Contents

1 European Trigonometry Comes of Age (1552–1613)

The subject we know today as trigonometry has a long, complex history that weaves through several major cultures and more than two millennia. Perhaps more than any other subject in the modern mathematics curriculum, trigonometry has been shaped, has been reconfigured, and gone through metamorphoses several times. Born of needs in ancient astronomy, it has been repurposed by many scientific disciplines and worked to serve several cultural and religious perspectives. It has been a participant, active or passive, in many of humanity's most significant scientific pursuits. The tidy, polished package found in today's high school and university textbooks camouflages a tangled story that interacts with many themes in the history of science, often with implications for some of the most transformative moments in our and other cultures.

I told the first half of this story in *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry.*[1] This volume narrates the second half, but we begin with a brief summary of what went before. Trigonometry began with Greek astronomers such as Hipparchus of Rhodes, who had constructed geometric models of the motions of the sun and moon that reproduced qualitatively the phenomena he witnessed in the sky. Converting these models into tools for prediction of events like eclipses required the translation of their geometric components into numerical measures. Since these components were lines and circles, it quickly became necessary to convert the magnitudes of circular arcs into lengths of line segments and vice versa. Hence the chord function was formulated,[2] giving the astronomer the ability to compute the length of a chord within a circle given the magnitude of the arc that it spans. The earliest table of chords of which we are aware was constructed by Hipparchus; the earliest account of the construction of chord tables is in Claudius Ptolemy's *Almagest*. The mathematical preparation for astronomy began with these chords and grew from there. However, since the geometric arena was often the celestial sphere rather than a flat surface, plane trigonometry was only the beginning. Perhaps already from the time of

[1] [Van Brummelen 2009].

[2] The term "function" has a long and complicated history. Properly speaking, according to the term's modern usage, it is an anachronism to refer to functions at all before the modern period. However, there is an affinity at least between ancient numerical tables and our use of the term: ancient astronomers found the length of the chord of a given arc by inputting the numerical value of that arc into a table and treating the value obtained as an output. In this book the word "function" is used in this loose sense, unless stated otherwise.

Hipparchus, astronomers quickly moved from the plane to the sphere, where much of the most important work was done.

The first major transformation occurred with the complicated and controversial transmission of mathematical astronomy from Greece to India. The early Indian astronomers' appropriation of the geometric models of the planets, much more than a simple transmission of knowledge (but a topic for another book), also extended to many new ways of thinking in trigonometry. The most obvious effect of the transformation of trigonometry in India is the introduction of the sine function: a slightly less intuitive quantity from a geometric point of view but a more efficient tool for astronomical computation. The versed sine followed quickly afterward. The inventions of new mathematical methods to work with these functions, such as iterative solutions to equations and higher-order interpolation within numerical tables, greatly enriched mathematical astronomy. In the fourteenth and fifteenth centuries, astronomers even employed infinitesimal arguments that we recognize today as related to calculus to derive several powerful results beneficial to astronomy, most famously the Taylor series for the sine and cosine.

The reception and naturalization of trigonometry in medieval Islam is no less complicated. In the eighth and ninth centuries Indian astronomy found its way through Persia to Baghdad. As interest grew, a translation movement brought a fresh crop of Greek texts to Islamic scholars. This produced the curious circumstance that two approaches to astronomy, both of which contained at least some trace of Greek origin, were in opposition to each other. The Greek texts gradually took precedence during the ninth and tenth centuries, but many of the Indian advances (including the sine and iterative methods) were retained. Around the end of the tenth century several advances streamlined eastern Islamic trigonometry. The tangent, invented in the process of sundial construction, became part of the trigonometric toolkit. New theorems reformulated the foundations of spherical trigonometry and delivered greater power to both astronomy and astrology. Trigonometry was also applied to new contexts, including ritual needs like determining the beginning of the month of Ramadan and the direction of prayer toward Mecca. Some of the work done on the latter problem became a standard tool in mathematical geography, bringing trigonometry down from the heavens to the earth for the first time.

