of, and Einstein’s equivalence principle, 712; spherically symmetric, time dependent, 311; around spherically symmetric mass distribution, 304–307, 310–311, 409; spinors in curved, 604–605; static, 61, 303–304; static isotropic, motion in, 306–307; stretching of, 615; thermodynamics of, 448–449; topology of, physics sensitive to, 720; as a triangle, 428, 434; twistor space point-to-line mapped to, 742; and twistors, 739–740; and variations of metric in, 716
- spacetime curvature. *See* curved spacetime
- spacetime derivative, two powers of, role in Einstein field equation, 402
- spacetime dimensions, four, 174
- spacetime events, light rays being more fundamental than, 741
- spacetime fluctuations, 762
- spacetime metric: around spherical mass distribution, Schwarzschild solution, 363–364; around stars, 62; formal similarity to rotation, 181; notation of, 183; perturbed, gravitational sources of, 569. *See also* metric
- spacetime picture, thinking in terms of, 28
- spatial boundaries, 655; in anti de Sitter spacetime, 649
- spatial coordinates: in continuum mechanics, 117; emerging in AdS/CFT correspondence, 787; growing from boundary, 660; and location of particles, difference between, 31
- spatial curvature: for closed, flat, and open universes, 634; effect on CMB fluctuations, 525–526
- spatial distance, in general curved spacetime, 290–292
- spatial metric, and cosmic expansion, 491
- special matrices, 40
- special relativity: abstract of, 20; accelerated particles, 193e; applied, 195–206; counterintuitivity, 204; electromagnetism from, 244–246; in everyday life, 205; geometrical view of, 582; pedagogically correct presentation, 203; performance of young Einstein, 783; problems in, foolproof method for solving, 195; and quantum mechanics, 437; time, different rates of, 196
- speed limit, existence of, 172
- speed of light. *See* light speed
- spheres: in 3-spaces, distances of, 610; curvature of surface of, by Gauss’s strategy, 105; d-dimensional, definition of, 624; in de Sitter spacetime, 624; determination of metric on, 65; as example for curved space, 83; generalized, 92; higher dimensional, metric of, 80e; and hyperbolic spaces, 93; “at infinity,” 428; intersecting, 647f; intrinsic and extrinsic curvature of, 6, 85; metric of surface of, 83–84; squashed, 469; stereographic projection of, 80–81e, 81f; tangent plane of, 98; topology of, 727; unfamiliar metrics of, 585
- spherical blobs, 725f; growing a trunk, 726
- spherical coordinates: change from Cartesian coordinates, Euclidean spaces, 63; introduction of, 108
- spherical shell of photons, 429; in Minkowski spacetime and Schwarzschild spacetime, 430f
- spherical symmetry, comoving coordinates, 298
- spherically symmetric mass distribution, 304–307; around black holes, 409; Christoffel symbols, 310–311; foliation, 305–306; Killing vectors for, 305; Schwarzschild solution for spacetime metric around, 363–364; time dependent, and Jebsen-Birkhoff theorem, 373–374. *See also* stars
- spherically symmetric spacetime: static, 61; time dependent, 311
- spin 1 particles, 256
- spin connection, for index transformation, 603
- spin fields, in terms of spinor field, 789n

854 | Index

- spin vector: parallel transport of, 549; precession of for particle in orbit, 550
- spinor fields, and gauge potential, 789n
- spinor indices, “metric” for, 742
- spinors: complexified, 732; in curved spacetime, 604–605
- splitting of energy levels. *See* energy level splitting
- spooky action, Newton’s, 146
- spring oscillations, equation of motion, 26
- square root: calculation of, 207; of Lorentz vector, 731
- squashed sphere, length of equator, 80e
- stacked entities, 56
- standard candles, 359
- standard model of particle physics, 683
- standard notation, of coordinates, 25
- standard relativistic wave equation, 565
