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# CHAPTER I

# *Étale Morphisms*

A flat morphism is the algebraic analogue of a map whose fibers form a continuously varying family. For example, a surjective morphism of smooth varieties is flat if and only if all fibers have the same dimension. A finite morphism to a reduced scheme is flat if and only if, over any connected component, all fibers have the same number of points (counting multiplicities). The image of a flat morphism of finite-type is open, and flat morphisms that are surjective on the underlying spaces are epimorphisms in a very strong sense.

An étale morphism is a flat quasi-finite morphism $Y \to X$ with no ramification (that is, branch) points. Locally $Y$ is then defined by an equation $T^m + a_1 T^{m-1} + \cdots + a_m = 0$, where $a_1, \ldots, a_m$ are functions on an open subset $U$ of $X$ and all roots of the equation over a point of $U$ are simple. An étale morphism induces isomorphisms on the tangent spaces and so might be expected to be a local isomorphism. This is true over the complex numbers if local is meant in the sense of the complex topology, but the Zariski topology is too coarse for this to hold algebraically. However, an étale morphism induces an isomorphism on the completions of the local rings at a point where there is no residue field extension. Moreover, it has all the uniqueness properties of a local isomorphism.

A local scheme is Henselian if, for any scheme étale over it, any section of the closed fiber extends to a section of the scheme. It is strictly Henselian, or strictly local, if any scheme étale and faithfully flat over it has a section. The strictly local rings play the same role for the étale topology as local rings play for the Zariski topology.

The fundamental group of a scheme classifies finite étale coverings of it. For a smooth variety over the complex numbers, the algebraic fundamental group is simply the profinite completion of the topological fundamental group. There are algebraic analogues for many of the results on the topological fundamental group.

## §1. Finite and Quasi-Finite Morphisms

Recall that a morphism of schemes $f: Y \to X$ is *affine* if the inverse image of any open affine subset $U$ of $X$ is an open affine subset of $Y$. If, moreover, $\Gamma(f^{-1}(U), \mathcal{O}_Y)$ is a finite $\Gamma(U, \mathcal{O}_X)$-algebra for every such $U$, then $f$ is said to be *finite*. These conditions need only be checked for all $U$ in some open affine covering of $X$ (Mumford [3, III.1, Prop. 5]).

Examples of finite morphisms abound. Let $X$ be an integral scheme with field of rational functions $R(X)$, and let $L$ be a finite field extension of $R(X)$. The *normalization* of $X$ in $L$ is a pair $(X', f)$ where $X'$ is an integral scheme with $R(X') = L$ and $f: X' \to X$ is an affine morphism such that, for all open affines $U$ of $X$, $\Gamma(f^{-1}(U), \mathcal{O}_{X'})$ is the integral closure of $\Gamma(U, \mathcal{O}_X)$ in $L$.

PROPOSITION 1.1. *If $X$ is normal and $f: X' \to X$ is the normalization of $X$ in some finite separable extension of $R(X)$, then $f$ is finite.*

*Proof.* One has only to show that $\Gamma(f^{-1}(U), \mathcal{O}_{X'})$ is a finite $\Gamma(U, \mathcal{O}_X)$-algebra for $U$ an open affine in $X$, but this is done in Atiyah-Macdonald [1, 5.17].

*Remark* 1.2. The above proposition holds for many schemes $X$ without the separability assumption, for example, for reduced excellent schemes and so for varieties ([EGA. IV.7.8] and Bourbaki [2, V.3.2]). (A field is excellent and a Dedekind domain $A$ is excellent if the completion $\hat{K}$ of its field of fractions $K$ at any maximal ideal of $A$ is separable over $K$; any scheme of finite type over an excellent scheme is excellent.)

PROPOSITION 1.3. (a) *A closed immersion is finite.*

(b) *The composite of two finite morphisms is finite.*

(c) *Any base change of a finite morphism is finite, that is, if $f: Y \to X$ is finite, then so also is $f_{(X')}: Y_{(X')} \to X'$ for any morphism $X' \to X$.*

*Proof.* These reduce easily to statements about rings, all of which are obvious.

The "going up" theorem of Cohen-Seidenberg has the following geometric interpretation.

PROPOSITION 1.4. *Any finite morphism $f: Y \to X$ is proper, that is, it is separated, of finite-type, and universally closed.*

*Proof.* For any open affine covering $(U_i)$ of $X$, $f$ restricted to $f^{-1}(U_i) \to U_i$ is separated for all $i$, and so $f$ is separated. (Hartshorne [2, II.4.6]). To show that finite morphisms are universally closed it suffices, according to (1.3c), to show that they are closed, and for this it suffices, according to (1.3a,b), to show that they map the whole space onto a closed set. Thus we must show that $f(Y)$ is closed. This reduces easily to the affine case with, for example, $f = {}^a g$ where $g: A \to B$ is finite. Let $\mathfrak{J} = \ker(g)$. Then $f$ factors into $\operatorname{spec} B \to \operatorname{spec} A/\mathfrak{J} \to \operatorname{spec} A$. The first map is surjective (Atiyah-Macdonald [1, 5.10]), and the second is a closed immersion.

For morphisms $X \to \operatorname{spec} k$, with $k$ a field, there is a topological characterization of finiteness.

PROPOSITION 1.5. *Let $f: X \to \operatorname{spec} k$ be a morphism of finite-type with $k$ a field. The following are equivalent:*

(a) *$X$ is affine and $\Gamma(X, \mathcal{O}_X)$ is an Artin ring;*

(b) *$X$ is finite and discrete (as a topological space);*

(c) *$X$ is discrete;*

(d) *$f$ is finite.*

*Proof.* See Atiyah-Macdonald [1, Chapter VIII, especially exercises 2, 3, 4].

A morphism $f: Y \to X$ is *quasi-finite* if it is of finite-type and has finite fibers, that is, $f^{-1}(x)$ is discrete (and hence finite) for all $x \in X$. Similarly an $A$-algebra $B$ is *quasi-finite* if it is of finite-type and if $B \otimes_A k(\mathfrak{p})$ is a finite $k(\mathfrak{p})$-algebra for all prime ideals $\mathfrak{p} \subset A$.

*Exercise* 1.6. (a) Let $A$ be a discrete valuation ring. Show that $A[T]/(P(T))$ is a quasi-finite $A$-algebra if and only if some coefficient of $P(T)$ is a unit, and that it is finite if and only if the leading coefficient of $P(T)$ is a unit.

(b) Let $A$ be a Dedekind domain with field of fractions $K$. Show that $\operatorname{spec} K \to \operatorname{spec} A$ is never finite, that it is quasi-finite if it is of finite-type, and that it is of finite-type if and only if $A$ has only finitely many prime ideals.

PROPOSITION 1.7. (a) *Any immersion is quasi-finite.*

(b) *The composite of two quasi-finite morphisms is quasi-finite.*

(c) *Any base change of a quasi-finite morphism is quasi-finite.*

*Proof.* (a) Let $f: Y \to X$ be an immersion. Clearly $f$ has finite fibers, and to show that it is of finite-type it suffices to show that $f^{-1}(U)$ is quasi-compact for any open affine $U$ in $X$. But $U$ is a Noetherian topological space (recall that all schemes are locally Noetherian), and $f^{-1}(U) = U \cap Y$ is a subset of $U$.

(b) This is obvious.

(c) Let $f: Y \to X$ be quasi-finite and $X' \to X$ arbitrary. If $x' \mapsto x$ under $X' \to X$, then the fiber

$$f_{(X')}^{-1}(x') = f^{-1}(x) \otimes_{k(x)} k(x')$$

and hence is discrete.

If $f: Y \to X$ is finite and $U$ is an open subscheme of $Y$, then it follows from the above proposition that $U \to X$ is quasi-finite. The remarkable thing is that essentially every quasi-finite morphism comes in this way.

THEOREM 1.8. (Zariski's Main Theorem). *If $X$ is a quasi-compact, then any separated, quasi-finite morphism $f: Y \to X$ factors as $Y \xrightarrow{f'} Y' \xrightarrow{g} X$ where $f'$ is an open immersion and $g$ is finite.*

*Proof*. The most elementary proof may be found in Raynaud [3, Chapter IV]. We sketch the deduction of (1.8) from the following affine form of it, proved in Raynaud [3, p. 42]: let $B$ be an $A$-algebra that is quasi-finite, and let $A'$ be the integral closure of $A$ in $B$; then the map spec $B \to$ spec $A'$ is an open immersion.

Consider a scheme $X$. Associated with any quasi-coherent sheaf $A$ of $\mathcal{O}_X$-algebras, there is a pair $(X', g)$ where $X'$ is a scheme and $g: X' \to X$ is an affine morphism such that $g_* \mathcal{O}_{X'} = A$ (Hartshorne [2, II. Ex. 5.17] and [EGA. I.9.1.4]). One writes $X' = \mathbf{spec}\ A$. For any $X$-scheme $Y \xrightarrow{f} X$, to give an $X$-morphism $Y \to X'$ is the same as to give a homomorphism $A \to f_* \mathcal{O}_Y$ of $\mathcal{O}_X$-algebras.

Now let $f: Y \to X$ be separated and of finite-type. Then $f_* \mathcal{O}_Y$ is a quasi-coherent $\mathcal{O}_X$-algebra [EGA. I.6.7.1], and the $\mathcal{O}_X$-algebra $A'$ such that $\Gamma(U, A')$ is the integral closure of $\Gamma(U, \mathcal{O}_X)$ in $\Gamma(U, f_* \mathcal{O}_Y)$ for all open affines $U \subset X$ is also quasi-coherent [EGA. II.6.3.4]. The associated $X$-scheme $X' = \mathbf{spec}\ A'$ is called the normalization of $X$ in $Y$.

Assume further that $f$ is quasi-finite. It follows easily from the affine result quoted above, that the morphism $Y \to X'$ induced by the inclusion $A' \subset f_* \mathcal{O}_Y$ is an open immersion. Now let $(A_i)$ be the family of all coherent $\mathcal{O}_X$-subalgebras of $A'$. One checks easily that the morphism $Y \to \mathbf{spec}\ A_i$, induced by the inclusion $A_i \subset f_* \mathcal{O}_Y$, is an open immersion for all sufficiently large $A_i$ (using the fact that $A' = \bigcup A_i$; compare the proof of Raynaud [3, p. 42, Cor. 2(2)]). Since $\mathbf{spec}\ A_i$ is finite over $X$, this proves (1.8).

*Remark* 1.9. Zariski's main theorem is, more correctly, the main theorem of Zariski [2]. There he was interested in the behavior of a singularity on a normal variety under a birational map. The original statement is essentially that if $f: Y \to X$ is a birational morphism of varieties and $\mathcal{O}_{X,x}$ is integrally closed, then either $f^{-1}(x)$ consists of one point and the inverse morphism $f^{-1}$ is defined in a neighborhood of $x$ or all components of $f^{-1}(x)$ have dimension $\geq 1$. To relate this to Grothendieck's version, note that if in (1.8) $X$ and $Y$ are varieties, $f$ is birational and $X$ is normal, then $g$ is an isomorphism. For a more complete discussion of the theorem, see Mumford [3, III.9].

COROLLARY 1.10. *Any proper, quasi-finite morphism $f: Y \to X$ is finite.*

*Proof.* Let $f = gf'$ be the factorization as in (1.8). As $g$ is separated and $f$ is proper, $f'$ is proper. (Use the factorization

$$f' = f_{(Y')} \circ \Gamma_{f'}: Y \to Y \times_X Y' \to Y'.)$$

Thus $f'$ is an immersion with closed image, that is, a closed immersion. Now both $f'$ and $g$ are finite.

*Remark* 1.11. The separatedness is necessary in both of the above results; for if $X$ is the affine line with the "origin doubled" (Hartshorne [2, II.2.3.6]), and $f: X \to \mathbb{A}^1$ is the natural map, then $f$ is universally closed and quasi-finite, but is not finite. (It is even flat and étale; see the next two sections.)

*Exercise* 1.12. Let $f: Y \to X$ be separated and of finite-type with $X$ irreducible. Show that if the fiber over the generic point $\eta$ is finite, then there is an open neighborhood $U$ of $\eta$ in $X$ such that $f^{-1}(U) \to U$ is finite.

