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Chapter 1

Introduction

The generalized surface quasigeostrophic (gSQG) equations describing the evolution of the potential temperature ω read as

$$\begin{cases} \omega_t + u \cdot \nabla\omega = 0, & \text{for } (t,x) \in [0,\infty) \times \mathbb{R}^2, \\ u = -\nabla^{\perp}(-\Delta)^{-1+\alpha/2}\omega, & \alpha \in [0,2], \\ \omega(0,\cdot) = \omega_0(\cdot). \end{cases} \tag{1.1}$$

Formally, these equations interpolate between the case of the Euler equation ($\alpha = 0$) and the case of stationary solutions ($\alpha = 2$). The case ($\alpha = 1$) is known as the SQG equation.

The SQG equation models the evolution of the temperature from a general quasi-geostrophic system for atmospheric and oceanic flows (see [CMT94, HPGS95, Ped82, MB02] for more details). The first rigorous mathematical study of the SQG equation was done by Constantin–Majda–Tabak [CMT94] where its mathematical importance due to its analogy with the incompressible 3D Euler equations was highlighted and the first numerical and analytical study of the equation was carried out. Córdoba–Fontelos–Mancho–Rodrigo in [CFMR05] proposed the gSQG or $(\mathrm{SQG})_\alpha$ model (1.1) as an interpolation between the Euler and surface quasigeostrophic equations. Nevertheless, very little is known for this family of equations, and specifically the question of global existence versus finite-time singularities is still open, for all $\alpha > 0$. In this monograph we aim to prove the existence of a large class of initial data for which there is time quasiperiodic behavior and thus global existence in the more singular case $\alpha \in (1,2)$.

1.1 MOTIVATION OF THE PROBLEM: FROM EULER TO GSQG

Among the fundamental equations in fluid mechanics are the three-dimensional Euler equations, given by

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\mathbf{p}, \quad \nabla\cdot\mathbf{v} = 0, \quad \mathbf{v}(0,x) = \mathbf{v}_0(x), \quad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3,$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is the velocity vector of the fluid and $\nabla \cdot \mathbf{v}$ denotes the divergence of the velocity field. The Euler equations describe the motion of an incompressible fluid with no viscosity (inviscid flow) and constant density. Specifically, in the case of incompressible flow (constant density), these equations govern the evolution of the fluid's velocity field in a three-dimensional space. The system consists of three components:

- **Momentum equation**, $\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla \mathbf{p}$. This equation represents the conservation of momentum and describes the acceleration of the fluid due to pressure gradients.
- **Continuity equation**, $\nabla \cdot \mathbf{v} = 0$. This equation ensures the incompressibility of the fluid, meaning that the volume of any fluid element remains constant over time. It guarantees that there is no net volumetric change in the fluid as it moves.
- **Initial condition**, $\mathbf{v}(0, x) = \mathbf{v}_0(x)$. This specifies the initial velocity field of the fluid. The evolution of the fluid's velocity can be completely determined if the initial condition, together with the momentum and continuity equations, can be uniquely solved.

The Euler equations describe idealized, frictionless flows. In real fluids, viscosity introduces additional terms, leading to the more general *Navier–Stokes equations*, which will not be covered in this monograph.

1.1.1 Singularity Formation vs Global Well-Posedness

Key challenges in understanding the behavior of solutions to the Euler equations are the possibility of *singularity formation*, where the solution or its derivatives become infinite in finite time, and the question of *global well-posedness*, which concerns whether a smooth solution exists for all time. These two concepts are central to the study of fluid mechanics as they determine whether mathematical models accurately reflect physical phenomena and whether they can reliably predict the behavior of real-world fluids over extended periods.

Singularity formation refers to the possibility that the velocity field $\mathbf{v}$ (or its derivatives) becomes unbounded or develops infinite values after a finite amount of time. Specifically, a singularity may manifest as a *blow-up* of the solution, where the velocity becomes infinite, leading to a breakdown of the mathematical model. Despite several potential scenarios in which singularities might arise, a rigorous proof of finite-time singularity formation for the 3D Euler equations remains an open problem. This is an area of ongoing research, with the famous *Navier–Stokes existence and smoothness problem* being one of the seven Millennium Prize Problems [Fef00]. Although the Euler equations are known to have smooth solutions in certain cases, a general proof of their behavior for arbitrary initial conditions remains elusive.

In contrast to singularity formation, in the context of the 3D Euler equations, *global well-posedness* refers to the existence of a unique, smooth solution $\mathbf{v}(t, x)$ for all times $t \geq 0$, given an initial velocity field $\mathbf{v}(0, x)$ that is smooth and incompressible. The question of global well-posedness remains an open problem. Although local well-posedness has been established under certain conditions, which means that for sufficiently smooth initial data, solutions exist and remain smooth for a short time, the potential for singularity formation suggests that global well-posedness may not hold for arbitrary initial conditions.

1.1.2 Generalized Surface Quasi-Geostrophic Equation

The primary mathematical model that motivates the main theorem in this monograph is the Surface Quasi-Geostrophic (SQG) equation. This equation is widely used in fluid dynamics, particularly in atmospheric and oceanic sciences, to describe the evolution of temperature or potential vorticity on the surface of a fluid under the influence of the Coriolis force. The SQG equation is given by

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = 0, \quad \mathbf{u} := -\nabla^\perp (-\Delta)^{-1/2} \omega, \quad \omega(0,x) = \omega_0(x), \quad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2,$$

where ω represents the temperature or potential vorticity in the quasigeostrophic system for atmospheric flows, and $\mathbf{u}$ is the velocity field.

The 3D Euler equations and the SQG equation are both fundamental models in fluid dynamics. While the 3D Euler equations govern the motion of an incompressible, inviscid fluid in a three-dimensional space, the SQG equation is a simplified model that describes the evolution of a scalar field on a surface, which is two-dimensional. Despite their differences, these two equations show several important similarities in their mathematical structures.

By formally taking the curl operator in the momentum equation of the 3D Euler equations, we can derive the vorticity equation for $\mathbf{\Theta} := \nabla \times \mathbf{v}$, namely

$$\partial_t \mathbf{\Theta} + (\mathbf{v} \cdot \nabla)\mathbf{\Theta} = (\mathbf{\Theta} \cdot \nabla)\mathbf{v}.$$

We can list several features of the vorticity equation:

E-(a) $\nabla \cdot \mathbf{\Theta} = \nabla \cdot \mathbf{v} = 0$, which follows from the fact that $\mathbf{\Theta}$ is the curl of a vector field.
E-(b) $\nabla \mathbf{v} = T(\mathbf{\Theta})$ for some singular integral operator T of order zero.
E-(c) Conservation of kinetic energy, $\|\mathbf{v}(t)\|_{L^2} = \|\mathbf{v}_0\|_{L^2}$.

On the other hand, the two-dimensional vector field $\nabla^\perp \omega$ for a solution to the SQG equation satisfies

$$\partial_t (\nabla^\perp \omega) + (\mathbf{u} \cdot \nabla)(\nabla^\perp \omega) = (\nabla^\perp \omega \cdot \nabla)\mathbf{u},$$

and exhibits similar features:

S-(a) $\nabla \cdot (\nabla^\perp \omega) = \nabla \cdot \mathbf{u} = 0$.
S-(b) $\nabla \mathbf{u} = S(\nabla^\perp \omega)$ for some singular integral operator S of order zero.
S-(c) Conservation of the L^2-norm of ω, $\|\omega\|_{L^2} = \|\omega_0\|_{L^2}$.

We also note that vortex lines in the Euler equation move along the flow, while the level curves of ω in the SQG equation also move along the flow. This observation suggests that the Euler and SQG equations have many structural similarities, and their behaviors are expected to resemble each other. As in the Euler equation, the question of finite-time singularity formation versus global well-posedness for the SQG equation remains open. Consequently, the well-posedness question for the SQG equation has attracted significant attention and is an active area of research.

Despite the close relationship between the 3D Euler and SQG equations, there are also important differences. For instance, while the 3D Euler equations do not conserve $\|\mathbf{v}(t)\|_{L^p}$ for $p \neq 2$, the transport nature of the SQG equation immediately guarantees the conservation of $\|\omega\|_{L^p}$ for all $p \in [1, \infty]$. Moreover, the SQG equation is spatially two-dimensional, which simplifies certain technical computations compared to the 3D Euler equations. These features are reminiscent of the vorticity form of the two-dimensional Euler equations:

$$\partial_t \theta + \mathbf{v} \cdot \nabla \theta = 0, \quad \mathbf{v} = -\nabla^\perp (-\Delta)^{-1} \theta, \quad \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2.$$

The difference between the 2D Euler equations and the SQG equation is that the velocity $\mathbf{v}$ is related to the transported scalar θ through an integral operator of order -1 in the 2D Euler case, while it is an integral operator of order zero in the SQG equation. This distinction suggests that the SQG equation encodes some similarities and differences with the 2D Euler equation, motivating the generalized SQG equations, as described in (1.1), which mathematically interpolate the relationship between the scalar ω and the velocity field $\mathbf{u}$. However, in contrast to the 2D Euler equation, for which global well-posedness is known, it is not yet known whether smooth solutions to the generalized SQG equations are globally well-posed or whether they may develop a singularity in finite time for any range of $\alpha > 0$.

1.1.3 Patch Problems

As mentioned previously, the motivation for studying the generalized Surface Quasi-Geostrophic (gSQG) equations arises from their analogies with the Euler equations. For the 2D Euler equations, an important class of solutions is known as vortex patch solutions. A patch solution takes the form

$$\omega(t, x) := 1_{D(t)}(x), \quad \text{for a bounded domain } D(t) \subset \mathbb{R}^2.$$

Here, $1_{D(t)}$ denotes the characteristic function of the domain $D(t)$, and by a solution we mean that ω satisfies the 2D Euler equations in the distributional sense:

$$\int_{\mathbb{R}^2} \omega(T, x)\eta(T, x)\, dx - \int_{\mathbb{R}^2} \omega_0(x)\eta(0, x)\, dx$$
$$= \int_0^T \int_{\mathbb{R}^2} \omega(t, x) \left(\partial_t \eta(t, x) + \mathbf{v}(t, x) \cdot \nabla \eta(t, x)\right)\, dx\, dt,$$

for all smooth, compactly supported test functions η. The fact that the solution remains a characteristic function relies on the fact that the 2D Euler equation is a scalar transport equation. This property suggests that such patch solutions can also be naturally formulated for the gSQG equations.

