

## Contents

Acknowledgments	xi
Chapter 1. Introduction	1
Chapter 2. A preview	5
2.1 Basic concepts	5
2.2 First example: $SU(n)$	8
2.3 Second example: $E_6$ family	12
Chapter 3. Invariants and reducibility	14
3.1 Preliminaries	14
3.2 Defining space, tensors, reps	17
3.3 Invariants	19
3.4 Invariance groups	22
3.5 Projection operators	23
3.6 Spectral decomposition	25
Chapter 4. Diagrammatic notation	27
4.1 Birdtracks	27
4.2 Clebsch-Gordan coefficients	29
4.3 Zero- and one-dimensional subspaces	31
4.4 Infinitesimal transformations	32
4.5 Lie algebra	35
4.6 Other forms of Lie algebra commutators	37
4.7 Classification of Lie algebras by their primitive invariants	38
4.8 Irrelevancy of clebsches	38
4.9 A brief history of birdtracks	39
Chapter 5. Recouplings	42
5.1 Couplings and recouplings	42
5.2 Wigner $3n-j$ coefficients	44
5.3 Wigner-Eckart theorem	46
Chapter 6. Permutations	49
6.1 Symmetrization	49
6.2 Antisymmetrization	51
6.3 Levi-Civita tensor	53
6.4 Determinants	55
6.5 Characteristic equations	57

6.6	Fully (anti)symmetric tensors	57
6.7	Identically vanishing tensors	58
<b>Chapter 7. Casimir operators</b>		<b>60</b>
7.1	Casimirs and Lie algebra	61
7.2	Independent casimirs	62
7.3	Adjoint rep casimirs	64
7.4	Casimir operators	64
7.5	Dynkin indices	66
7.6	Quadratic, cubic casimirs	69
7.7	Quartic casimirs	70
7.8	Sundry relations between quartic casimirs	72
7.9	Dynkin labels	75
<b>Chapter 8. Group integrals</b>		<b>76</b>
8.1	Group integrals for arbitrary reps	77
8.2	Characters	79
8.3	Examples of group integrals	80
<b>Chapter 9. Unitary groups</b>		<b>82</b>
<i>P. Cvitanović, H. Elvang, and A. D. Kennedy</i>		
9.1	Two-index tensors	82
9.2	Three-index tensors	83
9.3	Young tableaux	84
9.4	Young projection operators	90
9.5	Reduction of tensor products	94
9.6	$U(n)$ recoupling relations	97
9.7	$U(n)$ $3n-j$ symbols	98
9.8	$SU(n)$ and the adjoint rep	102
9.9	An application of the negative dimensionality theorem	104
9.10	$SU(n)$ mixed two-index tensors	105
9.11	$SU(n)$ mixed defining $\otimes$ adjoint tensors	106
9.12	$SU(n)$ two-index adjoint tensors	109
9.13	Casimirs for the fully symmetric reps of $SU(n)$	114
9.14	$SU(n)$ , $U(n)$ equivalence in adjoint rep	115
9.15	Sources	116
<b>Chapter 10. Orthogonal groups</b>		<b>118</b>
10.1	Two-index tensors	119
10.2	Mixed adjoint $\otimes$ defining rep tensors	120
10.3	Two-index adjoint tensors	121
10.4	Three-index tensors	124
10.5	Gravity tensors	127
10.6	$SO(n)$ Dynkin labels	130
<b>Chapter 11. Spinors</b>		<b>132</b>
<i>P. Cvitanović and A. D. Kennedy</i>		
11.1	Spinography	133
11.2	Fierzing around	136
11.3	Fierz coefficients	140