## §2. Flat Morphisms

A homomorphism $f: A \to B$ of rings is *flat* if $B$ is flat when regarded as an $A$-module via $f$. Thus, $f$ is flat if and only if the functor $- \otimes_A B$ from $A$-modules to $B$-modules is exact. In particular, if $\mathfrak{I}$ is any ideal of $A$ and $f$ is flat, then $\mathfrak{I} \otimes_A B \to A \otimes_A B = B$ is injective. The converse to this statement is also true.

PROPOSITION 2.1. *A homomorphism $f: A \to B$ is flat if $(a \otimes b \mapsto f(a)b)$: $\mathfrak{I} \otimes_A B \to B$ is injective for all ideals $\mathfrak{I}$ in $A$.*

*Proof.* Let $g: M' \to M$ be an injective map of $A$-modules where, following Atiyah-Macdonald [1, 2, 19], we may assume $M$ to be finitely generated.

*Case* (*a*) $M$ is free. We prove this case by induction on the rank $r$ of $M$. If $r = 1$, then we may identify $M$ with $A$ and $M'$ with an ideal in $A$; then the statement to be proved is the statement given. If $r > 1$, then $M = M_1 \oplus M_2$ with $M_1$ and $M_2$ free of rank $<r$. Consider the exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle g_1} & & \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle g_2} & & \\
0 & \longrightarrow & g^{-1}(M_1) & \longrightarrow & M' & \longrightarrow & pg(M') & \longrightarrow & 0
\end{array}$$

When tensored with $B$, the top row remains exact, and $g_1$ and $g_2$ remain injective. This implies that $g \otimes 1$ is injective.

*Case* (*b*) $M$ arbitrary (finitely generated). Let $x_1, \ldots, x_r$ generate $M$, let $M^*$ be the free $A$-module on $x_1, \ldots, x_r$, and consider the exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{j} & M^* & \xrightarrow{h} & M & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle g} & & \\ 0 & \longrightarrow & N & \longrightarrow & h^{-1}g(M') & \longrightarrow & M' & \longrightarrow & 0. \end{array}$$

By case (a) $i \otimes 1$ is injective, and it follows that $g \otimes 1$ is injective.

PROPOSITION 2.2. *If $f: A \to B$ is flat, then so also is $S^{-1}A \to T^{-1}B$ for all multiplicative subsets $S \subset A$ and $T \subset B$ such that $f(S) \subset T$. Conversely, if $A_{f^{-1}(\mathfrak{n})} \to B_{\mathfrak{n}}$ is flat for all maximal ideals $\mathfrak{n}$ of $B$, then $A \to B$ is flat.*

*Proof.* $S^{-1}A \to S^{-1}B$ is flat according to Atiyah-Macdonald [1, 2.20], and $S^{-1}B \to T^{-1}B$ is flat according to Atiyah-Macdonald [1, 3.6]. For the converse, let $M' \to M$ be an injective map of $A$-modules. To show that $B \otimes_A M' \to B \otimes_A M$ is injective, it suffices to show that

$$B_{\mathfrak{n}} \otimes_B (B \otimes_A M') \to B_{\mathfrak{n}} \otimes_B (B \otimes_A M)$$

is injective for all $\mathfrak{n}$, but this follows from the flatness of $A_{\mathfrak{p}} \to B_{\mathfrak{n}}$ with $\mathfrak{p} = f^{-1}(\mathfrak{n})$ and the isomorphism $B_{\mathfrak{n}} \otimes_B (B \otimes_A N) \approx B_{\mathfrak{n}} \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \otimes_A N)$, which exists for any $A$-module $N$.

*Remark* 2.3. If $a \in A$ is not a zero-divisor and $f: A \to B$ is flat, then $f(a)$ is not a zero-divisor in $B$ because the injectivity of $(x \mapsto ax): A \to A$ implies that of $(x \mapsto f(a)x): B \to B = A \otimes_A B$. Thus, if $A$ is an integral domain and $B \neq 0$, then $f$ is injective. Conversely, any injective homomorphism $f: A \to B$, where $A$ is a Dedekind domain and $B$ is an integral domain, is flat, In proving this, we may localize and hence assume that $A$ is principal. According to (2.1), it suffices to prove that for any ideal $\mathfrak{I} \neq 0$ of $A$, $\mathfrak{I} \otimes_A B \to B$ is injective, but $\mathfrak{I} \otimes_A B$ is a free $B$-module of rank one, and we know that the generator of $\mathfrak{I}$ is not mapped to zero in $B$.

A morphism $f: Y \to X$ of schemes is *flat* if, for all points $y$ of $Y$, the induced map $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is flat. Equivalently $f$ is flat if for any pair $V$ and $U$ of open affines of $Y$ and $X$ such that $f(V) \subset U$, the map $\Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_Y)$ is flat. From (2.2) it follows that the first condition needs only to be checked for closed points $y$ of $Y$.

PROPOSITION 2.4. (a) *An open immersion is flat.*

(b) *The composite of two flat morphisms is flat.*

(c) *Any base extension of a flat morphism is flat.*

*Proof.* (a) and (b) are obvious.

(c) If $f: A \to B$ is flat and $A \to A'$ is arbitrary, then to see that $A' \to B \otimes_A A'$ is flat, one may use the canonical isomorphism $(B \otimes_A A') \otimes_{A'} M \approx B \otimes_A M$, which exists for any $A'$-module $M$.

In order to get less trivial examples of flat morphisms we shall need the following criterion.

PROPOSITION 2.5. *Let $B$ be a flat $A$-algebra, and consider $b \in B$. If the image of $b$ in $B/\mathfrak{m}B$ is not a zero-divisor for any maximal ideal $\mathfrak{m}$ of $A$, then $B/(b)$ is a flat $A$-algebra.*

*Proof*. After applying (2.2), we may assume that $A \to B$ is a local homomorphism of local rings. By assumption, if $c \in B$ and $bc = 0$, then $c \in \mathfrak{m}B$. We shall show by induction that in fact $c \in \mathfrak{m}^r B$ for all $r$, and hence $c \in \bigcap \mathfrak{m}^r B \subset \bigcap \mathfrak{n}^r = (0)$, where $\mathfrak{n}$ is the maximal ideal of $B$. Assume that $c \in \mathfrak{m}^r B$, and write

$$c = \sum a_i b_i$$

where the $a_i$ form a minimal generating set for $\mathfrak{m}^r$. Then

$$0 = bc = \sum_i a_i b_i b,$$

and so, by one of the standard flatness criteria (proved in (2.10b′) below), there are equations

$$b_i b = \sum_j a_{ij} b'_j$$

with the $b'_j \in B$, $a_{ij} \in A$ such that

$$\sum_i a_i a_{ij} = 0$$

for all $j$. From the choice of the $a_i$, all $a_{ij} \in \mathfrak{m}$. Thus $b_i b \in \mathfrak{m}B$, and since $b$ is not a zero-divisor in $B/\mathfrak{m}B$, this implies that $b_i \in \mathfrak{m}B$. Thus $c \in \mathfrak{m}^{r+1}B$, which completes the induction. We have shown that $b$ is not a zero-divisor in $B$, and the same argument, with $A$ replaced by $A/\mathfrak{I}$ and $B$ by $B/\mathfrak{I}B$, shows that $b$ is not a zero-divisor in $B/\mathfrak{I}B$ for any ideal $\mathfrak{I}$ of $A$.

Fix such an ideal, and consider the exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathfrak{I} \otimes B & \longrightarrow & \mathfrak{I} \otimes B & \longrightarrow & \mathfrak{I} \otimes (B/(b)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \xrightarrow{b} & B & \longrightarrow & B/(b) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B/\mathfrak{I}B & \xrightarrow{b} & B/\mathfrak{I}B & \longrightarrow & (B/(b))/\mathfrak{I}(B/(b)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which $b$ means multiplication by $b$. An application of the snake lemma shows that $\mathfrak{I} \otimes B/(b) \to B/(b)$ is injective, which shows that $B/(b)$ is flat over $A$, according to (2.1).

*Remarks* 2.6. (a) For any ring $A$, $A[X_1, \ldots, X_n]$ is a free $A$-module, and so $\mathbb{A}^n_A$ is flat over $A$. Let $Z$ be a hypersurface in $\mathbb{A}^n_A$, that is, a scheme of the form spec $(A[X_1, \ldots, X_n]/(P))$, $P \neq 0$. Then (2.5) shows that $Z$ is flat over spec $A \Leftrightarrow$ for all maximal ideals $\mathfrak{m}$ of $A$, $Z \otimes_A k(\mathfrak{m}) \neq \mathbb{A}^n_{k(\mathfrak{m})} \Leftrightarrow$ the ideal generated by the coefficients of $P$ is $A$ (assuming that spec $A$ is connected). Similar statements hold for hypersurfaces in $\mathbb{P}^n_A$.

(b) We may restate (a) as follows: a hypersurface $Z$ is flat if and only if its closed fibers over spec $A$ all have the same dimension. This generalizes. Firstly, if $f: Y \to X$ is flat, then

$$\dim(\mathcal{O}_{Y_x, y}) = \dim(\mathcal{O}_{Y,y}) - \dim(\mathcal{O}_{X,x}),$$

where $x = f(y)$. For varieties this means that $\dim(Y_x) = \dim(Y) - \dim(X)$ for any closed point $x$ of $X$ with $Y_x$ nonempty. The proof, which is quite elementary, may be found in [EGA. IV.6.1] or Hartshorne [2, III.9.5]. Secondly, if $X$ and $Y$ are regular schemes and $f: Y \to X$ is such that

$$\dim(\mathcal{O}_{Y_x, y}) = \dim(\mathcal{O}_{Y,y}) - \dim(\mathcal{O}_{X,x})$$

for all closed points $y$ of $Y$, where $x = f(y)$, then $f$ is flat. The proof again may be found in [EGA. IV.6.1]. (See also Hartshorne [2, III. Ex. 10.9].)

(c) There is another criterion for flatness that is frequently very useful. It is easy to construct examples $Z \xrightarrow{f} Y \xrightarrow{g} X$ in which $g$ and $gf$ are flat, but $f$ is not flat. However, if one also knows that the maps on fibers $f_x: Z_x \to Y_x$ are flat for all closed $x \in X$, then $f$ is flat ([SGA. 1, IV.5.9], or Bourbaki [2, III.5.4 Prop. 2,3]).

(d) If $B$ is flat over $A$ and $b_1, \ldots, b_n$ is a sequence of elements of $B$ whose image in $B/\mathfrak{m}B$ is regular for each maximal ideal $\mathfrak{m}$ of $A$, that is, $b_i$ is not a zero-divisor in $B/\mathfrak{m} + (b_1, b_2, \ldots, b_{i-1})$ for any $i$, then $B/(b_1, \ldots, b_n)$ is flat over $A$. This follows by induction from (2.5).

(e) There is a second generalization of (a). Let $X$ be an integral scheme and $Z$ a closed subscheme of $\mathbb{P}^n_X$; for each $x \in X$, let $p_x \in \mathbb{Q}[T]$ be the Hilbert polynomial of the fiber $Z_x \subset \mathbb{P}^n_{k(x)}$; then $Z$ is flat over $X$ if and only if $p_x$ is independent of $x$ (Hartshorne [2, III.9.9]).

A flat morphism $f: A \to B$ is *faithfully flat* if $B \otimes_A M$ is nonzero for any nonzero $A$-module $M$. On taking $M$ to be a principal ideal in $A$, we see that such a morphism is injective.

PROPOSITION 2.7. *Let $f: A \to B$ be a flat morphism with $A \neq 0$. The following are equivalent:*

(a) *$f$ is faithfully flat;*

(b) *a sequence $M' \to M \to M''$ of $A$-modules is exact whenever $B \otimes_A M' \to B \otimes_A M \to B \otimes_A M''$ is exact;*

(c) *${}^a f$ :spec $B \to$ spec $A$ is surjective;*

(d) *for every maximal ideal $\mathfrak{m}$ of $A$, $f(\mathfrak{m})B \neq B$. In particular, a flat local homomorphism of local rings is automatically faithfully flat.*

*Proof.* (a) $\Rightarrow$ (b). Suppose that $M' \overset{g_1}{\to} M \overset{g_2}{\to} M''$ becomes exact after tensoring with $B$. Then im $(g_2 g_1) = 0$ because

$$B \otimes_A \operatorname{im}(g_2 g_1) = \operatorname{im}((1 \otimes g_2)(1 \otimes g_1)) = 0,$$

and im $(g_1) =$ ker $(g_2)$ because

$$B \otimes (\ker g_2/\operatorname{im} g_1) = \ker(1 \otimes g_2)/\operatorname{im}(1 \otimes g_1) = 0.$$

(b) $\Rightarrow$ (a). $M \overset{0}{\to} M \to 0$ is exact if and only if $M = 0$.

