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Chapter One

Introduction

The notion of height functions on projective varieties was created by Weil [Wei51] as a measurement for the complexity of solutions to Diophantine equations. Despite its modern name, "height" is closely related to the method of "infinite descent", and thus has its roots in Euclid's proof of the irrationality of $\sqrt{2}$ more than two thousand years ago and Fermat's proof that $x^4 + y^4 = z^4$ has no solutions in positive integers more than three hundred years ago. It was implicitly used in Weil's proof of the Mordell–Weil theorem in [Wei29] in 1928, Siegel's theorem on integral points on curves in 1929, and Northcott's further works on the Northcott property and arithmetic dynamical systems in [Nor49, Nor50] in 1949–1950. Since its creation, the theory of heights has been widely used in Diophantine geometry, such as in Roth's theorem and Schmidt's subspace theorem and the formulation of the Birch and Swinnerton-Dyer conjecture. For more details, see our reviews in §A.1.

In the 1970s, to translate the proof of Mordell's conjecture from function fields to number fields, Arakelov [Ara74] proposed an intersection theory for arithmetic surfaces. In this theory, the heights are interpreted as degrees of hermitian line bundles on arithmetic curves. Faltings [Fal83] used this interpretation to define a special height function in his proof of the Mordell conjecture in 1983. After quick developments by Faltings [Fal84], Deligne [Del85], Szpiro [Szp85], and Gillet–Soulé [GS90], the Arakelov theory was used again by Vojta [Voj91] for a second proof of the Mordell conjecture and by Faltings [Fal91] for an extension to the Mordell–Lang conjecture for the subvarieties of Abelian varieties. For more details, see our reviews in §A.2 and §A.3.

In the late 1980s, Szpiro [Szp90] proposed a program to prove the Bogomolov conjecture for small points on curves using arithmetic positivity. The Bogomolov conjecture was eventually reduced to a positivity statement after the development of arithmetic ampleness for adelic line bundles by Zhang [Zha92, Zha93, Zha95a, Zha95b], which was an extension of the classical Arakelov theory to handle bad reductions of abelian varieties and algebraic dynamical systems. Shortly after, the Bogomolov conjecture was eventually proved by Ullmo [Ull98] and Zhang [Zha98] using a new ingredient, the equidistribution theorem initiated by Szpiro–Ullmo–Zhang [SUZ97]. The arithmetic positivity and equidistribution theorem were further extended by the Yuan [Yua08] to big line bundles and were widely used to treat problems in arithmetic dynamical systems. For more details, see our reviews in §A.4, §A.5, and §A.6.

The goal of this book is to extend the theory of Zhang [Zha95b] from projective varieties over number fields to *quasi-projective* varieties over *finitely generated fields*. More precisely, let F be a finitely generated field over $\mathbb{Q}$ (or a constant field), and let X be a quasi-projective variety over F. We introduce a notion of *adelic line bundles* on X, consider their intersection theory, study their volumes for effective sections, and introduce heights associated to them. Fundamental properties behind these terms are various notions of positivity of adelic line bundles.

An immediate application of our framework is a theory of canonical heights on polarized algebraic dynamical systems over quasi-projective varieties over finitely generated fields. In particular, we introduce Néron–Tate heights of abelian varieties over finitely generated fields and extend the arithmetic Hodge index theorem of Faltings [Fal84] and Hriljac [Hri85] to this setting. Furthermore, we prove an equidistribution theorem of small points on quasi-projective varieties over number fields, generalizing the equidistribution theorems of Szpiro–Ullmo–Zhang [SUZ97], Chambert-Loir [CL06], and Yuan [Yua08].

The exposition of this book uses a combination of algebraic geometry, complex algebraic geometry, Arakelov theory (cf. [Ara74, GS90]), and Berkovich analytic spaces (cf. [Ber90, Ber09]). In the following, we sketch the main constructions and theorems of this book.

1.1 ADELIC LINE BUNDLES

To illustrate the concept quickly, we will take an approach different from that of the major parts of this book, but it will give equivalent constructions.

We will use a *uniform terminology*, which will be explained in detail in §1.5, to treat both algebraic varieties and arithmetic varieties. Namely, we fix a base scheme $\operatorname{Spec} k$, where k is either $\mathbb{Z}$ or a field. By a *variety X over k*, we mean an integral scheme X which is finite, flat, and of finite type over $\operatorname{Spec} k$. We say that X is an arithmetic (resp. algebraic) variety if $k = \mathbb{Z}$ (resp. k is a field). We say that X is a *projective variety over k* if the structure morphism $X \to \operatorname{Spec} k$ is projective; we say that X is a *quasi-projective variety over k* if X is an open subscheme of a projective variety over k. In particular, when X is a quasi-projective variety over $\mathbb{Z}$, we allow finitely many fibers of $X \to \operatorname{Spec} \mathbb{Z}$ to be empty.

In the arithmetic situation, we could take a fancier notation $k = \mathbb{F}_1$, the field of one element, and thus $\operatorname{Spec} \mathbb{Z}$ becomes an *affine arithmetic curve* over $\mathbb{F}_1$. The curve $\operatorname{Spec} \mathbb{Z}$ can be further compactified over $\mathbb{F}_1$ by adding an archimedean place. So projective varieties over $\mathbb{Z}$ can be further *compactified* by adding complex varieties over archimedean places. This is the main motivation for Arakelov to introduce hermitian line bundles on projective varieties; see our reviews in §A.2 and §A.3. Sometimes, we need to deal with projective varieties over $\mathbb{Z}[1/N]$ for some positive integer N, for example, when we treat abelian varieties and algebraic dynamical systems. In this case, we need to add p-adic metrics to the missing primes. This is the main motivation for Zhang to introduce adelic line bundles for projective varieties; see our reviews in §A.5.

