Contents

CHAPTER 1 Introduction

This book provides an intermediate introduction to investments, asset pricing, and derivatives. The difficulty level is aimed to be in the sweet spot for a quantitative master's program in finance, somewhere between the MBA and PhD levels. It has been used in, and grew out of, multiple courses in UC Berkeley's Master of Financial Engineering program, in the University of Lausanne's Master of Science in Finance program, and in PhD-level asset pricing theory courses at UC Berkeley and the Stockholm School of Economics. The book may also be useful for some advanced undergraduate courses.

The book mainly focuses on theories and models for how investors and financial markets behave that were introduced in the second half of the 20th century. It studies how to think about making decisions under uncertainty and over time, how to pick an investment portfolio in the financial market, and—most extensively—how to characterize the prices and returns of financial assets in equity (stock) markets, bond markets, and derivative markets.

A wide range of topics are covered, and as we move through the material, it may from time to time be challenging to see how the different pieces fit together. The so-called asset pricing pyramid, shown in figure 1.1, provides an overarching conceptual framework to help navigate the material in such situations. The vertical dimension in this pyramid describes how specific the assumptions made in a particular theory or model are. Concepts at the base of the pyramid are general/generic, and we expect them to hold widely in financial markets. For example, payoff linearity of financial assets, the principle that the dividends paid by two units of an asset are twice those of the dividends paid by one unit, is something we expect to hold in general. Implications for asset prices that we can derive solely based on such general assumptions will consequently hold in general. A key such result is the existence of a so-called stochastic discount factor, which can be used to determine prices of financial assets.

Concepts high up in the pyramid depend on more specific assumptions. For example, the so-called capital asset pricing model, the CAPM, makes strong assumptions about how agents (a synonym of investors in this context) behave in the market. It is a so-called equilibrium model, in which asset prices need to be such that agents' aggregate demands equal the assets' aggregate supply, so that markets clear. These assumptions have strong implications for how the returns of financial assets behave—much stronger than implied by the existence of a stochastic discount

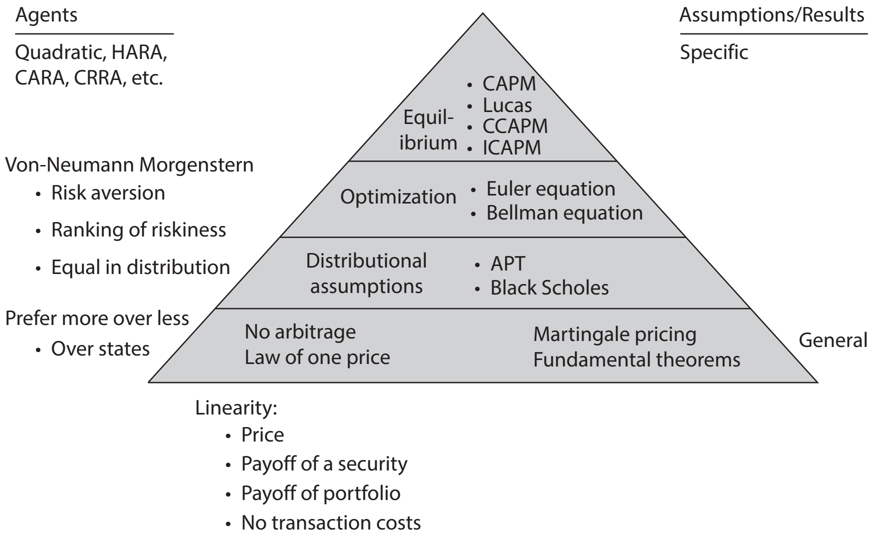

Figure 1.1: The asset pricing pyramid.

factor. It is great to have such strong results. Of course, the downside with being far up in the pyramid is that one can only trust that these results hold in environments where it is known that the specific underlying assumptions hold.

On the left-hand side of the asset pricing pyramid, we have listed assumptions made about the behavior of agents in financial markets. Again, at the bottom the assumptions are general, for example, that agents prefer more money over less. The assumptions at the top of the pyramid are much more specific, for example, with respect to their attitudes toward risk. We will frequently refer back to the asset pricing pyramid throughout the text.

This book is divided into three parts. Part I begins with an introduction to financial markets, leading to questions around how to make decisions under uncertainty, how to choose an investment portfolio, and how to determine what are appropriate prices and returns in financial markets. These questions are then the focus throughout the rest of part I. We first analyze investor preferences and decision-making under risk, cover fundamental concepts of expected utility theory, mean–variance utility, and other preference specifications, and introduce the concepts of risk aversion and risk rankings. In other words, we study the left-hand part in figure 1.1.

Building on the preference results, we then analyze the general portfolio choice problem in the mean–variance setting and in more general settings, leading to some workhorse equilibrium asset pricing models: the capital asset pricing model (CAPM), the Arrow–Debreu economy, the Lucas economy, the consumption CAPM, and arbitrage pricing theory (APT).