(a) $\Rightarrow$ (c). For any prime ideal $\mathfrak{p}$ of $A$, $B \otimes_A k(\mathfrak{p}) \neq 0$, and so ${}^a f^{-1}(\mathfrak{p}) =$ spec $(B \otimes_A k(\mathfrak{p}))$ is nonempty.

(c) $\Rightarrow$ (d). This is trivial.

(d) $\Rightarrow$ (a). Let $x \in M$, $x \neq 0$. Because $f$ is flat, it suffices to show that $B \otimes_A N \neq 0$, where $N = Ax \subset M$. But $N \approx A/\mathfrak{I}$ for some ideal $\mathfrak{I}$ of $A$, and hence $B \otimes N \approx B/\mathfrak{I}B$. If $\mathfrak{m}$ is a maximal ideal of $A$ containing $\mathfrak{I}$, then $\mathfrak{I}B \subset f(\mathfrak{m})B \neq B$, and so $B/\mathfrak{I}B \neq 0$.

COROLLARY 2.8. *Let $f: Y \to X$ be flat; let $y \in Y$, and let $x'$ be such that $x = f(y)$ is in the closure $\overline{\{x'\}}$ of $\{x'\}$. Then there is a $y'$ such that $y \in \overline{\{y'\}}$ and $f(y') = x'$.*

*Proof.* The $x'$ such that $x \in \overline{\{x'\}}$ are exactly the points in the image of the canonical map spec $\mathcal{O}_x \to X$. The corollary therefore follows from the fact that the map spec $\mathcal{O}_y \to$ spec $\mathcal{O}_x$ induced by $f$ is surjective.

A morphism $f: Y \to X$ is *faithfully flat* if it is flat and surjective. According to (2.7c), this agrees with the previous definition for rings.

We now consider the question of flatness for finite morphisms. The next theorem shows that, for such a morphism $f: Y \to X$, flatness has a very explicit interpretation in terms of the properties of $f_*\mathcal{O}_Y$ as an $\mathcal{O}_X$-module.

THEOREM 2.9. *Let $M$ be a finitely generated $A$-module. The following are equivalent:*

(a) *$M$ is flat;*

(b) *$M_\mathfrak{m}$ is a free $A_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ of $A$;*

(c) *$\tilde{M}$ is a locally free sheaf on spec $A$;*

(d) *$M$ is a projective $A$-module.*

*Moreover, if $A$ is an integral domain, they are equivalent to:*

(e) $\dim_{k(\mathfrak{p})}(M \otimes_A k(\mathfrak{p}))$ *is the same for all prime ideals $\mathfrak{p}$ of $A$.*

*Proof.* (d) ⇒ (a). This implication does not use the finite generation of $M$. As tensor products commute with direct sums, any free module is flat, and any direct summand of a flat module is flat.

(b) ⇒ (c). Let $\mathfrak{m}$ be a maximal ideal of $A$, and let $x_1, \ldots, x_r$ be elements of $M$ whose images in $M_\mathfrak{m}$ form a basis for $M_\mathfrak{m}$ over $A_\mathfrak{m}$. Then the homomorphism

$$g: A^r \to M, \qquad g(a_1, \ldots, a_r) = \sum a_i x_i,$$

induces an isomorphism $A_a^r \to M_a$ for some $a \in A$, $a \notin \mathfrak{m}$, because the kernel and cokernel of $g$ are zero at $\mathfrak{m}$ and, being finitely generated, have closed support in spec $A$.

(c) ⇒ (a). Let $a_1, \ldots, a_r$ be elements of $A$ such that the ideal $(a_1, \ldots, a_r) = A$ and $M_{a_i}$ is a free $A_{a_i}$-module for all $i$. Let $B = \prod A_{a_i}$. Then $B$ is faithfully flat over $A$, and $B \otimes_A M = \prod M_{a_i}$ is clearly a flat $B$-module. It follows that $M$ is a flat $A$-module (apply (2.7b)).

To prove the remaining implications, (a) ⇒ (d), (a) ⇒ (b), we shall need the following lemma.

LEMMA 2.10. *Let $0 \to N \to F \xrightarrow{g} M \to 0$ be an exact sequence of $A$-modules with $N$ a submodule of $F$.*

(a) *If $M$ and $F$ are flat over $A$, then $N \cap \mathfrak{I}F = \mathfrak{I}N$ for all ideals $\mathfrak{I}$ of $A$.*

(b) *Let $M$ be flat and $F$ free, with basis $(y_i)$ over $A$. If*

$$n = \sum a_i y_i \in N,$$

*then there exist $n_i \in N$ such that*

$$n = \sum a_i n_i.$$

(b′) *Let $M$ be any flat $A$-module. If*

$$\sum_i a_i x_i = 0,$$

*$a_i \in A$, $x_i \in M$, then there are equations*

$$x_i = \sum_j a_{ij} x'_j$$

*with $x'_j \in M$, $a_{ij} \in A$, such that*

$$\sum_i a_i a_{ij} = 0$$

*for all $j$.*

(c) *Let $M$ be flat and $F$ free. For any finite set $\{n_1, \ldots, n_r\}$ of elements of $N$, there exists an $A$-linear map $f: F \to N$ with $f(n_j) = n_j$, $j = 1, \ldots, r$.*

*Proof*. a. From the given exact sequence, we obtain exact sequences,

$$0 \longrightarrow N \cap \mathfrak{J}F \longrightarrow \mathfrak{J}F \longrightarrow \mathfrak{J}M \longrightarrow 0,$$

$$\mathfrak{J} \otimes N \longrightarrow \mathfrak{J} \otimes F \longrightarrow \mathfrak{J} \otimes M \longrightarrow 0.$$

As $M$ and $F$ are flat, $\mathfrak{J} \otimes F$ and $\mathfrak{J} \otimes M$ may be identified with $\mathfrak{J}F$ and $\mathfrak{J}M$, and then the image of $\mathfrak{J} \otimes N$ in $\mathfrak{J} \otimes F = \mathfrak{J}F$ becomes identified with $\mathfrak{J}N$. But from the first sequence, this is also $N \cap \mathfrak{J}F$.

(b) Let $\mathfrak{J}$ be the ideal generated by the $a_i$ occurring in

$$n = \sum a_i y_i.$$

Then $n \in N \cap \mathfrak{J}F = \mathfrak{J}N$, and so there are $n_i \in N$ such that

$$n = \sum a_i n_i.$$

(b′) Write $M$ as a quotient of a free module, as in (b), and let $a_1 x_1 + \cdots + a_r x_r = 0$. It is possible to choose $F$ so that it has a basis $(y_i)$ with $g(y_i) = x_i$, $i = 1, \ldots, r$. Then

$$n = \sum a_i y_i \in N,$$

and so it may be written

$$n = \sum a_i n_i,$$

$n_i \in N$. Write

$$n_i = y_i - \sum a_{ij} y_j,$$

some $a_{ij}$. Then

$$n = \sum a_i n_i = n - \sum_j \sum_i (a_i a_{ij}) y_j,$$

and so

$$\sum_i a_i a_{ij} = 0$$

each $j$. Also

$$x_i = \sum a_{ij} g(y_j),$$

and so $x_j'$ may be taken to be $g(y_j)$.

(c) We use induction on $r$. Assume first that $r = 1$, and write

$$n_1 = \sum_{j=1}^{s} a_j y_{i_j}$$

where $(y_i)$ is a basis for $F$. Then

$$n_1 = \sum_{j=1}^{s} a_j n_j'$$

for some $n_j' \in N$, and $f$ may be taken to be the map such that $f(y_{i_j}) = n_j'$, $j = 1, \ldots, s$, and $f(y_i) = 0$ otherwise. Now suppose that $r > 1$, and there are maps $f_1, f_2: F \to N$ such that $f_1(n_1) = n_1$ and

$$f_2(n_i - f_1(n_i)) = n_i - f_1(n_i), \qquad i = 2, \ldots, r.$$

Then

$$f: F \to N, \qquad f(y) = f_1(y) + f_2(y) - f_2 f_1(y)$$

has the required property.

We now complete the proof of (2.9).

(a) ⇒ (d). Embed $M$ into an exact sequence

$$0 \to N \to F \to M \to 0$$

in which $F$ is free and $N$ and $F$ are both finitely generated. According to (2.10c), this sequence splits, and so $M$ is projective.

(a) ⇒ (b). We may assume that $M$ is a finitely generated flat module over a local ring $A$. Let $x_1, \ldots, x_r \in M$ be such that their images in $M/\mathfrak{m}M$ form a basis for this over the field $A/\mathfrak{m}$. Embed $M$ in a sequence

$$0 \to N \to F \xrightarrow{g} M \to 0$$

where $F$ is free with basis $\{y_1, \ldots, y_r\}$ and $g(y_i) = x_i$. As $N \subset \mathfrak{m}F$, $\mathfrak{m}N = N \cap (\mathfrak{m}F) = N$, and $N$ is zero according to Nakayama's lemma.

(c) ⇒ (e). This is obvious.

(e) ⇒ (c). Fix a prime ideal $\mathfrak{p}$ of $A$, and choose elements $x_1, \ldots, x_r$ of $M_a$, some $a \notin \mathfrak{p}$, whose images in $M \otimes_A k(\mathfrak{p})$ form a basis. According to Nakayama's lemma the map

$$g: A_a^r \to M_a, \qquad g(a_1, \ldots, a_r) = \sum a_i x_i$$

defines a surjection $A_\mathfrak{p}^r \to M_\mathfrak{p}$. On changing $a$, we may assume that $g$ itself is surjective. For any prime ideal $\mathfrak{q}$ of $A_a$, the map $k(\mathfrak{q})^r \to M \otimes_A k(\mathfrak{q})$ is surjective, and hence is an isomorphism because $\dim(M \otimes_A k(\mathfrak{q})) = r$. Thus $\ker(g) \subset \mathfrak{q}A_a^r$ for any $\mathfrak{q}$, which implies that it is zero as $A_a$ is reduced. Thus $M_a$ is free.

*Remark* 2.11. Let $f: Y \to X$ be finite and flat. I claim that $f$ is open. Following (2.9), we may assume that $X = \operatorname{spec} A$, $Y = \operatorname{spec} B$, and $B \approx A^r$ as an $A$-module. Let $T^r + a_1 T^{r-1} + \cdots + a_r$ be the characteristic polynomial over $A$ of an element $b \in B$. A prime ideal $\mathfrak{p}$ of $A$ is in the image of $\operatorname{spec}(B_b) \to \operatorname{spec}(A)$ exactly when $B_b/\mathfrak{p}B_b$ is nonzero. But $B_b/\mathfrak{p}B_b \approx (B/\mathfrak{p}B)_{\bar{b}}$ and so this ring is nonzero exactly when $\bar{b}$ is not nilpotent in $B/\mathfrak{p}B$ or, equivalently, when some coefficient of $T^r + a_1 T^{r-1} + \cdots + a_r$ is nonzero in $A/\mathfrak{p}$. Thus the image of $\operatorname{spec} B_b$ in $\operatorname{spec} A$ is $\bigcup \operatorname{spec} A_{a_i}$, which is open. A much more general statement holds.

THEOREM 2.12. *Any flat morphism that is locally of finite-type is open.*

LEMMA 2.13. *Let $f: Y \to X$ be of finite-type. For all pairs $(Z, U)$ where $Z$ is a closed irreducible subset of $Y$ and $U$ is an open subset such that*

*$U \cap Z \neq \emptyset$, there is an open subset $V$ of $X$ such that $f(U \cap Z) \supset V \cap \overline{f(Z)} \neq \emptyset$. (Here, $\overline{f(Z)}$ denotes the closure of the set $f(Z)$).*

*Proof*. First note the following statements.

(a) The lemma is true for closed immersions.