CONTENTS	ix
11.4 $6$ - $j$ coefficients	141
11.5 Exemplary evaluations, continued	143
11.6 Invariance of $\gamma$ -matrices	143
11.7 Handedness	145
11.8 Kahane algorithm	146
<b>Chapter 12. Symplectic groups</b>	<b>148</b>
12.1 Two-index tensors	149
<b>Chapter 13. Negative dimensions</b>	<b>151</b>
<i>P. Cvitanović and A. D. Kennedy</i>	
13.1 $SU(n) = \overline{SU}(-n)$	152
13.2 $SO(n) = \overline{Sp}(-n)$	153
<b>Chapter 14. Spinors' symplectic sisters</b>	<b>155</b>
<i>P. Cvitanović and A. D. Kennedy</i>	
14.1 Spinsters	155
14.2 Racah coefficients	160
14.3 Heisenberg algebras	161
<b>Chapter 15. <math>SU(n)</math> family of invariance groups</b>	<b>162</b>
15.1 Reprs of $SU(2)$	162
15.2 $SU(3)$ as invariance group of a cubic invariant	164
15.3 Levi-Civita tensors and $SU(n)$	167
15.4 $SU(4)$ - $SO(6)$ isomorphism	168
<b>Chapter 16. <math>G_2</math> family of invariance groups</b>	<b>170</b>
16.1 Jacobi relation	171
16.2 Alternativity and reduction of $f$ -contractions	172
16.3 Primitivity implies alternativity	174
16.4 Casimirs for $G_2$	177
16.5 Hurwitz's theorem	178
<b>Chapter 17. <math>E_8</math> family of invariance groups</b>	<b>180</b>
17.1 Two-index tensors	181
17.2 Decomposition of $\text{Sym}^3 A$	183
17.3 Diophantine conditions	186
17.4 Dynkin labels and Young tableaux for $E_8$	186
<b>Chapter 18. <math>E_6</math> family of invariance groups</b>	<b>190</b>
18.1 Reduction of two-index tensors	190
18.2 Mixed two-index tensors	193
18.3 Diophantine conditions and the $E_6$ family	194
18.4 Three-index tensors	194
18.5 Defining $\otimes$ adjoint tensors	196
18.6 Two-index adjoint tensors	201
18.7 Dynkin labels and Young tableaux for $E_6$	202
18.8 Casimirs for $E_6$	205
18.9 Subgroups of $E_6$	207
18.10 Springer relation	208

18.11	Springer's construction of $E_6$	208
Chapter 19. $F_4$ family of invariance groups		210
19.1	Two-index tensors	210
19.2	Defining $\otimes$ adjoint tensors	213
19.3	Jordan algebra and $F_4(26)$	216
19.4	Dynkin labels and Young tableaux for $F_4$	217
Chapter 20. $E_7$ family and its negative-dimensional cousins		218
20.1	$SO(4)$ family	219
20.2	Defining $\otimes$ adjoint tensors	221
20.3	Lie algebra identification	222
20.4	$E_7$ family	224
20.5	Dynkin labels and Young tableaux for $E_7$	227
Chapter 21. Exceptional magic		229
21.1	Magic Triangle	229
21.2	A brief history of exceptional magic	231
21.3	Extended supergravities and the Magic Triangle	234
Epilogue		235
Appendix A. Recursive decomposition		237
Appendix B. Properties of Young projections		239
<i>H. Elvang and P. Cvetanović</i>		
B.1	Uniqueness of Young projection operators	239
B.2	Orthogonality	240
B.3	Normalization and completeness	240
B.4	Dimension formula	241
Bibliography		243
Index		259

# Chapter One

---

## Introduction

This monograph offers a derivation of all classical and exceptional semisimple Lie algebras through a classification of “primitive invariants.” Using somewhat unconventional notation inspired by the Feynman diagrams of quantum field theory, the invariant tensors are represented by diagrams; severe limits on what simple groups could possibly exist are deduced by requiring that irreducible representations be of integer dimension. The method provides the full Killing-Cartan list of all possible simple Lie algebras, but fails to prove the existence of  $F_4$ ,  $E_6$ ,  $E_7$  and  $E_8$ .

One simple quantum field theory question started this project; what is the group-theoretic factor for the following Quantum Chromodynamics gluon self-energy diagram

(1.1)

I first computed the answer for  $SU(n)$ . There was a hard way of doing it, using Gell-Mann  $f_{ijk}$  and  $d_{ijk}$  coefficients. There was also an easy way, where one could doodle oneself to the answer in a few lines. This is the “birdtracks” method that will be developed here. It works nicely for  $SO(n)$  and  $Sp(n)$  as well. Out of curiosity, I wanted the answer for the remaining five exceptional groups. This engendered further thought, and that which I learned can be better understood as the answer to a different question. Suppose someone came into your office and asked, “On planet  $Z$ , mesons consist of quarks and antiquarks, but baryons contain three quarks in a symmetric color combination. What is the color group?” The answer is neither trivial nor without some beauty (planet  $Z$  quarks can come in 27 colors, and the color group can be  $E_6$ ).