The goal of this section is to sketch our theory of adelic line bundles on a quasi-projective variety X over k. Adelic line bundles are roughly defined as the "limits" of "projective models" under our "boundary topology." There are two natural approaches to define these limits. The first approach is by an abstract notion of completion by Cauchy sequences (combined with two processes of direct limits), and this is the main approach of this book precisely realized in Chapter 2. The second approach is to "put" all "projective models" into the category $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ of metrized line bundles on a large analytic space X^{an}. It turns out that $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ is big enough to contain all the limiting line bundles. This is essentially treated in Chapter 3. Each approach has its advantages. We will take the second approach in this introduction.

1.1.1 Berkovich spaces

Let k be either $\mathbb{Z}$ or a field. Let X/k be a quasi-projective variety. There is a natural *Berkovich analytic space* X^{an} associated to X/k. In fact, if X has an open affine cover $\{\mathrm{Spec}\, A_i\}_i$, then $X^{\mathrm{an}} = \cup_i \mathcal{M}(A_i)$, where $\mathcal{M}(A_i)$ is the set of multiplicative semi-norms $|\cdot|$ on A_i; if k is a field, we further require the restriction of $|\cdot|$ to k trivial. A *metrized line bundle on* X^{an} is a pair $\overline{L} = (L, \|\cdot\|)$ consisting of a line bundle L on X and a continuous metric $\|\cdot\|$ of L on X^{an}. Denote by $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ the category of metrized line bundles on X, in which a morphism between two objects is defined to be an isometry. There is a forgetful functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}}) \longrightarrow \mathcal{P}\mathrm{ic}(X).$$

Here, $\mathcal{P}\mathrm{ic}(X)$ denotes the category of line bundles on X, in which a morphism between two objects is an isomorphism of line bundles.

1.1.2 Model adelic line bundles

Let k be either $\mathbb{Z}$ or a field. Let X/k be a quasi-projective variety. Objects of the category $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ are too general for intersection theory. Instead, we will define a full subcategory $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$ of adelic line bundles in $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$, and a full subcategory $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{int}}$ of integrable adelic line bundles in $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ for intersection theory. For this, we will start with model adelic line bundles and take a limit process to extend them to more general notions.

As a convention, we will write tensor products of various line bundles additively, so, for example, mL means $L^{\otimes m}$ for $L \in \mathcal{P}\mathrm{ic}(X)$ and $m \in \mathbb{Z}$.

An object of $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ with underlying line bundle $L \in \mathcal{P}\mathrm{ic}(X)$ is called a *model adelic line bundle* if it is induced by a projective model $(\mathcal{X}, \overline{\mathcal{L}})$ of (X, eL) over k for some positive integer e, where $\mathcal{X}$ is a projective variety over k with an open immersion $X \hookrightarrow \mathcal{X}$, and $\overline{\mathcal{L}}$ is as follows:

(1) if k is a field, then $\overline{\mathcal{L}} = \mathcal{L}$ is a line bundle on $\mathcal{X}$ extending eL;
(2) if $k = \mathbb{Z}$, then $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ is a hermitian line bundle on $\mathcal{X}$, which consists of a line bundle $\mathcal{L}$ on $\mathcal{X}$ extending eL and a continuous hermitian metric $\|\cdot\|$ of $\mathcal{L}(\mathbb{C})$ on $\mathcal{X}(\mathbb{C})$. The metric is required to be invariant under the complex conjugate.

Because of the integer e in the definition, $(\mathcal{X}, e^{-1}\overline{\mathcal{L}})$ is a projective model of (X, L) in terms of the notion of $\mathbb{Q}$-line bundles. Denote by $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{mod}}$ the full subcategory of $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ consisting of model adelic line bundles on X.

1.1.3 Limit process

Let k be either $\mathbb{Z}$ or a field. Let X/k be a quasi-projective variety. Choose a projective compactification $X \subset \mathcal{X}_0$ such that the boundary $\mathcal{X}_0 \setminus X$ is exactly equal to the support of an effective Cartier divisor $\mathcal{E}_0$ on $\mathcal{X}_0$. If k is a field, set $\overline{\mathcal{E}}_0 = \mathcal{E}_0$. If $k = \mathbb{Z}$, set $\overline{\mathcal{E}}_0 = (\mathcal{E}_0, g_0)$, where $g_0 > 0$ is a Green function of $\mathcal{E}_0(\mathbb{C})$ on $\mathcal{X}_0(\mathbb{C})$. Then $\overline{\mathcal{E}}_0$ induces a Green function $\widetilde{g}_0$ of $\mathcal{E}_0$ on $\mathcal{X}_0^{\mathrm{an}}$, which restricts to a continuous function $\widetilde{g}_0 : X^{\mathrm{an}} \to \mathbb{R}_{\geq 0}$.

Consider the space $C(X^{\mathrm{an}})$ of real-valued continuous functions on X^{an}. It is endowed with a *boundary topology* induced by the extended norm

$$\|f\|_{\widetilde{g}_0} := \sup_{x \in X^{\mathrm{an}},\ \widetilde{g}_0(x) > 0} \frac{|f(x)|}{\widetilde{g}_0(x)}.$$

We refer to [Bee15] for the basics of extended norms, which are allowed to take values in $[0, \infty]$ but still required to satisfy the triangle inequality. The boundary topology is independent of the choice of $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$. Moreover, $C(X^{\mathrm{an}})$ is complete under the boundary topology.

We say that a sequence $\overline{\mathcal{L}}_i = (\mathcal{L}_i, \|\cdot\|_i)$ in $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ *converges* to an object $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ in $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ if there are isomorphisms $\tau_i : \mathcal{L} \to \mathcal{L}_i$ such that the sequence $-\log(\tau_i^* \|\cdot\|_i / \|\cdot\|)$ converges to 0 in $C(X^{\mathrm{an}})$ under the boundary topology.