- Stark, Johannes, on Einstein, 216n
- stars: collapse into black holes, 455–456; first, 519; made of nothing, 456; pulsating, 304; relativistic interiors, 451–457; Riemann curvature tensor around, 362; Schwarzschild radius of sun, 409; stellar nucleosynthesis, 518–519
- static coordinates, 634; definition of, 652; time-independence of metric, 636
- static fields, classical theory, 119
- static isotropic spacetime, motion in, 306–307
- static solutions, for coupled Einstein and Maxwell equations, 482–483
- static spacetime, 303–304; spherically symmetric, 61; translation invariant physics in, 304f
- static universe, Einstein’s, 509–510, 514
- stationary limit surface, 461–463; angular velocity inside, 471; Kerr black hole, 462f; outer, 469
- stationary phase approximation, 770
- “stationed” observer, around black hole, 412
- stellar nucleosynthesis, and anthropic principle, 758
- stereographic projection: for anti de Sitter spacetime, 661; for de Sitter spacetime, 641; of sphere, 80–81e, 81f
- straight line: appearance of, curved coordinates, 130–131; distance of, in Minkowskian spacetime, 175; form dependence of coordinate systems, 127; geodesic problem, solutions of, 124; most complicated description of, 125; and parallel transport, 545; as shortest path between two points, 4; in twistor space, 742; between two points, 66, 90
- stress energy tensor, 386e; in outgoing brane wave model, 704–705. *See also* energy momentum tensor
- string: action of, 146; boundary conditions for energy of, 115; elastic, hanging under force of gravity, 113; hanging, and variational calculus, 113–123; with nonuniform force distribution, 117; relativistic, action for, 210n
- string action, invariance of, 147, 216e
- string theory: and anthropic principle, 757; Bekenstein-Hawking entropy and, 444; current of, 235; dilaton field in, 680; in early universe, 518; and extremal black holes, 467; and generalized uncertainty principle, 769; as higher dimensional theory, 695; and Kaluza-Klein / Yang-Mills theories, 682–683; large extra dimensions in, 696; minimal, 147; sheets created from strings, 216f
- string vibrations, speed of propagation of, 147
- strong energy condition, 557; and gravity attraction, 562n
- strong force, generated by pions, 205
- strong interaction, 526; understanding of, 785
- structural equations, Cartan’s, 684
- structure formation, in early universe, 520, 522–523
- subextremal black holes: charged, 478–479; Reissner-Nordström, 483
- subgroups, restriction of groups to, 57
- subscript, index notation, 32
- subtraction, of vectors, in Euclidean space, 101, 101f
- summation convention, 46, 184, 316; and general coordinate transformations, 71; in general relativity, 314; and Greek symbol notation, 63–64; Lorentz transformation of, 186; Minkowski metric, 182; and tensors, 52; and upper and lower indices, 64
- summation variables, dummy, 184n
- sums, notation of, Kronecker delta, 45
- sun: ratio of Schwarzschild radius to actual radius, 367; Schwarzschild radius, 266, 409
- superb theorems, Newton’s, 33
- superconductivity, high temperature, 789n
- superrenormalizable interactions, 711–712
- superscript, index notation, 32
- supersymmetry: Bekenstein-Hawking entropy and, 444; Yang-Mills theory, 621
- supertwistors, 739n
- suppressed angular coordinates, 422, 426
- surface curvature: compared to curved line, 89n; determination of, Gauss’s strategy, 104–105
- surface parametrization, 98
- surface vectors: basis for, in Euclidean space, 98; normal, 184; parallel transport of, 543
- surfaces: in 3-dimensional Euclidean space, 98–109; generated of light rays, 185; gravity at, 473; “inside” and “outside” of, 85; metric on, in Euclidean space, 99; normal to, at certain point, 99f; “one way” in spacetime metrics, 185; punctured, 726; spacelike, 184; stationary limit, 461–463, 469; tangent plane of, in Euclidean space, 98–99; trapped, 484, 789n; triangulation of, 726