From the tenth century onward, Islamic science gradually diversified according to cultural subgroups spread across its vast geographical area. The most prominent division was between eastern Islam and al-Andalus, in what is now Spain. Andalusian mathematical astronomy retained Indian and Greek influences, but after AD 1000 it developed without much conversation with the East. Rather, their knowledge spread northward into Europe, especially through the Toledan and Alfonsine Tables. Some innovations in trigonometry occurred in medieval Europe, sometimes through interactions with practical geometry and with astronomical instruments. However, the fifteenth century saw the beginning of tremendous growth through the theoretical astronomy of people such as Giovanni Bianchini (ca. 1410–1469) and Regiomontanus (1436–1476). This period set in motion the events that we shall survey in this chapter.

It is a reflection of the richness of the history of trigonometry that after more than one and a half millennia of years of progress, in the year 1550 the word itself was still 50 years away from being coined. Indeed, triangles did not really emerge as the primitive objects of study until Regiomontanus's *De triangulis omnimodis* ("Concerning Triangles of Every Kind") became popular in the mid-sixteenth century. This volume's title, *The Doctrine of Triangles*, is taken from one of the names that was given to trigonometry in the sixteenth and seventeenth centuries.

What's in a Name?

By 1550, the central problem of trigonometry—determining lengths in geometric diagrams from corresponding circular arcs and vice versa—had long been solved. European astronomers had within their grasp a somewhat compact theory that allowed them to solve every problem that they needed to solve, both on the plane and on the sphere. Regiomontanus's *De triangulis omnimodis*, written in the fifteenth century but published in 1533,[3] provided a unified source for the mathematical methods and most (although not quite all) of the fundamental theorems. Sine tables composed by Regiomontanus and others provided a straightforward tool for working out the practical calculations. Seemingly, there was not much left to do.

However, there was a great deal left to do. Over the next 50 years, the mathematical structure and even the basic notions of trigonometry were overhauled. New theorems were discovered, and more elegant and efficient ways of organizing the material were found. By the beginning of the seventeenth century, new ways to employ the subject, both within science and outside of it, were being devised with regularity. Even the basic functions, the fundamental building blocks of trigonometry, went through multiple reinventions. By 1613, the subject no longer looked much like Regiomontanus's *De triangulis omnimodis*.

We may begin to get a sense of the contrast by comparing basic definitions in the works of two of the dominant figures in the mid-sixteenth century, Regiomontanus and Rheticus. We start with Regiomontanus's *De triangulis omnimodis*.

[3] [Regiomontanus 1533]; see also the edition [Regiomontanus 1561]. *De triangulis* has been translated in [Regiomontanus (Hughes) 1967]. Finally, see [de Siebenthal 1993, chapter 5, 268–352] for an account of the mathematics in French.

Text 1.1

Regiomontanus, Defining the Basic Trigonometric Functions

(from *De triangulis omnimodis*)

Definitions:

. . .

An *arc* is a part of the circumference of a circle.

The straight line coterminous with the arc is usually called its *chord*.

When the arc and its chord are bisected, we call that half-chord the *right sine*[4] of the half-arc.

Furthermore, the *complement* of any *arc* is the difference between [the arc] itself and a quadrant.

The *complement* of an *angle* is the difference between [the angle] itself and a right angle.

Book I, Theorem 20: In every right triangle, one of whose acute vertices becomes the center of a circle and whose [hypotenuse] its radius, the side subtending this acute angle is the right sine of the arc adjacent to that [side and] opposite the given angle, and the third side of the triangle is equal to the sine of the complement of the arc.[5]