1.1.4 Adelic line bundles

There is a notion of nefness of hermitian line bundles on projective arithmetic varieties, and we refer to §A.4.1 for a quick definition.

An object of $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ is called an *adelic line bundle on* $\mathcal{U}$ if it is isomorphic to the limit of a sequence in $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{mod}}$. An adelic line bundle on X is called *strongly nef* if it is isomorphic to the limit of a sequence in $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{mod}}$ induced by projective models $(\mathcal{X}_i, \overline{\mathcal{L}}_i)$ over k such that $\overline{\mathcal{L}}_i$ is nef on $\mathcal{X}_i$. An adelic line bundle $\overline{\mathcal{L}}$ on X is called *nef* if there exists a strongly nef adelic line bundle $\overline{\mathcal{M}}$ on X such that $a\overline{\mathcal{L}} + \overline{\mathcal{M}}$ is strongly nef for all positive integers a. An adelic line bundle on X is called *integrable* if it is isometric to $\overline{\mathcal{L}}_1 - \overline{\mathcal{L}}_2$ for two strongly nef adelic line bundles $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ on X.

Denote by $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$ the full subcategory of $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ consisting of adelic line bundles on X. Denote by $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{nef}}$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{int}}$) the full subcategory of $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ consisting of nef (resp. integrable) adelic line bundles on X. Their objects are called *adelic line bundles* (resp. *nef adelic line bundles*, *integrable adelic line bundles*) on X/k.

We can further extend the definition to quasi-projective varieties over finitely generated fields. Namely, let F be a finitely generated field over k, i.e., a finitely generated field over the fraction field of k. Let X be a quasi-projective variety over F. Then we define

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k) := \varinjlim_{U\to V} \widehat{\mathcal{P}\mathrm{ic}}(U/k),$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{nef}} := \varinjlim_{U\to V} \widehat{\mathcal{P}\mathrm{ic}}(U/k)_{\mathrm{nef}},$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{int}} := \varinjlim_{U\to V} \widehat{\mathcal{P}\mathrm{ic}}(U/k)_{\mathrm{int}}.$$

Here, the limit is over all flat morphisms $U \to V$ of quasi-projective varieties over k whose generic fibers are isomorphic to $X \to \operatorname{Spec} F$. Denote by $\widehat{\mathrm{Pic}}(X/k)$ (resp. $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{nef}}$, $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{int}}$, $\widehat{\mathrm{Pic}}(X^{\mathrm{an}})$) the *group* of isomorphism classes of objects of $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{nef}}$, $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{int}}$, $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$). Note that the previous definitions of the analytic terms X^{an} and $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ are actually valid in the current situation.

As we have seen, our theory of adelic line bundles is valid for both quasi-projective varieties over k and quasi-projective varieties over finitely generated fields over k. We will introduce a natural notion of *essentially quasi-projective varieties over* k, which includes both of the above cases. For simplicity, we will not use this notion in this chapter.

1.1.5 Functoriality

Let E/F be an extension of finitely generated fields over k, and $f : X \to Y$ be an F-morphism of quasi-projective varieties X/E and Y/F. Then we have a pull-back functor

$$f^* : \widehat{\mathcal{P}\mathrm{ic}}(Y/k) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X/k).$$

When $k = \mathbb{Z}$, we can also have a base change of a quasi-projective scheme X/k to the generic fiber $X_{\mathbb{Q}}/\mathbb{Q}$. In this case, we denote the functor as

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X_{\mathbb{Q}}/\mathbb{Q}), \quad \overline{L} \longmapsto \widetilde{L}.$$

We call $\widetilde{L}$ *the geometric part of* $\widehat{L}$.

Both functors preserve the subcategories of the model (resp. nef, integrable) adelic line bundles.

1.2 INTERSECTION THEORY AND HEIGHTS

Our intersection theory includes an absolute intersection pairing of Gillet–Soulé and a relative intersection pairing that extends the Deligne pairing.

1.2.1 Intersection numbers and heights

Let k be either $\mathbb{Z}$ or a field. Let F be a finitely generated field over k. Let X be a quasi-projective variety over F. Then our absolute intersection pairing is a symmetric and multi-linear map

$$\widehat{\mathrm{Pic}}(X/k)^{d}_{\mathrm{int}} \longrightarrow \mathbb{R},$$

where $d = \dim X + \dim k + \operatorname{tr}\deg_k F$. Here $\dim k$ denotes the Krull dimension of k, and $\operatorname{tr}\deg_k F$ denotes the transcendence degree of F over the fraction field of k. This is the limit version of the intersection theory in algebraic geometry and the arithmetic intersection theory of Gillet–Soulé. See Proposition 4.1.1.

Now let K be a number field if $k = \mathbb{Z}$; let K be a function field of one variable over k if k is a field. Let X be a quasi-projective variety over K of dimension n. Let $\overline{L}$ be an integrable adelic line bundle on X. Define a *height function*

$$h_{\overline{L}} : X(\overline{K}) \longrightarrow \mathbb{R}$$

by

$$h_{\overline{L}}(x) := \frac{\widehat{\deg}(\overline{L}|_{x'})}{\deg(x')}.$$

Here x' denotes the closed point of X containing x, $\deg(x')$ denotes the degree of the residue field of x' over K, $\overline{L}|_{x'}$ denotes the *pull-back* of $\overline{L}$ to $\widehat{\mathrm{Pic}}(x'/k)_{\mathrm{int}}$, and $\widehat{\deg} : \widehat{\mathrm{Pic}}(x'/k)_{\mathrm{int}} \to \mathbb{R}$ is by the intersection theory.