The question of finite-time singularity formation in the gSQG equations can thus be rephrased in the context of patch solutions as: Does the boundary $\partial D(t)$ maintain smoothness throughout the evolution? Unfortunately, a complete answer to this question remains unresolved for any $\alpha > 0$, except in certain cases [KRYZ16, GP21, Zla23], where the authors constructed finite-time singularities in domains with boundaries (rather than $\mathbb{R}^2$).

In both the smooth and patch cases, determining whether a finite-time singularity can occur in the gSQG equation remains an open and challenging problem. However, a potentially more accessible question is whether a global solution (or more generally, a large family of global solutions) can be constructed. This is the primary focus of investigation in this monograph.

To approach the construction of a global solution, it is useful to first consider a steady solution. A steady solution refers to a time-independent solution in a certain reference frame, typically under a Galilean transformation. Radial functions are known to be steady solutions to the gSQG equations. For example, a patch solution with the domain D being a disk does not alter the shape of the patch during its evolution. However, one might argue that such a stationary solution is too trivial to capture the more complex features of a global solution. Therefore, the next step is to attempt the construction of a global solution that does not remain unchanged but instead exhibits mild evolution over time. To this end, we will explore the Hamiltonian structure of the gSQG equations.

1.1.4 Hamiltonian Systems and the gSQG Equations

Let us briefly digress from our discussion of the gSQG equations and review the basic notions of a Hamiltonian system.

For a manifold X, a symplectic form Ω is a nondegenerate, skew-symmetric 2-form on the tangent bundle TX. This means that for every point $p \in X$, the following properties hold:

$$\Omega_p(V, W) = -\Omega_p(W, V) \quad \text{for all } V, W \in T_pX,$$

and

$$\Omega_p(V, W) = 0 \quad \text{for all } W \in T_pX \implies V = 0.$$

Let us consider a functional $H : X \to \mathbb{R} \cup \{\infty\}$, which may take the value ∞ at some point $p \in X$. By the nondegeneracy of the symplectic form, there exists a unique vector field X_H, called the Hamiltonian vector field, such that

$$\Omega_p(X_H(p), W) = d_pH(W), \quad \text{for all } W \in T_pX,$$

where $d_pH \in T^*X$ denotes the gradient of H at p. When the Hamiltonian vector field is well defined, the differential equation

$$\partial_t r = X_H(r(t))$$

is called a Hamiltonian system.

Example 1: A Simple Harmonic Oscillator. Consider $X := \mathbb{R}^2 = \mathbb{R}\times\mathbb{R} \ni (y,p)$, with constants $m, k \in \mathbb{R}$. The Hamiltonian is given by

$$H(y,p) := \frac{p^2}{2m} + \frac{1}{2}ky^2,$$

which represents the sum of the kinetic and potential energies of a mass m attached to a spring with spring constant k. Here, (y,p) denotes the displacement and momentum of the mass. With the natural symplectic form

$$\Omega(V,W) := W \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} V, \quad \text{for } V, W \in \mathbb{R}^2,$$

the associated Hamiltonian vector field is given by

$$X_H(y,p) := \begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix} \begin{pmatrix} y \\ p \end{pmatrix} = \begin{pmatrix} \frac{p}{m} \\ -ky \end{pmatrix}.$$

Therefore, the Hamiltonian system is described by the equation

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \frac{p(t)}{m} \\ -ky(t) \end{pmatrix} = \begin{pmatrix} \text{velocity} \\ \text{spring force} \end{pmatrix},$$

which describes a simple harmonic oscillator.

Example 2: The Airy Equation. Let us now consider $X := L^2_0(\mathbb{T})$, the set of square-integrable functions with zero average on the torus $\mathbb{T}$. The Airy equation is a simple linear PDE given by

$$\partial_t f(t,x) + \partial_{xxx} f(t,x) = 0, \quad f(0,x) = f_0(x) \quad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{T}.$$

To formulate this as a Hamiltonian system, we consider a symplectic form Ω and a Hamiltonian functional H given by

$$\Omega(f,g) := \int (\partial_x^{-1} f)(x) g(x)\, dx, \quad H(f) := \frac{1}{2}\int_{\mathbb{T}} |\partial_x f(x)|^2\, dx.$$

The associated Hamiltonian vector field is $X_H(f) := -\partial_{xxx} f$, which corresponds to the Airy equation. This equation can be solved explicitly using the Fourier transform, yielding the solution

$$\widehat{f}(t,j) = \sum_{j\neq 0} \widehat{f_0}(j) e^{i(j^3 t + jx)},$$

where $\widehat{f}(j)$ denotes the jth Fourier mode.

One key observation from these examples is that the linear operators associated with the Hamiltonian vector fields, $\begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix}$ and ∂_{xxx}, have purely imaginary eigenvalues, which lead to oscillations in motion. This is characteristic of Hamiltonian systems, where near equilibrium, solutions tend to exhibit oscillatory behavior. Such behavior is typical in many Hamiltonian systems, where the linearized model approximates the full system near an equilibrium. In addition to periodic motion, as shown in the simple harmonic oscillator, systems with many particles or waves can exhibit quasiperiodic motion, where different components oscillate with different frequencies.

As we will see in Chapter 3, the gSQG equations can be formulated as a Hamiltonian system in an infinite-dimensional phase space. In light of our earlier discussion on constructing global solutions, this observation leads us to investigate the existence of periodic and quasiperiodic solutions near a steady state. To this end, we will first review several key research works related to global well-posedness of periodic and quasiperiodic solutions not only to the gSQG equations but also to various mathematical models. Subsequently, we will present our main theorem and proof strategy, which leverages KAM theory, a systematic method for constructing quasiperiodic solutions in general Hamiltonian systems.

1.2 OVERVIEW OF RELATED WORKS

1.2.1 Patch Problems

In this monograph we will work in the *patch* setting, where $\omega(\cdot, t) = 1_{D(t)}$ is an indicator function of a simply connected, bounded set that moves with the fluid. In such a situation, we parametrize $\partial D(t)$ as $z(\theta, t)$, $\theta \in [0, 2\pi]$ and the evolution equations are

$$\partial_t z(\theta, t) = \int_0^{2\pi} \frac{\partial_\theta z(\theta, t) - \partial_\theta z(\theta - \eta, t)}{|z(\theta, t) - z(\theta - \eta, t)|^\alpha} d\eta + c(\theta, t)\partial_\theta z(\theta, t), \qquad (1.2)$$

where $c(\theta, t)$ accounts for the reparametrization freedom of the curve.

Concerning well-posedness results for patch solutions, Rodrigo (in a C^∞ space) [Rod05] and Gancedo [Gan08] and Chae–Constantin–Córdoba–Gancedo–Wu [CCC+12] (in a Sobolev space) proved local existence for the case $0 < \alpha \leq 1$ and $1 < \alpha$, respectively. See also [KYZ17, GP21, AA24].

1.2.2 Steady Solutions and Global Existence of the gSQG Equation

The construction of nontrivial global solutions for the generalized SQG equations is a very challenging open problem for all parameters $\alpha \in (0, 2)$, both in the smooth and patch cases. For $\alpha = 0$ (the 2D Euler equations), global regularity of solutions was well understood a long time ago, both in the smooth and patch cases; see, for example, the classical papers of Wolibner [Wol33], Yudovich [Yud63], Burbea [Bur82], Chemin [Che93], and Bertozzi-Constantin [BC93]. However, the construction of global solutions in the case of $\alpha \in (0, 2)$ is much more challenging than when $\alpha = 0$, since the velocity is more singular, and only partial results have been obtained in recent years. We review some of these results below.

Most of the results around global existence of solutions to the gSQG equation have revolved around solutions that exhibited some rigid character (steady, uniformly rotating –V-states–, traveling). In the case where $0 < \alpha < 1$, Hassainia–Hmidi [HH15] proved the existence of V-states with C^k boundary regularity. Castro–Córdoba–Gómez-Serrano then expanded upon this result in [CCGS16a] by showing that V-states also exist with C^∞ boundary regularity in the remaining open cases of $\alpha \in [1, 2)$ for existence and $\alpha \in (0, 2)$ for regularity. This boundary regularity was later refined to be analytic in [CCGS16b]. Other notable works on rotating solutions include [CF22, Gar21, HM17, HW22], which discuss other families of rotating solutions or even more steady states, [dlHHH16, Ren17] which address the doubly connected case, and [CCGS20] which presents a construction in the smooth setting.

In [dlHHH16], de la Hoz–Hassainia–Hmidi showed that there exist nonradial patches bifurcating from annuli at negative angular velocities and Gómez-Serrano [GS19] constructed nonradial, doubly connected stationary patches. García [Gar20] proved the existence of a Kármán vortex street structure by desingularizing an infinite array of point vortices in the case $\alpha \in [0, 1)$. In [CCGSMZ14] it was ruled out by Castro–Córdoba–Gómez-Serrano–Martín Zamora that ellipses could be rotating solutions for $\alpha > 0$, as opposed to the case $\alpha = 0$. Gravejat–Smets [GS17], in the case $\alpha = 1$, constructed smooth translating solutions. Ao–Dávila–del Pino–Musso–Wei [ADdP+21], expanded the range to $\alpha \in (0, 2)$ as well as to rotating solutions; see also [GC21, GCGS23] and [CQZZ21a, CQZZ22, CQZZ21b] for alternative constructions. In [GSPSY21], Gómez-Serrano–Park–Shi–Yao proved that any smooth, nonnegative rotating solution with simply connected superlevel sets can only rotate with positive angular velocity and, in the case of a patch of fixed area, the authors derived, moreover, a sharp upper bound on the angular velocity.

The drawback of the aforementioned solutions is that they are *special* in the sense that general solutions will not have such behavior. Concerning results for general solutions, Córdoba–Gómez-Serrano–Ionescu [CGSI19] proved global existence of solutions for small patch data close to a halfplane in the case $\alpha \in (1, 2)$, using a different mechanism based on dispersion and decay. This was extended in [HSZ21, HSZ24]. The main idea was to show that general initial data that are small perturbations of the halfplane stationary patch solution lead to global solutions that decay in time (at an optimal rate of $t^{-1/2}$), thus converging back to the halfplane stationary patch. Unfortunately, the mechanism of dispersion and decay seems to require unbounded domains and, in particular, infinite energy solutions.