Once you know how to answer such group-theoretical questions, you can answer many others. This monograph tells you how. Like the brain, it is divided into two halves: the plodding half and the interesting half.

The plodding half describes how group-theoretic calculations are carried out for unitary, orthogonal, and symplectic groups (chapters 3–15). Except for the “negative dimensions” of chapter 13 and the “spinsters” of chapter 14, none of that is new, but the methods are helpful in carrying out daily chores, such as evaluating Quantum Chromodynamics group-theoretic weights, evaluating lattice gauge theory group integrals, computing  $1/N$  corrections, evaluating spinor traces, evaluating casimirs, implementing evaluation algorithms on computers, and so on.

The interesting half, chapters 16–21, describes the “exceptional magic” (a new construction of exceptional Lie algebras), the “negative dimensions” (relations between bosonic and fermionic dimensions). Open problems, links to literature, software and other resources, and personal confessions are relegated to the epilogue, monograph’s Web page [birdtracks.eu](http://birdtracks.eu). The methods used are applicable to field-theoretic model building. Regardless of their potential applications, the results are sufficiently intriguing to justify this entire undertaking. In what follows we shall forget about quarks and quantum field theory, and offer instead a somewhat unorthodox introduction to the theory of Lie algebras. If the style is not Bourbaki [29], it is not so by accident.

There are two complementary approaches to group theory. In the *canonical* approach one chooses the basis, or the Clebsch-Gordan coefficients, as simply as possible. This is the method which Killing [189] and Cartan [43] used to obtain the complete classification of semisimple Lie algebras, and which has been brought to perfection by Coxeter [67] and Dynkin [105]. There exist many excellent reviews of applications of Dynkin diagram methods to physics, such as refs. [312, 126].

In the *tensorial* approach pursued here, the bases are arbitrary, and every statement is

invariant under change of basis. Tensor calculus deals directly with the invariant blocks of the theory and gives the explicit forms of the invariants, Clebsch-Gordan series, evaluation algorithms for group-theoretic weights, *etc.*

The canonical approach is often impractical for computational purposes, as a choice of basis requires a specific coordinatization of the representation space. Usually, nothing that we want to compute depends on such a coordinatization; physical predictions are pure scalar numbers (“color singlets”), with all tensorial indices summed over. However, the canonical approach can be very useful in determining chains of subgroup embeddings. We refer the reader to refs. [312, 126] for such applications. Here we shall concentrate on tensorial methods, borrowing from Cartan and Dynkin only the nomenclature for identifying irreducible representations. Extensive listings of these are given by McKay and Patera [234] and Slansky [312].

To appreciate the sense in which canonical methods are impractical, let us consider using them to evaluate the group-theoretic factor associated with diagram (1.1) for the exceptional group  $E_8$ . This would involve summations over 8 structure constants. The Cartan-Dynkin construction enables us to construct them explicitly; an  $E_8$  structure constant has about  $248^3/6$  elements, and the direct evaluation of the group-theoretic factor for diagram (1.1) is tedious even on a computer. An evaluation in terms of a canonical basis would be equally tedious for  $SU(16)$ ; however, the tensorial approach illustrated by the example of section 2.2 yields the answer for all  $SU(n)$  in a few steps.

Simplicity of such calculations is one motivation for formulating a tensorial approach to exceptional groups. The other is the desire to understand their geometrical significance. The Killing-Cartan classification is based on a mapping of Lie algebras onto a Diophantine problem on the Cartan root lattice. This yields an exhaustive classification of simple Lie algebras, but gives no insight into the associated geometries. In the 19th century, the geometries or the invariant theory were the central question, and Cartan, in his 1894 thesis, made an attempt to identify the primitive invariants. Most of the entries in his classification were the classical groups  $SU(n)$ ,  $SO(n)$ , and  $Sp(n)$ . Of the five exceptional algebras, Cartan [44] identified  $G_2$  as the group of octonion isomorphisms and noted already in his thesis that  $E_7$  has a skew-symmetric quadratic and a symmetric quartic invariant. Dickson characterized  $E_6$  as a 27-dimensional group with a cubic invariant. The fact that the orthogonal, unitary and symplectic groups were invariance groups of real, complex, and quaternion norms suggested that the exceptional groups were associated with octonions, but it took more than 50 years to establish this connection. The remaining four exceptional Lie algebras emerged as rather complicated constructions from octonions and Jordan algebras, known as the *Freudenthal-Tits construction*. A mathematician’s history of this subject is given in a delightful review by Freudenthal [130]. The problem has been taken up by physicists twice, first by Jordan, von Neumann, and Wigner [173], and then in the 1970s by Gürsey and collaborators [149, 151, 152]. Jordan *et al.*’s effort was a failed attempt at formulating a new quantum mechanics that would explain the neutron, discovered in 1932. However, it gave rise to the Jordan algebras, which became a mathematics field in itself. Gürsey *et al.* took up the subject again in the hope of formulating a quantum mechanics of quark confinement; however, the main applications so far have been in building models of grand unification.