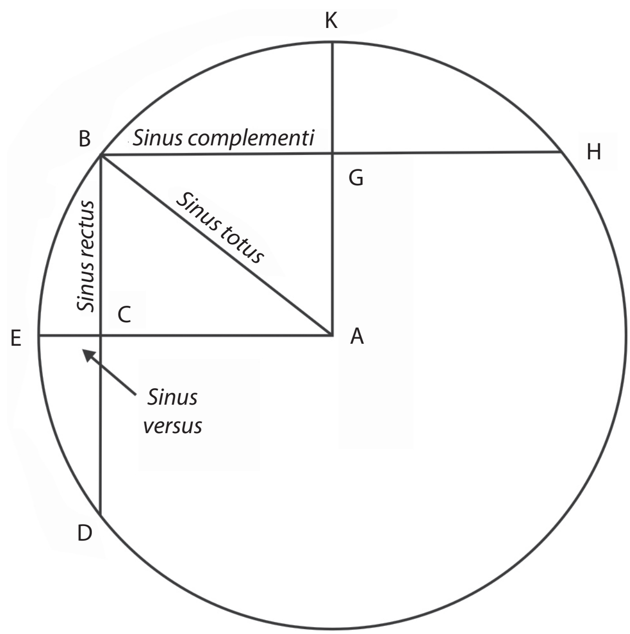


Figure 1.1 Regiomontanus's definitions of the primitive trigonometric functions.

[4] In Latin, *sinum rectum*.

[5] [Regiomontanus (Hughes) 1967, 31, 59].

Explanation: In right-angled triangle *ABC* (figure 1.1), draw a circle centered at *A* with radius *AB*. Draw *AK* vertically, *BC* parallel to *AK*, and *BH* parallel to *AC*; and extend *AC* to *E* and *BC* to *D*. Several differences between Regiomontanus's structure and the modern definition are apparent. Firstly, following his predecessors, he defines the trigonometric functions as lengths of line segments in the diagrams, not as ratios. Secondly, again following convention, he relies on the ancient Greek chord function by defining the sine *BC* (*sinus rectus*) as half the length of the chord *BD*. Thirdly, he allows the radius *R* of the base circle to be any chosen value. In the *De triangulis* Regiomontanus at times uses $R=60{,}000$ but at other times uses $R=10{,}000{,}000$. Such large radii were chosen to avoid having to work with decimal fractions.

Regiomontanus calls the circle's radius $R=AB$ the *sinus totus*, a term used already in medieval Islam that represents the greatest possible sine value. The right sine of $\widehat{BE}$ is *BC*; in modern terms, $\mathrm{Sin}(\widehat{BE}) = R\sin\widehat{BE} = BC$.[6] This is the only function used in most of the *De triangulis*. What we call the cosine is called simply the *sinus complementi*, the sine of the complement of the given arc. Near the end of the book Regiomontanus uses the versed sine, the *sinus versus EC*, the difference between the *sinus totus* and the *sinus rectus*. This function originated in India.

Just like Ptolemy's *Almagest* a millennium and a half earlier, the *De triangulis* lacks an equivalent to the tangent function. In I.28, Regiomontanus describes how to find an angle in a right triangle if the ratio between two sides is known, a simple but nontrivial process if one does not have a tangent. But Regiomontanus did not have long to wait. In his popular collection of tables for spherical astronomy, the *Tabulae directionum* ("Tables of directions"),[7] he borrowed several tables from his predecessor Giovanni Bianchini to solve stellar coordinate conversion problems.[8] One of these tables, repeatedly borrowed in turn by various successors, was recognized as useful in many other calculations, hence the name bestowed on it by Regiomontanus, the *tabula fecunda* ("fruitful table"). Mathematically equivalent to the tangent, it would become accepted gradually as a full-fledged trigonometric function on its own.

Regiomontanus was the most frequently quoted trigonometer of the sixteenth century, and we shall see more of his influence later in this chapter. His definitions and terms, most of them not original to him but spread by him, became the foundation of the field. One of his early adopters was Erasmus

[6] Here and throughout, we capitalize a trigonometric function if it is used with a circle with $R \neq 1$.
[7] See [Van Brummelen 2009, 261–263], as well as [Delambre 1819, 292–293] and [Folkerts 1977, 234–236].
[8] [Van Brummelen 2018].