- Sylvester, James Joseph, 210; law of inertia, 193e
- symbolic manipulation software, computation of curvature tensor, 607
- symmetric mass distribution. *See* spherically symmetric mass distribution
- symmetric spaces, maximally, 585–593, 588; curvature tensor in, 589; negatively curved, 610

- symmetric spacetimes, spherically, 611
symmetric tensors, character of, 55
symmetry: of angular momentum, 150; approach to fluid dynamics, 164; and conservation, 150–155; and curvature tensor, 561; cyclic, of Riemann curvature tensor, 351e; deduction of physics from, 254; dot notation, 129; and equivalence principle, 317–318; Fermi normal coordinates, 561; and Fermi normal coordinates, 561; gauge, in higher dimensional theories, 682; and gauge invariance, 249; hidden, of nature, 210; imposed on gravity, 254; and invariance, 242–243; local gauge, in higher dimensional theories, 682; Lorentz, restrictions on electromagnetism, 339; matter-antimatter, violation of, 528, 683; maximal, 592, 625, 626, 650; physical, definition of, 47; as property of tensors, 61; restrictions on Newtonian gravity, 339; of Riemann curvature tensor, 343, 561; in spatial indices, 609; of spheres in spacetime, 585; spherical, 298, 304–307, 310–311, 373, 409; supersymmetry, 444, 621. *See also* antisymmetry; rotations
symmetry breaking, spontaneous, 593, 784
symmetry group, Euclidean group as, 755
symmetry relations, investigations of, in locally flat coordinate system, 343–344
system of units, natural, 10–12
Szekeres, George. *See* Kruskal-Szekeres entries

't Hooft, Gerard: bound on entropy of black holes, 442–443; naturalness doctrine, 750; and Yang-Mills field, 789n
tangent plane: and curved surface of sphere, 83–84, 83f; and normal to surface, 99f; rotating around normal vector, 100; of surface in Euclidean space, 98–99
tangent vectors: of curves, 96, 327; to geodesic, 555; spacetime surfaces, 185; to straight lines, 130
tautochrone problem, Lagrange, 144
Taylor, Joseph H., detection of binary pulsar, 563
Taylor coefficients, and Riemann curvature, 91
Tegmark, Max, inflationary cosmology, 536
teleological discussions, in physics, 136
temperature: ambient, of universe, 504; concept of, 15; of cosmic microwave background, 515, 521–522; Hawking, 436, 441; inverse, 445; mystery of, 15; for nonrelativistic gas, 231; of photon gases, 495
temporal boundary, and Poincaré half plane, 632
temporal coordinate: in boundary theory, 660; dependence of spatial coordinate, 652
temporal ordering: in antimatter creation, 206; in different frames, 204
tennis ball trajectories, in space and spacetime, 33
tensor, notation, Greek symbols in, 63
tensor decomposition, 236e
tensor density, definition of, 75n
tensor fields, 243; electromagnetic, 244; gravity, 257; introduction of, 53–54
tensor notation: gravity potential, 57–59; Greek symbols in, 63; and Laplace's equation, 58; Newtonian orbits, 60; particle motion, 57–59
tensors: antisymmetric and symmetric character of, 55; construction of, 313; contraction, 316; covariant derivative as, 322; covariant derivative of, 324; covariant divergence of, 332; definition of, 52; differentiation, 318; fear of, 52–53; form invariant, 592–593; in general relativity, 312–319; and indices (upper and lower), 74; invariant, definition of, 59–60; Lie derivative, 328, 331; Lorentz, 188, 243; under Lorentz transformation, 193e; in Newtonian mechanics, 57–59; of polarization, gravitational waves, 565; and representation theory, 54; Ricci (*see* Ricci tensor); of slowly rotating bodies, 570; stress energy (*see* energy momentum tensor); symmetry properties of, 61, 343; trace of, 55; transformation of, 132; and vectors, interplay of, 53–54