More generally, for any closed $\overline{K}$-subvariety Z of X, define *the height of Z for $\overline{L}$* by

$$h_{\overline{L}}(Z) := \frac{(\overline{L}|_{Z'})^{\dim Z+1}}{(\dim Z+1)(\widetilde{L}|_{Z'_K})^{\dim Z}}.$$

Here Z' denotes the image of $Z \to X$ (which is a closed subvariety of X over K), and

$$\overline{L} \longmapsto \overline{L}|_{Z'} \longmapsto \widetilde{L}|_{Z'_K}$$

denotes the image of $\overline{L}$ via the functorial maps

$$\widehat{\mathrm{Pic}}(X/k)_{\mathrm{int}} \longrightarrow \widehat{\mathrm{Pic}}(Z'/k)_{\mathrm{int}} \longrightarrow \widehat{\mathrm{Pic}}(Z'_K/K)_{\mathrm{int}},$$

and the self-intersections are as in the above intersection theory. The height is well-defined only if the denominator is nonzero.

1.2.2 Deligne pairing and relative heights

Let k be either $\mathbb{Z}$ or a field. Let $f : X \to Y$ be a projective and flat morphism of relative dimension n between quasi-projective varieties over k. Assume that Y is normal, which is required in our proof.

Theorem 1.2.1 (Theorem 4.1.3)**.** *The Deligne pairing on model adelic line bundles induces a symmetric and multi-linear functor*

$$\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(Y)_{\mathrm{int}}.$$

When restricted to nef adelic line bundles, the functor induces a functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{nef}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(Y)_{\mathrm{nef}}.$$

Moreover, the functors are compatible with base changes of the form $Y' \to Y$, where Y' is a quasi-projective normal variety over k such that $X' = X \times_Y Y'$ is integral.

In the setting of the theorem, let $F = k(Y)$ be the function field of Y, and $X_F \to \operatorname{Spec} F$ the generic fiber of $X \to Y$. Let $\overline{L}$ be an object of $\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}}$. By this, we can define a *vector-valued* height function

$$\mathfrak{h}_{\overline{L}} : X(\overline{F}) \longrightarrow \widehat{\mathrm{Pic}}(F/k)_{\mathrm{int},\mathbb{Q}}.$$

Here the group

$$\widehat{\mathrm{Pic}}(F)_{\mathrm{int}} := \varinjlim_{U} \widehat{\mathrm{Pic}}(U/k)_{\mathrm{int}},$$

where U runs through all open subschemes of Y.

More generally, for any closed $\overline{F}$-subvariety Z of X_F, define *the vector-valued height of Z for $\overline{L}$* as

$$\mathfrak{h}_{\overline{L}}(Z) := \frac{\langle \overline{L}|_{Z'} \rangle^{\dim Z+1}}{(\dim Z+1)(\overline{L}|_{Z'_F})^{\dim Z}} \in \widehat{\mathrm{Pic}}(F)_{\mathrm{int},\mathbb{Q}}.$$

Here Z' denotes the image of $Z \to X$, Z'_F is the generic fiber of $Z' \to Y$, and

$$\overline{L} \longmapsto \overline{L}|_{Z'} \longmapsto \overline{L}|_{Z'_F}$$

denotes the image of $\overline{L}$ via the functorial maps

$$\widehat{\mathrm{Pic}}(X/k)_{\mathrm{int}} \longrightarrow \widehat{\mathrm{Pic}}(Z'/k)_{\mathrm{int}} \longrightarrow \mathrm{Pic}(Z'_F/F).$$

Note that the first self-intersection is the Deligne pairing, and the second self-intersection is just the degree on the projective variety Z'_F in the classical sense. The height is well-defined only if the denominator is nonzero.

When F is polarized in the sense of Moriwaki [Mor00], we can also define the Moriwaki heights. If K is a number field or a finite field, and if X is projective over F, we obtain a Northcott property of the Moriwaki heights from that of [Mor00]. In general, we obtain the fundamental inequality for the Moriwaki height following the strategy of [Mor00].

Hence, part of our height theory extends the previous works of Moriwaki [Mor00, Mor01]. In fact, [Mor00, Mor01] developed a height theory for projective varieties over finitely generated fields F over $\mathbb{Q}$, depending on the choice of an arithmetic polarization of $\mathrm{Spec}\, F$. His motivation was to apply Arakelov geometry to varieties over arbitrary fields (of characteristic 0), and he succeeded in formulating and proving the Bogomolov conjecture in that setting. His treatment was more on the numerical theory of heights, but ours is more on the geometric theory of adelic line bundles.

1.3 VOLUMES AND EQUIDISTRIBUTION

As in the projective case, we can define effective sections of adelic line bundles, study their volumes, and prove equidistribution theorems on quasi-projective varieties.

1.3.1 Volumes

Let X be a quasi-projective variety over k. Let $\overline{L} = (L, \|\cdot\|)$ be an adelic line bundle on X. Define

$$\widehat{H}^0(X, \overline{L}) := \{s \in H^0(X, L) : \|s(x)\| \leq 1,\ \forall\, x \in X^{\mathrm{an}}\}.$$

Elements of $\widehat{H}^0(X, \overline{L})$ are called *effective sections* of $\overline{L}$ on X. If $k = \mathbb{Z}$, denote

$$\widehat{h}^0(X, \overline{L}) := \log \# \widehat{H}^0(X, \overline{L});$$

if k is a field, denote

$$\widehat{h}^0(X, \overline{L}) := \dim_k \widehat{H}^0(X, \overline{L}).$$

We check that $\widehat{h}^0(X, \overline{L})$ is always a finite real number. In this setting, we have the following fundamental results.

Theorem 1.3.1 (Theorem 5.2.1, Theorem 5.2.2)**.** *Let X be a quasi-projective variety over k. Let $\overline{L}, \overline{M}$ be adelic line bundles on X. Denote $d = \dim X + \dim k$. Then the following holds.*

(1) The limit

$$\widehat{\mathrm{vol}}(X, \overline{L}) = \lim_{m\to\infty} \frac{d!}{m^d} \widehat{h}^0(X, m\overline{L})$$

exists.