Reinhold (1511–1553), one of the best quantitative astronomers of his generation. A colleague of Georg Rheticus at the University of Wittenberg, Reinhold was one of the first to receive a copy of Copernicus's work. Reinhold is most known for his very successful astronomical *Prutenic Tables*, but more relevant to us is his posthumous 1554 *Tabularum directionum*.[9] This collection of tables is an expansion of Regiomontanus's work of the same name and includes a tangent table ("*canon fecundus*") greatly expanded from Regiomontanus's. This table gives values to at least seven places for every minute of arc from 0° to 89° and for every 10 seconds of arc between 89° and 90° where the values change rapidly from entry to entry.[10] To give the reader a sense of calculations in typical astronomical work of the time, we provide a short passage of his commentary on Copernicus, one of many where Reinhold uses his tangent table.

Text 1.2
Reinhold, a Calculation in a Planetary Model Using Sines and Tangents
(from Reinhold's commentary on Copernicus's *De revolutionibus*)

Likewise, because angle *FEN* is 39°37′38″, therefore in right [triangle] *EPL* the remaining angle of *LEP* [that is, angle *ELP*] is 50°22′22″; and when *EL* is 100,000, then *LP* is 63,779 and *PE* is 77,021. And now when *EL* is taken to be 5,943, such that it is half the eccentricity, then *LP* is 3,790 and *EP* is 4,577. And from here, their doubles are $DQ = 7{,}580$ and $EQ = 9{,}154$, when *EN* . . . is 100,000. Therefore, the whole of these, *QEN*, is 109,154. And with *QN* taken to be 10,000,000, then *QD* is 694,432. And from our table, angle *DNQ* is 3°58′21″.[11]

Explanation: (See figure 1.2.) In the figure, *D* is the center of the universe and *E* is the center of the topmost eccentric deferent circle.

Reinhold knows that $\angle FEN = 39°37'38''$ and wants to find $\angle QND$. Firstly, since $\angle FEN = \angle PEL$ and $\angle EPL$ is a right angle, $\angle ELP = 90° - 39°37'38'' = 50°22'22''$. Next, in right-angled triangle *EPL*, Reinhold sets the hypotenuse $R = 100{,}000$. This allows him to use his Sine table; he finds $LP = \text{Cos}\,\angle ELP = R \cos \angle ELP = 63779$ and $PE = \text{Sin}\,\angle ELP = R \sin \angle ELP = 63779$. But *EL* is a known parameter with value 5,943, so *LP* and *EP* are scaled downward to 3,790 and 4,577, adjusting from the hypotenuse of 100,000 assumed by the Sine table to a hypotenuse of 5,943. Now, the astronomical model assumes that $EL = LD$, so the sides of triangle *DQE* are double those of ΔLPE, which

[9] [Reinhold 1554]. The "canon fecundus" may be found on folios 17 through 51.
[10] The values in the table stray significantly away from the correct ones as the argument approaches 90°, a problem that plagued both medieval Islamic and especially early European table makers. See the account of Rheticus, Romanus, and Pitiscus in [Van Brummelen 2009, 280–282]. See also the analysis of early European tangent tables in [Pritchard, forthcoming].
[11] [Nobis/Pastori 2002, 246–247]. Translated from the Latin.