(b) The lemma is true for $f$ if it is true for

$$f_{\text{red}}: Y_{\text{red}} \to X_{\text{red}}.$$

(c) The lemma is true for $gf$ if it is true for $f$ and $g$.

For, if $V'$ satisfies the conclusion of the lemma for the pair $(\overline{f(Z)}, V)$ and the map $g$, then it also satisfies the conclusion for the pair $(Z, U)$ and the map $gf$.

(d) It suffices to check the lemma locally on $Y$ and $X$.

(e) In checking the lemma for a given $Z$, we note that $X$ may be replaced by $\overline{f(Z)}$, and hence may be assumed to be irreducible.

Using (a), (c), and (d), we may reduce the proof to the case that $f$ is the projection $\mathbb{A}^n \times X \to X$ where $X$ is affine. Using (b) and (e), we reduce the proof further to the case that $X = \operatorname{spec} A$, $A$ an integral domain. Finally, using (c) again, we reduce the proof to the case that $f$ is the projection $\mathbb{A}^1 \times X \to X$.

Let $Z$ be a closed irreducible subset of $\mathbb{A}^1_X$, say $Z = \operatorname{spec} B$ where $B = A[T]/\mathfrak{q}$. We may assume that $\mathfrak{q} \neq 0$, for otherwise the lemma is easy. We may also assume, according to (e), that $\mathfrak{q} \cap A = (0)$, that is, that $f(Z) = X$. Let $K$ be the field of fractions of $A$, and let $t = T \pmod{\mathfrak{q}}$. Since $\mathfrak{q}$ contains a nonconstant polynomial, $t$ is algebraic over $K$, and so there is an $a \in A$, $a \neq 0$, such that $at$ is integral over $A$. Then $B_a$ is finite over $A_a$, and so $\operatorname{spec} B_a \to \operatorname{spec} A_a$ is surjective (Atiyah-Macdonald [1, 5.10]). Thus we are reduced to showing that the image of a nonempty open subset $U$ of $\operatorname{spec} B_a$ contains a nonempty open subset of spec A. But if $U$ contains $(\operatorname{spec} B_a)_b$, and $b$ satisfies the polynomial $T^m + a_1 T^{m-1} + \cdots + a_m = 0$, $a_i \in A_a$, then $f(U) \supset \bigcup (\operatorname{spec} A_a)_{a_i}$.

*Proof of* (2.12). (Compare Hartshorne [2, III. Ex. 9.1].) Let $f: Y \to X$ be as in the proposition. It suffices to show that $f(Y)$ is open. We may assume that $X$ is quasi-compact. Let $W = X - f(Y)$ and let $Z_1, \ldots, Z_n$ be the irreducible components of $\overline{W}$. Let $z_j$ be the generic point of $Z_j$. If $z_j \in f(Y)$, say $z_j = f(y)$, then (2.13) applied to $(\overline{\{y\}}, Y)$ shows that there exists an open $U$ in $X$ such that $f(Y) \supset U \cap Z_j \supset \{z_j\}$. But then

$$f(Y) \supset U \cap \left(X - \bigcup_{i \neq j} Z_i\right) \supset \{z_j\},$$

and, as $U$ and $(X - \bigcup_{i \neq j} Z_i)$ are open, this implies that $z_j \notin \overline{W}$, which is a contradiction. Thus $z_j \in W$, and, according to (2.8), all specializations of $z_j$ belong to $W$. Thus $W \supset Z_j$, $W \supset \bigcup Z_j = \overline{W}$, and $f(Y)$ is open.

*Remark* 2.14. If $f: Y \to X$ is finite and flat, then it is both open and closed. Thus, if $X$ is connected, then $f$ is surjective and hence faithfully flat (provided $Y \neq \emptyset$).

*Exercise* 2.15. Give an example to show that (2.12) is false without the finiteness condition, even if $f$ is surjective. (Start with the example in (1.6*b*)).

If $f: Y \to X$ is finite, and for some $y \in Y$, $\mathcal{O}_y$ is free as an $\mathcal{O}_{f(y)}$-module, then clearly $\Gamma(f^{-1}(U), \mathcal{O}_Y)$ is free over $\Gamma(U, \mathcal{O}_X)$ for some open affine $U$ in $X$ containing $f(y)$. (See the proof of (b) $\Rightarrow$ (c) in (2.9.) Thus the set of points $y \in Y$ such that $\mathcal{O}_y$ is flat over $\mathcal{O}_x$ is open in $Y$ and is even nonempty if $X$ is integral and $\overline{f(Y)} = X$. Again this holds more generally.

THEOREM 2.16. *Let $f: Y \to X$ be locally of finite-type. The set of points $y \in Y$ such that $\mathcal{O}_y$ is flat over $\mathcal{O}_{f(y)}$ is open in $Y$; it is nonempty if $X$ is integral.*

*Proof*. A reasonably self-contained proof of this may be found in Matsumura [1, Chapter VIII]. See also [EGA, IV.11.1.1].

Recall that, in any category with fiber products, a morphism $Y \to X$ is a *strict epimorphism* if the sequence

$$Y \times_X Y \underset{p_2}{\overset{p_1}{\rightrightarrows}} Y \to X$$

is exact, that is, if the sequence of sets

$$\mathrm{Hom}\,(X, Z) \to \mathrm{Hom}\,(Y, Z) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}\,(Y \times Y, Z)$$

is exact for all $Z$, that is, if the first arrow maps $\mathrm{Hom}\,(X, Z)$ bijectively onto the subset of $\mathrm{Hom}\,(Y, Z)$ on which $p_1^*$ and $p_2^*$ agree.

Clearly the condition that a morphism of schemes be surjective is not sufficient to imply that it is a (strict) epimorphism (consider spec $k \to$ spec $A$, where $A$ is a local Artin ring with residue field $k$), but for flat morphisms it is (almost).

THEOREM 2.17. *Any faithfully flat morphism $f: Y \to X$ of finite-type is a strict epimorphism.*

It is convenient to prove the following result first.

PROPOSITION 2.18. *If $f: A \to B$ is faithfully flat, then the sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{d^0} B^{\otimes 2} \to \cdots \to B^{\otimes r} \xrightarrow{d^{r-1}} B^{\otimes r+1} \to \cdots$$

*is exact, where*

$$B^{\otimes r} = B \otimes_A B \otimes \cdots \otimes_A B \qquad (r \text{ times})$$
$$d^{r-1} = \sum (-1)^i e_i$$
$$e_i(b_0 \otimes \cdots \otimes b_{r-1}) = b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{r-1}.$$

*Proof*. The usual argument shows that $d^r d^{r-1} = 0$. We assume first that $f$ admits a section, that is, that there exists a homomorphism $g: B \to A$ such that $gf = 1$, and we construct a contracting homotopy $k_r: B^{\otimes r+2} \to B^{\otimes r+1}$. Define

$$k_r(b_0 \otimes \cdots \otimes b_{r+1}) = g(b_0)b_1 \otimes b_2 \otimes \cdots \otimes b_{r+1}, \qquad r \geq -1.$$

It is easily checked that $k_{r+1}d^{r+1} + d^r k_r = 1$, $r \geq -1$, and this shows that the sequence is exact.

Now let $A'$ be an $A$-algebra, let $B' = A' \otimes_A B$, and let $f' = 1 \otimes f: A' \to B'$. The sequence corresponding to $f'$ is obtained from the sequence for $f$ by tensoring with $A'$ (because $B^{\otimes r} \otimes_A A' \approx B'^{\otimes r}$). Thus, if $A'$ is a faithfully flat $A$-algebra, it suffices to prove the theorem for $f'$. Take $A' = B$, and then $f' = (b \mapsto b \otimes 1): B \to B \otimes_A B$ has a section, namely, $g(b \otimes b') = bb'$, and so the sequence is exact.

*Remark* 2.19. A similar argument to the above shows that if $f: A \to B$ is faithfully flat and $M$ is an $A$-module, then the sequence

$$0 \to M \to M \otimes_A B \xrightarrow{1 \otimes d^0} M \otimes_A B^{\otimes 2} \to \cdots$$
$$\to M \otimes B^{\otimes r} \xrightarrow{1 \otimes d^{r-1}} M \otimes B^{\otimes r+1} \to \cdots$$

is exact. Indeed, one may assume again that $f$ has a section and construct a contracting homotopy as before.

*Proof of* 2.17. We have to show that for any scheme $Z$ and any morphism $h: Y \to Z$ such that $hp_1 = hp_2$, there exists a unique morphism $g: X \to Z$ such that $gf = h$.

*Case* (*a*) $X = \operatorname{spec} A$, $Y = \operatorname{spec} B$, and $Z = \operatorname{spec} C$ are all affine. In this case the theorem follows from the exactness of

$$0 \to A \to B \xrightarrow{e_0 - e_1} B \otimes_A B$$

(since ${}^a e_0 = p_2$, ${}^a e_1 = p_1$).

*Case* (*b*) $X = \operatorname{spec} A$ and $Y = \operatorname{spec} B$ affine, $Z$ arbitrary. We first show the uniqueness of $g$. If $g_1, g_2: X \to Z$ are such that $g_1 f = g_2 f$, then $g_1$ and $g_2$ must agree on the underlying topological space of $X$ because $f$ is surjective. Let $x \in X$; let $U$ be an open affine neighborhood of $g_1(x)\,(=g_2(x))$ in $Z$, and let $a \in A$ be such that $x \in X_a$ and $g_1(X_a) = g_2(X_a) \subset U$. Then $B_b$, where $b$ is the image of $a$ in $B$ is faithfully flat over $A_a$, and it therefore follows from case (a) that $g_1 | X_a = g_2 | X_a$.

Now let $h: Y \to Z$ have $hp_1 = hp_2$. Because of the uniqueness just proved, it suffices to define $g$ locally. Let $x \in X$, $y \in f^{-1}(x)$, and let $U$ be an open affine neighborhood of $h(y)$ in $Z$. Then $f(h^{-1}(U))$ is open in $X$ (2.12), and so it is possible to find an $a \in A$ such that $x \in X_a \subset f(h^{-1}(U))$. I claim $f^{-1}(X_a)$ is contained in $h^{-1}(U)$. Indeed, if $f(y_1) = f(y_2)$, there is a $y' \in Y \times Y$ such that $p_1(y') = y_1$ and $p_2(y') = y_2$; if $y_2 \in h^{-1}(U)$, then

$$h(y_1) = hp_1(y') = hp_2(y') = h(y_2) \in U,$$

which proves the claim. If now $b$ is the image of $a$ in $B$, then $h(Y_b) = h(f^{-1}(X_a)) \subset U$, and $B_b$ is faithfully flat over $A_a$. Thus the problem is reduced to case (a).

*Case* (*c*) General case. It is easy to reduce to the case where $X$ is affine. Since $f$ is quasi-compact, $Y$ is a finite union, $Y = Y_1 \cup \cdots \cup Y_n$, of open affines. Let $Y^*$ be the disjoint union $Y_1 \amalg \cdots \amalg Y_n$. Then $Y^*$ is affine and the obvious map $Y^* \to X$ is faithfully flat. In the commutative diagram,

$$\begin{array}{ccccc} \mathrm{Hom}(X, Z) & \longrightarrow & \mathrm{Hom}(Y, Z) & \rightrightarrows & \mathrm{Hom}(Y \times Y, Z) \\ \| & & \downarrow & & \downarrow \\ \mathrm{Hom}(X, Z) & \longrightarrow & \mathrm{Hom}(Y^*, Z) & \rightrightarrows & \mathrm{Hom}(Y^* \times Y^*, Z), \end{array}$$

the lower row is exact by case (b) and the middle vertical arrow is obviously injective. An easy diagram chase now shows that the top row is exact.

*Exercise* 2.20. Show that $\operatorname{spec} k[T] \to \operatorname{spec} k[T^3, T^5]$ is an epimorphism, but is not a strict epimorphism.

