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Introduction

The study of *exponential sums* over finite fields goes back to Gauss. The importance of estimating them goes back at least to Kloosterman’s 1926 paper [K1]. In the one-variable case, it was understood by Hasse and Davenport in 1934 [HD] that the good estimate would result from the proof of the Riemann Hypothesis for curves over finite fields. That proof was supplied by Weil in 1945 [Weil1]. See also Weil’s 1948 paper [Weil2], whose Math Review was written by Kloosterman. The following year, Weil explained [Weil3] what should be true for projective smooth varieties of any dimension over finite fields, in what came to be known as the Weil Conjectures. The next big advance came with Grothendieck’s invention, and the development by his school, of ℓ -adic cohomology and its sheaf-theoretic setting, cf. [SGA4, 7.2]. This setting allowed Deligne to prove the Riemann Hypothesis part of the Weil Conjectures in the general case, cf. [De1, 1.6]. Deligne then vastly generalized his result to the setting of *ℓ -adic sheaves* in [De2, 3.3.1], and used this generalization to prove the Sato-Tate Conjecture for elliptic curves over function fields, cf. [De2, 3.5.7]. To do this, Deligne brings to bear the *arithmetic and geometric monodromy groups* attached to a lisse sheaf which is “pure of weight zero,” and shows that determining these groups is precisely what leads to equidistribution theorems in the function field case.

At this point, let us clarify the notion of “pure of integer weight w ” for a lisse $\overline{\mathbb{Q}}_\ell$ sheaf \mathcal{F} on a smooth, geometrically connected X/\mathbb{F}_q . The requirement is that for **every** field embedding $\iota : \overline{\mathbb{Q}}_\ell \subset \mathbb{C}$, the following condition holds: for every finite extension L/\mathbb{F}_q , and every point $x \in X(L)$, the eigenvalues of $\text{Frob}_{x,L}$ on \mathcal{F} all have, via ι , complex absolute value $(\#L)^{w/2}$. Note that

if an element $\alpha \in \overline{\mathbb{Q}_\ell}$ has $|\iota(\alpha)|_{\mathbb{C}}$ independent of ι , then α is an algebraic number, all of whose conjugates (as algebraic numbers) have the same complex absolute value as each other.

Another key output of the ℓ -adic theory is the ability to interpret a parametrized family of exponential sums as the Frobenius traces of an ℓ -adic sheaf on the parameter space, and to control the open set on which this sheaf is a *local system*. Moreover, the results of Weil and Deligne will ensure that this local system, after a partial Tate twist, is pure of weight zero. One then obtains equidistribution results for the family of exponential sums in question, as soon as one computes the arithmetic and geometric monodromy groups of the local system in question.

The families of exponential sums we will deal with in this book will typically have parameter space either the affine line \mathbb{A}^1 or the multiplicative group $\mathbb{G}_m := \mathbb{A}^1 \setminus \{0\}$ over a finite field k , of characteristic $p > 0$. Their incarnating sheaves will be ℓ -adic local systems on the parameter space, for any choice we like of a prime $\ell \neq p$.

Given a prime p , it was conjectured by Abhyankar [**Abh**] and proven by Raynaud [**Ray**] (see also [**Pop**]) that any finite group G which is generated by its Sylow p -subgroups occurs as a quotient of the fundamental group of the affine line $\mathbb{A}^1/\overline{\mathbb{F}_p}$. The analogous result for the multiplicative group \mathbb{G}_m , also conjectured by Abhyankar and proven by Harbater [**Har**], is that any finite group G which, modulo the subgroup $\mathbf{O}^{p'}(G)$ generated by its Sylow p -subgroups, is cyclic, occurs as a quotient of the fundamental group of $\mathbb{G}_m/\overline{\mathbb{F}_p}$. In the ideal world, given such a finite group G , and a complex representation V of G , we would be able, for any prime $\ell \neq p$, to choose an embedding of \mathbb{C} into $\overline{\mathbb{Q}_\ell}$, and to write down an explicit $\overline{\mathbb{Q}_\ell}$ -local system on either $\mathbb{A}^1/\overline{\mathbb{F}_p}$ or on $\mathbb{G}_m/\overline{\mathbb{F}_p}$ whose geometric monodromy group is G , in the given representation.

Needless to say, we do not live in the ideal world. On $\mathbb{G}_m/\overline{\mathbb{F}_p}$, the simplest local systems are the *hypergeometric sheaves*. They are simplest in the sense that among irreducible local systems of rank > 1 , they are characterized by having their H_c^1 of minimum possible dimension, namely 1, cf. [**Ka-ESDE**, 8.5.3]. So it is natural to investigate their monodromy groups. A key step in this investigation is to show that monodromy groups of a wide class

of hypergeometric sheaves \mathcal{H} satisfy the group-theoretic *condition* **(S+)**, cf. Theorem 5.2.9 for the precise statement. [Condition **(S+)** is a slight strengthening of condition **(S)** introduced in **[GT3]**, and roughly speaking, corresponds to Aschbacher’s class **S** of maximal subgroups of classical groups **[Asch]**.] When **(S+)** holds, it imposes strong restrictions on the pair $(G_{\text{geom}}, \mathcal{H})$. If G_{geom} is infinite, then the identity component G_{geom}° of G_{geom} is a simple algebraic group, still acting irreducibly. If $G := G_{\text{geom}}$ is finite, then either G is *almost quasisimple* (that is, there is some non-abelian simple group S such that $S \triangleleft G/\mathbf{Z}(G) \leq \text{Aut}(S)$), or G is an “*extraspecial normalizer*,” in particular, the dimension of the representation is a prime power r^n and there is an extraspecial r -group E in G of order r^{1+2n} acting irreducibly.