In a different direction, one could hope to use the mechanism of inviscid damping to construct families of global-in-time solutions around explicit stationary solutions of finite energy, such as smooth shear flows or vortices. This has been successfully implemented in recent years in the 2D Euler case $\alpha = 0$, for perturbations of the Couette flow (by Bedrossian–Masmoudi [BM15] and Ionescu–Jia [IJ20]) and then general monotonic shear flows [IJ23a, MZ24]. It is tempting to try to adapt the mechanism of inviscid damping to construct families of nontrivial global solutions of the gSQG equations, at least for some small parameters $\alpha > 0$. The easiest would be to perturb around the Couette flow corresponding to $\theta(t, x, y) = -1$ on the bounded channel $\mathcal{D} = \mathbb{T} \times [0, 1]$. Unfortunately and surprisingly, recent work of Gómez-

Serrano–Ionescu–Jia (discussed in [IJ23b]) shows that this fails to produce global solutions for any parameter $\alpha > 0$, due to a forward cascade that leads to loss of regularity in finite time.

1.2.3 Quasiperiodic Solutions in PDE

Our main goal in this monograph is to demonstrate the existence of large families of global solutions of the generalized SQG equations. We do this using KAM theory, by constructing quasiperiodic solutions for almost all initial data in a neighborhood of the unit disk (the simplest stationary patch solution with finite energy).

The first application of KAM theory [Kol54, Ad63, Mos62] was to prove the existence of invariant tori that were small perturbations of finite-dimensional nearly integrable Hamiltonian systems. As for the upgrade to the infinite-dimensional (PDE) case, the first results are due to Kuksin [Kuk87], Wayne [Way90], Pöschel [P96] for 1D semilinear wave and Schrödinger equations with Dirichlet boundary conditions and Craig–Wayne [CW93], Bourgain [Bou05], Grébert–Kappeler [GK14], and Chierchia–You [CY00] with periodic boundary conditions; see also [Kuk00]. In the semilinear multidimensional case, we refer to the works of Bourgain [Bou94], Eliasson–Kuksin [EK10], Grébert–Paturel [GP16], Wang [Wan16], and Berti–Bolle [BB20], as well as the references therein; see also De la Llave–Sire [dlLS19]. Note that all the previous results only were able to deal with semilinear problems.

In the last decade there has been an emergence of results on quasiperiodic solutions for quasilinear PDEs, motivated by applications to the dynamics of confined fluids, building up and polishing the techniques and the methods and culminating with excellent theorems. Baldi–Berti–Montalto constructed quasiperiodic solutions to the Airy equation [BBM14a], as well as KdV and mKdV equations [BBM14b, BBM16a, BBM16b]; see also the results of Giuliani for gKdV [Giu17], and [Feo16, FP15, Mon17, FGP20], and the references therein for other relevant models. In the context of water waves, Baldi–Berti–Haus–Montalto [BBHM18] (gravity case), Berti–Montalto [BM20] (gravity–capillary case), Feola–Giuliani [FG24, FG20] (infinite depth) and Berti–Franzoi–Maspero [BFM21] (constant nonzero vorticity) constructed quasiperiodic solutions. Numerically, Wilkening–Zhao [WZ21a, WZ21b] computed quasiperiodic gravity–capillary water waves in the infinite depth case.

Berti–Hassainia–Masmoudi [BHM23] constructed quasiperiodic solutions close to elliptical vortex patches, introducing the angular momentum as a symplectic variable. Hassainia–Roulley [HR22] constructed quasiperiodic solutions of the 2D Euler equations in a bounded domain, Roulley [Rou23] proved its existence for the Euler-α equation and Hmidi–Roulley [HR21] did the same for the QGSW equations.

Other examples of quasiperiodic solutions in the context of the incompressible Euler and Navier–Stokes equations, even in high dimensions, were obtained by Crouseilles–Faou, Elgindi–Jeong, Enciso–Peralta-Salas–Torres de Lizaur [CF13, EJ20, EPSTdL23] for Euler, using non-KAM constructions, Baldi–Montalto [BM21] for forced Euler, using a KAM construction and Franzoi–Montalto, Montalto [FM24, Mon21] for forced Navier–Stokes, using a KAM construction. Finally, we would like to draw the attention to the recent results by Hassainia–Hmidi–Masmoudi [HHM21] who proved the existence of global quasiperiodic solutions for the generalized SQG equations, for a set of parameters $\alpha \in (0, 1/2)$. The set of acceptable parameters α is unknown, but is of full measure in $(0, 1/2)$.

We emphasize that most of these recent results in the quasilinear case (with the notable exception of the papers [FG24, Giu17, BBM16a]) rely on the use of *external parameters*. Quasiperiodic solutions are then constructed for all initial data, but for an unknown set of parameters, usually generic and of full measure. The point is that the presence of external parameters improves significantly the structure of the resonances of the system, which plays a key role in the analysis.

The drawback is that the family of acceptable parameters is not explicit, and one cannot guarantee that quasiperiodic solutions exist for a specific given equation. Our broad goal in this monograph is to develop a robust and flexible method to construct quasiperiodic solutions for certain fluid models, without requiring the presence of external parameters. The basic idea is to replace the genericity of the external parameters with genericity of the initial data. This leads, however, to very significant difficulties at the implementation level; see below for a more detailed discussion.

1.2.4 Weak Solutions and Finite-Time Singularities

The generalized SQG equations have been studied extensively by many authors. In this subsection we discuss two other areas of active research, and provide some references.

In his thesis [Res95], Resnick demonstrated the global existence of weak solutions in L^2 through the use of the oddness of the Riesz transform to achieve additional cancellation. Marchand [Mar08] later extended this result to include initial data belonging to L^p with p greater than $\frac{4}{3}$; see also [NPST18] for other existence results concerning weak solutions. Nonuniqueness of weak solutions of SQG remains a difficult problem, with progress being made through works such as Azzam–Bedrossian [AB15] and Isett–Vicol [IV15], and most importantly, Buckmaster–Shkoller–Vicol [BSV19], as well as alternative proofs by Isett–Ma [IM21] and the investigation of the stationary problem by Cheng–Kwon–Li [CKL21].

One of the most significant questions in mathematical fluid mechanics is whether the SQG and gSQG system exhibits finite-time singularities or global existence of solutions. Kiselev–Nazarov [KN12] created solutions that exhibited norm inflation, and Friedlander–Shvydkoy [FS05] demonstrated the presence of unstable eigenvalues in the spectrum. He–Kiselev [HK21] proved an exponential in time growth of the C^2-norm; see also the construction of singular solutions with infinite energy by Castro–Córdoba [CC10] and ill-posedness results by Córdoba–Martínez-Zoroa and Jeong–Kim [CMZ22, JK24].

In order to understand the possibility of a finite time blow-up scenario, numerical studies have been conducted. Constantin–Majda–Tabak [CMT94] suggested that a singularity in the form of a hyperbolic saddle may occur, closing in a finite amount of time. However, Ohkitani–Yamada [OY97] and Constantin–Nie–Schörghofer [CNS98] proposed that the growth was actually double exponential. Córdoba [Cor98] bounded the growth at quadruple exponential, and later Córdoba and Fefferman [CF02] proposed a double exponential bound, which was supported by numerical simulations from Deng–Hou–Li–Yu [DHLY06]. Constantin–Lai–Sharma–Tseng–Wu [CLS+12] later reexamined the hyperbolic saddle scenario using improved algorithms and found no evidence of blowup. Scott [Sco11] proposed a scenario in which filamentation occurs and blowup of $\nabla\theta$ occurs after several cascades, starting from elliptical configurations. This is currently the only scenario that remains valid in the smooth setting. In [GGS22], very recently, García–Gómez-Serrano constructed a big class of nontrivial self-similar spiral solutions close to radial ones with a mild singularity at the origin.

Even though the finite time singularity problem seems elusive, there exist several numerical scenarios suggesting such a singularity. The first, proposed by Córdoba–Fontelos–Mancho–Rodrigo [CFMR05], initially starts as two patches rolling onto each other and finally collapsing. At the intersection point, the curvature blows up (the curve should lose regularity due to the results by Gancedo and Strain [GS14], see also [KL23, JZ24]) and the collapse is suggested to be asymptotically self-similar. The second scenario was proposed by Scott–Dritschel [SD14], taking ellipses as initial condition; starting with an aspect ratio of 0.16, they report a self-similar cascade of filamentation. In [SD19], again taking ellipses as initial condition and combining numerical analysis with asymptotic calculations, they conjecture a scenario where the patch develops a corner in finite time, together with a self-similar spiral. Finally, Kiselev–Ryzhik–Yao–Zlatoš [KRYZ16] (for $0 < \alpha < \frac{1}{12}$) and later Gancedo–Patel [GP21] (for $0 < \alpha < \frac{1}{3}$) constructed finite-time singularities in the presence of a boundary.

1.3 MAIN RESULT

Before we state the main result, let us first recall the definition of a quasiperiodic function:

Definition 1.1. Let X be a Hilbert space and $\nu \in \mathbb{N}$ a fixed natural number. A function $f\colon \mathbb{R} \mapsto X$ is said to be quasiperiodic with frequency $\omega \in \mathbb{R}^\nu$ if there exists $i\colon \mathbb{T}^\nu \mapsto X$ such that $f(t) = i(\omega t)$.

In this monograph, we consider a patch solution to (1.2) of the form

$$z(x,t) := \sqrt{1 + f(x,t)}(\cos x, \sin x), \text{ for some } f(\cdot,t)\colon \mathbb{T} \mapsto (-1,\infty). \tag{1.3}$$

Note that one of the advantages of the use of the variable f, instead of a more natural parametrization $z(x,t) = R(x,t)(\cos x, \sin x)$ relies on the conservation of the area of the patch in the dynamics in (1.1); if the patch initially has area $|D(0)| = \pi$, then $|D(t)| = \pi$ for all $t \geq 0$, therefore

$$\pi = |D(t)| = \frac{1}{2}\int_{\mathbb{T}} R(x,t)^2 dx = \pi + \int_{\mathbb{T}} f(x,t)dx. \tag{1.4}$$

Thus, we can assume that f has zero average in the variable x.

Plugging (1.3) into (1.2), one can find that the evolution of f can be expressed as (we refer to Section 3.1 for more detailed computations)

$$\partial_t f(x,t) = \frac{2}{2-\alpha}\partial_\theta \left(\int_{\mathbb{T}} \frac{(z(x,t)-z(y,t))\cdot \partial_x z(y,t)^\perp}{|z(x,t)-z(y,t)|^\alpha} dy \right) =: X_{\mathrm{gSQG}}(f(x,t)). \tag{1.5}$$

As noted in [HH15, Res95, MW83, Saf92], Equation (1.5) can be seen as a Hamiltonian system with the associated Hamiltonian

$$\mathcal{H}(f) := \int_D 1_D * \frac{1}{|\cdot|^\alpha}(x)dx, \tag{1.6}$$

where D is the patch determined by the parametrization f as in (1.3) (see Chapter 3). More precisely, the vector field $X_{\mathrm{gSQG}}(f)$ is given by

$$X_{\mathrm{gSQG}}(f) = \partial_x \left(\nabla_{L^2}\mathcal{H}(f)\right), \tag{1.7}$$

where $\nabla_{L^2}\mathcal{H}(f)$ denotes the gradient vector field of $\mathcal{H}$ at f in the space $L^2(\mathbb{T})$.