*Remark* 2.21. Let $f: A \to B$ be a faithfully flat homomorphism, and let $M$ be an $A$-module. Write $M'$ for the $B$-module $f_*M = B \otimes_A M$. The module $e_{0*}M' = (B \otimes_A B) \otimes_B M'$ may be identified with $B \otimes_A M'$ where $B \otimes_A B$ acts by $(b_1 \otimes b_2)(b \otimes m) = b_1 b \otimes b_2 m$, and $e_{1*}M'$ may be identified with $M' \otimes_A B$ where $B \otimes_A B$ acts by $(b_1 \otimes b_2)(m \otimes b) = b_1 m \otimes b_2 b$. There is a canonical isomorphism $\phi\colon e_{1*}M' \to e_{0*}M'$ arising from

$$e_{1*}M' = (e_1 f)_* M = (e_0 f)_* M = e_{0*}M';$$

explicitly it is the map

$$M' \otimes_A B \to B \otimes_A M', \qquad (b \otimes m) \otimes b' \mapsto b \otimes (b' \otimes m), \qquad m \in M.$$

Moreover, $M$ can be recovered from the pair $(M', \phi)$ because

$$M = \{m \in M' \mid 1 \otimes m = \phi(m \otimes 1)\}$$

according to (2.19).

Conversely, every pair $(M', \phi)$ satisfying certain conditions does arise in this way from an $A$-module. Given $\phi: M' \otimes_A B \to B \otimes_A M'$ define

$$\phi_1: B \otimes_A M' \otimes_A B \to B \otimes_A B \otimes_A M',$$
$$\phi_2: M' \otimes_A B \otimes_A B \to B \otimes_A B \otimes_A M',$$
$$\phi_3: M' \otimes_A B \otimes_A B \to B \otimes_A M' \otimes_A B$$

by tensoring $\phi$ with $\mathrm{id}_B$ in the first, second, and third positions respectively. Then a pair $(M', \phi)$ arises from an $A$-module $M$ as above if and only if $\phi_2 = \phi_1\phi_3$. The necessity is easy to check. For the sufficiency, define $M = \{m \in M' \mid 1 \otimes m = \phi(m \otimes 1)\}$. There is a canonical map $(b \otimes m \mapsto bm) : B \otimes_A M \to M'$, and it suffices to show that this is an isomorphism (and that the map arising from $M$ is $\phi$). Consider the diagram

$$\begin{array}{ccc} M' \otimes_A B & \overset{\alpha \otimes 1}{\underset{\beta \otimes 1}{\rightrightarrows}} & B \otimes_A M' \otimes_A B \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_1} \\ B \otimes_A M' & \overset{e_0 \otimes 1}{\underset{e_1 \otimes 1}{\rightrightarrows}} & B \otimes_A B \otimes_A M' \end{array}$$

in which $\alpha(m) = 1 \otimes m$ and $\beta(m) = \phi(m \otimes 1)$. As the diagram commutes with either the upper or the lower horizontal maps (for the lower maps, this uses the relation $\phi_2 = \phi_1\phi_3$), $\phi$ induces an isomorphism on the kernels. But, by definition of $M$, the kernel of the pair $(\alpha \otimes 1, \beta \otimes 1)$ is $M \otimes_A B$, and, according to (2.19), the kernel of the pair $(e_0 \otimes 1, e_1 \otimes 1)$ is $M'$. This essentially completes the proof.

More details on this, and the following two results may be found in Murre [1, Chapter VII] and Knus-Ojanguren [1, Chapter II].

PROPOSITION 2.22. *Let $f: Y \to X$ be faithfully flat and quasi-compact. To give a quasi-coherent $\mathcal{O}_X$-module $M$ is the same as to give a quasi-coherent $\mathcal{O}_Y$-module $M'$ plus an isomorphism $\phi: p_1^* M' \to p_2^* M'$ satisfying*

$$p_{31}^*(\phi) = p_{32}^*(\phi)p_{21}^*(\phi).$$

*(Here the $p_{ij}$ are the various projections $Y \times Y \times Y \to Y \times Y$, that is $p_{ji}(y_1, y_2, y_3) = (y_j, y_i), j > i$).*

*Proof.* In the case that $Y$ and $X$ are affine, this is a restatement of (2.21).

By using the relation between schemes affine over a scheme and quasi-coherent sheaves of algebras (Hartshorne [2, II. Ex. 5.17]), one can deduce from (2.22) the following result.

THEOREM 2.23. *Let $f: Y \to X$ be faithfully flat and quasi-compact. To give a scheme $Z$ affine over $X$ is the same as to give a scheme $Z'$ affine over $Y$ plus an isomorphism $\phi: p_1^* Z' \to p_2^* Z'$ satisfying*

$$p_{31}^*(\phi) = p_{32}^*(\phi)p_{21}^*(\phi).$$

*Remark* 2.24. The above is a sketch of part of descent theory. Another part describes which properties of morphisms descend. Consider a

Cartesian square

$$\begin{array}{ccc} Y & \longleftarrow & Y' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ X & \longleftarrow & X' \end{array}$$

in which the map $X' \to X$ is faithfully flat and quasi-compact. If $f'$ is quasi-compact (respectively separated, of finite-type, proper, an open immersion, affine, finite, quasi-finite, flat, smooth, étale), then $f$ is also [EGA. IV.2.6,2.7]. The reader may check that this statement implies the same statement for faithfully flat morphisms $X' \to X$ that are locally of finite-type. (Use (2.12)).

Of a similar nature is the result that if $f: Y \to X$ is faithfully flat and $Y$ is integral (respectively normal, regular), then so also is $X$ [EGA. $0_{IV}$, 17.3.3].

Finally, we quote a result that may be regarded as a vast generalization of the Hilbert Nullstellensatz. The Nullstellensatz says that any morphism of finite-type $f: X \to \operatorname{spec} k$, $k$ a field, has a *quasi-section*, that is, that there exists a $k$-morphism $g: \operatorname{spec} k' \to X$ with $k'$ a finite field extension of $k$.

PROPOSITION 2.25. *Let $X$ be quasi-compact, and let $f: Y \to X$ be a faithfully flat morphism that is locally of finite-type. Then there exists an affine scheme $X'$, a faithfully flat quasi-finite morphism $h: X' \to X$, and an $X$-morphism $g: X' \to Y$.*

*Proof.* One has to show that, locally, there exist sequences satisfying the conditions of (2.6d) and of length equal to the relative dimension of $Y/X$. (See [EGA. IV.17.16.2] for the details.)

## §3. Étale Morphisms

Let $k$ be a field and $\bar{k}$ its algebraic closure. A $k$-algebra $A$ is *separable* if $\bar{A} = A \otimes_k \bar{k}$ has zero Jacobson radical, that is, if the maximal ideals of $\bar{A}$ have intersection zero.

PROPOSITION 3.1. *Let $A$ be a finite algebra over a field $k$. The following are equivalent:*

(a) *$A$ is separable over $k$;*

(b) *$\bar{A}$ is isomorphic to a finite product of copies of $\bar{k}$;*

(c) *$A$ is isomorphic to a finite product of separable field extensions of $k$;*

(d) *the discriminant of any basis of $A$ over $k$ is nonzero (that is, the trace pairing $A \times A \to k$ is nondegenerate).*

*Proof.* (a) ⇒ (b). From (1.5) we know that $\bar{A}$ has only finitely many prime ideals and that they are all maximal. Now (a) implies that their intersection is zero and (b) follows from the Chinese remainder theorem (Atiyah-Macdonald [1, 1.10]).

(b) ⇒ (c). The Chinese remainder theorem implies that $A/I_r$, where $I_r$ is the Jacobson radical of $A$, is isomorphic to a finite product $\prod k_i$ of finite field extensions of $k$. Write $[K:k]_s$ for the separable degree of a field extension $K/k$. Then $\mathrm{Hom}_{k\text{-alg}}(A, \bar{k})$ has

$$\sum [k_i:k]_s$$

elements. But

$$\mathrm{Hom}_{k\text{-alg}}(A, \bar{k}) \approx \mathrm{Hom}_{\bar{k}\text{-alg}}(\bar{A}, \bar{k}),$$

and this set has $[\bar{A}:\bar{k}]$ elements by (b). Thus

$$[\bar{A}:\bar{k}] = \sum [k_i:k]_s \leq \sum [k_i:k] = [A/I_r:k] \leq [A:k].$$

Since $[\bar{A}:\bar{k}] = [A:k]$, equality must hold throughout and we have (c).

(c) ⇒ (d). If $A = \prod k_i$, where the $k_i$ are separable field extensions of $k$, then $\mathrm{disc}(A) = \prod \mathrm{disc}(k_i)$, and this is nonzero by one of the standard criteria for a field extension to be separable.

(d) ⇒ (a). The discriminants of $A$ and $\bar{A}$ are the same. If $x$ is in the radical of $\bar{A}$, then $xa$ is nilpotent for any $a \in \bar{A}$, and so $Tr_{\bar{A}/\bar{k}}(xa) = 0$ all a. Thus $x = 0$.

A morphism $f: Y \to X$ that is locally of finite-type is said to be *unramified* at $y \in Y$ if $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$ is a finite separable field extension of $k(x)$, where $x = f(y)$. In terms of rings, this indicates that a homomorphism $f: A \to B$ of finite-type is unramified at $\mathfrak{q} \in \mathrm{spec}\, B$ if and only if $\mathfrak{p} = f^{-1}(\mathfrak{q})$ generates the maximal ideal in $B_{\mathfrak{q}}$ and $k(\mathfrak{q})$ is a finite separable field extension of $k(\mathfrak{p})$. Thus this terminology agrees with that in number theory.

A morphism $f: Y \to X$ is *unramified* if it is unramified at all $y \in Y$.

PROPOSITION 3.2. *Let $f: Y \to X$ be locally of finite-type. The following are equivalent:*

(a) *$f$ is unramified;*

(b) *for all $x \in X$, the fiber $Y_x \to \mathrm{spec}\, k(x)$ over $x$ is unramified;*

(c) *all geometric fibers of $f$ are unramified (that is, for all morphisms $\mathrm{spec}\, k \to X$, with $k$ separably closed, $Y \times_X \mathrm{spec}\, k \to \mathrm{spec}\, k$ is unramified);*

(d) *for all $x \in X$, $Y_x$ has an open covering by spectra of finite separable $k(x)$-algebras;*

(e) *for all $x \in X$, $Y_x$ is a sum $\coprod \mathrm{spec}\, k_i$, where the $k_i$ are finite separable field extensions of $k(x)$;*

*(If $f$ is of finite-type, then $Y_x$ itself is the spectrum of a finite separable $k(x)$-algebra in* (d), *and $Y_x$ is a finite sum in* (e)*; in particular $f$ is quasi-finite).*

*Proof*. (a) $\Leftrightarrow$ (b). This follows from the isomorphism $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y} \approx \mathcal{O}_{Y_x,y}$.

(b) $\Rightarrow$ (d). Let $U$ be an open affine subset of $Y_x$, and let $\mathfrak{q}$ be a prime ideal in $B = \Gamma(U, \mathcal{O}_{Y_x})$. According to (b), $B_\mathfrak{q}$ is a finite separable field extension of $k(x)$. Also

$$k(x) \subset B/\mathfrak{q} \subset B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q} = B_\mathfrak{q},$$

and so $B/\mathfrak{q}$ is also a field. Thus $\mathfrak{q}$ is maximal, $B$ is an Artin ring (Atiyah-Macdonald [1, 8.5]), and $B = \prod B_\mathfrak{q}$, where $\mathfrak{q}$ runs through the finite set spec $B$. This proves (d).

A similar argument shows that (c) $\Rightarrow$ (d), and (d) $\Rightarrow$ (e) $\Rightarrow$ (c) and (d) $\Rightarrow$ (b) are trivial consequences of (3.1).

Notice that according to the above definition, any closed immersion $Z \hookrightarrow X$ is unramified. Since this does not agree with our intuitive idea of an unramified covering, for example, of Riemann surfaces, we need a more restricted notion. A morphism of schemes (or rings) is defined to be *étale* if it is flat and unramified (hence also locally of finite-type).

PROPOSITION 3.3. (a) *Any open immersion is étale.*

(b) *The composite of two étale morphisms is étale.*

(c) *Any base change of an étale morphism is étale.*

*Proof*. After applying (2.4), we only have to check that the three statements hold for unramified morphisms. Both (a) and (b) are obvious (any immersion is unramified). Also, (c) is obviously true according to (3.1) if the base change is of the form $k \to k'$, where $k$ and $k'$ are fields, but, according to (3.2), this is all that has to be checked.