(2) If $\overline{L}$ is the limit of a sequence of model adelic line bundles induced by a sequence $\{(\mathcal{X}_i, \overline{\mathcal{L}}_i)\}_{i\geq 1}$ of projective models of (X, L) over k, then

$$\widehat{\mathrm{vol}}(X, \overline{L}) = \lim_{i\to\infty} \widehat{\mathrm{vol}}(\mathcal{X}_i, \overline{\mathcal{L}}_i).$$

(3) If $\overline{L}$ is nef, then

$$\widehat{\mathrm{vol}}(X, \overline{L}) = \overline{L}^d.$$

(4) If $\overline{L}, \overline{M}$ are nef, then

$$\widehat{\mathrm{vol}}(X, \overline{L} - \overline{M}) \geq \overline{L}^d - d\,\overline{L}^{d-1}\overline{M}.$$

Part (1) generalizes the classical result of Fujita (cf. [Laz04b, 11.4.7]) for line bundles on projective varieties and the result of [Che08, Che10, Yua09] for hermitian line bundles on projective arithmetic varieties. Part (2) allows us to transfer many previous results in the projective case to the quasi-projective case. Part (3) generalizes the classical Hilbert–Samuel formula in algebraic geometry and the arithmetic Hilbert–Samuel formula proved by Gillet–Soulé [GS92], Bismut–Vasserot [BV89], and Zhang [Zha95a]. Part (4) generalizes the classical theorem of Siu [Siu93] and the arithmetic bigness theorem of Yuan [Yua08].

In the setting of the theorem, we say that $\overline{L}$ is *big* if $\widehat{\mathrm{vol}}(X, \overline{L}) > 0$. We will see that, in this case, we will have nice lower bounds of the height function associated to $\overline{L}$.

1.3.2 Height inequality

Let K be a number field if $k = \mathbb{Z}$; let K be a function field of one variable over k if k is a field. Let X be a quasi-projective variety over K. For an adelic line bundle $\overline{L}$ on X/k, we usually denote by $\widetilde{L}$ the *geometric part* of $\overline{L}$ on X/K, i.e., the image of $\overline{L}$ under the functorial map $\widehat{\mathcal{P}\mathrm{ic}}(X/k) \to \widehat{\mathcal{P}\mathrm{ic}}(X/K)$.

As a quick consequence of the above fundamental results on volumes, we have the following height inequality.

Theorem 1.3.2 (Theorem 5.3.7). *Let $\pi : X \to S$ be a morphism of quasi-projective varieties over K. Let $\overline{L} \in \widehat{\mathrm{Pic}}(X)$ and $\overline{M} \in \widehat{\mathrm{Pic}}(S)$ be adelic line bundles. If $\overline{L}$ is nef on X/k and $\widetilde{L}$ is big on X/K, then for any $c > 0$, there exist $\epsilon > 0$ and a non-empty open subvariety U of X such that*

$$h_{\overline{L}}(x) \geq \epsilon\, h_{\overline{M}}(\pi(x)) - c, \qquad \forall\, x \in U(\overline{K}).$$

We refer to Theorem 5.3.7 for various versions of the height inequality and to Theorem 5.3.8 for a partial converse to the height inequality.

In a series of works, Dimitrov–Gao–Habegger [GH19, DGH21] and Kühne [Kuh21] proved a uniform Bogomolov conjecture over number fields, and combined the work of Vojta [Voj91] on the Mordell conjecture to confirm Mazur's uniform Mordell conjecture. A key result of [DGH21] is a height inequality in a setting of abelian schemes, which also plays a fundamental role in the further work [Kuh21]. Our current height inequality can be viewed as a theoretical version of that of [DGH21]. We will come back to this connection later, and we refer to the context of Theorem 6.2.2 for more details on this connection and for the application of our height inequality to dynamical systems.

In recent work, Yuan [Yua21] has used our theory of adelic line bundles to prove the uniform Bogomolov conjecture over global fields. This gives a new proof of the main results of [DGH21, Kuh21], and it works uniformly for both number fields and function fields of any characteristic. The key ingredient of [Yua21] is proving the bigness of the admissible canonical bundle of the universal curve using the Deligne pairing and apply the bigness to obtain a certain height inequality to control the number of points of small heights. We refer to §2.6.5 for a brief introduction to the admissible canonical bundle.

In recent work, Gao–Zhang [GZh24] proved a Northcott property for Gross–Schoen cycles and Ceresa cycles parametrized by a non-empty open subset of moduli spaces of curves of genus at least 3. One key ingredient in their proof is applying our height inequality to convert the positivity properties of adelic line bundles to a Northcott property.

1.3.3 Equidistribution

One of the most important theorems of this book is an equidistribution theorem for small points of a quasi-projective variety over a number field or a function field of one variable.

Theorem 1.3.3 (Theorem 5.4.3). *Let k be either $\mathbb{Z}$ or a field. Let K be a number field if $k = \mathbb{Z}$; let K be the function field of one variable over k if k is a field. Let X be a quasi-projective variety over K. Let $\overline{L}$ be a nef adelic line bundle on X such that $\deg_{\widetilde{L}}(X) > 0$. Let $\{x_m\}_m$ be a generic sequence in $X(\overline{K})$ such that $\{h_{\overline{L}}(x_m)\}_m$ converges to $h_{\overline{L}}(X)$. Then the Galois orbit of $\{x_m\}_m$ is equidistributed in $X_{K_v}^{\mathrm{an}}$ for $d\mu_{\overline{L},v}$ for any place v of K.*

Here $d\mu_{\overline{L},v}$ is a canonical probability measure on $X_{K_v}^{\mathrm{an}}$, defined using the recent theory of Chambert-Loir and Ducros in [CLD12] if v is non-archimedean. This generalizes the Monge–Ampère measure and the Chambert-Loir measure from the projective case to the quasi-projective case.