Although beautiful, the Freudenthal-Tits construction is still not practical for the evaluation of group-theoretic weights. The reason is this: the construction involves  $[3 \times 3]$  octonionic matrices with octonion coefficients, and the 248-dimensional defining space of  $E_8$  is written as a direct sum of various subspaces. This is convenient for studying subgroup embeddings [291], but awkward for group-theoretical computations.

The inspiration for the primitive invariants construction came from the axiomatic approach of Springer [314, 315] and Brown [34]: one treats the defining representation as a single

vector space, and characterizes the primitive invariants by algebraic identities. This approach solves the problem of formulating efficient tensorial algorithms for evaluating group-theoretic weights, and it yields some intuition about the geometrical significance of the exceptional Lie groups. Such intuition might be of use to quark-model builders. For example, because  $SU(3)$  has a cubic invariant  $\epsilon^{abc}q_aq_bq_c$ , Quantum Chromodynamics, based on this color group, can accommodate 3-quark baryons. Are there any other groups that could accommodate 3-quark singlets? As we shall see,  $G_2$ ,  $F_4$ , and  $E_6$  are some of the groups whose defining representations possess such invariants.

Beyond its utility as a computational technique, the primitive invariants construction of exceptional groups yields several unexpected results. First, it generates in a somewhat magical fashion a triangular array of Lie algebras, depicted in figure 1.1. This is a classification of Lie algebras different from Cartan's classification; in this new classification, all exceptional Lie groups appear in the same series (the bottom line of figure 1.1). The second unexpected result is that many groups and group representations are mutually related by interchanges of symmetrizations and antisymmetrizations and replacement of the dimension parameter  $n$  by  $-n$ . I call this phenomenon "negative dimensions."

For me, the greatest surprise of all is that in spite of all the magic and the strange diagrammatic notation, the resulting manuscript is in essence not very different from Wigner's [345] 1931 classic. Regardless of whether one is doing atomic, nuclear, or particle physics, all physical predictions ("spectroscopic levels") are expressed in terms of Wigner's  $3n-j$  coefficients, which can be evaluated by means of recursive or combinatorial algorithms.

Parenthetically, this book is *not* a book about diagrammatic methods in group theory. If you master a traditional notation that covers all topics in this book in a uniform way, more elegantly than birdtracks, more power to you. I would love to learn it.



---

## Index

- 2-index tensor
  - $SO(n)$ , 119
  - $SO(n)$  adjoint, 121
  - $SU(n)$  adjoint, 109
  - $U(n)$ , 82
  - Young tableau, 89
- 3-index tensor
  - $SO(n)$ , 124
  - $U(n)$ , 83
- 3- $j$  coefficient, 44
- 3- $j$  symbol, 44
  - $SU(n)$ , **98–105**
- 3 $n$ - $j$ ,  $U(n)$ , 98
- 3 $n$ - $j$  coefficient, 40
- 3 $n$ - $j$  symbol, 44
- 3-vertex, spinster, 157
- 6- $j$  coefficient
  - spinorial, **141–143**
- 6- $j$  coefficient, 44
- 6- $j$  symbol, 44
  
- abelian group, 15
- adjoint rep, 32, 33
  - $SU(n)$ , 109
  - dimension, 32
  - reality, 35
  - $SU(n)$ , 102, 106, 115
- algebra, **17**
  - associative, 17
  - of invariants, 22, 108
- alternativity relation, 172, 174, 179
- Angelopoulos, E., 232
- antisymmetric tensor, 58
- antisymmetrization operator, 51, 83
- AS relation, 41
- associative algebra, 17
  