*Example* 3.4. Let $k$ be a field and $P(T)$ a monic polynomial over $k$. Then the monogenic extension $k[T]/(P)$ is separable (or unramified or étale) if and only if $P$ is separable, that is, has no multiple roots in $\bar{k}$.

This generalizes to rings. A monic polynomial $P(T) \in A[T]$ is *separable* if $(P, P') = A[T]$, that is, if $P'(T)$ is a unit in $A[T]/(P)$ where $P'(T)$ is the formal derivative of $P(T)$. It is easy to see that $P$ is separable if and only if its image in $k(\mathfrak{p})[T]$ is separable for all prime ideals $\mathfrak{p}$ in $A$.

Let $B = A[T]/(P)$, where $P$ is any monic polynomial in $A[T]$. As an $A$-module, $B$ is free of finite rank equal to the degree of $P$. Moreover, $B \otimes_A k(\mathfrak{p}) = k(\mathfrak{p})[T]/(\bar{P})$ where $\bar{P}$ is the image of $P$ in $k(\mathfrak{p})[T]$. It follows from (3.2b) that $B$ is unramified and so étale over $A$ if and only if $P$ is separable. More generally, for any $b \in B$, $B_b$ is étale over $A$ if and only if $P'$ is a unit in $B_b$.

For example, $B = A[T]/(T^r - a)$ is étale over $A$ if and only if $ra$ is invertible in $A$ (for $ra \in A^* \Leftrightarrow r\bar{a} \in k(\mathfrak{p})^*$, all $\mathfrak{p} \Leftrightarrow T^r - \bar{a}$ is separable in $k(\mathfrak{p})[T]$ all $\mathfrak{p}$).

For algebras generated by more than one element, there is the following Jacobian criterion: let $C = A[T_1, \ldots, T_n]$, let $P_1, \ldots, P_n \in C$, and let $B = C/(P_1, \ldots, P_n)$; then $B$ is étale over $A$ if and only if the image of $\det(\partial P_i/\partial T_j)$ in $B$ is a unit. That $B$ is unramified over $A$ if and only if the condition holds follows directly from (3.5b) below. (The $B$-module $\Omega^1_{B/A}$ has generators $dT_1, \ldots, dT_n$ and relations $\sum$ "$(\partial P_i/\partial T_j)$" $dT_j = 0$.) That $B$ is flat over $A$ may be proved by repeated applications of (2.5). (See Mumford [3, III. §10. Thm. 3′] for the details.)

Note that if $Y = \operatorname{spec} B$ and $X = \operatorname{spec} A$ were analytic manifolds, then this criterion would indicate that the induced maps on the tangent spaces were all isomorphisms, and hence $Y \to X$ would be a local isomorphism at every point of $Y$ by the inverse function theorem. It is clearly not true in the geometric case that $\operatorname{spec} B \to \operatorname{spec} A$ is a local isomorphism (unless local is meant in the sense of the étale topology: see later). For example, consider $\operatorname{spec} \mathbb{Z}[T]/(T^2 - 2) \to \operatorname{spec} \mathbb{Z}$, which is étale on the complement of $\{(2)\}$.

PROPOSITION 3.5. *Let $f: Y \to X$ be locally of finite-type. The following are equivalent:*

(a) *$f$ is unramified;*

(b) *the sheaf $\Omega^1_{Y/X}$ is zero;*

(c) *the diagonal morphism $\Delta_{Y/X}: Y \to Y \times_X Y$ is an open immersion.*

*Proof*. (a) ⇒ (b). Since $\Omega^1_{Y/X}$ behaves well with respect to base change, one can reduce the proof to the case that $Y = \operatorname{spec} B$ and $X = \operatorname{spec} A$ are affine, then to the case that $A \to B$ is a local homomorphism of local rings and using Nakayama's lemma to the case where $A$ and $B$ are fields. Then $B$ is a separable field extension of $A$, and it is a standard fact that this implies that $\Omega^1_{B/A} = 0$.

(b) ⇒ (c). Since the diagonal is always at least locally closed, we may choose an open subscheme $U$ of $Y \times_X Y$ such that $\Delta_{Y/X}: Y \to U$ is a closed immersion and regard $Y$ as a subscheme of $U$. Let $I$ be the sheaf of ideals on $U$ defining $Y$. Then $I/I^2$, regarded as a sheaf on $Y$, is isomorphic to $\Omega^1_{Y/X}$ and hence is zero. Using Nakayama's lemma, one sees this implies that $I_y = 0$ for all $y \in Y$, and it follows that $I = 0$ on some open subset $V$ of $U$ containing $Y$. Then $(Y, \mathcal{O}_Y) = (V, \mathcal{O}_V)$ is an open subscheme of $Y \times Y$.

(c) ⇒ (a). By passing to the geometric fiber over a point of $X$, we may reduce the problem to the case of a morphism $f: Y \to \operatorname{spec} k$ where $k$

is an algebraically closed field. Let $y$ be a closed point of $Y$. Because $k$ is algebraically closed, there exists a section $g$:spec $k \to Y$ whose image is $\{y\}$. The following square is Cartesian:

$$\begin{array}{ccc} Y & \xrightarrow{\Delta} & Y \times_X Y \\ \uparrow g & & \uparrow (gf, 1) \\ \{y\} & \xrightarrow{g} & Y. \end{array}$$

Since $\Delta$ is an open immersion, this implies that $\{y\}$ is open in $Y$. Moreover, spec $\mathcal{O}_y = \{y\} \to$ spec $k$ still has the property that spec $\mathcal{O}_y \xrightarrow{\Delta}$ spec $(\mathcal{O}_y \otimes_k \mathcal{O}_y)$ is an open immersion. But $\mathcal{O}_y$ is a local Artin ring with residue field $k$, and so spec $\mathcal{O}_y \otimes_k \mathcal{O}_y$ has only one point, and $\mathcal{O}_y \otimes_k \mathcal{O}_y \to \mathcal{O}_y$ must be an isomorphism. By counting dimensions over $k$, one sees then that $\mathcal{O}_y = k$. Thus, by applying (3.1) and (3.2), we have (a).

COROLLARY 3.6. *Consider morphisms* $f: X \to S$, $g: Y \to X$. *If* $fg$ *is étale and* $f$ *is unramified, then* $g$ *is étale.*

*Proof*. Write $g = p_2\Gamma_g$ where $\Gamma_g: Y \to Y \times_S X$ is the graph of $g$ and $p_2: Y \times_S X \to X$ is the projection on the second factor. $\Gamma_g$ is the pull-back of the open immersion $\Delta_{X/S}: X \to X \times_S X$ by $g \times 1: Y \times_S X \to X \times_S X$, and $p_2$ is the pull-back of the étale map $fg: Y \to S$ by $f: X \to S$. Thus, by using (3.3), we see that $g$ is étale.

*Remark* 3.7. Let $f: Y \to X$ be locally of finite-type. The annihilator of $\Omega^1_{Y/X}$ (an ideal in $\mathcal{O}_Y$) is called the *different* $\mathfrak{d}_{Y/X}$ of $Y$ over $X$. That this definition agrees with the one in number theory is proved in Serre [7, III.7].

The closed subscheme of $Y$ defined by $\mathfrak{d}_{Y/X}$ is called the *branch locus* of $Y$ over $X$. The open complement of the branch locus is precisely the set on which $\Omega^1_{Y/X} = 0$, that is, on which $f: Y \to X$ is unramified. The theorem of the purity of branch locus states that the branch locus (if nonempty) has pure codimension one in $Y$ in each of the two cases: (a) when $f$ is faithfully flat and finite over $X$; or (b) when $f$ is quasi-finite and dominating, $Y$ is regular and $X$ is normal. (See Altman and Kleiman [1, VI.6.8], [SGA. 1, X.3.1], and [SGA. 2, X.3.4].)

PROPOSITION 3.8. *If* $f: Y \to X$ *is locally of finite-type, then the set of points* $y$ *of* $Y$, *such that* $\mathcal{O}_{Y,y}$ *is flat over* $\mathcal{O}_{X,f(y)}$ *and* $\Omega^1_{Y/X,y} = 0$, *is open in* $Y$. *Thus there is a unique largest open set* $U$ *in* $Y$ *on which* $f$ *is étale.*

*Proof*. This follows immediately from (2.16).

*Exercise* 3.9. Let $f: Y \to X$ be finite and flat, and assume that $X$ is connected. Then $f_*\mathcal{O}_Y$ is locally free, of constant rank $r$ say. Show that there is a sheaf of ideals $\mathfrak{D}_{Y/X}$ on $X$, called the *discriminant* of $Y$ over

$X$, with the property that if $U$ is an open affine in $X$ such that $B = \Gamma(f^{-1}(U), \mathcal{O}_Y)$ is free with basis $\{b_1, \ldots, b_r\}$ over $A = \Gamma(U, \mathcal{O}_X)$, then $\Gamma(U, \mathfrak{D}_{Y/X})$ is the principal ideal generated by det $(Tr_{B/A}(b_i b_j))$. Show that $f$ is unramified, hence étale, at all $y \in f^{-1}(x)$ if and only if $(\mathfrak{D}_{Y/X})_x = \mathcal{O}_{X,x}$ (use (3.1d)). Use this to show that if $f$ is unramified at all $y \in f^{-1}(x)$ for some $x \in X$, then there exists an open subset $U \subset X$ containing $x$ such that $f : f^{-1}(U) \to U$ is étale. Show that if $B = A[T]/(P(T))$ with $P$ monic, then the discriminant $\mathfrak{D}_{B/A} = (D(P))$, where $D(P)$ is the discriminant of $P$, that is, the resultant, res $(P, P')$, of $P$ and $P'$. Show also that the different $\mathfrak{d}_{B/A} = (P'(t))$ where $t = T \pmod{P}$. (See Serre [7, III.§6].)

The next proposition and its corollaries show that étale morphisms have the uniqueness properties of local isomorphisms.

PROPOSITION 3.10. *Any closed immersion $f : Y \to X$ that is flat (hence étale) is an open immersion.*

*Proof.* According to (2.12), $f(Y)$ is open in $X$ and so, after replacing $X$ by $f(Y)$, we may assume $f$ to be surjective. As $f$ is finite, $f_*\mathcal{O}_Y$ is locally free as an $\mathcal{O}_X$-module (2.9). Since $f$ is a closed immersion, this implies that $\mathcal{O}_X \approx f_*\mathcal{O}_Y$, that is, that $f$ is an isomorphism.

*Remark* 3.11. By using Zariski's main theorem, we may prove a stronger result, namely, that any étale, universally injective, separated morphism $f : Y \to X$ is an open immersion. (*Universally injective* is equivalent to *injective and all maps $k(f(y)) \to k(y)$ radicial* [EGA. I.3.7.1].) In fact, by proceeding as above, one can assume that $f$ is universally bijective, hence a homeomorphism (2.12), hence proper, and hence finite (1.10). Now $f$ being étale and radicial implies that $f_*\mathcal{O}_Y$ must be free of rank one.

COROLLARY 3.12. *If $X$ is connected and $f : Y \to X$ is étale (respectively étale and separated), then any section $s$ of $f$ is an open immersion (respectively an isomorphism onto an open connected component). Thus there is a one-to-one correspondence between the set of such sections and the set of those open (respectively open and closed) subschemes $Y_i$ of $Y$ such that $f$ induces an isomorphism $Y_i \to X$. In particular, a section is known when its value at one point is known if $f$ is separated.*

*Proof.* Only the first assertion requires proof. Assume first that $f$ is separated. Then $s$ is a closed immersion because $fs = 1$ is a closed immersion, and $f$ is separated (compare with the proof of (3.6)). According to (3.6) $s$ is étale, and hence it is an open immersion. Thus $s$ is an isomorphism onto its image, which is both open and closed in $Y$. If $f$ is only assumed to be étale, then it is separated in a neighborhood of $y$ and $x = f(y)$, and hence the above argument shows that $s$ is a local isomorphism at $x$.

