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CHAPTER 1

Relativistic Kinematics; Time and Space

Sometime in the early 1970's, Eugene Wigner, one of the architects of Quantum Mechanics and also a very thoughtful philosopher of science [3], was invited to the University of Chicago to give a talk. The conference room, with high windows and ancient stuffed leather chairs used for the weekly faculty-only seminar, was in the old Research Institutes building that housed the Franck and Fermi Institutes. Wigner's talk had been publicly advertised in Chicago; unusually, in the audience there were many old men wearing Eastern European (read Hungarian) suits from the early 20th century.

Wigner gave his talk (I don't remember the topic—it may have been Civil Defense). When he concluded the host asked for questions. From the back of the room an ancient man in an ancient suit asked, "Professor Wigner, do you think we will ever understand it all—that is, we will have a Theory of Everything?" Wigner replied: "Let me tell you a story."

"I had a dog once—a very smart dog. He was so smart: I taught him to beg, to shake hands, to roll over. He learned so quickly that I decided I should teach him to solve Diophantine equations. But you know, it was just beyond that dog."

1.1 Foundations of Classical Mechanics

1.1.1 Introduction: Classical Mechanics as the Limiting Cases of SR and QM

Classical Mechanics gives only an approximate description of motion, as it is a limiting case of each of two *theories*, Special Relativity and Quantum Mechanics (Figure 1.1). Starting with the relativistic expressions for energy and momentum we will derive in just a few lines the classical (approximate) expressions under the assumption that the speed of light is infinite. The classical limit of Quantum Mechanics (QM) is more subtle as QM is non-local in space and time. The limit is

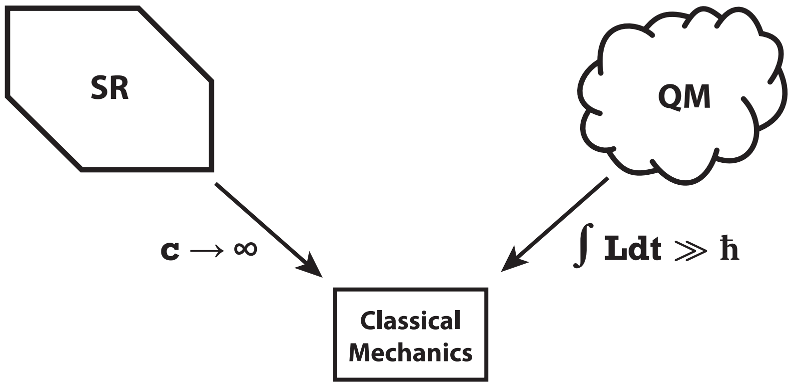

Figure 1.1. Classical Mechanics is approximate, as it is a limiting case of each of two theories, Special Relativity (SR) and Quantum Mechanics (QM). For Special Relativity, the classical approximation corresponds to a world in which the speed of light is infinite, so that we see "everything happening everywhere all at once," i.e., we ignore the time it takes light to arrive from its multiple sources. For Quantum Mechanics, the approximations hold when the energies of a moving object are high enough so that, speaking loosely, effects due to the wave nature of matter integrate to zero along all paths that differ measurably from the classical path.

unfortunately beyond the scope of a one-quarter introductory course, but may be a good subject for an introductory talk in a Discussion Session or office hours after we have learned about path integrals.

1.1.2 Principles of Invariance

Our introduction to classical mechanics rests on three principles. To state them correctly will take developing a small vocabulary for the conditions under which they apply. However, we can loosely state them here now: 1) the laws of physics should be the same for all non-accelerating observers (Lorentz invariance); 2) the laws of physics should be the same at all spatial locations (invariance under translations in space); 3) the laws of physics should be the same at all times (invariance under translations in time). These principles lead to a remarkably concise and elegant description of motion of objects in the classical physical world. Developing this description and a corresponding physical intuition in a concise mathematical language is the subject of the course.

1.2 Inertial Frames

Newton's First Law[1] states that:

> *All bodies at rest remain at rest or if in uniform linear motion continue in that motion unless compelled to change their state by an applied external force.*

[1] Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare [8].

A reference frame in which Newton's First Law is true is called an Inertial Frame. This may seem like a tautology (true by definition), but isn't. If in a frame any object follows Newton's First Law, then all other objects will also obey the Law, and the frame is inertial.

It is easy to think of counter-examples to Newton's First Law, e.g., rotating frames such as merry-go-rounds, cars going around a curve, or a train accelerating smoothly from a stop. If you have ever tried to walk in a straight line across a merry-go-round in an arcade, you will appreciate George Atwood's First Law of Classical Mechanics:

If asked to work in a non-inertial frame, just say "NO."

1.3 Lorentz Invariance: The Principle of Special Relativity

We state the Principle of Special Relativity as:

The Laws of Physics are the same in all inertial frames, i.e., there are no preferred inertial frames of reference.

Let's take this for now to mean that the mathematical description of motion, for example the equations of motion for a particle, are the same in all inertial frames. The transformation of quantities from one inertial frame to another is called a Lorentz transformation, and invariance under a Lorentz transformation we will take to be a requirement to be a Law of Physics.

1.3.1 The Lorentz Invariance of Electric Charge and the Speed of Light

You may have been taught that the Principle of Relativity is "The speed of light is the same for all observers" [9]. Remarkably, electromagnetic waves (e.g., light), gravitational waves, and massive elementary particles all share the same value of c. We habitually call it "The Speed of Light," but it is a much more general phenomenon than purely electro-magnetic, with c being the ratio of length in space to length in time [6]. We have a highly parochial view of the Universe, limiting it to the very small fraction[2] that interacts electromagnetically [5].

1.4 Einstein's Three Gedanken Experiments

To describe motion of an object we need to answer the question *motion with respect to what?* We define a "Frame of Reference" as the surroundings in which position

[2] Known matter is only 5%. More than 98% of that 5% is in the binding energy of nucleons (protons and neutrons) that we cannot observe directly.

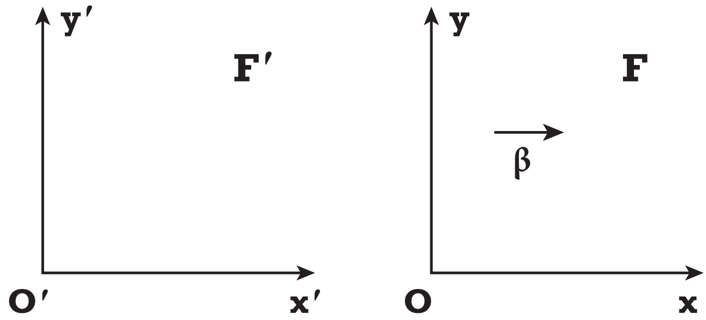

Figure 1.2. The frame convention is that frame F, described by unprimed variables, is moving to the right in frame F', described by primed variables, at velocity $\beta \equiv v/c$. The two origins coincide at $t' = t = 0$. To convert to the alternative convention in which frame F' is moving to the left in frame F, we have only to change β to $-\beta$ in the transformation equations.

and motion are measured. The relative positions of all points in the frame are assumed to be fixed, and are quantified in a coordinate system with an origin, coordinate axes, and scales (the units along an axis). The interior of the train and the station platform in the following Einstein thought experiments below are examples of two frames moving relative to each other (Figure 1.2).