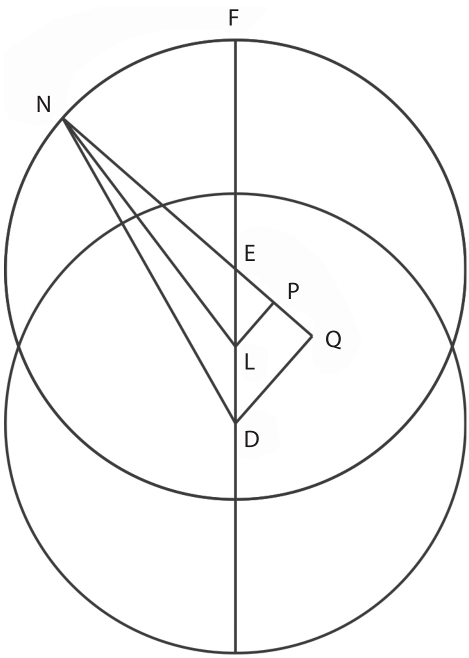


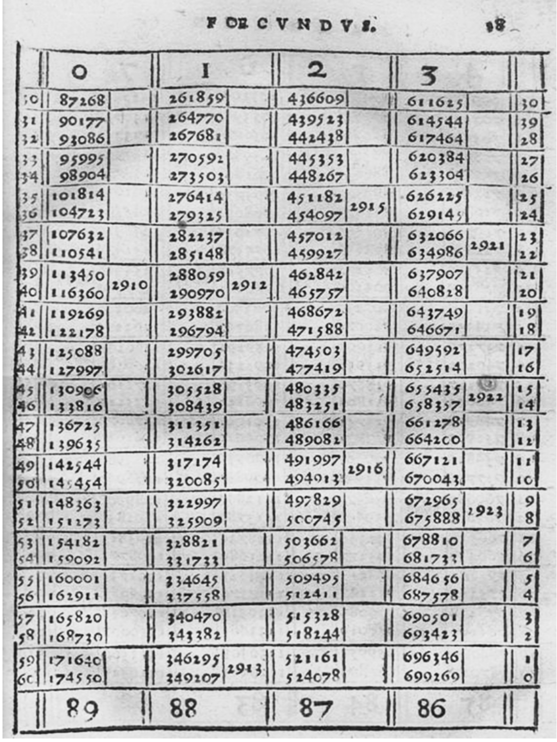
Figure 1.2
Reinhold's calculation with a planetary model using the tangent.

make $DQ = 2LP = 7{,}580$ and $EQ = 2EP = 9{,}154$. But EN is the radius of the circle, previously set to 100,000; therefore $QEN = 109{,}154$. Finally, consider right triangle NQD. Reinhold's *Canon fecundus* uses a radius of 10,000,000, so he sets QEN (the side adjacent to the angle we seek) equal to that value rather than 109,154. This requires him to adjust DQ's value accordingly, from 7,580 upward to 694,432. He can now look up this value in the *Canon fecundus* (figure 1.3); we can see for ourselves that $\angle QND$ is between 3°58′ and 3°59′.

Clearly the tangent has come a long way from its initial role as a helper to Bianchini and Regiomontanus in solving stellar coordinate problems. Reinhold is now using his *Canon fecundus* as a general purpose tool for dealing with arbitrary right triangles.

The approach shared by Regiomontanus and Reinhold, dominant in the sixteenth century, was opposed by Georg Rheticus (1514–1574). Known as the man who discovered Copernicus and convinced him to publish his heliocentric theory, Rheticus hailed from the region of Rhaetia, which overlaps Austria, Switzerland, and Germany.[12] In his mid-twenties he visited Copernicus and became his student; he announced the heliocentric theory in his

[12] We have already discussed Rheticus and Copernicus in [Van Brummelen 2009, 273–282]. For more on Rheticus, see [Burmeister 1967–1968] and [Danielson 2006].



FOECVNDVS. 98

	0		1		2		3		
30	87268		261859		436609		611625		30
31	90177		264770		439523		614544		39
32	93086		267681		442438		617464		28
33	95995		270592		445353		620384		27
34	98904		273503		448267		623304		26
35	101814		276414		451182	2915	626225		25
36	104723		279325		454097		629145		24
37	107632		282237		457012		632066	2921	23
38	110541		285148		459927		634986		22
39	113450	2910	288059	2912	462842		637907		21
40	116360		290970		465757		640828		20
41	119269		293882		468672		643749		19
42	122178		296794		471588		646671		18
43	125088		299705		474503		649592		17
44	127997		302617		477419		652514		16
45	130906		305528		480335		655435	2922	15
46	133816		308439		483251		658357		14
47	136725		311351		486166		661278		13
48	139635		314262		489082		664200		12
49	142544		317174		491997	2916	667121		11
50	145454		320085		494913		670043		10
51	148363		322997		497829		672965	2923	9
52	151273		325909		500745		675888		8
53	154182		328821		503662		678810		7
54	159092		331733		506578		681733		6
55	160001		334645		509495		684656		5
56	162911		337558		512411		687578		4
57	165820		340470		515328		690501		3
58	168730		343382		518244		693423		2
59	171640		346295	2913	521161		696346		1
60	174550		349207		524078		699269		0
	89		88		87		86		

Figure 1.3
A page from Reinhold's *Canon fecundus*. This section gives tangents from 0° to 4°, and cotangents from 86° to 90°. This page includes tangent values for arcs with minute values between 30′ and 60′; the grid on the facing page gives values for arcs with minute values between 0′ and 30′.