If $k = \mathbb{Z}$ and X is projective over K, the equidistribution theorem is proved by Szpiro–Ullmo–Zhang [SUZ97], Chambert-Loir [CL06], and Yuan [Yua08]. Our current theorem still follows the variational principle of the pioneering work [SUZ97], applying our adelic Hilbert–Samuel formula and adelic bigness theorem.

We can further generalize our equidistribution theorem in two different ways, which give us an equidistribution theorem (Theorem 5.4.6) and an equidistribution conjecture (Conjecture 5.4.1). The equidistribution theorem considers a projective and flat morphism of quasi-projective varieties over a number field or a function field of one variable, and its proof follows a strategy of Moriwaki [Mor00]. The equidistribution conjecture considers projective varieties over finitely generated fields, and is stated as follows.

Conjecture 1.3.4 (Conjecture 5.4.1)**.** *Let k be either $\mathbb{Z}$ or a field. Let F be a finitely generated field over k. Let v be a non-trivial valuation of F. Assume that the restriction of v to k is trivial if k is a field. Let X be a projective variety over F. Let $\overline{L}$ be a nef adelic line bundle on X such that L is big on X. Let $\{x_m\}_m$ be a generic and small sequence in $X(\overline{F})$. Then the Galois orbit of $\{x_m\}_m$ is equidistributed in $X_{F_v}^{\mathrm{an}}$ for $d\mu_{\overline{L},v}$.*

We refer to the context of Conjecture 5.4.1 for the notion of "small sequence" and the equilibrium measure $d\mu_{\overline{L},v}$.

1.4 ALGEBRAIC DYNAMICS

Here we apply the theory of adelic line bundles to algebraic dynamics.

1.4.1 Algebraic dynamics

Let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective variety over k with function field F. Let (X, f, L) be a *polarized algebraic dynamical system* over S, i.e., X is a flat and projective integral scheme over S, $f : X \to X$ is an endomorphism over S, and L is an f-ample $\mathbb{Q}$-line bundle satisfying $f^*L \simeq qL$ for some rational number $q > 1$.

By Tate's limiting argument, we can construct a canonical adelic $\mathbb{Q}$-line bundle $\overline{L}_f \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q},\mathrm{nef}}$ extending L which is f-invariant in that $f^*\overline{L}_f \simeq q\overline{L}_f$. Here $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q},\mathrm{nef}}$ denotes the sub-semigroup of $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ consisting of positive rational multiples of elements of $\widehat{\mathrm{Pic}}(X)_{\mathrm{nef}}$.

For any closed $\overline{F}$-subvariety Z of X_F, we have the canonical height

$$\mathfrak{h}_f(Z) = \mathfrak{h}_{\overline{L}_f}(Z) := \frac{\langle \overline{L}_f|_{Z'}^{\dim Z+1}\rangle}{(\dim Z+1)\deg_L(Z'_F)} \in \widehat{\mathrm{Pic}}(F)_{\mathbb{Q},\mathrm{nef}}.$$

In particular, we have a height function

$$\mathfrak{h}_f : X(\overline{F}) \longrightarrow \widehat{\mathrm{Pic}}(F)_{\mathbb{Q},\mathrm{nef}}.$$

Tate's limiting argument also explains these heights.

The height function $\mathfrak{h}_f$ is f-invariant. As a consequence, $\mathfrak{h}_f(x) = 0$ for a preperiodic point $x \in X(\overline{F})$. In the minimal case that K is a number field or a function field of one variable over a finite field k, $\mathfrak{h}_f$ satisfies the Northcott property. In this case, $\mathfrak{h}_f(x) = 0$ for a point $x \in X(\overline{F})$ implies that x is preperiodic under f.

1.4.2 Equidistribution of small points

Our equidistribution conjecture naturally implies an equidistribution conjecture of preperiodic points.

Conjecture 1.4.1 (Conjecture 6.1.5)**.** *Let k be either $\mathbb{Z}$ or a field. Let F be a finitely generated field over k. Let (X, f, L) be a polarized algebraic dynamical system over F. Let v be a non-trivial valuation of F. Assume that the restriction of v to K is trivial if k is a field. Let $\{x_m\}_m$ be a generic sequence of preperiodic points in $X(\overline{F})$. Then the Galois orbit of $\{x_m\}_m$ is equidistributed in $X_{F_v}^{\mathrm{an}}$ for the canonical measure $d\mu_{L,f,v}$.*

As an example of our equidistribution theorem (cf. Theorem 5.4.3), we deduce the following equidistribution theorem of small points on non-degenerate subvarieties in a family of polarized algebraic dynamical systems.

Theorem 1.4.2 (Theorem 6.2.3)**.** *Let S be a quasi-projective variety over a number field K. Let (X, f, L) be a polarized algebraic dynamical system over S. Let Y be a non-degenerate closed subvariety of X over K. Let $\{y_m\}_{m\geq 1}$ be a generic sequence of $Y(\overline{K})$ such that $h_{\overline{L}_f}(y_m) \to 0$. Then for any place v of K, the Galois orbit of $\{y_m\}_{m\geq 1}$ is equidistributed on the analytic space Y_v^{an} for the canonical measure $d\mu_{\overline{L}_f|_Y,v}$.*

In the theorem, a closed subvariety Y of X is called *non-degenerate* if $\deg_{\widetilde{L}}(Y) > 0$. This is equivalent to the property that $\widetilde{L}|_Y$ is *big*. If X is an abelian scheme and K is a number field, our definition of "non-degenerate" agrees with that of [DGH21], which uses Betti maps in the complex analytic setting.