Einstein illustrated the effects of the constant value of the speed of light as measured in different inertial frames through three Gedanken (thought) Experiments [7], each consisting of a train moving at constant speed past a station platform (or vice versa, if you are on the train).[3] Mr. Casals is stationary on the train, which is consequently his frame of reference,[4] denoted as F. Mr. Primrose is standing on the platform near the track; we will denote the platform and station as the primed frame, F'.

We will take the speed of the train in frame F' as $+v$ along the x-axis, i.e., frame F is moving to the right in frame F' at velocity v, and frame F' is moving to the left in frame F, i.e., at velocity $-v$ as shown in Fig. 1.2. For simplicity we assume the two origins in time, t and t', and in space, x and x', of the coordinate systems coincide, i.e., $x' = x = 0$ when $t' = t = 0$.

1.4.1 First Gedanken Experiment: Time Dilation

In the first thought experiment, Casals is traveling on a train. He has constructed a "clock" from a flashing light source on one side of the train and a mirror mounted

[3] Einstein is apocryphally said to have asked a train conductor, "Excuse me, could you please tell me when Zurich will arrive?"

[4] Our convention assigns the unprimed frame to the untransformed system in which the initial events occur, and the prime to the transformed frame. To use the opposite convention, replace β with $-\beta$ everywhere. Note that the Lorentz factor γ, which will be introduced shortly in Section 1.4.1, is unchanged. For rotations, the convention corresponds to rotating the coordinate system rather than rotating the vector.

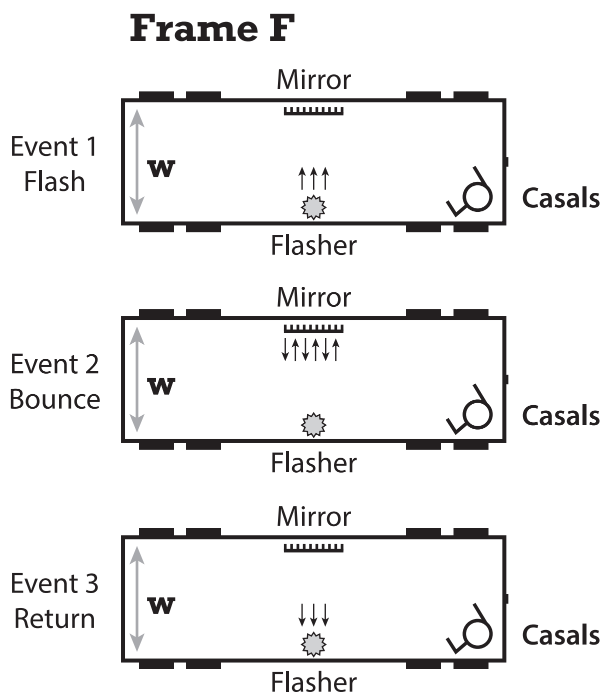

Figure 1.3. Gedanken Experiment 1, the measurement of time, as recorded by Casals. In his reference frame, F, a tick of the clock corresponds to a cycle of 3 events, shown in the 3 panels: (1) A flash of light at the source; (2) the bounce of the light at the mirror; and (3) the return of the light to the source, initiating another flash. The time between ticks of the clock is twice the light travel time across the train.

directly across the train from the light, as shown in Figure 1.3. A "tick" of the clock corresponds to a cycle of three events: (1) The source flashes a short pulse of light; (2) the light bounces off the mirror on the other side of the train; and (3) the light initiates another flash at the source after returning across the train. The clock continuously cycles, with Casals observing the flashes of the clock as his basis for measuring time.

The time it takes light to travel across the train is the width of the train, w, divided by the velocity of light, c. To go and return takes twice that, so the period (time between ticks) of the clock according to Casals is t = 2w/c.

Primrose, however, has a different story, as shown in Figure 1.4. He agrees that the sequence starts when the light flashes. However, the train is moving at velocity v, and so when the light arrives at the mirror, the mirror is not directly across from the light, but has moved along the x'-axis by a distance $x' = vt'$. The distance traveled by the light consequently is longer than w/c. Since light travels at the same velocity in Primrose's frame as in Casals' frame, the time between ticks will be longer, i.e., Primrose records the clock running slower than does Casals. Time is "dilated" in his frame—the separation in time between events in frame F, recorded by Casals, is recorded as longer in frame F' by Primrose.

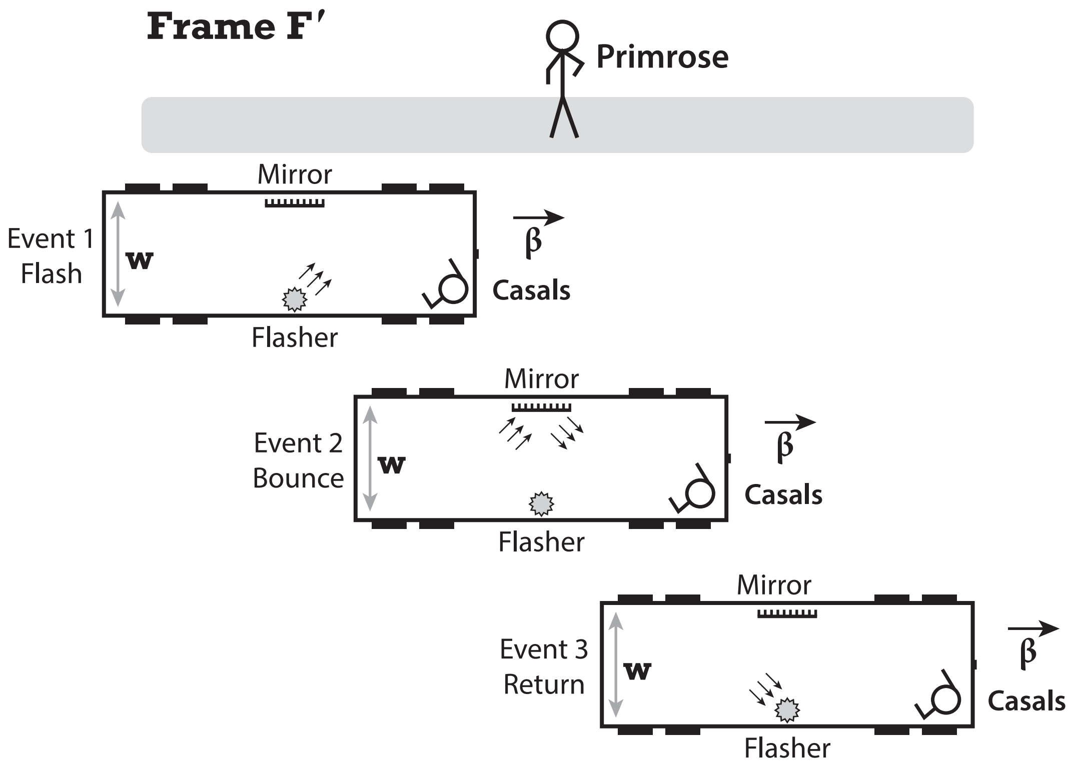

Figure 1.4. Gedanken Experiment 1 as seen in Primrose's frame on the station platform as the train goes by. Primrose records the same set of events as Casals: Event 1 is a flash; Event 2 is the bounce from the mirror; and Event 3 is another flash when the light returns to the source. However, the events occur at different places and times in Primrose's frame than in Casals' frame, in which for example, Events 1 and 3 occur at the same space point.