The converse question of which (complex or modular) representations of almost quasisimple groups satisfy condition **(S)** is of great importance to the Aschbacher-Scott program of classifying maximal subgroups of finite classical groups, and ultimately to primitive permutation group theory. We refer the reader to **[T]** for a detailed account of this problem. The complex representations of almost quasisimple groups that can arise in the hypergeometric context have been classified in **[KT5]**, see §3.1; for these representations condition **(S+)** is established in Theorem 3.1.6. We also note that the full extraspecial normalizers in $\text{GL}_{r^n}(\mathbb{C})$, respectively in $\text{Sp}_{r^n}(\mathbb{C})$ and $\text{O}_{r^n}(\mathbb{C})$, satisfy **(S)**; see **[KIL, Proposition 7.6.2]** for the result in the more general situation of ℓ -modular representations with $\ell \neq r$.

In studying local systems and their monodromy groups, there are two kinds of natural questions which arise. The first is this: given a simple (in the sense of simple to remember) local system, determine its monodromy group.

One of the main themes of this book, along the lines of the first kind of question, is to investigate what are arguably the simplest one-parameter families $\mathcal{F}(A, B, \chi)$ of exponential sums, those of the form

$$(0.0.0.1) \quad t \mapsto - \sum_x \psi(x^A + tx^B) \chi(x),$$

for given prime to p integers $A > B > 0$ with $\text{gcd}(A, B) = 1$, a fixed additive character ψ , and a given multiplicative character χ .

It turns out that these families are Kummer pullbacks of hypergeometric sheaves, cf. Theorem 10.1.1. This relation allows us, in §§10.2, 10.3, to completely determine their monodromy groups. In turn, building on these one-parameter results, in Chapter 11 we complete the classification of all multi-parameter families $\mathcal{F}(A, B_1, \dots, B_r, \chi)$ of exponential sums

$$(0.0.0.2) \quad (t_1, t_2, \dots, t_r) \mapsto - \sum_x \psi(x^A + t_1 x^{B_1} + \dots + t_r x^{B_r}) \chi(x)$$

that admit finite monodromy, and the determination of the corresponding geometric monodromy groups G_{geom} .

The second kind of natural question is this: given a finite group G together with a faithful irreducible representation V satisfying **(S+)**, construct a simple (again, in the sense of simple to remember) local system whose monodromy is (G, V) , if such a local system exists. This second question, when G is almost quasisimple, has already been the subject of a number of papers by the authors, some jointly with Antonio Rojas-León, cf. the Bibliography. Investigation of the other **(S+)** case, when G is an extraspecial normalizer, is a second main theme of this book.

Let us now turn to a more detailed description of the contents of this book. We work with geometrically irreducible hypergeometric sheaves \mathcal{H} on \mathbb{G}_m , i.e., those that are lisse on \mathbb{G}_m and whose G_{geom} acts irreducibly. At the possible expense of interchanging 0 and ∞ on \mathbb{G}_m by inversion, we may and will assume \mathcal{H} is of type (D, m) with $D > m$. One knows [Ka-ESDE, 8.4.11] that if G_{geom} is finite, then a generator of local monodromy at 0 is an element of G_{geom} which has all distinct eigenvalues in the given representation (a *simple spectrum element*). In general, such a generator has *regular spectrum*, in the sense of Definition 1.1.5.

Our first main result, Theorem 2.4.4, shows that if such a sheaf \mathcal{H} in characteristic p has wild part of dimension w with $1 \leq w < (p-1)/2$, then its geometric monodromy group G_{geom} is either infinite, or finite but imprimitive (unless \mathcal{H} has rank 2 and $p = 5$). This result can be viewed as a hypergeometric version of the celebrated result [FT] of Feit and Thompson on linear groups of degree $< (p-1)/2$.

Building on [KT5, Theorem 7.4], our Theorem 3.1.10 shows that if $D \geq 11$ and \mathcal{H} has a finite geometric monodromy group

G_{geom} which is almost quasisimple of Lie type in some characteristic r , then the characteristic of \mathcal{H} must necessarily be r , aside from three exceptions for $D = 12$ and $D = 14$. A similar result for hypergeometric sheaves with G_{geom} an extraspecial normalizer was established in [KT5, Theorem 9.19].

Our next result, Theorem 3.3.4, extending prior work of Howe [HS, Theorem 4.6.3], gives a full classification of representations of (not necessarily connected) simple algebraic groups that admit elements with regular spectrum.