tentacles, consisting of spacelike geodesics, 558
terrestrial and celestial mechanics, Newton's unification of, 28
test, "1–2," 326
test particle, 302; PPN approximation, 309
tetrahedra: glued together, 725f; topology of, 725
Theorema Egregium, 90–91
theorems. *See* specific theorems
theoretical physics: and cosmological constant paradox, 753; Einstein mode of, 778; fundamentals of, 783; "golden" guiding principle in, 338; impact of Einstein gravity, 777; unified perspective on, 170. *See also* physics; quantum physics
theories. *See* specific theories
thermal radiation, from de Sitter horizon, 637
thermocouples, Einstein's ether detection, experimental set-up, 163
thermodynamics: first and second law of, for black holes, 472–473; first law of, and pressure of universe, 360n; of spacetime, 448–449
Thomson, J. J., praise for solar eclipse expeditions, 369
Thomson, Benjamin (Count Rumford), energy conservation, 387n
Thoreau, Henry David, deeds for old and young people, 788
thought experiments. *See* gedanken experiments
tidal forces, 554; and finite sized objects, 716–717; gravitational waves, 567, 567f; introduction of, 59
tilting light cones, at Schwarzschild radius, 420–421, 421f
time: in 4-dimensional matrix, 210; connection with gravity, 579; cosmic, 295, 530, 632; cosmological, in outgoing brane wave model, 706; cosmological problem of not enough, 521–522, 531; creation of, 787; different rates of, in special relativity, 196; and gravity, 257–258; imaginary, in derivation of

856 | Index

- time (*continued*)
Hawking temperature, 445–446; lines of constant, 637; in Minkowskian sphere, 631; mystery of, 787; in Newtonian universe, 7; psychological, 175n; and space, unifying, 174–175; specific, in integrals, 228–229; transit time, minimization, 139; translation invariance in, 303–304; units for, 10; unwound, 653
time coordinates: multiple, 666n; notation, 25; two, 652
time delta function, 229
time dependence: disappearing in static coordinates, 635; Lagrangian without explicit, 153; of metric, 455; in physics, 137–138; of spherically symmetric mass distributions, 373–374; of spherically symmetric spacetime, 311
time dilation, 197; gravitational, 258–259, 284, 412, 304; lifetime of particles, 198
time evolution, of universe, 511f
time evolution equations, importance in Newtonian mechanics, 400–401
time reversal, strong gravitational sources, 574
time reversal invariance, 416–417; accelerated expansion, 500
time translation, in Penrose diagram, 620
timelike curves, closed, 484; violating physics, 653
timelike distances, 175
timelike geodesics, 645; behavior of, 554–555; congruence of, 555; dense collection of, 555
timelike infinities, 428, 428f
timelike Killing vectors, 631, 637
timelike physical singularity, 479
Tinseau, D’Amondans Charles de, introduction of osculating plane, 97
Tolman-Oppenheimer-Volkoff equation, 453, 457
top ten worst physics terms, 767
topological action, 720–721
topological cylinder, anti de Sitter spacetime, 654
topological field theory, 719–728
topological invariants, 725–727
topological quantization, 723
topological terms, in gauge theories, 720–721
topology. *See* differential forms
torsion, of curves, 97
torsion pendulum, and non-quantized gravity, 771
torus, systems on, 723n
total action, Newtonian world, 145
total energy: conservation of, 35; Hamiltonian, 144
total energy momentum tensor, disappearance of, 394
total momentum, conservation of, 37
totally antisymmetric symbol, definition of, 50
toy model, of spherical cluster of noninteracting particles, 376n
trace: of matrix, and intrinsic curvature, 84; of tensor, 55
trans-Planckian cosmology, 518
transextremal charged black holes, 478
transformation invariance: of action principle, 147; of Poincaré coordinates, 657
transformation matrix, 312; linearity, 313