- basis vector, 15
- Betti number, 64, 233
- binor, 151, 154, 160, 161, 218
- birdtracks, **27–37**
  - history, 39
  - named, 41
- Bourbaki, 1, 232
- bra-ket formalism, 47
- Brauer, R., 40
  
- Cartan
  - canonical basis, 35, 38
  - roots, 43
  - spinor, 132
- Cartan, E., 132
- Cartan-Killing
  - classification, 1, 2, 60, 75, 186, 235
  - form, 35, 64, 181
- casimir, **60–75**
  - orthogonality, 68
  - quartic, **70–75**
  - $SU(n)$ , 114
  - symmetrized, 62
- Casimir operator, *see* casimir
- character, **79**, 231
  - orthonormality, 79
- characteristic equation, 7, 9, 23, 57, 62
  - $E_6$ , 193, 197
  - $E_7$ , 222
  - $E_8$ , 182, 229
  - $F_4$ , 215, 217
  - $SU(3)$ , 166
  - $U(n)$  2-index tensor, 82
  - $U(n)$  3-index tensor, 84
- class algebra, 26
- clebsch, 7, **29–32**
  - irrelevancy, 38
- Clebsch-Gordan
  - coefficient, *see* clebsch
  - series, 44, 78
- Clifford algebra
  - Grassmann extension, 155, 159
- color algorithm, 93
- commutator
  - Lie algebra, 36, 37
  - Lorentz group, 38
- completeness
  - relation, 24, 31, 43
  - spinster, 157
  - Wigner 3- $j$ , 160
- conjugate, hermitian, 18
- coordinate reflection, 32
- Coxeter
  - dual number, 233
  - graph, 186
- Coxeter, H. S. M., 75, 232

- cubic invariant
  - $SU(3)$ , **164–166**
- curvature scalar, 129
- decomposition
  - irreducible, 26
  - $S_n$  reps, 94
- defining
  - irrep, 25
  - rep, 18
  - vector space, 17
- Deligne, P., 232
- determinant
  - birdtrack, 57
- diagonalizing matrix, 23
- diagrammatic notation, *see* birdtracks
  - history, 39
- $d_{ijk}$  tensor, **111–113**
  - $F_4$ , 211
- dimension
  - adjoint rep, 32
  - color algorithm, 93
  - group, 32
  - Lie algebra, 32
  - $SO(n)$ , 131
  - $U(n)$ , 89, 92
- Dirac, P. A. M., 132, 155
- Dirac  $\gamma$  matrix, 43, **132–147**, 168, 174
  - Grassmann valued, 155
- direct product, Young tableau, 95
- dual
  - Coxeter number, 233
  - rep, 16, 18
  - space, 16
  - vector space, 18
- Dynkin diagram, 75
- Dynkin index, **66–68**
  - cubic, 69
  - $F_4$ , 213
  - quadratic, 67, 69
  - $SO(n)$ , 120
  - sum rules, 68
  - $U(n)$  2-index tensor, 83
- Dynkin label, 39, 75, 87
  - $SO(n)$ , 130
  - $SU(n)$ , 103
- $E_6$ , **190–209**
  - Springer construction, **208–209**
  - triality, 205
- $E_7$ , **218–228**, 234
- $E_8$ , **180–190**
- $E_8$  primitiveness assumption, 180
- El Houari, M., 232
- Elvang, E., xi
- $F_4$ , **210–217**
- Feynman diagram, 27, 40
- Fierz coefficients, **136–141**
- Fierz identity, 139
- Frege, F.L.G., 39
- Frobenius's theorem, 178
- $G_2$ , **170–179**
- Gell-Mann  $d_{ijk}$  tensor, *see*  $d_{ijk}$  tensor
- $\lambda$  matrix, 33, 38, 43
- generator
  - transformation, 32
- $GL(n, \mathbb{F})$ , 16
- grand unified theories, 70
- Grassmann, 155
  - Clifford algebra, 155
  - extension, Clifford algebra, 159
- gravity tensors, 127
- Gross, B. H., 234
- group, **14–15**
  - abelian, 15
  - dimension, 32
  - general linear, 16
  - integral, **76–81**
  - integral,  $SU(n)$ , 80
  - invariance, 22
  - order, 15
  - $SU(3)$ , **164**
  - symmetric, **49–58**
- Hamilton-Cayley, *see* characteristic equation
- handedness
  - spinorial, **145–146**
- Harter, W. G., 14, 26
- Heisenberg algebra, 155, 161
- hermitian
  - conjugation, 18, 28
  - matrix, 19
- history
  - birdtracks, 39
- Hurwitz's theorem, 179
- IHX relation, 41
- index permutation, 49
- index summation, repeated, 15
- infinitesimal transformation, **32–38**
- invariance condition, 33
  - antisymmetric tensor, 58
  - symmetric tensor, 58
- invariance group, 22
- invariant, 20
  - algebra, 22, 108
  - composed, 20
  - matrix, 19
  - primitive tensor, 21
  - tensor, 19
  - tensor operator, 48
  - tree, 21
  - vector, 19
- irreducible

