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The h-cobordism theorem can be generalized in several directions. No one has succeeded in removing the restriction that V and V' have dimension > 4. (See page 113.) If we omit the restriction that V and (hence) V' be simply connected, the theorem becomes false. (See Milnor [34].) But it will remain true if we at the same time assume that the inclusion of V (or V') into W is a simple homotopy equivalence in the sense of J. H. C. Whitehead. This generalization, called the s-cobordism theorem, is due to Mazur [35], Barden [33] and Stallings. For this and further generalizations see especially Wall [36]. Lastly, we remark that analogous h- and s-cobordism theorems hold for piecewise linear manifolds.

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Section 1. The Cobordism Category

First some familiar definitions. Euclidean space will be denoted by $R^n = \{(x_1,\ldots,x_n) | x_i \in R,\ i = 1,\ldots,n\}$ where R = the real numbers, and Euclidean half-space by $R^n_+ = \{(x_1,\ldots,x_n) \in R^n | x_n \geq 0\}$.

Definition 1.1. If V is any subset of R^n, a map $f: V \longrightarrow R^m$ is smooth or differentiable of class C^∞ if f can be extended to a map $g: U \longrightarrow R^m$, where $U \supset V$ is open in R^n, such that the partial derivatives of g of all orders exist and are continuous.

Definition 1.2. A smooth n-manifold is a topological manifold W with a countable basis together with a smoothness structure $\mathcal{S}$ on M. $\mathcal{S}$ is a collection of pairs (U,h) satisfying four conditions:

(1) Each $(U,h) \in \mathcal{S}$ consists of an open set $U \subset W$ (called a coordinate neighborhood) together with a homeomorphism h which maps U onto an open subset of either R^n or R^n_+ .

(2) The coordinate neighborhoods in $\mathcal{S}$ cover W.

(3) If (U_1,h_1) and (U_2,h_2) belong to $\mathcal{S}$, then

$$h_1 h_2^{-1}: h_2(U_1 \cap U_2) \longrightarrow R^n \text{ or } R^n_+$$

is smooth.

(4) The collection $\mathcal{S}$ is maximal with respect to property (3); i.e. if any pair (U, h) not in $\mathcal{S}$ is adjoined to $\mathcal{S}$, then property (3) fails.

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The <u>boundary of W</u>, denoted $\mathrm{Bd}\, W$, is the set of all points in W which do not have neighborhoods homeomorphic to R^n (see Munkres [5, p.8]).

<u>Definition 1.3</u>. $(W; V_0, V_1)$ is a <u>smooth manifold triad</u> if W is a compact smooth n-manifold and $\mathrm{Bd}\, W$ is the disjoint union of two open and closed submanifolds V_0 and V_1 .

If $(W; V_0, V_1)$, $(W'; V_1', V_2')$ are two smooth manifold triads and $h: V_1 \longrightarrow V_1'$ is a diffeomorphism (i.e. a homeomorphism such that h and h^{-1} are smooth), then we can form a third triad $(W \cup_h W'; V_0, V_2')$ where $W \cup_h W$ is the space formed from W and W' by identifying points of V_1 and V_1' under h, according to the following theorem.

<u>Theorem 1.4</u>. <u>There exists a smoothness structure</u> $\mathcal{S}$ <u>for</u> $W \cup_h W'$ <u>compatible with the given structures</u> (i.e. <u>so that each inclusion map</u> $W \longrightarrow W \cup_h W'$, $W' \longrightarrow W \cup_h W'$ <u>is a diffeomorphism onto its image</u>.)

$\mathcal{S}$ is unique up to a diffeomorphism leaving V_0, $h(V_1) = V_1'$, <u>and</u> V_2' <u>fixed</u>.

The proof will be given in § 3 .

<u>Definition 1.5</u>. Given two closed smooth n-manifolds M_0 and M_1 (i.e. M_0 , M_1 compact, $\mathrm{Bd}\, M_0 = \mathrm{Bd}\, M_1 = \emptyset$), a <u>cobordism</u> from M_0 to M_1 is a 5-tuple, $(W; V_0, V_1; h_0, h_1)$, where $(W; V_0, V_1)$ is a smooth manifold triad and $h_i: V_i \longrightarrow M_i$ is a diffeomorphism, $i = 0, 1$. Two cobordisms $(W; V_0, V_1; h_0, h_1)$ and $(W'; V_0', V_1'; h_0', h_1')$ from M_0 to M_1 are <u>equivalent</u> if there exists a diffeomorphism $g: W \longrightarrow W'$ carrying V_0 to V_0' and V_1 to

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V_1' such that for $i = 0,1$ the following triangle commutes:

$$\begin{tikzcd}
V_i \arrow[rr, "g|V_i"] \arrow[dr, "h_i"'] & & V_i' \arrow[dl, "h_i'"] \\
& M_i &
\end{tikzcd}$$

Then we have a category (see Eilenberg and Steenrod, [2,p.108]) whose objects are closed manifolds and whose morphisms are equivalence classes c of cobordisms. This means that cobordisms satisfy the following two conditions. They follow easily from 1.4 and 3.5, respectively.

(1) Given cobordism equivalence classes c from M_0 to M_1 and c' from M_1 to M_2, there is a well-defined class cc' from M_0 to M_2. This composition operation is associative.