transformations, 80–81e; compared to variations, 389; conformal, 614, 616; coordinate, 62, 68–70, 564; Galilean, 18–20; gauge, as 5-dimensional coordinate transformation, 673; importance of, in theoretical physics, 75; infinitesimal, 187, 615; in Kaluza-Klein theory, 672; as pervasive theme of theoretical physics, 68; under $SO(3)$, 57f; and vectors, 42
transit time, minimization of, 139
translation, generators of, 644
translation invariance, 242; of physics, in static spacetime, 304f; in time, 303–304
translation operator, introduction of, 340
transport, Lie, 328
transpose: of matrix, 45; of vector or matrix, definition of, 39
transverse-traceless (TT) gauge, 565
trapped surface, 484; presence of, 789n
triangulation, of a surface, 726
trihedron, moving, of smooth curve, 97f
trunk, grown from spherical blob, 726
Tsai, Ming-liang, *What Time Is It over There?* 514
TT (transverse-traceless) gauge, 565
tunneling, quantum, and Hawking radiation, 449
Twain, Mark, on truth of knowledge, 410n
twin paradox, 189, 194e
twistor space: analogs of Euclidean space objects in, 742; geometry of, 741–742; point-to-line mapped to spacetime, 742; points in, 741f
twistors: ambitwistor representation, 736; complexification of variables, 732; covered Lorentz group, 729–730; and Einstein-Hilbert action, 739; freedom to rescale, 733; geometric essence of, 739–740; and interaction among gravitons, 738–739; introduction to, 730–745; Lorentz invariance, 734; motivation for studying, from quantum field theory, 731; polarization and helicity, 734; and power of helicity spinors, 735; and Roger Penrose, 730–731; and $SL(4, R)$ group, 737; and spacetime, 739–740
“Tycho Brahe day,” 369n
ultimate theory, dream of, 789n
ultrarelativistic particles. *See* massless particles
ultraviolet catastrophe, 781; Planck and, 789n
ultraviolet completion, of quantum gravity, 765
ultraviolet regime, linkage to infrared regime, 752
umveg test, 9n
uncertainty principle, 206n; antimatter creation, 205; generalized, 769; and Kaluza-Klein theory, 674; and minimum length, 763; quantum field theory, 437; and quantum gravity, 762; and the three natural units, 11–12; and zero point energy, 745–746
unification: fundamental interactions (*see* grand

- unified theory, string theory); of gravity and other interactions, 767–768, 780; relativistic, 247; weak interaction and electromagnetic interaction, 765
- unified language, for different physical phenomena, 186
- unified notation, of Lorentz transformation, 186
- unimodular gravity, and cosmological constant paradox, 755–756
- unit circle, length element on, 80e
- unit determinants, 40
- unit matrix, definition of, 39
- unit spheres, metric on, 80e
- unit tangent vector, of a curve, 96
- unitarization, and ultraviolet completion, 765
- units: change of using, 16n; of distance, 168; Hubble, 293; for length and time, 10; natural system of, 10–12; royal and “revolutionary,” 163n. *See also* Planck units
- universal clock: in Newtonian physics, 25; set-up of, 172
- universality of gravity, 258, 269–270; curved spacetime, 275–276
- universe: 2-dimensional map of, 506–507; acausality of, 754; acceleration or deceleration of expansion, 506–507; action of, 346, 356; age of, 512–513; ambient temperature as cosmic clock, 504; critical density, 497–498; curvature of, 490–491, 526, 748; dominated by (nonrelativistic) matter, 495–496, 514; dominated by radiation, 495–496; dynamic, 489–501; early (*see* early universe); energy density of, 504; entropy of, 527; equation of motion for, 357; equation of state of, 359; expanding (*see* expanding universe); fate of, 507–509; filled with constant energy density, 356; filled with perfect fluids, 492–493; filtered through human mind, 779; foamlike structure of, 754, 758n; and gravity, 778; hidden acausality of, 783; history of, 496, 502, 503f, 515–529; homogeneity and isotropy problem, 531; Hubble radius of, and photon mean free path, 517; inflationary, 534–535; intrinsic curvature, 6; length scale of, characteristic, 788n; mass of, 747–748; obesity index of, 13; open, 629; open or flat, troubling Wheeler, 779; as perfect fluid, 231; with positive cosmological constant, 633; scale factor of (*see* scale factor of universe); Schwarzschild radius of, 514; time evolution of, 511f