COROLLARY 3.13. *Let $f, g: Y' \to Y$ be $X$-morphisms where $Y'$ is a connected $X$-scheme and $Y$ is étale and separated over $X$. If there exists a point $y' \in Y'$ such that $f(y') = g(y') = y$ and the maps $k(y) \to k(y')$ induced by $f$ and $g$ coincide, then $f = g$.*

*Proof*. The graphs of $f$ and $g$, $\Gamma_f, \Gamma_g: Y' \to Y' \times_X Y$, are sections to $p_1: Y' \times_X Y \to Y'$ The conditions imply that $\Gamma_f$ and $\Gamma_g$ agree at a point, and so $\Gamma_f$ and $\Gamma_g$ are equal (3.12). Thus $f = p_2\Gamma_f = p_2\Gamma_g = g$.

We saw in (3.4) above that given a monic polynomial $P(T)$ over $A$, it is possible to construct an étale morphism spec $C \to$ spec $A$ by taking $C = B_b$ where $B = A[T]/(P)$ and $b$ is such that $P'(T)$ is a unit in $B_b$. We shall call such an étale morphism *standard*. The interesting fact is that locally every étale morphism $Y \to X$ is standard. Geometrically this means that in a neighborhood of any point $x$ of $X$, there are functions $a_1, \ldots, a_r$ such that $Y$ is locally described by the equation $T^r + a_1T^{r-1} + \cdots + a_r = 0$, and the roots of the equation are all simple (at any geometric point).

THEOREM 3.14. *Assume that $f: Y \to X$ is étale in some open neighborhood of $y \in Y$. Then there are open affine neighborhoods $V$ and $U$ of $y$ and $f(y)$, respectively, such that $f|V: V \to U$ is a standard étale morphism.*

*Proof*. Clearly, we may assume that $Y =$ spec $C$ and $X =$ spec $A$ are affine. Also, by Zariski's main theorem (1.8), we may assume that $C$ is a finite $A$-algebra. Let $\mathfrak{q}$ be the prime ideal of $C$ corresponding to $y$. We have to show that there is a standard étale $A$-algebra $B_b$ such that $B_b \approx C_c$ for some $c \notin \mathfrak{q}$. It is easy to see (because everything is finite over $A$) that it suffices to do this with $A$ replaced by $A_\mathfrak{p}$, where $\mathfrak{p} = f^{-1}(\mathfrak{q})$, that is, that we may assume that $A$ is local and that $\mathfrak{q}$ lies over the maximal ideal $\mathfrak{p}$ of $A$.

Choose an element $t \in C$ whose image $\bar{t}$ in $C/\mathfrak{p}C$ generates $k(\mathfrak{q})$ over $k(\mathfrak{p})$, that is, $\bar{t}$ is such that $k(\mathfrak{p})[\bar{t}] = k(\mathfrak{q}) \subset C/\mathfrak{p}C$. Such an element exists because $C/\mathfrak{p}C$ is a product $k(\mathfrak{q}) \times C'$, and $k(\mathfrak{q})/k(\mathfrak{p})$ is separable. Let $\mathfrak{q}' = \mathfrak{q} \cap A[t]$. I claim that $A[t]_{\mathfrak{q}'} \to C_\mathfrak{q}$ is an isomorphism. Note first that $\mathfrak{q}$ is the only prime ideal of $C$ lying over $\mathfrak{q}'$ (in checking this, one may tensor with $k(\mathfrak{p})$). Thus the semilocal ring $C \otimes_{A[t]} A[t]_{\mathfrak{q}'}$ is actually local and so equals $C_\mathfrak{q}$. As $A[t] \to C$ is injective and finite, it follows that

$$A[t]_{\mathfrak{q}'} \to C \otimes_{A[t]} A[t]_{\mathfrak{q}'} = C_\mathfrak{q}$$

is injective and finite. It is surjective because $k(\mathfrak{q}') \to k(\mathfrak{q})$ is surjective, and Nakayama's lemma may be applied.

$A[t]$ is finite over $A$ (it is a submodule of a Noetherian $A$-module), and the isomorphism $A[t]_{\mathfrak{q}'} \to C_\mathfrak{q}$ extends to an isomorphism $A[t]_{c'} \xrightarrow{\approx} C_c$ for some $c \notin \mathfrak{q}$, $c' \notin \mathfrak{q}'$. Thus $C$ may be replaced by $A[t]$, that is, we may assume that $t$ generates $C$ over $A$.

Let $n = [k(\mathfrak{q}):k(\mathfrak{p})]$, so that $1, \bar{t}, \ldots, \bar{t}^{n-1}$ generate $k(\mathfrak{q})$ as a vector space over $k(\mathfrak{p})$. Then $1, t, \ldots, t^{n-1}$ generate $C = A[t]$ over $A$ (according to Nakayama's lemma), and so there is a monic polynomial $P(T)$ of degree $n$ and a surjection $h: B = A[T]/(P) \to C$. Clearly $\bar{P}(T)$ is the characteristic polynomial of $\bar{t}$ in $k(\mathfrak{q})$ over $k(\mathfrak{p})$ and so is separable. Thus $B_b$ is a standard étale $A$-algebra for some $b \notin h^{-1}(\mathfrak{q})$. With a suitable choice of $b$ and $c$ we get a surjection $h': B_b \to C_c$ with both $B_b$ and $C_c$ étale $A$-algebras. According to (3.6), $h'$ is étale, and ${}^a h'$: spec $C_c \to$ spec $B_b$ is a closed immersion. Hence, according to (3.10), ${}^a h'$ is an open immersion, which completes the proof.

*Remark* 3.15. The fact that $f$ was flat was used only in the last step of the above proof. Thus the argument shows that locally every unramified morphism is a composite of a closed immersion with a standard étale morphism.

COROLLARY 3.16. *A morphism $f: Y \to X$ is étale if and only if for every $y \in Y$, there exist open affine neighborhoods $V =$ spec $C$ of $y$ and $U =$ spec $A$ of $x = f(y)$ such that*

$$C = A[T_1, \ldots, T_n]/(P_1, \ldots, P_n)$$

*and* det $(\partial P_i/\partial T_j)$ *is a unit in $C$.*

*Proof*. Because of (3.4), we only have to prove the necessity. From the theorem, we may assume that $Y \to X$ is standard étale, for example $X =$ spec $A$, $Y =$ spec $C$, $C = B_b$, $B = A[T]/(P)$. Then $C \approx A[T, U]/(P(T), bU - 1)$, and the determinant corresponding to this is $P'(T)b$. Since the image of $P'(T)b$ is a unit in $C$, this proves the corollary with the added information that $n$ may be taken to be two.

With this structure theorem, it is relatively easy to prove that if $Y \to X$ is étale, then $Y$ inherits many of the good properties of $X$. (For the opposite inheritance, see (2.24).)

PROPOSITION 3.17. *Let $f: Y \to X$ be étale.*

(a) Dim $(\mathcal{O}_{Y,y}) =$ dim $(\mathcal{O}_{X,f(y)})$ *for all $y \in Y$.*

(b) *If $X$ is normal, then $Y$ is normal.*

(c) *If $X$ is regular, then $Y$ is regular.*

*Proof*. (a) We may assume that $X =$ spec $A$ where $A$ $(=\mathcal{O}_x)$ is local and that $Y =$ spec $B$. The proof uses only the assumption that $B$ is quasi-finite and flat over $A$. Let $\mathfrak{q}$ be the prime ideal of $B$ corresponding to $y$ (so $\mathfrak{q}$ lies over $\mathfrak{p}$, the maximal ideal of $A$). Then spec $B_\mathfrak{q} \to$ spec $A$ is surjective (2.7), so dim $(B_\mathfrak{q}) \geq$ dim $(A)$. Conversely we may assume $B = B'_b$, where $B'$ is finite over $A$ (1.8). Then dim $(A) \geq$ dim $(B')$ $(\geq$ dim $B_\mathfrak{q})$ (Atiyah-Macdonald [1, 5.9]).

(b) We may assume that $X =$ spec $A$ where $A$ is local (hence normal) and that $T =$ spec $C$ where $C = B_b$ is a standard étale $A$-algebra with

$B = A[T]/(P(T))$. Let $K$ be the field of fractions of $A$, let $L = C \otimes_A K = K[T]/(P(T))$, and let $A'$ be the integral closure of $A$ in $L$. Note that $L$ is a product of separable field extensions of $K$. Then we have the inclusions

$$\begin{array}{ccccc} & & C & \subset A'_b & \subset L \\ & & \cup & \cup & \\ A & \subset & B & \subset A' & \end{array}$$

Write $t = T \pmod{P(T)}$. Choose an $a \in A'$. We have to show that $a/b^s$, or equivalently just $a$, is in $C$.

Let $\bar{K}$ be the algebraic closure of $K$ and $\phi_1, \ldots, \phi_r$ the homomorphisms $L \to \bar{K}$ over $K$ such that $\phi_1(t), \ldots, \phi_r(t)$ are the roots of $P(T)$ (so $r =$ degree $P$). Write

$$a = a_0 + a_1 t + \cdots + a_{r-1} t^{r-1}, \qquad a_i \in K.$$

Then we have $r$ equations,

$$\phi_j(a) = a_0 + a_1 t_j + \cdots + a_{r-1} t_j^{r-1}$$

where $t_j = \phi_j(t)$. Let $D$ be the determinant of these equations, regarding the $a_i$ as unknowns, so that $D = \pm\prod_{i<j}(t_i - t_j)$, that is, $D^2 =$ discriminant of $P(T) = \mathfrak{D}_{B/A}$ (compare (3.9)). Since the $\phi_j(a)$ and $t_j^i$ are integral over $A$, it follows from Cramer's rule that the $Da_i$, $i = 1, \ldots, r$, are also integral over $A$. Since the $Da_i \in K$ and $A$ is normal, they belong to $A$, and this implies that $Da \in B \subset C$. Since $D$ is a unit in $C$, it follows that $a \in C$.

(c) Let $y \in Y$. Then $\dim(\mathcal{O}_{Y,y}) = \dim(\mathcal{O}_{X,f(y)})$, and $m_y = m_x\mathcal{O}_{Y,y}$ can be generated by $\dim(\mathcal{O}_{X,f(y)})$ elements.

*Remark* 3.18. An argument, similar to that in (b), shows that if $X$ is reduced, then $Y$ is reduced (Raynaud [3, p. 74]).

We now determine the structure of étale morphisms $Y \to X$ when $X$ is normal.

PROPOSITION 3.19. *Let $f: Y \to X$ be étale, where $X$ is normal. Then locally $f$ is a standard étale morphism of the form* spec $C \to$ spec $A$ *where $A$ is an integral domain, $C = B_b$, $B = A[T]/(P(T))$, and $P(T)$ is irreducible over the field of fractions of $A$.*

*Proof*. The only new fact to be shown is that $P(T)$ may be chosen to be irreducible over the field of fractions $K$ of $A$. Clearly we may reduce the problem to the case that $X =$ spec $A$ where $A$ is a local ring and assume that $Y =$ spec $C$ with $C$ a standard étale $A$-algebra, say $C = B_b$, $B = A[T]/(P(T))$ with $P(T)$ possibly reducible. Fix a prime ideal $\mathfrak{q}$ in $C$ such that $\mathfrak{p} = \mathfrak{q} \cap A$ is the maximal ideal of $A$.

Note that any monic factor $Q(T)$ of $P(T)$ in $K[T]$ automatically has coefficients in $A$. (Let $K'$ be a splitting field for $Q(T)$; the roots of $Q(T)$ in $K'$ are also roots of $P(T)$ and hence are integral over $A$; it follows that the coefficients of $Q(T)$ are also integral over $A$ since they can be expressed in terms of the roots.) Choose $P_1(T)$ to be a monic irreducible factor of $P(T)$ whose image in $k(\mathfrak{q})$ is zero, and write $P(T) = P_1(T)Q(T)$ with $P_1, Q \in A[T]$. Then the images $\bar{P}_1$ and $\bar{Q}$ of $P_1$ and $Q$ in $k(\mathfrak{p})[T]$ are coprime since $\bar{P}(T)$ is separable and so has no multiple roots. It follows that $(P_1, Q) = A[T]$ (compare (4.1a) below), and the Chinese remainder theorem shows that $B \approx A[T]/(P_1) \times A[T]/(Q)$. Let $b_1$ be the image of $b$ in $B_1 = A[T]/(P_1)$. Obviously $C_1 = (B_1)_{b_1}$ is the standard $A$-algebra sought.