Casals and Primrose agree that there is a sequence of periodic flashes, but they differ on where and when in their respective frames the flashes occur. We can make this quantitative as follows.

In Figure 1.2 we defined two frames of reference each with a coordinate system: Casals observes life (lives!) in frame F, and Primrose observes life in frame F'. For each of them we will define an "event" as a point in time and space in their reference frame, concisely written as 4 numbers. For example, each flash is an event. In Casals' frame, F is unprimed, and hence a flash has unprimed coordinates (t, x, y, z). In Primrose's frame, a flash occurs at (t', x', y', z').

Figure 1.5 shows the "clock" made from the flasher and mirror in Primrose's frame F'. Let's define t' to be the time it takes the flash to get to the mirror.[5] The distance traveled by the light is ct'. Applying Pythagoras' Theorem to the right triangle (Fig. 1.5) formed by the train width w as one side, the distance ct' traveled by the light at velocity c in the time t' as the hypotenuse, and the distance vt' traveled

[5] The time between ticks of Primrose's clock will be twice this, as the light has to go across and come back.

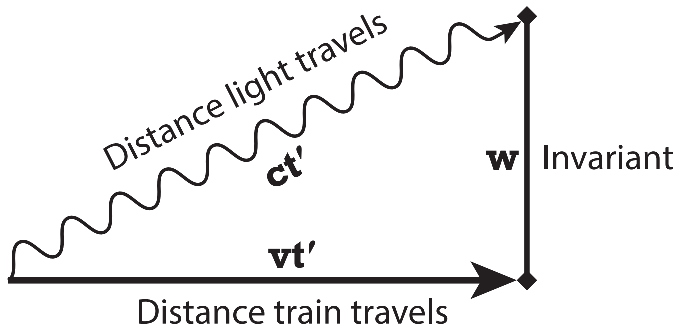

Figure 1.5. The right triangle in Primrose's frame formed by the train width w as one side, the distance ct' traveled by the light in the time t' it takes to get to the mirror as the hypotenuse, and the distance vt' traveled by the train along the x-axis in the time t'. By Pythagoras's Theorem (see Eq. 1.1), the time between flashes seen in Primrose's frame is longer than that in Casals' frame by the Lorentz factor $\gamma = \frac{1}{\sqrt{(1-(v/c)^2)}}$.

by the train in the time t' as the other side.

$$
\begin{aligned}
(ct')^2 &= w^2 + (vt')^2 \\
(ct')^2 - (vt')^2 &= w^2 \\
(t')^2\ (c^2 - v^2) &= w^2 \\
(t')^2 &= \frac{w^2}{(c^2 - v^2)} \\
(t')^2 &= \frac{(w/c)^2}{(1 - (v/c)^2)} \qquad (1.1) \\
t' &= \frac{(w/c)}{\sqrt{1 - (v/c)^2}} \\
t' &= \gamma \frac{w}{c}
\end{aligned}
$$

And thus Time Dilation $t' = \gamma t$

where γ (gamma) is the Lorentz factor,

$$\gamma \equiv \frac{1}{\sqrt{1 - (v/c)^2}}. \qquad (1.2)$$

Primrose sees the 'clock' of regular flashes running more slowly (e.g., a second takes *longer*) than Casals does by the factor of γ. This effect is traditionally referred to as Time Dilation; time intervals in the moving frame F are measured as "dilated," i.e., lengthened in the laboratory frame F'.

The Lorentz factor γ corresponds to the ratio of the hypotenuse to the side transverse to the direction of motion in the Pythagorean triangle of Figure 1.5. Since the hypotenuse cannot be shorter than a side, the Lorentz factor γ is always equal to or greater than 1. Because the ratio is to the transverse side, which is invariant, the Lorentz factor is (in principle) unbounded from above.

Note that if the speed of light were infinite as it is in the non-relativistic approximation, both Casals and Primrose would measure the time between ticks of their clocks as zero, i.e., all of the events would be simultaneous.[6]

1.4.2 Second Gedanken Experiment: Lorentz Contraction

In the second Gedanken Experiment, the train is again moving at velocity v relative to a station platform. However this time the light travels the length of the train from the flasher at the rear to the mirror at the front, where it is reflected back to the flasher, as shown in Fig. 1.6. On arrival of the light at the flasher, the cycle repeats, making a “clock” as in the first Gedanken Experiment. However, the difference is that the light travels along the direction of motion of the train rather than transverse to it. Casals is on the train[7] and measures the time between flashes and the length of the train L between the flasher and the mirror. As before we call Casals’ frame F.

Primrose is on the station platform as shown in Fig. 1.7; we call this the lab (for laboratory) frame, and denote it by F'. We will calculate the length of the train, L', measured by Primrose in F' relative to the length L measured in frame F by Casals.

The first step is to calculate the time in both frames. In F, the time from the flasher to the mirror is $t_{out} = L/c$. The time back from the mirror to the flasher is $t_{back} = L/c$. The total time for one cycle is $t_{Tot} = 2L/c$.

As seen by Primrose in F', however, the train moves as the light is traveling, and so the distance from the flasher to the mirror is longer. Similarly the distance back from the mirror to the flasher will be shorter. Calculating the total time as the sum of the time to go out and the time to come back:

$$\text{Time out}: \quad t'_o = (L' + vt'_o)/c$$
$$\text{Time back}: \quad t'_b = (L' - vt'_b)/c$$

$$\text{Solving for } t'_o: \quad ct'_o = (L' + vt'_o)$$
$$(c - v)t'_o = L'$$
$$t'_o = \frac{L'}{(c - v)}$$

[6] If this seems obvious, you have already acquired a relativistic intuition.
[7] It doesn’t matter where Casals is in the train, as long as he is not moving in it.