- decomposition, 26
  - representation, *see* irrep
- irrep, **25–26**, 77
  
- Jacobi relation, **37–180**
- Johansen, A., xi
  
- Kahane algorithm, **146–147**
- Kamiya, N., 232
- Klein-Nishina crosssection, 132
- Kronecker delta, 16, 22, 49
- Kronecker product
  - Young tableau reduction, 95
  
- Landsberg, J. M., 233
- lattice gauge theories, 70
- Levi-Civita tensor, 21, **53–55**, 63, 102, 104,  
106, 116, 131, 152, 153
  - $E_6$ , 207
  - $SO(3)$ , 172
  - $SU(3)$ , 166
  - $SU(4)$  -  $SO(6)$ , 168
  - $SU(n)$ , 167
  - supersymmetric invariants, 218
- Levinson, I. B., 47
- Lie algebra, **35–38**
  - $SO(n)$ , 38
  - $U(n)$ , 38
  - commutator, 36, 37
  - dimension, 32
- Lie product, 17
- linear space, 15
- Lorentz group commutator, 38
  
- Magic Triangle, **229–234**
- Mandelstam variables, 41, 44
- Manivel, L., 233
- matrix
  - diagonalizing, 23
  - hermitian, 19
  - invariant, 19
  - product, 17
  - rep, 17
- metaplectic reps of  $Sp(n)$ , 155
- Minkowski group  $SO(1, 3)$ , 118
- multiparticle state, 89
  
- negative dimensions, 149, **151–154**
  - $E_7$ , 218
  - spinsters, 159
  - $SU(n)$  3- $j$ , 104
- normalization
  - Young projection operator, 240
  
- observables, simultaneous, 25
- Okubo, S., 232
- order of a group, 15
- orthogonal group, *see*  $SO(n)$
  
- orthogonality
  - casimir, 68
  - relation, 24
  - spinor, 136
  - spinster, 157
  - Wigner 3- $j$ , 160
  - Young projection operators, 240
- orthonormality
  - relation, 31
  
- Pauli matrix, 33
- Penrose
  - binor, 154, 218
  - Levi-Civita tensor, 53
  - negative dimensions, 151
  - strand network, 141
  - Young projection operators, 116, 127
- Penrose, R., xi, 40, 160, 161
- permutations, **49–58**
- phase convention, 44
- plethysm, 49
- primitive
  - invariant tensor, 21
- primitiveness assumption, 21
  - $E_8$  family, 180
- product
  - Lie, 17
  - matrix, 17
- projection operator, **23–26**, **29–32**, 50
- propagator, 27, 39
  
- quartic casimir, **70–75**
  - relations, 72
  
- Racah coefficient, 160
- recoupling coefficient, spinster, 158
- recoupling relations, 44
  - $U(n)$ , 97
- reduced matrix elements, **47–48**
- reflection, coordinate, 32
- rep
  - character, 79
  - defining, 18
  - dual, 16, 18
  - irreducible, 26, 77
  - matrix, 17
  - spinster, dimension, 159
  - standard, 16
  - $SU(n)$ , 89
  - tensor, 18
- repeated index summation, 15
- representation, *see* rep
- representation space, 16
- Ricci tensor, 129
- Riemann-Christoffel tensor, 127
  