We relate these models to asset pricing puzzles and anomalies, and to factor models. Our main focus is on equity markets and to some extent bond markets, but we also mention other asset classes. Moreover, we touch upon the implications of the theory for real investments within publicly and privately held companies, providing a link to the corporate finance field. We use mathematical results from optimization theory, and so-called fixed-point theorems, to prove the existence of, and characterize, equilibrium. This part of the material lies in the upper part of the asset pricing pyramid.

Part II analyzes no-arbitrage asset pricing, with application to derivatives and bond markets. This theory lies at the very base of the asset pricing pyramid, and the results are therefore very general. We begin in a static discrete setting, and then gradually move to continuous-time models with diffusion processes. We cover forward contracts, futures, swaps, and options. Many of these financial instruments have been around for a long time, but have become increasingly popular over the past 50 years. They allow individuals and firms to manage their amount and kind of risk exposure, be it associated with changes in interest rates, exchange rates, stock prices, commodity prices, or default probabilities. We discuss how derivatives can be used to achieve various hedging and speculative objectives, introduce a general framework for derivatives pricing, and study several applications.

We cover the fundamental mathematical tools needed for the continuous-time theory: Itô processes, Itô's lemma, the martingale representation theorem, ordinary differential equations, stochastic differential equations, partial differential equations, Feynman–Kac's theorem, the Kolmogorov equations, the Black–Scholes PDE, and Girsanov's theorem. Finally, we discuss numerical methods. We cover three main classes of such numerical methods: binomial tree methods, finite difference methods, and Monte Carlo methods. We provide MATLAB code for some standard option problems, to understand the performance and general features of these methods. The mathematical level of chapters 6–8 is similar to part I (advanced undergraduate), whereas chapters 9–11 are more challenging. Consequently, a significant part of these later chapters is spent on understanding and developing the mathematical tools needed.

Part III covers a selection of more advanced topics. We discuss portfolio choice and equilibrium models in continuous time, with a focus on the Lucas one-tree exchange economy and the Cox–Ingersoll–Ross production economy. We introduce extensions of derivative pricing models: the constant elasticity of variance model, Merton's jump diffusion model, Heston's stochastic volatility model, and the variance gamma model. We also discuss the volatility index, VIX. Next, we introduce the Ross (2015) recovery theorem, in both the discrete- and continuous-time settings. We then discuss models with heterogeneous information and beliefs, with a focus on Hellwig's (1980) noisy rational expectations equilibrium model and Kyle's (1985) insider-trading model.

Finally, in the appendix, various mathematical concepts and results from set theory, topology, linear algebra, and calculus are stated.

1.1 Teaching Using This Book

The target audience for the material in this book is students in advanced quantitative master's programs in finance. The material may also be useful for advanced undergraduate courses and PhD courses.

The first main application of the book for teaching is for a three-unit (45 hours) asset pricing (or asset pricing theory) course. Part I of the book (chapters 2–5) together with chapters 6–7 in part II can be conveniently used for such a course, starting with a brief introduction to financial markets (chapter 2), followed by a treatment of different models for how agents make decisions under uncertainty and over time (chapter 3), which naturally leads to an analysis of the portfolio choice problem (chapter 4). These preliminaries are then used to introduce the main "classical," workhorse, equilibrium asset pricing models, the CAPM, and consumption-based models (static Arrow–Debreu model, Lucas model, consumption CAPM). In addition, the arbitrage pricing theory is introduced, showing how portfolio choice naturally leads to a factor asset pricing structure, even in the absence of equilibrium assumptions. Finally, chapters 6–7, cover the static no-arbitrage model, and especially the first and second fundamental theorems of asset pricing, portfolio replication, and risk-neutral pricing.

Variations of this course curriculum are possible. Sections 4.6 (challenges of empirically estimating expected returns), 4.7 (dynamic portfolio choice), and 5.3 (Black–Litterman model) are somewhat specific, and may be excluded. For students with knowledge of continuous-time finance, some pricing models from part II may also be covered, for example, section 10.3 on one-factor bond pricing models, and sections 12.1–12.3 on portfolio choice and equilibrium models in continuous time. Section 14.1, on recovery, does not require knowledge of continuous-time finance, and may easily be added for students interested in more recent developments, as may chapter 15 on information and disagreement in financial markets.

A note: In writing this part of the book, I have chosen a "classical" approach, where the initial focus is on individual decision-making and attitudes toward risk, followed by portfolio choice and equilibrium arguments, which eventually leads to asset pricing models and results. I am a very fond of this approach, rather than an alternative that pushes the analysis as far as possible under minimal assumptions, starting with absence of arbitrage, before making additional assumptions about agent behavior. It is also possible to take such an alternative approach, by first covering chapters 6–7, before moving on to part I.

The second main application of the book for teaching is for a two- or three-unit derivatives course (30–45 hours). Part II of the book (chapters 6–11) is designed for such a course. Chapter 7 begins with the static no-arbitrage model, and introduces as many of the fundamental concepts (absence of arbitrage, state prices, risk-neutral probabilities, replicating portfolios, etc.) in a simple setting. Chapter 8 introduces the multiperiod no-arbitrage model, providing a link between the discrete- and continuous-time models. Significant space is spent on the fundamental concepts of dynamic portfolio replication and filtrations as a way of representing information diffusion. These concepts are easier understood in a discrete model than in continuous time. Chapter 9 gradually introduces theory on differential equations and stochastic calculus, covering workhorse results that allow the derivation of the continuous-time model. Chapter 10 applies and extends the continuous-time methodology to analyze various additional assets, like bonds, and basket options. Finally, chapter 11 provides an introduction to numerical methods within continuous-time finance.