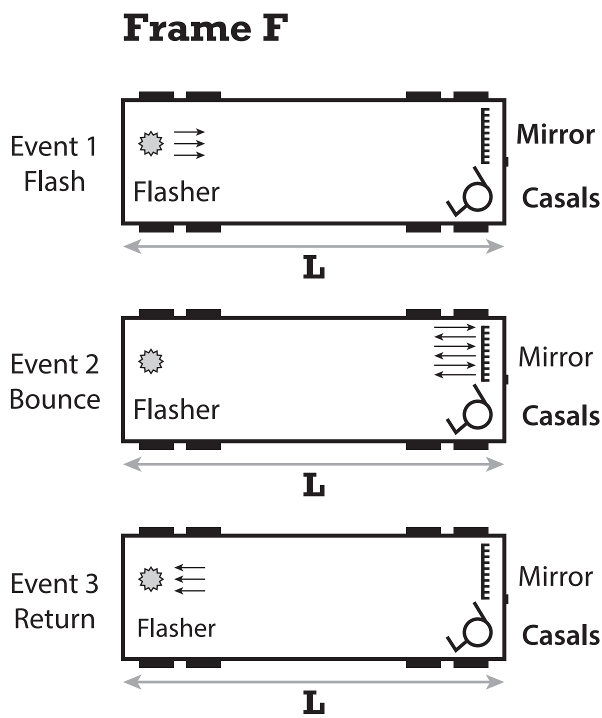

Figure 1.6. Gedanken Experiment 2, as recorded by Casals in his reference frame *F*. In Gedanken 2 the light travels down-and-back along the direction of motion of the train rather than across it. The 3 events that make up one cycle of the "clock," shown in the 3 panels, are: (1) a flash of light at the source; (2) the bounce of the light at the mirror; and (3) the return of the light to the source, initiating another flash. In frame F, the events occur at fixed locations, i.e., the flasher and the mirror are not moving. The time between flashes is twice the time it takes light to travel the length of the train between the source and the mirror.

$$\text{Solving for } t'_b: \quad ct'_b = (L' - vt'_b)$$

$$(c+v)t'_b = L'$$

$$t'_b = \frac{L'}{(c+v)} \tag{1.3}$$

$$\text{Total time}: \quad t'_T = \frac{L'}{(c-v)} + \frac{L'}{(c+v)}$$

$$= \frac{L'[(c+v)+(c-v)]}{(c^2-v^2)}$$

$$= \frac{2L'c}{(c^2-v^2)}$$

$$= \frac{2L'c}{c^2(1-v^2/c^2)}$$

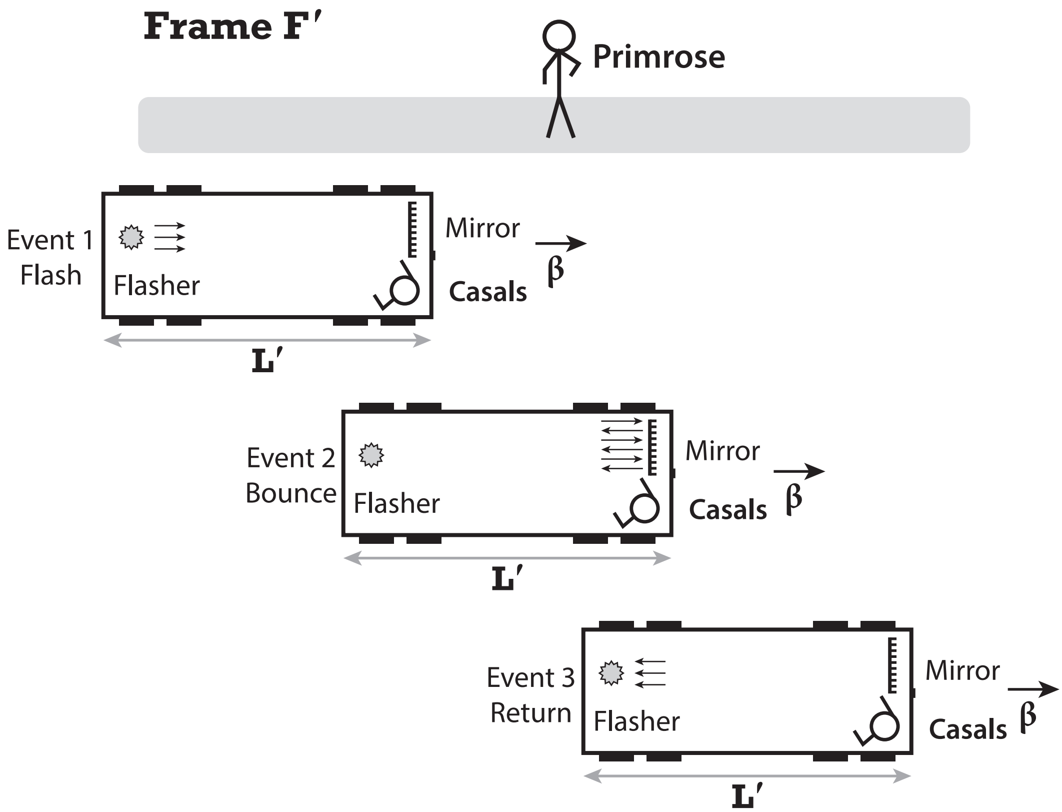

Figure 1.7. Gedanken Experiment 2 as recorded in frame F' by Primrose on the station platform. Top image, Event 1: the source emits a light flash in the direction of the train's motion toward the mirror, which is moving away from it. Middle image, Event 2: the light flash bounces off the mirror, which has moved away during the light transit time, and heads back toward the source, which is moving toward it. Bottom image, Event 3: the flash returns to the flasher, which has moved toward it. Both the times *and* space positions of the events recorded by Primrose in F' differ from those recorded by Casals in F.

$$= \frac{2L'}{c(1 - v^2/c^2)}$$

$$t'_T = \frac{2L'}{c}\gamma^2$$

$$\text{Solving for } L' \text{ in terms of } t'_T: \quad L' = \frac{c}{2\gamma^2}t'_T$$

where γ is the Lorentz factor.

However,[8] from Gedanken 1 we found that the time interval in F' was a factor of γ larger than in F:

[8] This is the final step that I always have to go back to my notes to remember.

$$t'_T = \gamma t_T = \gamma(2L/c).$$

$$\text{Substituting } t'_T \text{ into Eq.1.3}: \quad L' = \frac{c}{2\gamma^2} t'_T = \frac{c}{2\gamma^2}\gamma(2L/c) \qquad (1.4)$$

$$\text{And thus Lorentz Contraction}: \quad L' = L/\gamma.$$

The length of the train in F' is shorter than in F by a factor of γ. This is Lorentz Contraction. Time intervals in F' are longer by a factor of γ (Time Dilation); space intervals along the direction of motion are shorter by a factor of γ (Lorentz Contraction).

What about lengths measured perpendicular to the direction of motion? These are invariant (unchanged) under the transformation between frames. Why are these different from lengths along the direction of motion? In the direction of motion *both* the times and positions of the end-points of the space interval between the two events on the moving train (in this case a light flash and a bounce) are different in the two frames. For the transverse measurement, Casals and Primrose can arrange a simultaneous measurement of the width of the train by, for example, placing metal electrical contacts with one pair jutting out from the train and the other pair on fixed posts on both sides of the tracks, respectively. The two sets of contacts will make contact at the same time and place. They can measure the distance between the contacts in their own frame, and so will agree on the width of the train.