Narratio prima and helped Copernicus bring his *De revolutionibus* (and separately its trigonometry under the title *De lateribus et angulis triangulorum*, "On the Sides and Angles of Triangles")[13] to press.

Rheticus's accomplishments after Copernicus's death in 1543 are primarily trigonometric, especially in the design and production of tables. His short 1551 tract *Canon doctrinae triangulorum* ("Table of the Doctrines of Triangles"),[14] consisting of nothing more than a short introductory poem, 14 pages of tables, and a six-page dialogue, seems at first glance unassuming. But within its pages one finds not only tables of all six trigonometric func-

[13] [Copernicus 1542]. For an account of the trigonometry in this treatise (which is not very original), see [Swerdlow/Neugebauer 1984, part 1, 99–104]. See also [Rosińska 1983], which argues that the sine table in this work was computed by Copernicus himself but corrected by Rheticus based on Regiomontanus's tables.

[14] [Rheticus 1551]. This treatise has an unusual history. Since it was placed on the *Index expurgatorius* (and since Rheticus's later work, the *Opus palatinum*, rendered it obsolete), it disappeared from view after the sixteenth century. It was rediscovered by Augustus De Morgan in the mid-nineteenth century. See [De Morgan 1845], [Hunrath 1899], [Archibald 1949b], and [Archibald 1953]. [Roegel 2011d] contains a recomputation of all of its tables.

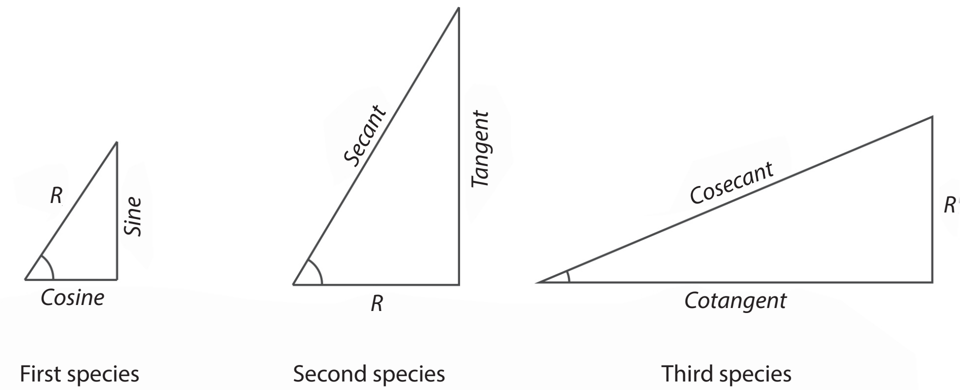


Figure 1.4
Rheticus's six trigonometric functions.