The linearized equation of (1.5) at the unit disk ($f = 0$) can be written as (see Proposition 3.2)

$$f_t = \frac{d}{dt}X_{\mathrm{gSQG}}(tf)\bigg|_{t=0} = \partial_x\left(-\frac{1}{2}\Lambda^{\alpha-1}f + \frac{T_\alpha}{4}f\right), \tag{1.8}$$

where

$$\Lambda^{\alpha-1}f(x) := \int_{\mathbb{T}}(2-2\cos(x-y))^{-\frac{\alpha}{2}}(f(x)-f(y))dy, \tag{1.9}$$

$$\text{and } T_\alpha := \frac{2\pi\Gamma(3-\alpha)}{\Gamma(2-\frac{\alpha}{2})\Gamma(2-\frac{\alpha}{2})}.$$

One can also rewrite the linearized equation (1.8) as

$$\partial_t f = Op(\mathrm{i}W(j))[f](x,t), \tag{1.10}$$

where $Op(W(j))$ denotes the pseudodifferential operator associated to the Fourier multiplier $W(j)$, defined as

$$W(j) := j\left(-\frac{1}{2}C_\alpha\left(\frac{\Gamma(|j|+\frac{\alpha}{2})}{\Gamma(1+|j|-\frac{\alpha}{2})} - \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}\right) + \frac{\pi(-1)^j\Gamma(3-\alpha)}{2\Gamma(2-\frac{\alpha}{2})\Gamma(2-\frac{\alpha}{2})}\right), \tag{1.11}$$

where $C_\alpha := -\frac{2\pi\Gamma(1-\alpha)}{\Gamma(\frac{\alpha}{2})\Gamma(1-\frac{\alpha}{2})}$. A classical asymptotic analysis for the Gamma function tells us that $W(j)$ exhibits an asymptotic behavior like $j|j|^{\alpha-1}$ (e.g., [LN12, Theorem 2.1]), more precisely,

$$W(j) = C(\alpha)j|j|^{\alpha-1} + O(1), \text{ for some constant } C(\alpha) \in \mathbb{R} \text{ for } \alpha \in (0,2)\setminus\{1\}.$$

Given a set of natural numbers $S^+ := \{j_1, \dots, j_\nu\} \subset \mathbb{N}$ (also denoting $S := \{\pm j : j \in S^+\}$), the linear equation (1.10) possesses time-quasiperiodic solutions of the form

$$f(x,t) = \sum_{j_k \in S} \sqrt{|j_k|\zeta_k} e^{\mathrm{i}(W(j_k)t + j_k x)} = \sum_{j_k \in S^+} 2\sqrt{|j_k|\zeta_k} \cos(W(j_k)t + j_k x), \quad (1.12)$$

for some $\zeta_1, \dots, \zeta_\nu > 0$, for which the jth Fourier coefficient is oscillating in time with frequency $W(j)$. Indeed, according to Definition 1.1, the solution (1.12) to the linearized equation can be expressed as

$$f(t,\cdot) = f(t) = i_{\text{linear}}(\overline{\omega} t),$$
$$\text{where } i_{\text{linear}}(\varphi) := \sum_{j_k \in S^+} 2\sqrt{\zeta_k} \cos(\varphi_k + j_k x) \text{ and } \overline{\omega}_k := W(j_k). \quad (1.13)$$

This naturally leads to the question whether there exists such a time-quasiperiodic solution to the full nonlinear problem (1.5) around the steady state $f = 0$.

In our analysis, we make use of several invariance properties of Equation (1.3). One is the so-called time-reversibility with respect to the involution $\rho : f(x) \mapsto f(-x)$, namely,

$$\mathcal{H}(\rho(f)) = \mathcal{H}(f), \text{ where } \mathcal{H} \text{ is defined as in (1.6).}$$

We say that a solution $f(x,t)$ to (1.5) is reversible, if $\rho(f)(x,-t)$ is also a solution. Another invariance property is the rotational invariance of solutions. More precisely, given an integer $\mathtt{M} \in \mathbb{N}$, if the initial data of the gSQG equation is invariant under a $\frac{2\pi}{\mathtt{M}}$-rotation, then the solution at any time is also invariant under a $\frac{2\pi}{\mathtt{M}}$-rotation. Such an $\mathtt{M}$-fold symmetric patch can be associated to a parametrization f in (1.3) being invariant under a $\frac{2\pi}{\mathtt{M}}$-translation of the variable x:

$$f(t, x + \frac{2\pi}{\mathtt{M}}) = f(t,x). \quad (1.14)$$

Those properties of the gSQG equation will be studied in detail in Chapter 3.

The main theorem we prove in this monograph is the following (stated informally, we refer to Theorem 6.5 for a precise statement):

Theorem 1.2 (= Theorem 6.5). *Let $\alpha \in (1,2)$, $S^+ \subset \mathbb{N}$, and a symmetry class $\mathtt{M}$ satisfying some nonresonance conditions (cf. Section 4.1 and 5.1) be fixed. Then, for all sufficiently small $\epsilon > 0$, there exists a set of amplitudes $A_\epsilon \subset [1,2]^{|S^+|} \subset \mathbb{R}^{|S^+|}$ such that for each $\vec{\zeta} \in A_\epsilon$, there exist a frequency vector $\omega = \omega(\vec{\zeta})$ and a time-quasiperiodic solution to* (1.5) *of the form*

$$f(\theta,t) = 2\epsilon \sum_{j_k \in S^+} \sqrt{|j_k|\vec{\zeta}_k} \cos(\omega_k t + j_k x) + o(\epsilon). \quad (1.15)$$

The set A_ϵ *is a Cantor-like set of asymptotically full measure, in the sense that* $\lim_{\epsilon\to 0}\frac{|A_\epsilon|}{|[1,2]^{|S^+|}|} = 1$. *The solution* (1.15) *is in some Sobolev space* H^{s_0} *for some* $s_0 \gg 1$, *and it is reversible and invariant under* $\frac{2\pi}{\mathtt{M}}$*-translation in the variable* θ. *Lastly, the solution is linearly stable under* $\frac{2\pi}{\mathtt{M}}$*-translation invariant perturbations.*

Some remarks are in order:

Remark 1.3. As stated above, our proof does not make use of any external parameters (α would be the natural candidate) as opposed to [HHM21] and, indeed, this results in needing the Diophantine constant γ to be $\gamma = o(\epsilon^2)$, which in turn requires Normal Form expansions (cf. Chapter 4), and also the computation of the explicit terms of size $O(\epsilon)$ and $O(\epsilon^2)$. Relaxing this constraint would significantly shorten the length and the complexity of this monograph.

Remark 1.4. It is conceivable that our proof of Theorem 6.5 would also work in the case $\alpha < 1$, changing the relevant sections and estimates. In the case of the SQG equation ($\alpha = 1$), the analysis in Section 10.3 breaks down since the sum of pseudodifferential symbols is not finite anymore and the regularity losses coming from the Egorov method are not finite. Most of the other parts of the reduction also hold for all cases of α, possibly with minimal changes.

Remark 1.5. The closer α is to 1, the more conjugations are required in the reduction of the linearized operator to a constant-coefficient operator. In the adaptation of the Egorov method, inspired by [BM20], we use a slightly more general flow, compared to those in [BM20, HH15, FG24], to avoid a large number of iterations that might cause potential complexity; see Remark 1.7.

Remark 1.6. All the amplitudes in the set A_ϵ in Theorem 1.2, which has asymptotically full measure, can possess quasiperiodic solutions, if the choice of tangential sites S^+ can be made properly so that some nonresonance conditions are satisfied. The precise conditions are stated in Section 5.1. Then a natural question is whether such a set $S^+ = \{j_1, \dots, j_\nu\} \subset \mathbb{N}$ is generic or not. The nonresonance conditions that we require can be roughly expressed as

$$P(j_1, j_2, \dots, j_\nu) \neq 0, \text{ for some function } P\colon \mathbb{Z}^\nu \mapsto \mathbb{R}.$$

Compared to previous works (e.g., [Giu17, BBM16a, BBM16b]), our P involves Gamma functions and the verification of the nonresonance condition is much more complicated. While we expect that such conditions can be satisfied by "generic" choices of S with small $\mathtt{M}$, we will give a rigorous proof only for the case where the frequencies are supported on multiples of a sufficiently large $\mathtt{M}$. This allows us to focus on the asymptotic behavior of the Gamma function in the analysis.

Before describing the idea of the proof in more detail, we give more explanation about the internal parameter, which we think of the most crucial part of the proof.

We consider a finite number of Fourier modes $S^+ \subset \mathbb{N}$ such that $|S^+| = \nu$. Setting $S := \{\pm j : j \in S^+\}$ and decomposing

$$L^2(\mathbb{T}) = H_S \oplus (H_S)^\perp, \quad H_S := \left\{ f \in L^2 : f_j = 0, \text{ if } j \notin S \right\}, \quad H_{S^\perp} = (H_S)^\perp,$$

one can see the SQG dynamics of the tangential component in H_S, and the normal component in $H_{S^\perp}$. While the dynamics of the tangential component is finite-dimensional, the dynamics of the normal component will be given as a slight variation of the SQG equation (1.5), due to the influence of the tangential component. Then under a suitable symplectic transformation Φ^{WB} (see Proposition 4.4) and the use of angle–action variables to reparametrize the tangential component (see (5.2)), the SQG Hamiltonian $\mathcal{H}$ in (1.6) on $L^2(\mathbb{T})$ can be reformulated in terms of another Hamiltonian H_ζ in (5.3) on $\mathbb{T}^\nu \times \mathbb{R}^\nu \times H_{S^\perp}$ taking the form (see Proposition 5.1)

$$H_\zeta(\theta, y, z) = C_\epsilon(\zeta) + 2\pi\omega(\zeta) \cdot y + \mathcal{N}(\theta)(z, z) + P(\theta, y, z),$$

for some θ-dependent bilinear map $\mathcal{N}(\theta)$, and a perturbative term $P(\theta, y, z)$, which must be sufficiently small. Note that $\zeta \in \mathbb{R}^\nu$ in the change of variable in (5.2) denotes a rescaled amplitude vector of the tangential component. Without the perturbative term P, the corresponding Hamiltonian system gives a quasiperiodic motion with the ζ-dependent frequency $\omega(\zeta) = \overline{\omega} + \epsilon^2 \mathbb{A}\zeta$ for some constant frequency $\overline{\omega}$ and $\mathbb{A} \in \mathbb{R}^{\nu\times\nu}$. The key point is that under such a process, we see that the amplitude ζ can modulate the linear frequency $\omega(\zeta)$, which will serve as an internal parameter to perform the KAM theory. Furthermore, since the size of the modulation at the linear level is $O(\epsilon^2)$, the perturbation P must be $o(\epsilon^2)$.