More dramatically, there's a proof by contradiction. Suppose transverse dimensions are shrunk by a factor of γ. Consider a train approaching a tunnel that is narrower than the train. Casals isn't worried, as he says, "Not a problem—I'll slow down and the tunnel will get wider." Primrose, on the other hand, is jumping up and down shouting, "Pablo, for Heaven's sake speed up! The train needs to be narrower!" Whether or not the train crashes would definitively identify either F' or F as a preferred frame. Happily, lengths transverse to the motion are invariant.

1.4.3 Third Gedanken Experiment: Simultaneity Is Not Lorentz Invariant

The third of Einstein's Gedanken Experiments demonstrates that two events measured as being simultaneous in a frame F are typically not simultaneous in a frame F' moving with respect to F, due to the speed of light being finite and equal in the two frames. Because of the motion of F' with respect to F, the simultaneous arrival of light from two events in F requires the two events to be an equal distance from Casals. However Primrose has a different story as we will discuss below.

We have set up the experiment here, but have assigned the solution to the Problem Set. Figure 1.8 shows the same moving train and platform as in the previous two Gedanken Experiments. Casals is now in the center of the train, standing at the window on the side closest to the platform. There are two flashers, one at each end of the train. The flashers are synchronized by Casals so that they flash simultaneously; the flash from each consequently simultaneously arrives at Casals at time $t = L/2c$.

Primrose is standing on the edge of the platform so that his head and Casals' head are very close at the moment when the flashes arrive. Consequently Primrose also sees two simultaneous flashes, one from the rear of the train and one from the front, at the same time as Casals. However, his story is that since the train was moving while the flashes were traveling, to arrive at the same time the flash at the rear of the train had to be earlier than the flash at the front, since the rear was further away and the front closer to the place where he and Casals both saw both flashes. The two flashes are simultaneous in frame F and *not* simultaneous in F'.

1.4.4 The Velocity β and Lorentz Factor γ; Identities and Generalizing to 3-Dimensions

The velocity β and Lorentz factor γ occur in so many situations that it is useful to write out the identities between them so that if given β one can find γ and vice versa. We also give the approximation for β in the limit of large γ (See Section A.1.3 of Appendix A).

$$\begin{aligned} \beta &\equiv v/c \\ \gamma &\equiv \frac{1}{\sqrt{1-\beta^2}} \\ \gamma^2 &= \frac{1}{1-\beta^2} \\ \beta^2 &= \frac{\gamma^2-1}{\gamma^2} \end{aligned} \tag{1.5}$$

$$\text{For } \gamma >> 1, \quad \beta \approx 1 - \frac{1}{2\gamma^2}.$$

In 3 dimensions, each of x, y, and z have their respective velocities $\beta_x, \beta_y, \beta_z$, and consequently their respective Lorentz factors $\gamma_x, \gamma_y, \gamma_z$. We will largely stick with uniform motion in one dimension, which always can be taken as along the x-axis.

1.5 A Coordinate System that Accounts for Light Travel Time

Einstein constructed his three Gedanken Experiments as brilliant pedagogy for the lay public. However, we will now leave them to develop a precise mathematical

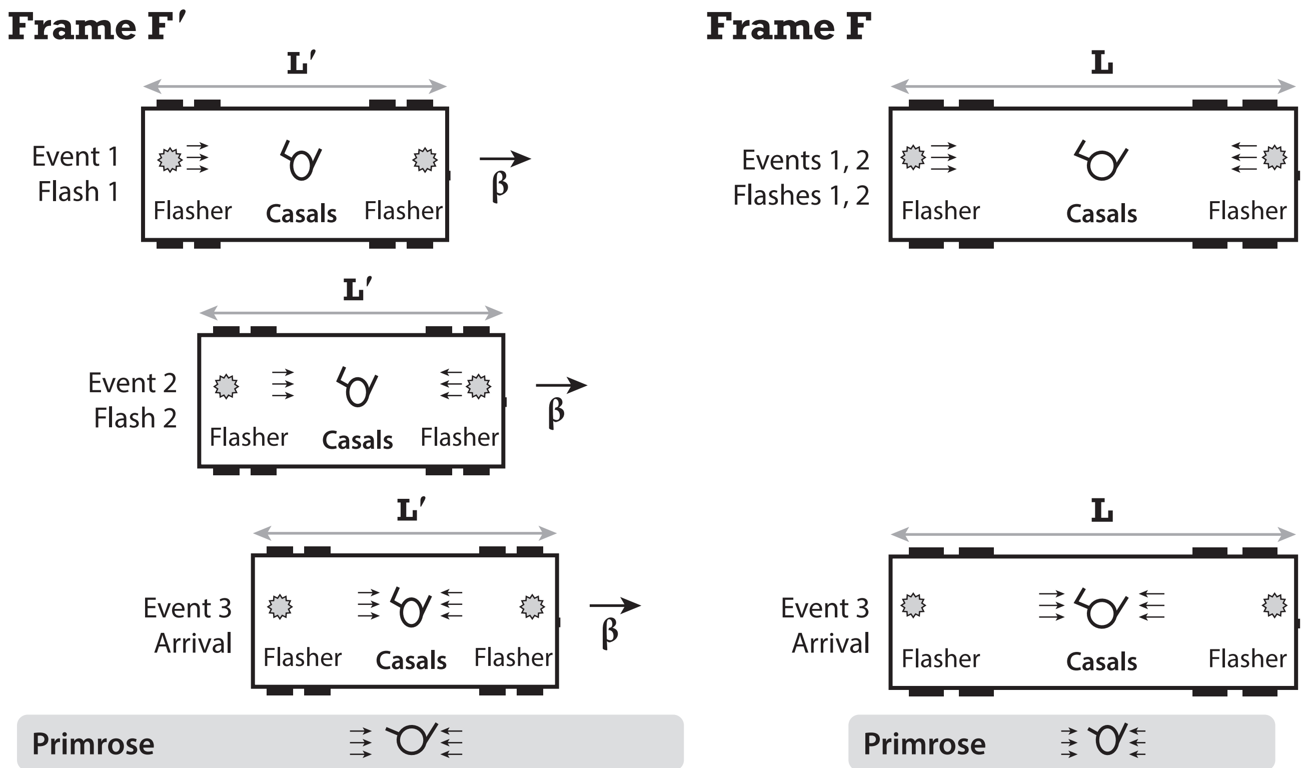

Figure 1.8. Gedanken Experiment 3. Right: Events recorded in frame F. Casals, who is on the train, puts a light at each end of the train, set to flash simultaneously. He verifies this by standing in the middle of the train and seeing the flashes arrive at the same time. Left: Events recorded in frame F'. Primrose is standing on the platform close to the tracks and sees the same two flashes simultaneously at the moment that Casals is directly opposite him. However, Primrose knows he has to take into account the distance the train traveled while the light was propagating. The light from the rear end of the train had to travel further than the light from the front end to arrive at the same time. He concludes that the rear light flashed before the front light, i.e., the two flashes were not simultaneous.