THEOREM 3.20. *Let $X$ be a normal scheme and $f: Y \to X$ an unramified morphism. Then $f$ is étale if and only if, for any $y \in Y$, $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is injective.*

*Proof.* If $f$ is flat, then $\mathcal{O}_{f(y)} \to \mathcal{O}_y$ is injective according to (2.3). For the converse, note that locally $f$ factors into $Y \xrightarrow{f'} Y' \xrightarrow{g} X$ with $f'$ a closed immersion and $g$ étale (3.15). Write $A = \mathcal{O}_{X,f(y)}$; following (3.19), we may write $\mathcal{O}_{Y',f'(y)} = C_\mathfrak{q}$ where $C = A[T]/(P(T))$ with $P(T)$ irreducible over the field of fractions $K$ of $A$. We have $A \to C_\mathfrak{q} \to \mathcal{O}_{Y,y}$, which, when tensored with $K$, becomes $K \to C_\mathfrak{q} \otimes_A K \to \mathcal{O}_{Y,y} \otimes_A K$. As $A \to \mathcal{O}_{Y,y}$ is injective, $K \to \mathcal{O}_{Y,y} \otimes_A K$ is injective, which shows that $C_\mathfrak{q} \otimes_A K \to \mathcal{O}_{Y,y} \otimes_A K$ is not the zero map. But $C_\mathfrak{q} \otimes_A K = K[T]/(P)$ is a field, and so this last map is injective. Hence $C_\mathfrak{q} \to \mathcal{O}_{Y,y}$ is injective, and we already know that it is surjective because $f'$ is a closed immersion. Thus $\mathcal{O}_{Y,y} = C_\mathfrak{q}$ is flat over $A$.

THEOREM 3.21. *Let $X$ be a connected normal scheme, and let $K = R(X)$. Let $L$ be a finite separable field extension of $K$, let $X'$ be the normalization of $X$ in $L$, and let $U$ be any open subscheme of $X'$ that is disjoint from the support of $\Omega^1_{X'/X}$. Then $U \to X$ is étale, and conversely any separated étale morphism $Y \to X$ of finite-type can be written $Y = \coprod U_i \to X$ where each $U_i \to X$ is of this form.*

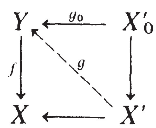
*Proof.* $\Omega^1_{U/X} = \Omega^1_{X'/X}|U = 0$, and so $U \to X$ is unramified according to (3.5). It is étale according to (3.20).

Conversely, let $Y \to X$ be separated, étale, and of finite-type. The connected components $Y_i$ of $Y$ are irreducible (because the irreducible components of $Y$ containing $y$ are in one-to-one correspondence with the minimal prime ideals of $\mathcal{O}_{Y,y}$ and $Y$ is normal). If $\operatorname{spec} L_i \to \operatorname{spec} K$ is the generic fiber of $Y_i \to X$ and $X_i$ is the normalization of $X$ in $L_i$, then Zariski's main theorem implies that $Y_i \to X_i$ is an open immersion (see (1.8), especially the proof).

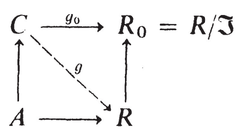
*Remark* 3.22. In [EGA. IV.17] the following functorial definitions are made. Let $X$ be a scheme and $F$ a contravariant functor $\mathbf{Sch}/X \to \mathbf{Sets}$. Then $F$ is said to be *formally smooth* (*lisse*) (respectively, *formally unramified* (*net*), *formally étale*) if for any affine $X$-scheme $X'$ and any subscheme $X'_0$ of $X'$ defined by a nilpotent ideal $\mathfrak{J}$, $F(X') \to F(X'_0)$ is surjective (respectively, injective, bijective).

A scheme $Y$ over $X$ is said to be *formally smooth*, *formally unramified*, or *formally étale* over $X$ when the functor $h_Y = \operatorname{Hom}_X(-, Y)$ it defines has the corresponding property. If, in addition, $Y$ is locally of finite presentation over $X$, then one says simply that $Y$ is smooth, unramified, or étale over $X$.

We show that a morphism $f: Y \to X$ that is étale in our sense is also étale in the above sense. (The converse, which is more difficult, may be found, for example, in Artin [9, I.1.1].) Thus, given an $X$-morphism $g_0: X'_0 \to Y$, we must show that there is a unique $X$-morphism $g: X' \to Y$ lifting it:




The uniqueness implies that it sufficies to do this locally. Thus we may assume that $f$ is standard, for example, $X = \operatorname{spec} A$, $Y = \operatorname{spec} C$, $C = B_b$, $B = A[T]/(P) = A[t]$. Let $X' = \operatorname{spec} R$, $X'_0 = \operatorname{spec} R_0$ and $R_0 = R/\mathfrak{J}$. Then we are given an $A$-homomorphism $g_0: C \to R_0$, and we want to find a unique $g: C \to R$ lifting it:




By using induction on the length of $\mathfrak{J}$, we may reduce the question to the case that $\mathfrak{J}^2 = 0$. Let $r \in R$ be such that $g_0(t) = r \pmod{\mathfrak{J}}$. We have to find an $r' \in R$ such that $r' \equiv r \pmod{\mathfrak{J}}$ and $P(r') = 0$. Write $r' = r + h$, $h \in \mathfrak{J}$. Then $h$ must satisfy the equation $P(r + h) = 0$. But $P(r + h) = P(r) + hP'(r)$, where $P(r) \in \mathfrak{J}$ and $P'(r)$ is a unit (since $P'(t) \in C^* \Rightarrow P'(r) \in R_0^*$), and so there is a unique $h$.

Alternatively, this may be proved by applying (3.12) to $Y \times_X X'/X'$.

THEOREM 3.23. (*Topological invariance of étale morphisms.*) *Let $X_0$ be the closed subscheme of a scheme $X$ defined by a nilpotent ideal. The functor $Y \mapsto Y_0 = Y \times_X X_0$ gives an equivalence between the categories of étale $X$-schemes and étale $X_0$-schemes.*

*Proof.* To give an $X$-morphism $Y \to Z$ of étale $X$-schemes is the same as to give its graph, that is, a section to $Y \times_X Z \to Y$. According to (3.12), such sections are in one-to-one correspondence with the open subschemes of $Y \times_X Z$ that map isomorphically onto $Y$. Since the same is true for $X_0$-morphisms $Y_0 \to Z_0$, it is easy to see using (3.10) or (3.11) that our functor is faithfully full. Thus it remains to show that it is essentially surjective on objects. Because of the uniqueness assertion for morphisms, it suffices to locally lift an étale $X_0$-scheme $Y_0$ to an $X$-scheme $Y$. But then we may assume that $Y_0 \to X_0$ is standard, and the assertion is obvious.

For completeness, we list some conditions equivalent to smoothness.

PROPOSITION 3.24. *Let $f: Y \to X$ be locally of finite-type. The following are equivalent:*

(a) *$f$ is smooth* (in the sense of (3.22));

(b) *for any $y \in Y$, there exist open affine neighborhoods $V$ of $y$ and $U$ of $f(y)$ such that $f|V$ factors into $V \to V' \to U \hookrightarrow X$ where $V \to V'$ is étale and $V'$ is affine $n$-space over $U$;*

(c) *for any $y \in Y$, there exist open affine neighborhoods $V =$ spec $C$ of $y$ and $U =$ spec $A$ of $x = f(y)$ such that*

$$C = A[T_1, \ldots, T_n]/(P_1, \ldots, P_m), \qquad m \leq n,$$

*and the ideal generated by the $m \times m$ minors of $(\partial P_i/\partial T_j)$ is $C$;*

(d) *$f$ is flat and for any algebraically closed geometric point $\bar{x}$ of $X$, the fiber $Y_{\bar{x}} \to \bar{x}$ is smooth;*

(e) *$f$ is flat and for any algebraically closed geometric point $\bar{x}$ of $X$, $Y_{\bar{x}}$ is regular;*

(f) *$f$ is flat and $\Omega^1_{Y/X}$ is locally free of rank equal to the relative dimension of $Y/X$.*

*Proof.* See [SGA. 1, II] or Demazure-Gabriel: [1, I. §4.4].

*Remark* 3.25. (a) In the case that $f$ is of finite-type, conditions (d) and (e) may be paraphrased by saying that $Y$ is a flat family of nonsingular varieties over $X$.

(b) Condition (b) shows that for a morphism of finite-type *étale* is equivalent to *smooth and quasi-finite.*

Finally we note that (2.25) has an analogue for smooth morphisms.

PROPOSITION 3.26. *Let $f: Y \to X$ be smooth and surjective, and assume that $X$ is quasi-compact. Then there exists an affine scheme $X'$, a surjective étale morphism $h: X' \to X$, and an $X$-morphism $g: X' \to Y$.*

*Proof.* See [EGA. IV.17.16.3].

*Exercise* 3.27. (Hochster). Let $A$ be the ring $k[T^2, T^3]$ localized at its maximal ideal $(T^2, T^3)$ (that is, $A$ is the local ring at a cusp on a curve); let $B = A[S]/(S^3Y^2 + S + T^2)$, and let $C$ be the integral closure of $A$ in $B$. Show that $B$ is étale over $A$, but that $C$ is not flat over $A$. (Hint: show that $TS$ and $T^2S$ are in $C$; hence $TS \in (T^2:T^3)_C$. If $C$ were flat over $A$, then

$$(T^2:T^3)_C = (T^2:T^3)_A C = (T^2, T^3);$$

but $TS \in (T^2, T^3)$ would imply $S \in C$.)

*Exercise* 3.28. Let $Y$ and $X$ be smooth varieties over a field $k$; show that a morphism $Y \to X$ is étale if and only if it induces an isomorphism on tangent spaces for any closed point of $Y$.

*Exercise* 3.29. Do Hartshorne [2, III. Ex. 10.6].

## §4. Henselian Rings

Throughout this section, $A$ will be a local ring with maximal ideal $\mathfrak{m}$ and residue field $k$. The homomorphisms $A \to k$ and $A[T] \to k[T]$ will be written as $(a \mapsto \bar{a})$ and $(f \mapsto \bar{f})$.

Two polynomials $f(T)$, $g(T)$ with coefficients in a ring $B$ are *strictly coprime* if the ideals $(f)$ and $(g)$ are coprime in $B[T]$, that is, if $(f, g) = B[T]$. For example, $f(T)$ and $T - a$ are coprime if and only if $f(a) \neq 0$ and are strictly coprime if and only if $f(a)$ is a unit in $B$.

If $A$ is a complete discrete valuation ring, then Hensel's lemma (in number theory) states the following: if $f$ is a monic polynomial with coefficients in $A$ such that $\bar{f}$ factors as $\bar{f} = g_0 h_0$ with $g_0$ and $h_0$ monic and coprime, then $f$ itself factors as $f = gh$ with $g$ and $h$ monic and such that $\bar{g} = g_0$, $\bar{h} = h_0$. In general, any local ring $A$ for which the conclusion of Hensel's lemma holds is said to be *Henselian.*

*Remark* 4.1. (a) The $g$ and $h$ in the above factorization are strictly coprime. More generally, if $f, g \in A[T]$ are such that $\bar{f}, \bar{g}$ are coprime in $k[T]$ and $f$ is monic, then $f$ and $g$ are strictly coprime in $A[T]$. Indeed, let $M = A[T]/(f, g)$. As $f$ is monic, this is a finitely generated $A$-module; as $(\bar{f}, \bar{g}) = k[T]$, $(f, g) + \mathfrak{m}A[T] = A[T]$ and $\mathfrak{m}M = M$, and so Nakayama's lemma implies that $M = 0$.

(b) The factorization $f = gh$ is unique, for let $f = gh = g'h'$ with $g$, $h$, $g'$, $h'$ all monic, $\bar{g} = \bar{g}'$, $\bar{h} = \bar{h}'$, and $\bar{g}$ and $\bar{h}$ coprime. Then $g$ and $h'$ are strictly coprime in $A[T]$, and so there exist $r, s \in A[T]$ such that $gr + h's = 1$. Now

$$g' = g'gr + g'h's = g'gr + ghs,$$

and so $g$ divides $g'$. As both are monic and have the same degree, they must be equal.

THEOREM 4.2. *Let $x$ be the closed point of $X =$ spec $A$. The following are equivalent:*

(a) *$A$ is Henselian;*

(continued...)

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