- $S_p$  symmetric group, **49–58**
- scalar multiplication, 15

- Schur's Lemma, 6, 47, 135
- secular equation, *see* characteristic equation
- sextonians, 177
- simultaneous observables, 25
- singlet, 77
- skew-symmetric invariant, 63, 148, 153, 155, 157, 167, 225
- $SO(1, 3)$  Minkowski group, 118
- $SO(n)$ , **118–131**
  - casimirs, 62, 63
  - dimensions, 131
  - Dynkin labels, 130
  - Lie algebra, 38
  - spinor reps, **132–147**
- space
  - adjoint, 33
  - defining vector, 17
  - dual, 16, 18
  - linear, 15
  - vector, 15
- span, 15
- special unitary group, *see*  $SU(n)$
- spectral decomposition, 14, 25
- spinography, **133–147**
- spinor, **132–147**
  - dimension sum rule, 138
  - handedness, **145–146**
  - Kahane algorithm, **146–147**
  - orthogonality, 136
- spinster, **155–161**
  - completeness, 157
  - orthogonality, 157
  - recoupling coefficient, 158
  - rep dimension, 159
  - trace, 157
- $Sp(n)$ , **148–150**
  - casimirs, 63
  - metaplectic reps, 155
  - spinster reps, **155–161**
- Springer  $E_6$  construction, **208–209**
- standard representation space, 16
- standard tableau, 87
- strand network, 141
- structure constant, 17, 36
- STU relation, 41
- $s, t, u$  variables, 44
- $SU(3)$ , **164–166**
- subgroup, 15
  - embedding, 39
- sum rule
  - spinor dimensions, 138
  - $SU(n)$  3- $j$ , 6- $j$ , 101
- $SU(n)$ , **162–169**
  - 3- $j$ , 98
  - adjoint rep, 102, 106, 115
  - casimirs, 114
  - Dynkin label, 103
  - Lie algebra, 38
  - Young tableaux, 89
- symmetric
  - group, **49–58**
  - tensors, 57
  - tensors, invariance, 58
- symmetrization operator, 50, 83
- symmetry breaking, 39
- symplectic group, *see*  $Sp(n)$
- symplectic invariant, 148
- tensor, 18
  - 2-index,  $U(n)$ , 82
  - 3-index,  $SO(n)$ , 124
  - 3-index,  $U(n)$ , 83
  - adjoint  $\times$  defining,  $SU(n)$ , 105
  - decomposition, 94
  - fully antisymmetric, 58
  - fully symmetric, 57
  - gravity, 127
  - invariant, 19
  - mixed adjoint  $\otimes$  defining,  $SO(n)$ , 120
  - operator
    - invariant, 48
  - rep, 18
  - vanishing, 58
- trace
  - birdtrack, 56
  - spinster, 157
- transformation
  - generator, 32
  - infinitesimal, **32–38**
- tree invariant, 21
- triality,  $E_6$ , 205
- trivalent graph, 40, 41
- $U(2)$ 
  - Young tableau, 89
- $U(n)$ , **82–115**
  - 2-index tensor, 82
  - 3-index tensor, 83
  - decomposition, 94
  - dimension, 89
  - Lie algebra, 38
  - recoupling relations, 97
  - rep dimension, 92, 93
  - Young proj. oper, **239–242**
- unitary group, *see*  $U(n)$
- vacuum bubbles, 45, 47
- Vanagas, V. V., 47
- vanishing tensors, 58
- vector
  - basis, 15
  - invariant, 19
  - space, 15
    - defining, 17
    - dual, 18

Vogel, P., 232

weak coupling expansions, 70

Weyl tensor, 129

Weyl, H., 14, 132, 138

Wigner

6- $j$  coefficient, 65

3- $j$  completeness, 160

3- $j$  orthogonality, 160

3 $n$ - $j$  symbol, 44

3 $n$ - $j$  coefficient, 40

Wigner, E. P., 3, 40

Wigner-Eckart theorem, 6, 42, **46–139**

Young polynomial, 92

Young projection operator, **90–92**

3- $j$ , 98

completeness, 94

decomposition, 94

normalization, 91, 240

orthogonality, 240

properties, 92, **239–242**

Young tableau, **84–89**

2-index tensor, 89

3- $j$ , 98

conjugate, 103

definition, 87

Kronecker product reduction, 95

length, 91

standard, 87

transpose, 87

$U(2)$ , 89

$U(n)$ , 89

$U(n)$  dimension, 92

Yutsis, A. P., 47