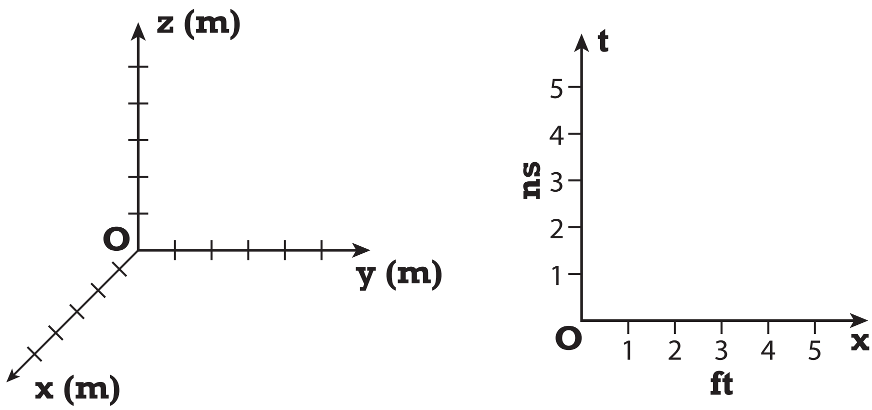

Figure 1.9. The conventions for Cartesian coordinates. Left: in 3-dimensions, a right-handed system with the positive z-axis up, x out of the page, and y to the right. Right: The 2-dimensional xt plane. In both panels note the labeling of the origin, axes, scales, and units.

language that allows us to transform any event, i.e., a time and location, from one frame into another frame.

Once we have the events in the new frame it is straightforward to calculate the intervals in space and time between one event and another event. We will find that there are invariant lengths, numerically the same in both frames. Wonderfully, the same mathematical framework applies directly to the transformation of energy and momentum from one frame to another, and to the transformations for electric charge and the Coulomb potential of classical Electromagnetism.

1.5.1 A Cartesian Coordinate System in 3-Space

Kinematics is the science of motion in the absence of applied force. To be quantitative, we need a coordinate system consisting of axes that span the space (i.e., every point in the space can be reached), and scales on the axes that provide a numerical value of the position along each axis. We will work initially in Cartesian coordinates, with three orthogonal axes. Figure 1.9 shows our conventions: a right-handed system[9] with the positive z-axis up; x out of the page; and y to the right.

1.5.2 Extending the 3D Coordinate System to Account for Light Travel Time

We have seen in the three Einstein Gedanken experiments that both the time and the place of events are needed to characterize what is seen in a given frame. In

[9] See Section A.1.1.3 of Appendix A for a definition of "right-handed" in vector notation.

3-dimensions vector notation provides a remarkably powerful calculational framework.[10] The fixed ratio of travel time to distance in every inertial frame, which we denote by the letter c, allows the incorporation of time into a dimensionally-correct vector framework.

1.5.3 The 4-Vector $x^\mu = (ct, x, y, z)$

We will define the time and place of an event in a given frame by a "4-vector" x^μ:

$$x^\mu = (ct, x, y, z) \tag{1.6}$$

where μ is an index[11] that runs from 0 to 3, with x^0 being ct, and x^i being x,y,z for $i = 1, 2, 3$, respectively.[12]

1.5.4 The Invariant Length of a 4-Vector

We saw in the first Einstein Gedanken experiment that the times and places of the events—the flashes of light and the bounce from the mirror—were different for Casals and Primrose. However, we can define a "distance" in 4-dimensions that they agree on, i.e., is invariant under the transformation from one frame to another. We call this the "invariant length" of the 4-vector.[13] It corresponds to the side transverse to the direction of motion; both Casals and Primrose agree on the length of the side. The relationship of the sides of the triangle in Figure 1.5 is given by the Pythagorean Theorem: the square of the hypotenuse is equal to the sum of the squares of the two sides. The square of the transverse side is thus the hypotenuse squared minus the square of the side in the direction of travel. Referring to Figure 1.5, in 4-vector notation, both Casals and Primrose agree on the invariant length squared:

$$|x^\mu|^2 = (ct)^2 - x^2. \tag{1.7}$$

More generally,

$$|x^\mu|^2 = (ct)^2 - x^2 - y^2 - z^2. \tag{1.8}$$

The invariant length formally is the square root of this; however, square roots are a pain for a quantity that can be positive or negative, and so I usually refer to the square as the invariant length, with the understanding that to get the correct numerical value and units one should take the square root.

[10] For examples see Section A.1.1 of Appendix A.

[11] For practice with indices see Problem 2.

[12] There are many conventions for 4-vectors; the choice doesn't matter as long as one is consistent. Another convention we are using includes using Greek letters for 4-vectors (μ is the Greek "m" (pronounced *myou* rather than the bovine *moo*) and Roman letters for 3-vectors, for example i, j, and k. If you don't know the Greek alphabet now would be a good time to familiarize yourself with at least some of it.

[13] The following treatment is adequate for our purposes but ignores the (unnecessary here) complexity of covariant and contravariant vectors. However, a good reference is Ref. [12] if one feels the need.

1.5.5 The Non-Cartesian Metric: The Minus Sign

The 4-dimensional space of time and 3-space dimensions is quite different from the Cartesian 3-dimensional space we wander in. One obvious difference is that in space one can go back to where one came from; one cannot go back in time. A related feature of the world is that not all points in 4-space are "reachable" from a given point; the finite velocity of light restricts communication to points for which the value of $|x^\mu|^2 = (ct)^2 - x^2 - y^2 - z^2$ is greater than or equal to zero. For these points there is time enough for light to travel, i.e., one can communicate. For negative values, the distance in space is larger than the light travel time, and so one cannot. These two cases are respectively called "time-like" ($(ct)^2 > |\vec{x}|^2$), i.e., the time difference is larger than the space distance, and "space-like," the spatial distance is the larger. When the two terms are equal, light from the earlier reaches the later; the surface of these points in 4-space is called "the light cone." See Problem 3 of this chapter.

1.6 Lorentz Transformations Between Frames

We close the chapter on Einstein's heuristic Gedanken Experiments with an introduction to Lorentz transformations, in this case transforming the events in Casals' frame to Primrose's frame.[14] In the next chapters we will develop a more elegant and powerful mathematical language to be able to address the transformations of energy and momentum as well as of time and space.

1.6.1 The Transformation Equations From Casals' Frame to Primrose's Frame

Consider an event in one spatial dimension in Casals' frame F, specified by position x and time t. The events as measured by Primrose in frame F' are specified by:

$$\begin{aligned} ct' &= \gamma\ (ct)\ + \beta\gamma x \\ x' &= \beta\gamma\ (ct) + \ \gamma x. \end{aligned} \tag{1.9}$$

1.6.2 Conventions for the Units of Time and Space

Units, to be frank, can be painful, and most texts spend far too much time on them. Here we address the units of time and space.