The theorem generalizes the equidistribution theorem of DeMarco–Mavraki [DMM20] for families of elliptic curves, and confirms the conjecture (REC) of

Kühne [Kuh21] for abelian schemes. A weaker version of the theorem for abelian schemes is proved by [Kuh21, Thm. 1] independently and applied to prove a uniform Bogomolov conjecture after the work of Dimitrov–Gao–Habegger [DGH21], as mentioned above. The proof of [Kuh21] is a limit version of the original proof in [SUZ97] and uses a result of Dimitrov–Gao–Habegger [DGH21] for uniformity in the limit process. Inspired by our formulation, Gauthier [Gau21] has extended the equidistribution theorem of [Kuh21] to more general settings, which has a large overlap with our equidistribution theorem.

1.4.3 Heights of points of a non-degenerate subvariety

Let k be either $\mathbb{Z}$ or a field. Let K be a number field if $k = \mathbb{Z}$ or a function field of one variable if k is a field. Let S be a quasi-projective variety over K. Let (X, f, L) be a polarized algebraic dynamical system over S. Let Y be a closed subvariety of X over K.

Suppose Y is a section of $X \to S$. In that case, our vector-valued height of adelic line bundles generalizes and reinterprets the Tate–Silverman specialization theorem of [Tat83, Sil92, Sil94a, Sil94b], and the work [DMM20] from families of elliptic curves to families of algebraic dynamical systems. See Lemma 6.2.1 for more details.

As mentioned above, if X is an abelian scheme and Y is non-degenerate in X, there is a height inequality of points of Y by [GH19, DGH21], which plays a fundamental role in the treatment of the uniform Mordell conjecture in [DGH21, Kuh21]. In terms of our theory, we have a simple interpretation of the height inequality, which is also valid in families of algebraic dynamical systems. As the non-degeneracy is interpreted as the bigness of $\widetilde{L}|_Y$, the height inequality is also interpreted by the bigness of some adelic line bundle. Applying Theorem 1.3.2(2) to the morphism $Y \to S$ and the adelic line bundle $\overline{L}_f|_Y$ on Y, we can have a lower bound of the canonical height of points on Y by Weil heights on S. See Theorem 6.2.2 for more details.

1.4.4 Equidistribution of PCF maps

Let S be a smooth and quasi-projective variety over a number field K. Let $X = \mathbb{P}^1_S$ be the projective line over S, and let $f : X \to X$ be a finite morphism over S of degree $d > 1$. A point $y \in S(\overline{K})$ is called *post-critically finite* (PCF) if all the ramification points (i.e., critical points) of $f_y : X_y \to X_y$ are preperiodic under f_y.

Denote by $\mathcal{M}_d$ the moduli space over K of endomorphisms of $\mathbb{P}^1$ of degree d. Inside $\mathcal{M}_d$, there is a closed subvariety corresponding to flexible Lattés maps. By the moduli property, there is a morphism $S \to \mathcal{M}_d$.

The main result here is the following equidistribution theorem of Galois orbits of PCF points.

Theorem 1.4.3 (Theorem 6.3.5). *Assume that the morphism $S \to \mathcal{M}_d$ is generically finite and its image is not contained in the flexible Lattès locus. Let $\{y_m\}_m$ be a generic sequence of PCF points of $S(\overline{K})$. Then the Galois orbit of $\{y_m\}_m$ is equidistributed in $S_{K_v}^{\mathrm{an}}$ for $d\mu_{\overline{M},v}$ for any place v of K.*

If S is a family of *polynomial* maps on $\mathbb{P}^1$, the theorem was previously proved by Favre–Gauthier [FG15]. Their strategy was to reduce the problem to the equidistribution of Yuan [Yua08].

Now we explain our proof of the theorem, which will also introduce the key term $\overline{M}$ in the statement. Denote by R the ramification divisor of the finite morphism $f : X \to X$, viewed as a (possibly non-reduced) closed subscheme in X. Then R is finite and flat of degree $2d-2$ over S, and the fiber R_y of R above any point $y \in S$ is the ramification divisor of $f_y : X_y \to X_y$.

Let L be a $\mathbb{Q}$-line bundle on X, of degree 1 on fibers of $X \to S$, such that $f^*L \simeq dL$. Denote by $\overline{L} = \overline{L}_f$ the f-invariant extension of L in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q},\mathrm{nef}}$ such that $f^*\overline{L} \simeq d\overline{L}$. Denote

$$\overline{M} := N_{R/S}(\overline{L}|_R) \in \widehat{\mathrm{Pic}}(S)_{\mathbb{Q},\mathrm{nef}}.$$

Here the norm map is the Deligne pairing of relative dimension 0.

Consider the height function

$$h_{\overline{M}} : S(\overline{K}) \longrightarrow \mathbb{R}_{\geq 0}.$$

For any $y \in S(\overline{K})$, the height $h_{\overline{M}}(y) = 0$ if and only if y is PCF in S. Then we are in the situation to apply the previous equidistribution theorem (Theorem 5.4.3) to $(S, \overline{M})$, except that we need to check the condition $\deg_{\widetilde{M}}(S) > 0$.

This requires the bifurcation measure originally introduced by DeMarco [DeM01, DeM03] and further studied by Bassanelli–Berteloot [BB07]. The degree $\deg_{\widetilde{M}}(S)$ is exactly equal to the total volume of the bifurcation measure on $S_\sigma(\mathbb{C})$ for any embedding $\sigma : K \to \mathbb{C}$. Then $\deg_{\widetilde{M}}(S) > 0$ is eventually equivalent to the condition on $S \to \mathcal{M}_g$ by the works of [BB07, GOV20]. This proves the theorem and confirms that the equilibrium measure $d\mu_{\overline{M},\sigma}$ is a constant multiple of the bifurcation measure for any embedding $\sigma : K \to \mathbb{C}$.

Recently, Ji–Xie [JX23] proved the dynamical Andre–Oort conjecture for one-dimensional families, which relies on our theory of adelic line bundles and especially our equidistribution theorem of PCF points.