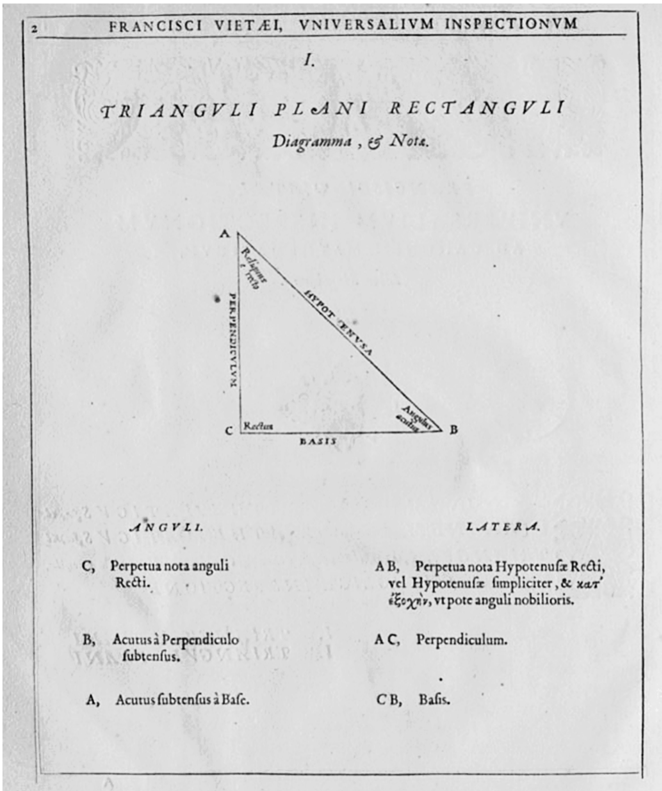
tions now considered standard (sine, cosine, secant, tangent, cosecant, and cotangent) but also a completely new and elegant set of terminology to describe them. Consider three "species" of right triangles (figure 1.4) described with respect to a given radius *R*.[15] In the first species the hypotenuse is set equal to *R*; in the second, the base; and in the third, the perpendicular. Then:

- in the first species, we have two functions, the perpendicular and the base (equivalent to Sine and Cosine respectively);
- in the second species, we have the hypotenuse and the perpendicular (Secant and Tangent); and
- in the third species, we have the hypotenuse and the base (Cosecant and Cotangent).

When Rheticus solves triangles, circles play no role. Thus, Rheticus's system not only defines all six trigonometric functions compactly but also divorces them from circular arcs: the arguments are now simply angles within the triangles, as they are today.

Rheticus found posthumous support for his design in the writings of the possibly the most well-known mathematician of the sixteenth century, François Viète (1540–1603). Viète's career was in the French civil service—not mathematics, on which he worked in his spare time. As a Huguenot during a time of unrest between Catholics and Protestants in France, his position was often hardly stable. He lived through an authorized massacre of Huguenots (which claimed the life of his older colleague Peter Ramus) and five years of

[15] In the *Canon doctrinae triangulorum* Rheticus sets $R = 10{,}000{,}000$; in the *Opus palatinum*, $R = 10{,}000{,}000{,}000$.


2 FRANCISCI VIETÆI, VNIVERSALIVM INSPECTIONVM

I.

TRIANGVLI PLANI RECTANGVLI

Diagramma, & Notæ.

ANGVLI.

C, Perpetua nota anguli Recti.

B, Acutus à Perpendiculo subtensus.

A, Acutus subtensus à Base.

LATERA.

AB, Perpetua nota Hypotenusæ Recti, vel Hypotenusæ simpliciter, & κατ' ἐξοχὴν, vt pote anguli nobilioris.

AC, Perpendiculum.

CB, Basis.

Figure 1.5
A page from Viète's *Canon mathematicus seu ad triangula* (1579), naming the sides and angles of a right triangle.

banishment from Paris, during which he worked on his mathematics. His interests were diverse, including astronomy and cryptography; but today he is recognized most for his contributions to the revolution of symbolic algebra, especially his *In artem analyticam isagoge.*[16]

While Viète's role in transforming algebra was fundamental, he was also deeply involved in the evolution of trigonometry. His first mathematical work, *Canon mathematicus seu ad triangula* ("Mathematical Canon, or On Trian-

[16] For editions and translations of Viète's mathematical treatises, see [Viète 1646; 1983] and [Viète/Girard/de Beaune 1986]; [van Egmond 1985] is a catalog of his works. None of these books contains *Canon mathematicus seu ad triangula* [Viète 1579], which occupies our attention here. See also [Ritter 1895] and [Reich/Gericke 1973]; the latter contains accounts of several of Viète's works in algebra. The secondary literature on Viète's role in the transformation of algebra is too large to be summarized here.