We will predominantly use two systems: 1) the International System (SI), also known as MKS, for meters, kilograms, and seconds. In SI, the unit of time is the

[14] We omit derivations of Lorentz transformations beyond the heuristic demonstrations. However, see Problem 4 for a "derivation" of two of the matrix elements and one constraint on the matrix.

second and the unit of space is the meter. Each of these has developed historically and independently; consequently in SI the constant of proportionality c in the 4-vector $(ct, \vec{x})$ has units of meters/seconds.[15] For purely historical reasons the conversion constant between distances in time (in seconds) and space (in meters) has the numerical value 3.0×10^8.[16]

In Natural Units the units of length and time are chosen so that the velocity of light is identically equal to 1, $c \equiv 1$. This may be familiar; astronomers have long chosen the unit of time Δt to be a year, and the unit of distance in space Δl to be a light-year, the distance light travels in one year. The conversion factor c for the speed of light in these units is then unity by definition: $c = \Delta l/\Delta t \equiv 1$.

Working particle physicists exploit an approximate relationship between 1 foot as the unit of distance and 1 nanosecond (10^{-9}s—i.e., one-billionth of a second), giving a value[17] for c within 2% of 1. We consequently will work in nsec and feet for most terrestrial relativistic problems. Note that with $c = 1$ you can measure time in feet or length in nsec; an object 6 feet away is also 6 nsec of light-travel time away, meaning that the light from it had to leave that much earlier to arrive at the same time as light from a nearby object. You are always seeing in the past while awaiting the future.[18]

1.6.3 Putting the Factors of *c* Back in by Dimensional Analysis

We will use natural units for relativistic problems such as occur naturally in particle physics, cosmology, and astronomy. Note that since $c = 1$, expressions such as v/c and ct become v and t, respectively. "Okay," you say; "it's much cleaner, but how to know where to put the c's back in after you have finished a calculation and want to convert to SI units?" It becomes natural from the context. If you have a t in an expression for length, you need to make it ct. If you have a v in an expression that is dimensionless (i.e., not a length or a time, as occurs in γ), you can make the dimensionless velocity β by dividing by c: $\beta \equiv v/c$.

As an example, the transformation of an event in Casals' frame F to Primrose's frame F' of Eq. 1.5 is given in natural units by:

$$\begin{aligned} t' &= \gamma t + \beta\gamma x \\ x' &= \beta\gamma t + \gamma x. \end{aligned} \tag{1.10}$$

In summary, in natural units $c \equiv 1$ and we will not write it explicitly when it is a multiplier or divisor. If you want to convert back to SI, wherever there is a t as a distance multiply by $c = 3 \times 10^8$ m/s to convert seconds to meters, and likewise wherever there is a β multiply by c to get v in m/s. Not hard.

[15] So that when the time in seconds is multiplied by c one gets the number in meters.

[16] We work to 2 significant figures, one more than is actually needed here.

[17] One foot is 30.48 cm; light travels 29.98 cm in 1 nsec; the ratio is 1.017. Working to two significant figures, we take c as 1.0.

[18] Which may already have left its source and be on its way.

1.7 Problem Set 1: Vectors, Time Dilation, Lorentz Contraction, Simultaneity, and the Lorentz Transformation

Time Management and Study Groups: You need to work with your study group. The problem sets will go faster if you discuss the problems, with friends/colleagues, and you will have a deeper understanding. However, the work you hand in **has to be your own.**[19]

Problems with answers, and recycled problems: There are a limited number of easily-solved mechanics problems, and so one can find answers to most by searching on the web. We trust you to instead work them yourself; ask your fellow students, the TA's, and/or your instructor for help if you need it. Browsing other texts is recommended; however, you should write out the solution with the book closed.

Getting help: Yes, if you need help ask for it. Bring your study group with you. Do it more than once if needed.

Formulae: For the velocity β and the Lorentz factor γ:

$$\beta = v/c; \quad \gamma^2 = 1/(1-\beta^2); \quad \beta^2 = (\gamma^2-1)/\gamma^2; \quad For \ \ \gamma >> 1, \quad \beta \approx 1 - \frac{1}{2\gamma^2}.$$

The invariant length of the 4-vector x_0, x_1, x_2, x_3: $|x^\mu| = \sqrt{x_0^2 - x_1^2 - x_2^2 - x_3^2}$. Lorentz transformation for a "Boost" of frame F along the x direction relative to frame F':

$$t' = \gamma t + \beta\gamma x \qquad (1.11)$$

$$x' = \beta\gamma t + \gamma x \qquad (1.12)$$

$$y' = y \qquad (1.13)$$

$$z' = z \qquad (1.14)$$

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}. \qquad (1.15)$$

Problems: Solutions will be provided.[20] *Please do not plug in any numerical values until the end*. In Problems 1 and 2 not all parts need be assigned if the set is deemed too long.

[19] I once required two students who inadvertently strayed to read Egil Krogh's book *Integrity* (see Skills and Guidelines).

[20] Having high quality solutions available at the problem set submission deadline is essential feedback. Do not settle for less.

Problem 1: Practice with 3-Vectors[21]

Consider the two vectors $\vec{A} = (-3, 1, -2)$ and $\vec{B} = (2, -2, 3)$ respectively:

1. Calculate the length of $\vec{A}$; (don't bother explicitly taking the square root, it's quicker to leave the length squared under the sqrt sign);
2. Calculate the length of $\vec{A} + \vec{B}$;
3. Calculate the length of $\vec{A} - \vec{B}$;
4. Draw a diagram of the reference frame showing the x, y, and z axes and the position vectors $\vec{A}$ and $\vec{B}$;
5. On your diagram show $\vec{A} - \vec{B}$ and $\vec{A} + \vec{B}$;
6. Calculate $\vec{A} \cdot \vec{B}$;
7. Calculate the angle between $\vec{A}$ and $\vec{B}$;
8. Calculate the projection of $\vec{A}$ on $\vec{B}$;
9. Calculate $\vec{A} \times \vec{B}$;
10. Find $(\vec{A} \times \vec{B}) \cdot \vec{A}$;
11. Find $(\vec{A} \times \vec{B}) \times (\vec{A} \times \vec{B})$.

Problem 2: Indices and Conventions

1. Define "Index" in the context of vectors and matrices and write down examples with 0, 1, 2, 3, and 4 indices, respectively (not trivial—discuss with your group).
2. Prepare a 2-minute semi-formal talk for your study group on what an index is and isn't. If you use Powerpoint or equivalent it should be no more than 1 slide.
3. Show that
$$\vec{A} \cdot \vec{B} = \sum_{i=1,3} A_i B_i. \tag{1.16}$$
4. Show that
$$\vec{A} \cdot \vec{B} = \sum_{i=1,3} \sum_{j=1,3} A_i B_j \delta_{ij}, \tag{1.17}$$
where δ_{ij} is the "Kronecker delta" (see Section A.1.8 of Appendix A.) If you are bothered or confused by the problem, write out all 9 terms.
5. Show that
$$(\vec{A} \times \vec{B})_i = A_j B_k - A_k B_j \tag{1.18}$$

[21] See Section A.1.1 of Appendix A for the scalar product. Also, if you are bold, Appendices A.1.8 and A.1.9 for the vector product and an elegant notation for both.

and cyclic ($i \to j \to k \to i$).

6. Show that

$$(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k \tag{1.19}$$

and cyclic ($i \to j \to k \to i$) where ϵ_{ijk} is the Levi-Civita tensor (see Section A.1.8 of Appendix A). If you are bothered or confused by the problem, write out all 27 terms.