(2) For every closed manifold M there is the identity cobordism class ι_M = the equivalence class of $(M \times I;\ M \times 0,\ M \times 1;\ p_0,\ p_1)$, $p_i(x,i) = x$, $x \in M$, $i = 0,1$. That is, if c is a cobordism class from M_1 to M_2, then

$$\iota_{M_1} c = c = c\,\iota_{M_2} .$$

Notice that it is possible that $cc' = \iota_M$, but c is not ι_M. For example

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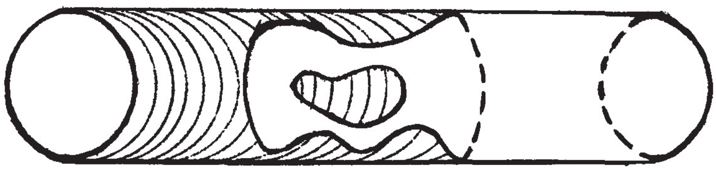
FIGURE 1



c is shaded. c' is unshaded.

Here c has a right inverse c', but no left inverse. Note that the manifolds in a cobordism are not assumed connected.

Consider cobordism classes from M to itself, M fixed. These form a monoid H_M, i.e. a set with an associative composition with an identity. The invertible cobordisms in H_M form a group G_M. We can construct some elements of G_M by taking $M = M'$ below.

Given a diffeomorphism $h: M \longrightarrow M'$, define c_h as the class of $(M \times I; M \times 0, M \times 1; j, h_1)$ where $j(x,0) = x$ and $h_1(x,1) = h(x)$, $x \in M$.

<u>Theorem 1.6.</u> $c_h c_{h'} = c_{h'h}$ <u>for any two diffeomorphisms</u> $h: M \longrightarrow M'$ <u>and</u> $h': M' \longrightarrow M''$.

<u>Proof</u>: Let $W = M \times I \cup_h M' \times I$ and let $j_h: M \times I \longrightarrow W$, $j_{h'}: M' \times I \longrightarrow W$ be the inclusion maps in the definition of $c_h c_{h'}$. Define $g: M \times I \longrightarrow W$ as follows:

$$g(x,t) = j_h(x,2t) \qquad 0 \leq t \leq \tfrac{1}{2}$$

$$g(x,t) = j_{h'}(h(x),2t-1) \qquad \tfrac{1}{2} \leq t \leq 1 .$$

Then g is well-defined and is the required equivalence.

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Definition 1.7. Two diffeomorphisms $h_0, h_1: M \longrightarrow M'$ are (smoothly) isotopic if there exists a map $f: M \times I \longrightarrow M'$ such that

(1) f is smooth,

(2) each f_t, defined by $f_t(x) = f(x,t)$, is a diffeomorphism,

(3) $f_0 = h_0$, $f_1 = h_1$.

Two diffeomorphisms $h_0, h_1: M \longrightarrow M'$ are pseudo-isotopic* if there is a diffeomorphism $g: M \times I \longrightarrow M' \times I$ such that $g(x,0) = (h_0(x),0)$, $g(x,1) = (h_1(x),1)$.

Lemma 1.8. Isotopy and pseudo-isotopy are equivalence relations.

Proof: Symmetry and reflexivity are clear. To show transitivity, let $h_0, h_1, h_2: M \longrightarrow M'$ be diffeomorphisms and assume we are given isotopies $f, g: M \times I \longrightarrow M'$ between h_0 and h_1 and between h_1 and h_2 respectively. Let $m: I \longrightarrow I$ be a smooth monotonic function such that $m(t) = 0$ for $0 \leq t \leq 1/3$, and $m(t) = 1$ for $2/3 \leq t \leq 1$. The required isotopy $k: M \times I \longrightarrow M'$ between h_0 and h_1 is now defined by $k(x,t) = f(x,m(2t))$ for $0 \leq t \leq 1/2$, and $k(x,t) = g(x,m(2t-1))$ for $1/2 \leq t \leq 1$. The proof of transitivity for pseudo-isotopies is more difficult and follows from Lemma 6.1 of Munkres [5,p.59].

* In Munkres' terminology h_0 is "I-cobordant" to h_1 . (See [5,p.62].) In Hirsch's terminology h_0 is "concordant" to h_1.

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It is clear that if h_0 and h_1 are isotopic then they are pseudo-isotopic, for if $f: M \times I \longrightarrow M'$ is the isotopy, then $\hat{f}: M \times I \longrightarrow M' \times I$, defined by $\hat{f}(x,t) = (f_t(x),t)$, is a diffeomorphism, as follows from the inverse function theorem, and hence is a pseudo-isotopy between h_0 and h_1. (The converse for $M = S^n$, $n \geq 8$ is proved by J. Cerf [39].) It follows from this remark and from 1.9 below that if h_0 and h_1 are isotopic, then $c_{h_0} = c_{h_1}$.

<u>Theorem 1.9.</u> $c_{h_0} = c_{h_1} \Longleftrightarrow h_0$ <u>is pseudo-isotopic to</u> h_1

<u>Proof</u>: Let $g: M \times I \longrightarrow M' \times I$ be a pseudo-isotopy between h_0 and h_1. Define $h_0^{-1} \times 1: M' \times I \longrightarrow M \times I$ by $(h_0^{-1} \times 1)(x,t) = (h_0^{-1}(x),t)$. Then $(h_0^{-1} \times 1) \circ g$ is an equivalence between c_{h_1} and c_{h_0}.

The converse is similar.

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