Such a derivation of the Hamiltonian H_ζ exhibiting the dependence on the amplitude, while ensuring $P = o(\epsilon^2)$ has been successfully implemented in earlier works in the literature, especially for the first time in [BBM14b] to the best of our knowledge. We emphasize that the technique that we adapt in this chapter requires a careful choice of the tangential modes S to exclude possible resonances. All the sufficient conditions on the choice of S are described in Section 5.1.

1.4 STRATEGY OF THE PROOF AND THE STRUCTURE OF THE MONOGRAPH

We first outline the main ideas of the proof of Theorem 1.2 and link them with the sections of this monograph afterwards.

In order to describe the strategy of the proof, let us fix

$$\nu \in \mathbb{N}, \quad S^+ := \{j_1 < \cdots < j_\nu\} \subset \mathbb{N}, \tag{1.16}$$

and denote

$$S := \left\{\pm j : j \in S^+\right\}, \quad S^\perp := \mathbb{Z}\backslash(S \cup \{0\}). \tag{1.17}$$

In view of (1.12), one can think of S as a set of Fourier modes of the solution at the linear level, and $S^\perp$ as the support of the orthogonal correction term for the solution to the nonlinear problem, while the 0th mode is excluded, since we look for a solution with zero average (see (1.4)). We also denote the linear frequency by

$$\overline{\omega} \in \mathbb{R}^\nu, \quad (\overline{\omega})_k = W(j_k), \text{ for } k = 1, \dots, \nu. \tag{1.18}$$

Using the notation in (1.5) and (1.10), we can rewrite (1.5) as

$$f_t = X_{\mathrm{gSQG}}(f) = Op(\mathrm{i}W(j))[f] + P_{\mathrm{gSQG}}(f), \tag{1.19}$$

where $P_{\mathrm{gSQG}}(f)$ collects the nonlinear contribution of the vector field $X_{\mathrm{gSQG}}(f)$. Since we are interested in the solutions near $f = 0$, replacing f by ϵf for a small $\epsilon > 0$, we are led to study the equation of the form

$$f_t = Op(\mathrm{i}W(j))[f] + P_{\epsilon,\mathrm{gSQG}}(f), \text{ where } P_{\epsilon,\mathrm{gSQG}}(f) := \frac{1}{\epsilon} P_{\mathrm{gSQG}}(\epsilon f). \tag{1.20}$$

As we observed in Section 1.3, we have an embedding i_{linear} (see (1.13)) for which $f_{\mathrm{lin}}(t,\theta) := i_{\mathrm{linear}}(\overline{\omega}t)$ solves the linear equation $\partial_t f_{\mathrm{lin}} = Op(\mathrm{i}W(j))[f_{\mathrm{lin}}]$. The question is whether such an embedding can persist under the nonlinear perturbation as in (1.19). Perhaps, one of the most naive attempts could be plugging the ansatz

$$f(t,x) = i(\overline{\omega}t, x), \text{ for some } i\colon \mathbb{T}^\nu \times \mathbb{T} \mapsto \mathbb{R},$$

into (1.19), which leads us to find i such that

$$\mathcal{F}_{\mathrm{gSQG}}(i) := \overline{\omega} \cdot \partial_\varphi i(\varphi) - Op(\mathrm{i}W(j))[i(\varphi)] - P_{\epsilon,\mathrm{gSQG}}(i(\varphi)) = 0, \text{ for } \varphi \in \mathbb{T}^\nu. \tag{1.21}$$

We can think of $\mathcal{F}_{\mathrm{gSQG}}$ as a map between spaces of functions of (φ, θ). Having the explicit solution i_{linear} at the linear level given in (1.13) and noting that $P_{\epsilon,\mathrm{gSQG}}$ is "small" depending on $\epsilon > 0$, we might expect the sequence of embeddings i_n, formally defined in the spirit of Newton's method,

$$i_0 = i_{\mathrm{linear}}, \quad i_{n+1} := i_n - \left(d_i \mathcal{F}_{\mathrm{gSQG}}(i_n)\right)^{-1} [\mathcal{F}_{\mathrm{gSQG}}(i_n)], \text{ for } n \geq 0, \tag{1.22}$$

where

$$\begin{aligned} d_i \mathcal{F}_{\mathrm{gSQG}}(i)[\hat{\mathbf{i}}] &:= \frac{d}{dt} \mathcal{F}_{\mathrm{gSQG}}(i + t\hat{\mathbf{i}}) \Big|_{t=0} \\ &\overset{(1.21)}{=} \overline{\omega} \cdot \partial_\varphi \hat{\mathbf{i}} - Op(\mathrm{i}W(j))[\hat{\mathbf{i}}] - d_i P_{\epsilon,\mathrm{gSQG}}(i)[\hat{\mathbf{i}}] \end{aligned} \tag{1.23}$$

to converge to a solution for (1.21). Clearly, the above argument is far less rigorous, and we will investigate how to modify the strategy.

1.4.1 Sketch of the Proof, Part 1: Internal Parameter and the Weak Birkhoff Normal Form

Loss of Derivatives and the Nash–Moser Scheme

As a rule of thumb in usual perturbative problems, the invertibility of the linearized operator $d_i\mathcal{F}_{\text{gSQG}}(i)$ in (1.22) would rely on the invertibility of the linear part

$$L[\hat{\mathbf{i}}] := \overline{\omega}\cdot\partial_\varphi\hat{\mathbf{i}} - Op(\mathrm{i}W(j))[\hat{\mathbf{i}}], \tag{1.24}$$

assuming that the contribution of the perturbative part is negligible. While looking for an embedding i in Sobolev spaces $H^s(\mathbb{T}^{\nu+1})$ (for large $s \gg 1$), it is not trivial whether the operator L can be invertible between two fixed Sobolev spaces. However, the classical KAM theory tells us that the invertibility of L can be achieved depending on the frequency vector $\overline{\omega}$. Indeed, for the frequency vectors that satisfy the so-called "Melnikov condition" with some $\gamma,\tau > 0$,

$$\{\omega : |\mathrm{i}\omega\cdot l - \mathrm{i}W(j)| \ge \gamma|l|^\tau \text{ for all } (l,j)\in\mathbb{Z}^\nu\times\mathbb{Z}\}, \tag{1.25}$$

one can formally invert L using the Fourier series, that is,

$$L[\hat{\mathbf{i}}](\varphi,x) = g(\varphi,x) \overset{(1.24)}{\iff} \hat{\mathbf{i}}(\varphi,x) = \sum_{(l,j)\in\mathbb{Z}^\nu\times\mathbb{Z}} \frac{\widehat{g}(l,j)}{\mathrm{i}(\overline{\omega}\cdot l - W(j))} e^{\mathrm{i}(\varphi\cdot l + jx)}, \tag{1.26}$$

where $\hat{g}(l,j) := \frac{1}{(2\pi)^{\nu+1}}\int_{\mathbb{T}^{\nu+1}} g(\varphi,x)e^{\mathrm{i}(\varphi\cdot l+jx)}d\varphi dx$. Note that the expression of the inverse in (1.26) tells us that there is a regularity mismatch between the image and the domain spaces. If $\hat{\mathbf{i}}\in H^s$ for some $s>0$, we see that there is a loss of derivatives due to the differential operators ∂_φ and $Op(W(j))$, while inverting L as in (1.26) does not gain the same amount of the regularity, and it actually causes another loss of derivatives by τ; the best estimate one can expect under the condition is that

$$\|L^{-1}[g]\|_{H^{s-\tau}} \lesssim \gamma^{-1}\|g\|_{H^s} \text{ for } g\in H^s(\mathbb{T}^{\nu+1}). \tag{1.27}$$

Therefore, the formal sequence of i_n in (1.22) does not seem to be closed, since i_{n+1} must be less regular than i_n at each iteration. Hence, the crude iteration procedure in (1.22) needs to be replaced by the Nash–Moser scheme, projecting each approximate solution i_n into a finite-dimensional space so that i_n remains in $C^\infty(\mathbb{T}^{\nu+1})$ for each $n\ge 0$.

Internal Parameter

We have observed above that the condition on the frequency vector in (1.25) is one of the necessary conditions to perform the iteration (1.22). Then a very natural question is how to check whether the linear frequency $\overline{\omega}$ satisfies such a condition. In general (for fixed $\gamma,\tau > 0$), it is very hard to determine whether a given vector $\omega\in\mathbb{R}^\nu$

satisfies even a more relaxed condition (the so-called Diophantine condition)

$$|\omega \cdot l| > \gamma |l|^{\tau}, \text{ for all } l \in \mathbb{Z}^{\nu}. \tag{1.28}$$

However, it is well known that given an open set $\Omega \subset \mathbb{R}^{\nu}$, "almost all" $\omega \in \Omega$ satisfy (1.28), more precisely, such nonresonance frequency vectors comprise a set of asymptotically full measure in Ω as $\gamma \to 0$. Indeed, the KAM theory does not tell us exactly which frequency vector can possess a quasiperiodic solution, but rather it tells us that the set of frequency vectors that possess a quasiperiodic solution has nonzero measure in a given set of frequencies. This is why we need parameter-dependent equations to perform the KAM theory; if our equation depends on a parameter, and the parameter can "properly" modulate the linear frequency $\overline{\omega}$, then for almost all parameter values, we might expect to invert the operator L. The gSQG equation (1.1) certainly involves a parameter $\alpha \in (0, 2)$, therefore one might be tempted to use α to modulate the linear frequency by looking at $\overline{\omega}$ in (1.18) and (1.11) as a function of α, that is, $\overline{\omega} = \overline{\omega}(\alpha)$. This attempt would enable us to obtain quasiperiodic solutions for "almost every" α (without knowing precisely which α satisfies the condition), while such a result cannot be, in principle, obtained for *every* α. For this reason, we follow the strategy in [BBM16a, BBM16b, FG24] and derive a weak Birkhoff normal form of the Hamiltonian $\mathcal{H}$, from which we can see a modulation effect of the linear frequency by the amplitude. In this regard, a bit more precise explanation will follow.