Problem 3: 4-Vectors and the Invariant Length

1. Consider the 4-vectors $x^\mu = (t, x, y, z) = (13, 0, 12, 5)$, $(0, 3, -6, -5)$, and $(-6, 0, -3, -2)$ where time is measured in nsec and space coordinates in feet. What is the invariant length squared of each?
2. Consider two events at space-time points $A^\mu = (15, 4, -16, 7)$ and $B^\mu = (2, 4, -4, 2)$ respectively, where time is measured in nsec and space coordinates in feet. What is the invariant distance squared in space-time between them, $|B^\mu - A^\mu|^2$? What is the distance in space between them? In time?
3. Suppose event A happened at time $t_A = 16$ nsec rather than 15. What is the distance in space between A and B? In time?
4. Suppose event A happened at time $t_A = 14$ nsec rather than 15. What is the distance in space between A and B? In time?
5. In each of the above three examples can event A cause event B?

Problem 4: The Lorentz Transformation of an Event in Space-Time

The Lorentz transformation for a boost of an event in frame F along the x direction relative to frame F' is given by[22]

$$t' = \gamma t + \beta\gamma x \tag{1.20}$$

$$x' = \beta\gamma t + \gamma x \tag{1.21}$$

where γ is the Lorentz factor $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, and β is the velocity in Natural Units, $\beta = v/c$.

1. Find the time t' in frame F' for an event at the location $(t, 0)$ (i.e., at time t at the origin) in frame F. Which of the Einstein Gedanken experiments does this

[22] We (naturally) work in natural units (NU). Distances in the first coordinate are measured in nanoseconds (10^{-9} seconds); distances in the next 3 coordinates are measured in feet. The speed of light (good enough for government work) is 1 ft/nsec, i.e., $c = 1$. If this troubles you, put a "c" next to every "t" where $c = 3 \times 10^8\ m/sec$, and work in SI units. You will get over it.

correspond to, and to which special point in frame F does $x = 0$ correspond? (A trivial question, but meaningful.)

2. Find the location x' in frame F' for an event at the location $(t, 0)$ (i.e., at time t at the origin) in frame F. Please parse this in terms of distance = velocity times time.

3. The structure of the transformation matrix for the Lorentz transformation, represented by Eq. 1.20 with γ as the diagonal elements and $\beta\gamma$ as the off-diagonal elements, is not easy to wrap one's mind around. Show that the invariant length of the event 4-vector position is the same in frames F' and F, i.e.,

$$\begin{aligned} |x'^{\mu}|^2 &= |x^{\mu}|^2 \\ t'^2 - x'^2 &= t^2 - x^2. \end{aligned} \tag{1.22}$$

Problem 5: Cosmic Rays[23]

Consider a muon (a heavy cousin of the electron), identical in the form of its interactions with matter except for effects due to its being 200 times heavier,[24] created in the atmosphere by a cosmic ray coming from far away. Assume that in its own rest frame, this individual muon has a lifetime of $\tau = 2200$ nanoseconds,[25] after which it decays to a muon neutrino and an electron/anti-neutrino pair. This muon is traveling with velocity $\beta = v/c = 0.9999995$ ($\gamma = 1000$) with respect to the Earth.

1. Draw a clear (not too small) diagram of the process in the muon rest frame and another diagram in your own frame. Be sure to label the respective origins and axes.
2. Write down the 4-vector for the decay point in the coordinate frame of the muon.
3. Starting with the value of β, calculate the Lorentz factor γ for the transformation from the muon frame to the Earth frame.
4. Lorentz transform the 4-vector representing the decay point in the muon frame to get the 4-vector for the decay point in the Earth's frame.
5. How long is the lifetime as measured in the Earth's frame?
6. How far did the muon travel from where it was created to where it decayed in the Earth's frame?
7. How far would the muon have traveled without the factor of γ?
8. Calculate the proper time (the invariant length of the 4-vector) from the coordinates of the decay event in both the muon and Earth's frame.

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[23] On a personal note, I recommend the (oldie) film: Time Dilation an Experiment with Mu Mesons 1962 PSSC David Frisch, James Smith, MIT. I found it on YouTube.

[24] I. I. Rabi (Columbia Univ.) is famously quoted as saying, "Who ordered *that?!*"

[25] The distribution in how long muons live goes as $e^{-\frac{t}{\tau}}$, where $\tau = 2200$ nanoseconds.

Problem 6: Time Dilation

Consider the first Einstein Gedanken Experiment. A simple clock is constructed on a *very* fast Chicago Metra Electric train by mounting an LED (light-emitting diode) and a photo-diode together inside the train on one wall, and a mirror on the wall across the train and directly opposite. The LED and photodiode are pointed at the mirror, and are electrically connected so that a short LED pulse reflected from the mirror triggers the photodiode to make the LED flash. The result is that the LED flashes repeatedly at a fixed interval that corresponds to twice the light transit time across the width of the train. The train is an Express, moving at $\beta = 0.99995$, ($\gamma = 100$) relative to the station.

1. Set up the problem and define the relevant events in the frame of the train. (Be sure to draw a well-labeled clear diagram.)
2. Transform the coordinates of each event into the frame of the station.
3. Draw a carefully-labeled diagram of the geometry of the light path in the frame of the station.
4. Find the time between flashes as seen in the frame of the station.
5. Find the distance between flashes as seen in the frame of the station.

Problem 7: Einstein Gedanken Experiment 3: The Frame-dependence of Simultaneity

Casals is on a train moving at speed corresponding to a Lorentz factor of $\gamma = 1000$ down a set of tracks past a platform on which Primrose is standing. Casals is in the middle of the train, i.e., equidistant from both ends. Just as Casals is opposite Primrose[26] each of them sees two simultaneous flashes of light that were produced by a light at each end of the train. Casals measures the length of the train to be L. Ignore the width of the train as the length is much longer than the width.

1. Draw a picture and label the frames and axes.
2. Taking the origins of the two coordinate systems and clocks to be the point where Primrose and Casals are when they see the flash, write down the 4-vector in Casals' reference frame corresponding to the position of each light when it flashed.
3. Use the Lorentz transformation to find the 4-vectors of each light when it flashed in Primrose's frame.
4. Primrose can calculate the length of the train from the following reasoning: the spatial separation of the two flashes is the distance the back of the train moved

[26] Take them to be so close as to effectively be at the same location.

while the light was propagating plus the length of the train. In symbols,

$$\Delta x' = \beta \Delta t' + L'. \tag{1.23}$$

Find the length of the train as measured by Primrose, L' in terms of L and γ. (Remember (learn) the identity $1/\gamma^2 = (1 - \beta^2)$).

5. Casals deduces that the two lights flashed simultaneously. In contrast, Primrose claims they had to have flashed at different times for him to have seen the flashes simultaneously. What is the time interval between the two lights flashing in Primrose's frame?" (The perils of translating into English—a better way to have asked is "Transform the two light-flashing events into Primrose's frame and find the time difference.")

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