- universes: closed/open/flat, 296–297, 491, 493–494, 497–498; as curved spacetime, 288–300; different from de Sitter spacetime, 633; with different laws of physics, 757; Friedmann-Robertson-Walker, 296, 491, 704; mathematical, 634; static, 509–510, 514
- unprimed coordinates, 18, 38; metric with, 71–73
- Unreasonable Effectiveness of Mathematics in Physics, The* (Wigner), 446
- Unruh effect, 446–447
- upper indices, 314–316; and introduction of lower indices, 64; transformations in change of coordinates, 71–73
- ur-vector, 312; definition of, 43; with lower index, 318; spacetime metrics, 181
- vacuum: as boiling sea of quantum fluctuations, 745–746; restless, 436–438
- vacuum Einstein equation, solution of, 647e
- vacuum energy, 746; driving inflation, 751; explanation of, 752–753; in outgoing brane wave model, 706; proofs of, 748
- vacuum energy density, upper bound to, 749
- vacuum state, 447
- variables: dynamical, 249; “free,” in variational calculus, 116; in functional variations, 121–122
- variation: of action: for electromagnetism, 244, 250–251, 380; of basis vectors, 100; compared to transformation, 389
- variational calculus, 155; of brachistochrone problem, 120; compromises in finding extreme values, 115; functional, 114–115; and hanging string, 113–123; integration by parts, 116; of several unknown functions, 123; solution of geodesic problem, 125
- variational principle: equation of motion from, 137; for gravity, Einstein and Grossmann, 396
- vector fields: constant, covariant derivative of, 331; differentiation of, 100–101; index-free representation of, 319; introduction of, 46; movement through, 544; studied by observers, 47f; visualized as fluids, 327f
- vector potential, Lorentz, 243, 248
- vector subtraction, in Euclidean space, 101, 101f
- vectors: and arrays, 51n; basic or ur-, definition of, 43; column, notation of, 45; and construction of tensors, 313; contravariant, 183; covariant, 183, 340; definition of, 39; definition of, representation theory, 54; differentiation, 318; displacement of, in curved rectangle, 341f; and indices (upper and lower), 73–74; lightlike, 731; Mother of All, 312–313; notation for, 182; parallel transport of, 101–102, 545f; projected on tangent plane, 102; solution of isometric condition, 586; of spacetime metrics, 181; on surface, parallel transport of, 543; and tensors, interplay of, 53–54; and transformations, 42; transporting via alternative routes, 548
- velocities: addition of, 160–161, 163, 171, 173e; angular: around rotating black holes, 460, 471; completion and promotion of, 218–219; Fermi-Walker transported, 193e; Galilean law for addition of, 19; low limit of Lorentz transformation, 169; measurements in trains, 166; of objects in cars, 162–163; observed in Galileo transformation, 161; rotating bodies, 570
- velocity vector: of curves, 327; along geodesic, 330
- vertices, in topology, 725–727

858 | Index

- Vicious and Nasty, dueling thinkers experiment, 7–9, 8f
- vielbein: 1-form, 600; and differential forms, 594–606; Kaluza-Klein metric, 690–691; as square roots of metric, 596
- VIRGO, gravitational wave detector, 577n
- virial theorem, relativistic generalization, 255
- visibility problem: in Kaluza-Klein theory, 673–674; with large extra dimensions, 696–697
- Voigt, W., Lorentz transformation, 169n
- Volovik, G., solution of cosmological constant paradox, 759n