Action–Angle Variables

According to the decomposition in (1.17), we denote

$$H_S := \left\{ f : f_j = 0, \text{ if } j \notin S \right\}, \quad H_{S^{\perp}} := \left\{ f : f_j = 0, \text{ if } j \notin S^{\perp} \right\}, \quad f_j := \int_{\mathbb{T}} e^{\mathrm{i} j x} dx, \tag{1.29}$$

and we will refer to H_S and $H_{S^{\perp}}$ as the *tangential space* and *normal space*, respectively. We introduce the amplitude variable ζ,

$$\zeta \in [1, 2]^{\nu} \subset \left(\mathbb{R}^{+}\right)^{\nu}, \tag{1.30}$$

and consider a ζ-dependent change of variables, $U_{\zeta} : \mathbb{T}^{\nu} \times \mathbb{R}^{\nu} \times H_{S^{\perp}} \mapsto L^2(\mathbb{T})$, given by

$$U_{\zeta}(\theta, y, z) := \epsilon \left(\sum_{j_k \in S} \sqrt{|j_k|(\zeta_k + \epsilon^{2(b-1)} y_k)} e^{\mathrm{i}(\theta_k + j_k x)} + \epsilon^{(b-1)} z \right), \tag{1.31}$$

$$\text{for some } b \in (1, 1 + 1/12), \text{ with } \zeta_{-k} := \zeta_k, \quad \theta_{-k} := -\theta_k, \quad y_{-k} := y_k.$$

The variables (θ, y) are the so-called action–angle variables and the above change of variables can be thought of as a reparametrization of functions in H_S. Also, in order to see the motivation of the constant b in (1.31), we note that in view of (1.13), the map

$$\mathbb{T}^\nu \ni \varphi \mapsto U_\zeta(\varphi, 0, 0) \in L^2(\mathbb{T}) \tag{1.32}$$

corresponds to i_{linear} up to the rescaling factor ϵ, therefore the terms $\epsilon^{2(b-1)}y$ and $\epsilon^{b-1}z$ in (1.31) can be thought of as correction terms to solve the nonlinear problem, which justifies the requirement $b > 1$.

Now we define a ζ-dependent Hamiltonian $\mathcal{H}_\zeta$ on $\mathbb{T}^\nu \times \mathbb{R}^\nu \times H_{S^\perp}$ as

$$\mathcal{H}_\zeta(\theta, y, z) := \epsilon^{-2b}\mathcal{H} \circ U_\zeta(\theta, y, z). \tag{1.33}$$

We note that the factor ϵ^{-2b} in (1.33) naturally arises in the rescaling of the change of variables to describe the Hamiltonian equation in the new phase space $\mathbb{T}^\nu \times \mathbb{R}^\nu \times H_{S^\perp}$. Indeed, the gSQG dynamics in $L^2(\mathbb{T})$ in (1.5) can be easily rewritten as an evolution equation in the new phase space by pulling back the vector field by U_ζ, and one can obtain the equivalent equation is given by (one can follow the same computations given in Chapter 5 after the proof of Proposition 5.1. Note that the factor $\frac{1}{2\pi}$ is due to our definition for the symplectic form σ in (2.128) and the gradient in (2.126), but does not play a crucial role throughout the proof)

$$\frac{d}{dt}\begin{pmatrix}\theta(t)\\ y(t)\\ z(t)\end{pmatrix} = \begin{pmatrix}\frac{1}{2\pi}\partial_y \mathcal{H}_\zeta(\theta, y, z)\\ -\frac{1}{2\pi}\partial_\theta \mathcal{H}_\zeta(\theta, y, z)\\ \partial_x\left(\nabla_z \mathcal{H}_\zeta(\theta, y, z)\right)\end{pmatrix}, \tag{1.34}$$

where $\nabla_z \mathcal{H}_\zeta$ is the gradient vector field of $\mathcal{H}_\zeta$ restricted to the subspace $H_{S^\perp}$. Indeed, one can show that if $(\theta(t), y(t), z(t))$ is a solution to (1.34), then $f(t) := U_\zeta(\theta(t), y(t), z(t))$ is a solution to the gSQG equation (1.6) (again, see Chapter 5).

- **Toy Model 1: A Perturbed Airy Equation**
  Recall that our goal at this moment is to see whether we can extract a dependence of the linear frequency on ζ. Therefore the question becomes whether the new system (1.34) has a linear frequency that can be modulated by ζ. Of course, the answer depends on the structure of $\mathcal{H}$. To this end, let us consider a simpler example, where we can see that the modulation of the linear frequency relies on the quartic homogeneous term of the Hamiltonian.
  As a toy model, let us consider a perturbed Airy equation (see [BBM14a, BBM16a, BBM16b, Giu17] for the results of relevant but more complicated models). We define a Hamiltonian $\mathcal{G}\colon L^2(\mathbb{T}) \mapsto \mathbb{R} \cup \{\infty\}$ of the form

$$\begin{aligned}
&\mathcal{G}(f) := \mathcal{G}_2(f) + \mathcal{G}_4(f), \text{ where}\\
&\mathcal{G}_2(f) := \frac{1}{2}\int_{\mathbb{T}} (\partial_x f)^2(x)dx,\\
&\mathcal{G}_4(f) := \sum_{j_1, j_2, j_3, j_4 \in \mathbb{Z}\setminus\{0\}} G(j_1, j_2, j_3, j_4) f_{j_1} f_{j_2} f_{j_3} f_{j_4}, \text{ for some } G\colon \mathbb{Z}^4 \mapsto \mathbb{C}.
\end{aligned} \tag{1.35}$$

The associated nonlinear Hamiltonian equation to $\mathcal{G}$ is written as (see the comparison with (1.7))

$$f_t = \partial_x(\nabla_{L^2}\mathcal{G}(f)) = -\partial_{xxx}f + \partial_x(\nabla_{L^2}\mathcal{G}_4(f)). \tag{1.36}$$

The linearized Hamiltonian equation associated to $\mathcal{G}$ only depends on the quadratic term $\mathcal{G}_2$ and it corresponds to the Airy equation

$$f_t = -\partial_{xxx}f. \tag{1.37}$$

It is trivial to see that the Airy equation possesses quasiperiodic solutions with the linear frequency $\omega^{\text{Airy}} \in \mathbb{R}^\nu$ given by

$$(\overline{\omega}^{\text{Airy}})_k := j_k^3, \text{ for } S^+ := \{j_1 < \cdots < j_\nu\}. \tag{1.38}$$

At this point, the linearized equation does not reveal the modulation of the linear frequency by the amplitude. To make the computations easier, let us make the following assumptions on the quartic term $\mathcal{G}_4$:

(A1) $G(j_1, j_2, j_3, j_4)$ is invariant under any permutation on $\{j_1, j_2, j_3, j_4\}$. This assumption is simply to make the computations easier and can be assumed for a general quartic Hamiltonian, since we can take the average of the summation in (1.35) over all the permutations.

(A2) G is supported only on modes S, that is, $G(j_1, j_2, j_3, j_4) = 0$, if $j_i \notin S$ for some $= 1, 2, 3, 4$. This is to focus on the contribution of ζ through this example, since ζ presents only in the modes in the set S (see (1.31)).

(A3) If $G(j_1, j_2, j_3, j_4) \neq 0$, then $j_a = -j_b$ and $j_c = -j_d$ for a permutation (a, b, c, d) of $(1, 2, 3, 4)$. In other words, there is no nontrivial resonance in $\mathcal{G}_4$. Therefore, using the assumption (A1) and (A2), we can define a $\nu \times \nu$ matrix $\mathbb{G}$ as

$$\mathbb{G}_k^i := G(j_i, -j_i, j_k, -j_k) \ \text{ for } i, k = 1, \ldots, \nu \text{ and } j_i, j_k \in S^+. \tag{1.39}$$

To see the modulation of the frequency by ζ, we compute the composition with the change of variables U_ζ,

$$\mathcal{G}_\zeta(\theta, y, z) := \epsilon^{-2b}\mathcal{G} \circ U_\zeta(\theta, y, z). \tag{1.40}$$

For the quadratic term, we see that

$$\begin{aligned}
&\epsilon^{-2b}\mathcal{G}_2 \circ U_\zeta(\theta, y, z) \\
&\overset{(1.31)}{=} \epsilon^{-2b}\left(2\pi \sum_{j_k \in S^+} j_k^2 \epsilon^2 (j_k\zeta_k + \epsilon^{2(b-1)} j_k y_k) + \frac{1}{2}\int_{\mathbb{T}} \epsilon^{2b}(\partial_x z)^2(x)dx\right) \\
&= C_{\epsilon,\zeta} + 2\pi \sum_{j_k \in S^+} (j_k)^3 y_k + \frac{1}{2}\int_{\mathbb{T}} (\partial_x z)^2(x)dx
\end{aligned}$$

$$\overset{(1.38)}{=} C_{\epsilon,\zeta} + 2\pi\left(\overline{\omega}^{\text{Airy}} \cdot y\right) + \frac{1}{2}\int_{\mathbb{T}} (\partial_x z)^2(x)dx, \tag{1.41}$$

for some $C_{\epsilon,\zeta}$ that does not depend on (θ, y, z). For the quartic term, we have

$$\begin{aligned}
\mathcal{G}_4(f) &\overset{(1.35),(A2)}{=} \sum_{j_1,j_2,j_3,j_4 \in S} G(j_1,j_2,j_3,j_4) f_{j_1} f_{j_2} f_{j_3} f_{j_4} \\
&\overset{(A1),(A3)}{=} 6 \sum_{j_1=j_2 \in S^+} G(j_1,-j_1,j_2,-j_2)|f_{j_1}|^2|f_{j_2}|^2 \\
&\qquad + 12 \sum_{j_1,j_2 \in S^+,\ j_1 \neq j_2} G(j_1,j_2,-j_1,-j_2)|f_{j_1}|^2|f_{j_2}|^2 \\
&\overset{(1.39)}{=} 6\sum_{i=1}^{\nu} \mathbb{G}_i^i |f_{j_i}|^4 + 12 \sum_{i,k=1,\ i\neq k}^{\nu} \mathbb{G}_k^i |f_{j_i}|^2 |f_{j_k}|^2.
\end{aligned}$$