The next main result, Theorem 5.2.9, vastly generalizing earlier related results in [KT5], shows that any geometrically irreducible hypergeometric sheaf of type (D, m) with $D > m$ satisfies $(\mathbf{S}+)$, as long as it is primitive and has rank $\neq 4, 8, 9$.

In Chapter 6, we determine, in Theorem 6.2.14, the possible identity component G_{geom}° of G_{geom} for a hypergeometric sheaf \mathcal{H} of type (D, m) with $D > m$ satisfying $(\mathbf{S}+)$ whose G_{geom} is infinite. Recall from [GT3, Proposition 2.8] that $(\mathbf{S}+)$ (which by Theorem 5.2.9 is automatic so long as $D \neq 4, 8, 9$ and \mathcal{H} is primitive) implies that G_{geom}° is a simple algebraic group acting irreducibly. In [Ka-ESDE, 7.2.7], it is proved that in *sufficiently large* (depending on $w := D - m$) characteristic p , the only such possibilities for the given representation of G_{geom}° are one of the classical groups SL_D , SO_D , or Sp_D for even D , in the standard D -dimensional representation or its dual; or G_2 in its 7-dimensional representation; or SL_3 in its 8-dimensional adjoint representation; or Spin_7 in its 8-dimensional spin representation. Removing the constraint on size of the characteristic p , Theorem 6.2.14, shows that, aside from a few possible low-rank exotic exceptions in characteristics $p = 2, 3$, these are the only possibilities.

Chapter 7 is devoted to the study of the extraspecial normalizer case in odd characteristic, with Theorem 7.3.5 as the principal result. Perhaps not surprisingly, the study of the extraspecial normalizer case in characteristic $p = 2$ is hugely more complicated, and takes up Chapters 8 and 9. Among other results, in parallel with the approach of [KT7], we are able to realize in Theorem 9.1.11 the extraspecial normalizers $2_+^{1+2nf} \cdot \Omega_{2n}^+(2^f)$ as geometric monodromy groups of hypergeometric sheaves, whereas type $-$ extraspecial normalizers $2_-^{1+2nf} \cdot \Omega_{2n}^-(2^f)$ are realized in Theorem 8.5.5 following the approach of [KT6]. Furthermore,

a novel use of Witt vectors allows us to produce, for the first time, explicit local systems with geometric monodromy groups of shape $(4 * 2_-^{1+2nf}) \cdot \mathrm{Sp}_{2n}(2^f)$, see Theorem 9.3.9.

Chapter 10 is devoted to computing the monodromy groups of the one-parameter families $\mathcal{F}(A, B, \chi)$ in (0.0.0.1). The main results are Theorems 10.2.4 and 10.2.7 (for exponents $A > B = 1$), and Theorems 10.3.13, 10.3.14, and 10.3.21 (for exponents $A > B > 1$). In particular, the list of (A, χ) for which the local system $\mathcal{F}(A, 1, \chi)$ in (0.0.0.1) has finite monodromy, previously conjectured in [KT1] and [R-L], is proved to be complete. We also show (see Lemmas 10.3.15, 10.3.16, 10.3.17, and 10.3.18) that the exotic possibilities for $p = 2, 3$ in Theorem 6.2.14 do not occur in the context of the one-parameter systems $\mathcal{F}(A, B, \chi)$. Multi-parameter analogues of these results for the families $\mathcal{F}(A, B_1, \dots, A_r, \chi)$ in (0.0.0.2) are obtained in Chapter 11

Chapter 12 is devoted to treating some of the very few cases of families with *non-monomial* perturbing terms where we can say anything at all. This is very much an area in which much remains to be done. The proofs of the main results in this chapter, Theorems 12.2.3 and 12.3.6, once again highlight the importance of the moment $M_{2,2}$ in the study of the G_{geom} of local systems. In addition, Theorems 12.5.4, 12.5.5, 12.5.11, and 12.5.12 determine geometric monodromy groups for some special classes of two-parameter local systems with non-monomial coefficients. This theme will be further explored in the forthcoming paper [KT9].

The appendices consist of two Magma programs.

A word about notation. Throughout the book, we use \mathcal{F} for a local system which is pure of some integer weight, and we use \mathcal{G} to denote a suitable constant field twist of \mathcal{F} which is pure of weight zero. The two are geometrically isomorphic, so have the same geometric monodromy group G_{geom} , but their arithmetic monodromy groups $G_{\mathrm{arith}, \mathcal{F}}$ and $G_{\mathrm{arith}, \mathcal{G}}$ may differ. [They will coincide if \mathcal{F} is itself pure of weight zero and we take $\mathcal{G} := \mathcal{F}$.] When \mathcal{F} has nonzero weight, the group $G_{\mathrm{arith}, \mathcal{F}}$ is never finite, indeed never has a semisimple identity component, simply because its determinant is pure of nonzero weight. It is only $G_{\mathrm{arith}, \mathcal{G}}$ which can ever be finite.

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