Variations are certainly also possible when designing the curriculum for such a derivatives course. The sections in chapter 10 are basically self-contained and independent, and may therefore be excluded at will. Chapter 11, on numerical methods, is also self-contained. More advanced material from chapters 12–14 may be included.

A note: Part II includes a lot of material focusing on understanding the theory and intuition that lead to derivative pricing formulas. In my experience, this is the right approach for a derivatives pricing course at the master's of finance level and above. Once the theory and intuition are understood, it is straightforward to look up and use specific derivative pricing formulas. But the specific formulas themselves do little to facilitate the understanding of the theory and underlying intuition.

The material in this book may also be used in several other courses. Chapters 3–4, and sections 5.1–5.3 may be used as the theoretical core in courses on investments, asset management, and portfolio management. Material in part I may be used for advanced readings in several core courses at the undergraduate and master's levels, including introduction to finance, financial markets, capital markets, and economic foundations of finance (chapters 2–5). The material may also be useful in several advanced undergraduate electives, for example, on asset pricing (chapters 2–5), derivatives (chapters 6–8), investments (chapters 3–5), and introduction to financial engineering (chapters 6–10).

1.2 On the Book's Origins and Approach

This book covers material I have taught over the years at UC Berkeley and elsewhere. The investments and asset pricing material in part I grew from the MFE Investments and Derivatives course in UC Berkeley's Master of Financial Engineering (MFE) program, MFE230A; the Asset Pricing and Long-Term Portfolio Management course, AP/LTPM, at University of Lausanne (UNIL), Switzerland; and Berkeley's PhD course in Asset Pricing Theory, PHD239A. I have also covered material from part I in the Theoretical Asset Pricing course, PhD404, at the Stockholm School of Economics, Sweden. Some of the discussions, on company valuation and real investments, relate to the core MBA Introduction to Finance course, MBA203, I taught at Berkeley, and also to the UNIL Valuation course.

The continuous-time material, part II, is most closely related to the Stochastic Calculus with Asset Pricing Applications course, MFE230Q, I taught at Berkeley, and to the Continuous Time Finance PhD course, PhD401, I taught at the Stockholm School of Economics, Sweden.

The material in part III has mainly been covered in PHD239A. Some of the mathematical preliminaries are covered in the Mathematical Foundations for Financial Engineers pre-program course for incoming Berkeley MFE students and, finally, some of the material in chapter 11, on numerical methods, has been covered in the pre-program computer programming course for incoming MFE students.

My goal has been to provide a theoretically rigorous treatment of the topics that are covered, combined with a lot of intuition. In Berkeley's MFE program, we are very proud of our students, who are quantitatively highly skilled, but also understand the underlying financial and economic intuition behind models in quantitative finance. Indeed, finding the right balance here is a key success factor for an MFE program. This has been my guiding principle when developing courses as well as for this text, and it has had multiple consequences.

In the trade-off between breadth and depth, I have often decided to cover fewer topics in more depth. Indeed, many of the theoretical results are proved in the main text. In my view, understanding how results come about—at a fairly detailed technical level—is key to *really* understanding a model also at the intuitive level. So I include proofs, not only for academic rigor but also to facilitate the learning process. In this spirit, I have tried to write proofs in a way that provides as much intuition as possible. A similar trade-off exists between going for the most general version of results versus making the mechanism behind them more transparent. Again, this is a question of finding the right balance. Here, I have often opted for less general results that are easier to grasp.

It will quickly become clear for anyone who reads this text that I *love* models. Models are invaluable in explaining the world around us, in structuring one's thinking, and in understanding causality—how one thing leads to another. Models are also excellent as pedagogical tools. A simple model is often worth a thousand pictures. Finally, models are fun! In the book, we dig into some workhorse models in quantitative finance and sort out how they work. I am personally not completely optimistic about the prospects of ever being able to "prove" which of many competing models is the "right one" for explaining the real world in practice. I leave it to others to make such value judgments, which are consequently largely absent from this text.

When writing the book, I have been guided by what works in the classroom. In my experience, the most efficient way of learning quantitative finance includes a lot of problem solving—going through examples and doing exercises. Consequently, this is how I have written the text, which has plenty of such examples and exercises. I often approach a topic by first discussing what we would like to achieve, and what the associated challenges are, rather than stating results up-front. For the learning process, this approach is highly efficient. Indeed, one of the core lessons I have learned, as a teacher, writer of academic papers, and management consultant, is how important it is to *always* know *why* I am working on something. My aim when writing this text has been that the reader at any point should be able to do a mini "elevator test," stepping back and explaining for themselves *what* we are doing and *why* we are doing it, in less than 30 seconds.

Index