Therefore, using (1.31), we can see that

$$\begin{aligned}
&\epsilon^{-2b}\mathcal{G}_4 \circ U_\zeta(\theta, y, z) \\
&\overset{(1.31)}{=} 6\epsilon^{-2b} \sum_{i=1}^{\nu} \mathbb{G}_i^i \epsilon^4 (j_i\zeta_i + \epsilon^{2(b-1)} j_i y_i)^2 \\
&\quad + 12\epsilon^{-2b} \sum_{i,k=1,\ i\neq k}^{\nu} \mathbb{G}_k^i \epsilon^4 (j_i\zeta_i + \epsilon^{2(b-1)} j_i y_i)(j_k\zeta_k + \epsilon^{2(b-1)} j_k y_k) \\
&= C_{\epsilon,\zeta} + 12\epsilon^2\left(\sum_{i=1}^{\nu} \mathbb{G}_i^i j_i^2 \zeta_i y_i + 2\sum_{i,k=1,\ i\neq k}^{\nu} \mathbb{G}_k^i j_i j_k \zeta_i y_k\right) \\
&\quad + 6\epsilon^{2b}\left(\sum_{i=1}^{\nu} \mathbb{G}_i^i j_i^2 y_i^2 + 2\sum_{i,k=1,\ i\neq k}^{\nu} \mathbb{G}_k^i j_i j_k y_i y_k\right) \\
&= C_{\epsilon,\zeta} + 12\epsilon^2 G^{\text{mod}}\zeta \cdot y + 6\epsilon^{2b} G^{\text{mod}} y \cdot y,
\end{aligned} \tag{1.42}$$

where G^{mod} is a $\nu \times \nu$ symmetric matrix defined as

$$(G^{\text{mod}})_k^i := \begin{cases} j_i^2 \mathbb{G}_i^i & \text{if } i = k, \\ 2 j_i j_k \mathbb{G}_k^i & \text{if } i \neq k, \end{cases} \quad \text{for } j_i, j_k \in S^+ = \{j_1 < \cdots < j_\nu\}. \tag{1.43}$$

Plugging (1.42) and (1.41) into (1.40), we see a normal form of the nonlinear Hamiltonian $\mathcal{G}_\zeta$, namely

$$\begin{aligned}
\mathcal{G}_\zeta(\theta, y, z) = C_{\epsilon,\zeta} &+ \left(2\pi\overline{\omega}^{\text{Airy}} + 12\epsilon^2 G^{\text{mod}}\zeta\right) \cdot y \\
&+ 6\epsilon^{2b} G^{\text{mod}} y \cdot y + \frac{1}{2}\int_{\mathbb{T}} (\partial_x z)^2(x)dx
\end{aligned}$$

$$= C_{\epsilon,\zeta} + 2\pi\omega^{\text{Airy}}(\zeta) \cdot y + 6\epsilon^{2b} G^{\text{mod}} y \cdot y + \frac{1}{2}\int_{\mathbb{T}} (\partial_x z)^2(x)\,dx, \tag{1.44}$$

where

$$\omega^{\text{Airy}}(\zeta) := \overline{\omega}^{\text{Airy}} + \frac{6}{\pi}\epsilon^2 G^{\text{mod}}\zeta \in \mathbb{R}^\nu. \tag{1.45}$$

Recall that we can write the Hamiltonian equation (1.36) in the phase space $L^2(\mathbb{T})$ as another Hamiltonian equation in the new phase space $\mathbb{T}^\nu \times \mathbb{R}^\nu \times H_{S^\perp}$ by pulling back the vector field by U_ζ, which should be written as (compare to (1.34))

$$\frac{d}{dt}\begin{pmatrix}\theta(t)\\ y(t)\\ z(t)\end{pmatrix} = \begin{pmatrix}\frac{1}{2\pi}\partial_y \mathcal{G}_\zeta(\theta,y,z)\\ -\frac{1}{2\pi}\partial_\theta \mathcal{G}_\zeta(\theta,y,z)\\ \partial_x\left(\nabla_z \mathcal{G}_\zeta(\theta,y,z)\right)\end{pmatrix} \overset{(1.44)}{=} \begin{pmatrix}\omega^{\text{Airy}}(\zeta)\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}\frac{6}{\pi}\epsilon^{2b}G^{\text{mod}}y(t)\\ 0\\ -\partial_{xxx}(z(t))\end{pmatrix}. \tag{1.46}$$

From the above equation, we can easily see that the trivial embedding $i_{\text{triv}} : \mathbb{T}^\nu \mapsto \mathbb{T}^\nu \times \mathbb{R}^\nu \times H_{S^\perp}$, defined as

$$i_{\text{triv}}(\varphi) := (\varphi, 0, 0), \tag{1.47}$$

is invariant under the vector field in (1.46), and the solution to (1.46) can be written as

$$(\theta(t), y(t), z(t)) = i_{\text{triv}}(\omega^{\text{Airy}}(\zeta)t) = (\omega^{\text{Airy}}(\zeta)t, 0, 0). \tag{1.48}$$

Clearly, the solution (1.48) is time-quasiperiodic because each "angular component" $\theta_i(t)$ oscillates with frequency $(\omega^{\text{Airy}}(\zeta))_i$ for each $i = 1, \ldots, \nu$ and the frequency vector $\omega^{\text{Airy}}(\zeta)$ is modulated by the amplitude ζ through the relation in (1.45), which cannot be observed by just looking at the linear Airy equation (1.37). Therefore, a quasiperiodic solution to (1.36) can be obtained as $f(t) := U_\zeta(i_{\text{triv}}(\omega^{\text{Airy}}(\zeta)t))$. As shown in this example, our strategy to study Equation (1.34) is to derive a "normal form" where we can see a modulation of the frequency by the amplitude ζ at the linear level of the equation (that is, quadratic level of the Hamiltonian).

Weak Birkhoff Normal Form

Our Hamiltonian $\mathcal{H}$ in (1.6) does not possess a simple structure as in the toy-model example, therefore it is hopeless to expect $\mathcal{H}_\zeta$ in (1.33) to have a simple form as in (1.44). However, we will construct a symplectic transformation $\Phi^{\text{WB}} : L^2(\mathbb{T}) \mapsto L^2(\mathbb{T})$ so that, defining another Hamiltonian H as

$$H(f) := \mathcal{H} \circ \Phi^{\text{WB}}(f), \tag{1.49}$$

we can rewrite H, under the composition with U_ζ, as (compare below with (1.44))

$$\begin{aligned} H_\zeta(\theta, y, z) &:= \epsilon^{-2b} H \circ (U_\zeta(\theta, y, z)) \\ &= C_{\epsilon,\zeta} + 2\pi\omega^{\mathrm{gSQG}}(\zeta) \cdot y + 6\epsilon^{2b}\mathbb{A}y \cdot y + \mathcal{N}(\theta)(z, z) + P(\theta, y, z), \end{aligned} \tag{1.50}$$

where

$$\omega^{\mathrm{gSQG}}(\zeta) = \overline{\omega} + \frac{6}{\pi}\epsilon^2\mathbb{A}\zeta \text{ (see (1.18) for the definition of } \overline{\omega}), \tag{1.51}$$

for some $\nu \times \nu$ symmetric matrix $\mathbb{A}$, θ-dependent bilinear form $\mathcal{N}(\theta)$ on $H_{S^\perp}$, and perturbation P satisfying some smallness condition. Note that in (1.50), the bilinear form $\mathcal{N}$ and the perturbation P depend on ζ, while we do not denote this dependence to avoid notational complication.

Certainly, the following concerns need to be taken into account:

(C1) What structure of $\omega^{\mathrm{gSQG}}(\zeta)$ do we need? More precisely, what do we require from the matrix $\mathbb{A}$?

(C2) How to construct Φ^{WB} so that H defined by (1.49) has the structure in (1.50) with a sufficiently small P?

We postpone more detailed comments on the above concerns to the next paragraph but for now focus on how to transform the functional equation (1.21), according to the new Hamiltonian (1.50). By requiring Φ^{WB} to be symplectic, we have that the Hamiltonian equation associated to $\mathcal{H}$ is equivalent to the Hamiltonian equation associated to H, therefore, again pulling back the Hamiltonian vector field of H by U_ζ. Thus we are led to study the equation

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix}\theta(t)\\ y(t)\\ z(t)\end{pmatrix} &= \begin{pmatrix} \frac{1}{2\pi}\partial_y H_\zeta(\theta, y, z) \\ -\frac{1}{2\pi}\partial_\theta H_\zeta(\theta, y, z) \\ \partial_x\left(\nabla_z H_\zeta(\theta, y, z)\right)\end{pmatrix} \\ &\overset{(1.50)}{=} \begin{pmatrix}\omega^{\mathrm{gSQG}}(\zeta)\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}\frac{6}{\pi}\epsilon^{2b}\mathbb{A}y \\ -\frac{1}{2\pi}\partial_\theta\mathcal{N}(\theta)(z,z) \\ \partial_x((\mathcal{N}(\theta))^T[z])\end{pmatrix} + \begin{pmatrix}\frac{1}{2\pi}\partial_y P(\theta, y, z) \\ -\frac{1}{2\pi}\partial_\theta P(\theta, y, z) \\ \partial_x\left(\nabla_z P(\theta, y, z)\right)\end{pmatrix} \\ &=: X_{H_\zeta}(\theta(t), y(t), z(t)), \end{aligned} \tag{1.52}$$

where $z \mapsto \mathcal{N}(\theta)^T[z]$ is the linear map on $H_{S^\perp}$ such that

$$\int_{\mathbb{T}} \mathcal{N}(\theta)^T[z](x)h(x)dx = \nabla_z(\mathcal{N}(\theta)(z,z))[h], \text{ for all } h \in S^\perp.$$

If $t \mapsto i_\infty(\omega^{\mathrm{gSQG}}(\zeta)t) \in \mathbb{T}^\nu \times \mathbb{R}^\nu \times H_{S^\perp}$ is a quasiperiodic solution to (1.52), then the quasiperiodic solution to the gSQG equation (1.5) will be recovered by

$$f(t) = \Phi^{\mathrm{WB}}(U_\zeta(i_\infty(\omega^{\mathrm{gSQG}}(\zeta)t))).$$

Assuming $P \equiv 0$, Equation (1.52) indeed possesses the trivial embedding (1.47) as a quasiperiodic solution with the frequency vector $\omega^{\text{gSQG}}(\zeta)$. Therefore, our main goal of this monograph becomes to study whether such an embedding can persist under a perturbation P in the system (1.52). Note that taking into account the dependence of the frequency on ζ, more precise statements to be proved are that "for almost every ζ" in (1.30), the quasiperiodic solution with frequency $\omega^{\text{gSQG}}(\zeta)$ can survive under the perturbation. Making an ansatz,

$$t \mapsto i(\omega^{\text{gSQG}}(\zeta)t) \text{ solves (1.52)}, \quad \text{for some } i : \mathbb{T}^\nu \mapsto \mathbb{T}^\nu \times \mathbb{R}^\nu \times H_{S^\perp},\ i(\varphi) =: (\theta(\varphi), y(\varphi), z(\varphi)), \tag{1.53}$$

we are led to find i such that

$$\mathcal{F}(i) := \omega^{\text{gSQG}}(\zeta) \cdot \partial_\varphi i(\varphi) - X_{H_\zeta}(i(\varphi)) = 0. \tag{1.54}$$

Now, let us discuss the concerns (C1)–(C2).