- volume element, generalized, determination for any curved space, 75–76
- Vulcan (predicted planet), 368
- Walker, Arthur G. *See* Friedmann-Robertson-Walker universes
- “wanting the cake and eating it too” syndrome, 751
- war, ancient art of, 103–104
- warp function, 701
- warped polar coordinates, 613e
- wave equation: derivation of speed of sound, 235; standard relativistic, 565
- wave function, phase angle of, in Kaluza-Klein theory, 678
- wave guide, 694
- wave vectors, by different observers, 185
- wavelength, de Broglie, particles at Schwarzschild radius, 442
- waves: bulk, to brane, 703f; gravitational (*see* gravitational waves); understanding of, 783–784
- weak energy condition, 57
- weak field, 564
- weak field action, determination of, without Riemannian geometry, 572
- weak field approximation, for gravitational sources, 569–570
- weak interaction, 526; CP violation in, 528, 683; Fermi’s theory of, 765; of massive particles, 522; ultraviolet completion of, 765
- Weinberg, Steven: primeval nucleosynthesis, 528; quantum gravity governed by attractive ultraviolet fixed point, 773n; upper bound for cosmological constant, 757; very weak version of anthropic principle, 752; from weak field to Einstein gravity, 580
- Weinberg-Witten theorem, 787
- Weingarten, Julius, equation of, 106
- wet dog, effect of inertia, 276
- Weyl, Hermann: corrections to de Sitter metric, 289, 642; and Kaluza-Klein theory, 693–694; “Raum und Zeit,” 175n; way to Schwarzschild solution, 374
- Weyl approach, to Kerr black holes, 473
- Weyl-Eddington terms, in effective field theory approach, 710
- Weyl equation, commutation relations, 192
- Weyl tensor, properties of, 352e
- Weyl transformation, introduction of, 94
- “What is greater than God?” puzzle, 789n
- What Time Is It over There?* (film, Tsai), 514
- Wheeler, John A.: Einstein’s late comments on space and time, 787; geometrodynamics, 693; Hawking radiation, 440; Kruskal-Szekeres coordinates, 434; mentorship of, 435; no-hair theorems, 482; “Pushing forward the many fingers of Time,” 691; spacetime picture, thinking in terms of, 28; tossed ball test, 501; troubled by open or flat universe, 779; wormholes, 433
- whiskey, ball of, in spaceship, 269–270
- White, T. H. (Terence Hanbury), *The Once and Future King*, 361n
- Wick rotation, 192, 640n
- Wigner, Eugene, *The Unreasonable Effectiveness of Mathematics in Physics*, 446
- Williams, George C., on Newton’s gravity, 31
- Wilson, Ken, effective field theory approach, 709
- Wilson, Robert, cosmic microwave background, 517
- wimps, 522
- world: mysteries of three copies of, 7; non-flatness of, 66f
- world indices, 608; conversion with vielbein, 603; definition of, 594; versus Lorentz indices, 595
- world sheets, created from strings, 216f
- worldline action, 207–217
- worldline length, for pointlike particle in baby string theory, 215
- worldlines: and causality in de Sitter spacetime, 639–640; and events, in special relativity, 196; of individual charged particles, 715; of particles, 175, 177f, 211f; for problems in special relativity, 201
- wormholes: and coordinate singularities, 91–92; in Kruskal-Szekeres coordinates, 432–433
- worst physics terms, top ten, 767
- Wright, Edward, and Mercator map coordinate transformation, 79e
- Yang-Mills action, 681
- Yang-Mills field: complexity of, 584n; strength of, 342n, 691–692, 694
- Yang-Mills gauge potential, 682
- Yang-Mills theory, 672; connection to Einstein gravity, 782; emergence of, 688–689, 691–693; and graviton interaction, 744n; higher dimensional, 680–682; supersymmetric, 621
- Yau, Shing-Tung, Calabi-Yau manifolds, 695
- Yukawa, Hideki, mass prediction of pion, 205
- Zel’dovich, Y. B., vacuum energy, 749
- zero-g environment, 266
- zero point energy, 745–746; proofs of, 748
- zero-sized objects, 717n