In the end, we note that the theorem also holds for a family of morphisms on $\mathbb{P}^n$ with a slightly weaker statement. In particular, the construction of the adelic line bundle $\overline{M}$ works in the same way. We refer to Theorem 6.3.4 for more details.

1.4.5 Hodge index theorem on curves

In the end, we present our generalization of the arithmetic Hodge index theorem of Faltings [Fal84] and Hriljac [Hri85] to finitely generated fields. We refer to Theorem 6.5.1 for a detailed account.

Let k be either $\mathbb{Z}$ or a field. Let F be a finitely generated field over k, and let $\pi : X \to \operatorname{Spec} F$ be a smooth, projective, and geometrically connected curve of genus $g > 0$. Denote by $J = \underline{\mathrm{Pic}}^0_{X/F}$ the Jacobian variety of X and by Θ the symmetric theta divisor on J. By the dynamical system $(J, [2], \Theta)$, we have a Néron–Tate height function

$$\hat{\mathfrak{h}} : \mathrm{Pic}^0(X_{\overline{F}}) \longrightarrow \widehat{\mathrm{Pic}}(F/k)_{\mathbb{Q},\mathrm{nef}}.$$

The height function is quadratic, as in the classical case.

Theorem 1.4.4 (Theorem 6.5.1)**.** *Let k be either $\mathbb{Z}$ or a field. Let F be a finitely generated field over k, and let $\pi : X \to \operatorname{Spec} F$ be a smooth, projective, and geometrically connected curve. Let M be a line bundle on X with* $\deg M = 0$. *Then there is an adelic line bundle $\overline{M}_0 \in \widehat{\mathrm{Pic}}(X/k)_{\mathrm{int},\mathbb{Q}}$ with underlying line bundle M such that*

$$\pi_*\langle \overline{M}_0, \overline{V}\rangle = 0, \quad \forall \overline{V} \in \widehat{\mathrm{Pic}}(X/k)_{\mathrm{vert},\mathbb{Q}}.$$

Moreover, for such an adelic line bundle,

$$\pi_*\langle \overline{M}_0, \overline{M}_0\rangle = -2\,\hat{\mathfrak{h}}(M).$$

In the theorem, $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{vert},\mathbb{Q}}$ is the space of vertical adelic line bundles defined as the kernel of the forgetful map $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{int},\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}}$.

1.5 NOTATION AND TERMINOLOGY

We will introduce a uniform system of terminology and notations for both the arithmetic and geometric cases. To achieve this, we need to abuse terminology frequently.

Our base ring k is either $\mathbb{Z}$ or an arbitrary field. This is divided into two cases:

(1) (arithmetic case) $k = \mathbb{Z}$. In this case, the adelic line bundles will be limits of hermitian line bundles on projective integral schemes over $\mathbb{Z}$. This limit process obtains the intersection theory.
(2) (geometric case) k is an arbitrary field of arbitrary characteristic. In this case, the adelic line bundles will be the limit of usual line bundles on projective varieties over k. This limit process obtains the intersection theory.

By a *finitely generated field F over k*, we mean a field F which is finitely generated over the fraction field of k. For any integral scheme X over k, denote by $k(X)$ the *function field* of X.

By a *projective variety* over k, we mean an integral, flat, and projective scheme over k. By a *quasi-projective variety* over k, we mean an open subscheme of projective variety over k. For a quasi-projective variety $\mathcal{U}$ over k, a *projective model* means a projective variety $\mathcal{X}$ over k endowed with an open immersion $\mathcal{U} \to \mathcal{X}$ over k. In the arithmetic case ($k = \mathbb{Z}$), we may also use the terms *quasi-projective arithmetic variety* and *projective arithmetic variety* to emphasize the situation.

In the arithmetic case, for a projective arithmetic variety $\mathcal{X}$ over $\mathbb{Z}$, we have the group $\widehat{\mathrm{Div}}(\mathcal{X})$ of arithmetic divisors on $\mathcal{X}$, and the group $\widehat{\mathrm{Pic}}(\mathcal{X})$ and the category $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ of hermitian line bundles on $\mathcal{X}$.

In the geometric case, for a projective variety $\mathcal{X}$ over a field k, an arithmetic divisor means a Cartier divisor, a hermitian line bundle means a line bundle, and we write $\widehat{\mathrm{Div}}(\mathcal{X})$, $\widehat{\mathrm{Pic}}(\mathcal{X})$, $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ for $\mathrm{Div}(\mathcal{X})$, $\mathrm{Pic}(\mathcal{X})$, $\mathcal{P}\mathrm{ic}(\mathcal{X})$. We take this convention in other similar situations.

This abuse of notation is only one-way. For example, by Div, Pic, or $\mathcal{P}$ic in the arithmetic case, we still mean the ones without the archimedean components.

Below are a few conventions that are not directly related to the base k but are adopted throughout this book.

(1) Denote $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ for any abelian group M. Adopt similar conventions for $M_{\mathbb{R}}$ and $M_{\mathbb{C}}$.
(2) For any field K, we fix an algebraic closure $\overline{K}$ of K throughout this book.
(3) Except in §2.7 and §3.6, all schemes are assumed to be noetherian.
(4) By a *variety* over a field, we mean an integral scheme, separated and of finite type over the field. We do not require it to be geometrically integral.
(5) By a *curve* over a field, we mean a variety over the field of dimension 1.
(6) All *divisors* in this book are *Cartier divisors*, unless otherwise instructed.
(7) By a *line bundle* on a scheme, we mean an invertible sheaf on the scheme. We often write or mention tensor products of line bundles additively, so $a\mathcal{L} - b\mathcal{M}$ means $\mathcal{L}^{\otimes a} \otimes \mathcal{M}^{\otimes(-b)}$.
(8) All the categories of (adelic, metrized) line bundles are groupoids, so the morphisms are isomorphisms.
(9) A *functor* between two categories may also be called a map or a homomorphism sometimes.

(continued...)

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