Item (C1): Frequency Vector $\omega^{\text{gSQG}}(\zeta)$ and Use of ω As a Parameter. Let us first make it clear why we need to care about the structure of $\omega^{\text{gSQG}}(\zeta)$. We recall from (1.30) that our parameter ζ lies in a fixed subset $[1,2]^\nu$, and we want to perform the iteration (1.61) for sufficiently many ζ in $[1,2]^\nu$. Denoting

$$\Omega := \left\{\omega \in \mathbb{R}^\nu : \omega = \omega^{\text{gSQG}}(\zeta),\ \zeta \in [1,2]^\nu\right\}, \tag{1.55}$$

we can only hope that "almost every" ω in Ω satisfies all the necessary nonresonance conditions, such as (1.28). This does not necessarily imply that for "almost every" $\zeta \in [1,2]^\nu$, $\omega^{\text{gSQG}}(\zeta)$ satisfies the necessary nonresonance conditions, especially in case $\zeta \mapsto \omega^{\text{gSQG}}(\zeta)$ is not one-to-one. Thus, we require that the matrix $\mathbb{A}$ in (1.51) is invertible. The explicit form of $\mathbb{A}$ is not important at this point, but it is important to note that $\mathbb{A}$ is completely determined by the choice of the set S^+ in (1.16). The invertibility of $\mathbb{A}$ is one of the "nonresonance conditions" described in the statement of Theorem 1.2. In the proof, this condition will be verified (see Section 5.1 and Proposition 5.5).

Once we have the invertibility of the map $\zeta \mapsto \omega^{\text{gSQG}}(\zeta)$, we will use the frequency ω as a parameter of the system (1.52) and think of ζ as a quantity determined by ω. More precisely, defining

$$\Omega_\epsilon := \left\{\omega \in \mathbb{R}^\nu : \omega = \overline{\omega} + \frac{6}{\pi}\epsilon^2 \mathbb{A}\zeta,\ \text{for } \zeta \in [1,2]^\nu\right\}, \tag{1.56}$$

we think of the amplitude ζ to be a function of ω for $\omega \in \Omega_\epsilon$. With a slight abuse of notation, we will still denote ζ as if it is an independent variable, but it is actually a function of ω, determined by

$$\zeta = \zeta(\omega) := \left(\frac{6}{\pi}\epsilon^2 \mathbb{A}\right)^{-1} (\omega - \overline{\omega}),\ \text{for } \omega \in \Omega_\epsilon. \tag{1.57}$$

Then the Hamiltonian functional (1.50), the system (1.52) and the functional equation (1.54) become

$$H_\zeta(\theta, y, z) = C_{\epsilon,\zeta} + 2\pi\omega \cdot y + 6\epsilon^{2b}\mathbb{A}y \cdot y + \mathcal{N}(\theta)(z, z) + P(\theta, y, z), \tag{1.58}$$

$$\frac{d}{dt}\begin{pmatrix}\theta(t)\\ y(t)\\ z(t)\end{pmatrix} = \begin{pmatrix}\omega\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}\frac{6}{\pi}\epsilon^{2b}\mathbb{A}y\\ -\frac{1}{2\pi}\partial_\theta\mathcal{N}(\theta)(z,z)\\ \partial_x((\mathcal{N}(\theta))^T[z])\end{pmatrix} + \begin{pmatrix}\frac{1}{2\pi}\partial_y P(\theta, y, z)\\ -\frac{1}{2\pi}\partial_\theta P(\theta, y, z)\\ \partial_x(\nabla_z P(\theta, y, z))\end{pmatrix}$$
$$=: X_{H_\zeta}(\theta(t), y(t), z(t)), \tag{1.59}$$

and

$$\mathcal{F}_\omega(i) := \omega \cdot \partial_\varphi i(\varphi) - X_{H_\zeta}(i(\varphi)) = \begin{pmatrix}\omega \cdot \partial_\varphi\theta(\varphi)\\ \omega \cdot \partial_\varphi y(\varphi)\\ \omega \cdot \partial_\varphi z(\varphi)\end{pmatrix} - X_{H_\zeta}(i(\varphi)) = 0. \tag{1.60}$$

Here $X_{H_\zeta} = X_{H_{\zeta(\omega)}}$ is now thought of as an ω-dependent vector field (instead of a ζ-dependent vector field) and the Hamiltonian H_ζ, as well as $\mathcal{N}$ and P, depends on ω implicitly through (1.57). Clearly, if an embedding i_∞ solves (1.60) with some $\omega \in \Omega_\epsilon$, then i_∞ solves (1.54) with $\zeta(\omega)$. The reason why we use ω as a parameter is that by doing so, it is easier to check the nonresonance conditions such as (1.25) or (1.28) throughout the proof.

Now, we transform the initial iteration scheme given in (1.22), since our new system (1.59) has a slightly different form, compared to (1.20). The main scheme is quite same as described in (1.22) based on Newton's method (more precisely, Nash–Moser scheme as we discussed before): Noting that the trivial embedding (1.47) solves (1.59) without the perturbation P, we set up a formal sequence of approximate solutions $i_n : \mathbb{T}^\nu \mapsto \mathbb{T}^\nu \times \mathbb{R}^\nu \times H_{S^\perp}$,

$$\begin{cases} i_0 := i_{\text{triv}}, \quad i_{n+1} := i_n - (d_i\mathcal{F}_\omega(i_n))^{-1}[\mathcal{F}_\omega(i_n)], \text{ for } n \geq 0,\\ d_i\mathcal{F}_\omega(i)[\hat{\mathbf{i}}] := \frac{d}{dt}\mathcal{F}_\omega(i + t\hat{\mathbf{i}})\Big|_{t=0} \overset{(1.60)}{=} \omega \cdot \partial_\varphi\hat{\mathbf{i}} - d_i X_{H_\zeta}(i)[\hat{\mathbf{i}}], \end{cases} \tag{1.61}$$

and study the convergence of i_n to a solution to (1.60). In the iteration scheme (1.61), we do not expect the inverse of the linearized operator $d_i\mathcal{F}_\omega(i)[\hat{\mathbf{i}}]$ to be obtained for every $\omega \in \Omega_\epsilon$, but we select ω such that ω satisfies all the necessary nonresonance conditions to obtain an inverse of the linearized operator.

Before we close our discussion concerning the use of ω as a parameter, we emphasize that the constant γ arising in the Diophantine condition (1.28) needs to be small depending on ϵ. We wish to select nonresonant frequencies from the set Ω_ϵ, that is, we wish the set (for some fixed $\tau > 0$)

$$\Omega_0 := \{\omega \in \Omega_\epsilon : |\omega \cdot l| > \gamma|l|^\tau, \text{ for all } l \in \mathbb{Z}^\nu\} \tag{1.62}$$

had asymptotically full measure in Ω_ϵ. However, Ω_ϵ in (1.56) is an ϵ^2-neighborhood of $\overline{\omega}$. Therefore, in case $\overline{\omega}$ is resonant, that is, $\overline{\omega} \cdot l_* = 0$ for some $l_* \in \mathbb{Z}^\nu$, we have

$$|\omega \cdot l_*| \leq |(\omega - \overline{\omega}) \cdot l_*| + |\overline{\omega} \cdot l_*| \leq |\omega - \overline{\omega}||l_*| \lesssim \epsilon^2 |l_*|, \text{ for all } \omega \in \Omega_\epsilon.$$

Thus, it is not, in general, possible for Ω_0 to have asymptotically full measure in Ω_ϵ, unless $\gamma = o(\epsilon^2)$. For this reason, we will fix γ to be

$$\gamma := \epsilon^{2b}, \text{ where } b > 1 \text{ is chosen as in (1.31).} \tag{1.63}$$

Item (C2): Construction of Φ^{WB}. Now, we discuss how to construct the transformation Φ^{WB} so that we can obtain (1.50) through (1.49).

Before we start, we first fix some notation. In view of (1.29), we denote by v and z the variables in spaces H_S and $H_{S^\perp}$, respectively, so that a given $f \in L^2(\mathbb{T})$ can be written as

$$f = v + z, \text{ for some } v \in H_S \text{ and } z \in H_{S^\perp} \text{ in a unique way.} \tag{1.64}$$

The variables v and z will be called a "tangential variable" and a "normal variable." We define $v_\zeta : \mathbb{T}^\nu \times \mathbb{R}^\nu \mapsto H_S$ by

$$v_\zeta(\theta, y) := \sum_{j_k \in S} \sqrt{|j_k|(\zeta_k + y_k)} e^{\mathrm{i}(\theta_k + j_k x)}, \tag{1.65}$$

so that U_ζ in (1.31) can be written as

$$U_\zeta(\theta, y, z) = \epsilon v_\zeta(\theta, \epsilon^{2(b-1)} y) + \epsilon^b z =: \epsilon v_\epsilon(\theta, y) + \epsilon^b z. \tag{1.66}$$

Let us consider a homogeneous expansion of H defined by (1.49),

$$H(f) = H_2(f) + H_3(f) + H_4(f) + H_5(f) + H_{\geq 6}(f), \tag{1.67}$$

where H_i is homogeneous of degree i for $i = 1, \ldots 5$, and $H_{\geq 6}$ collects all the terms that are homogeneous of degree at least 6. Also, for each H_i and $0 \leq k \leq i$, we denote by $H_{i,k}$ the term in H_i that is homogeneous of degree k in the variable z. For example, recalling the Airy equation in (1.35), we can write

$$\begin{aligned} \mathcal{G}_2(f) &\overset{(1.35)}{=} \frac{1}{2}\int_{\mathbb{T}} (\partial_x f)^2(x)dx = \frac{1}{2}\int_{\mathbb{T}} (\partial_x v)^2(x)dx + \frac{1}{2}\int_{\mathbb{T}} (\partial_x z)^2(x)dx \\ &= \mathcal{G}_{2,0}(f) + \mathcal{G}_{2,2}(f). \end{aligned}$$

The reason of introducing the above notation is to see the contribution of v and z in each homogeneous term H_i separately.

Using the above notation, we can rearrange the expansion in (1.67) as

$$H(f) = \underbrace{(H_{2,0} + H_{4,0})}_{=:Z_0(f)} + \underbrace{(H_{2,1} + H_{3,0} + H_{3,1} + H_{4,1} + H_{5,0} + H_{5,1})}_{=:Z_1(f)}$$

(continued...)

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