	Hypotenusa	*Perpendiculum*	*Basis*
I.	Totus	Sinus Anguli, vel Peripheriae (*sine*)	Sinus anguli Reliqui, seu Residuae peripheriae (*cosine*)
II.	Hypotenusa Faecundi Anguli, vel Peripheriae (*secant*)	Faecundus Anguli, vel Peripheriae (*tangent*)	Totus
III.	Hypotenusa Faecundi anguli Reliqui, vel Residuae peripheriae (*cosecant*)	Totus	Faecundus anguli Reliqui, vel Residuae peripheriae (*cotangent*)

Figure 1.6
Viète's nomenclature for the six trigonometric functions, taken from page 16 of *Universalium inspectionum* of his *Canon mathematicus seu ad triangula*. The Roman numerals on the left refer to Rheticus's triangle species.

gles," 1579),[17] is an unusual volume—as close as it comes to being a coffee table book on trigonometry. For instance, the first page of text (figure 1.5) lays out the names of the sides and angles of a right-angled triangle with an eye to filling the page in a pleasing way. The book begins with a set of trigonometric tables designed according to the methods of Rheticus's *Canon doctrinae triangulorum*, with all six functions grouped according to the three triangle species we saw in figure 1.4. Although his names for the various functions often vary (see figure 1.6) and borrow the term *fecunda* from Regiomontanus, the structure clearly imitates that of Rheticus.[18]

Most of Viète's colleagues and contemporaries, however, were content to stick with the language of Regiomontanus.[19] For instance, only eight years after the *De triangulis omnimodis* was published, the great German astronomical

[17] See [Viète 1579], [Hunrath 1899], and [Rosenfeld 1988, 24–27]. See also [Roegel 2011g] for a recomputation of the tables.

[18] See page 16 of the *Universalium inspectionum* within [Viète 1579], and [Ritter 1895, 40]. Viète applies the term *fecunda* to several quantities.

[19] [Von Braunmühl 1900/1903, vol. 1, 183] suggests that Viète's unique notation here and elsewhere, brilliant as it was, may have contributed to his colleagues' lack of appetite for his trigonometric inventions. But Rheticus and Viète were not without followers; Adrianus Romanus's *Canon triangulorum* [Romanus 1609], for instance, adopts some of Viéte's structure and terminology, including the terms "transsinuousae" for the secant and "prosinus" for the tangent (even though the standard terms are on the title page).

and geographical instrument maker Peter Apian (1495–1552)[20] had followed with his 1541 *Instrumentum sinuum seu primi mobilis*, a well-known treatise on trigonometric instruments and their use in solving various astronomical problems, which we shall consider later. Apian uses names that would have been familiar to Regiomontanus and his colleagues: the *sinus rectus primus* for the sine and the *sinus rectus secundus* for the cosine.[21] There is no reference to Regiomontanus's *tabula fecunda* or indeed to anything resembling a tangent function.

Apian's traditional names for the sine and cosine are found again in the 1558 collection of works on spherical astronomy[22] by Francesco Maurolico (1494–1575). A Sicilian priest, Maurolico held a variety of civil positions over the course of his life, including master of the mint, and was eventually appointed professor at the University of Messina. He was active in a wide variety of areas of mathematics and science, including optics and music; within astronomy he was especially prolific in spherical astronomy and edited several Greek works on the subject. Although he does not define the tangent and cotangent directly in his book on spherics, they do appear as *umbra versa* and *umbra recta* in Book II, Proposition 30,[23] as they often had before. These terms derive from ancient and medieval references to "shadows" in sundials, and Maurolico himself defines the *umbra versa* and *umbra recta* in this way in his astronomical treatise *De sphaera,* a work infamous for his vicious condemnation of Copernicus.[24] However, as we noted earlier, it was not from the *umbra versa* and *umbra recta* that the modern tangent and cotangent evolved.

We do find one innovation in Maurolico's work on spherics. Near the end he describes a new table as follows: "In imitation of the *tabula fecunda* of Johannes Regiomontanus, we made another table which we have named *benefica*, because certain calculations become easy by means of this table."[25]

[20] For a general introduction to Apian's mathematics see [Kaunzner 1997]; for his trigonometry see [Folkerts 1997].

[21] See the third page of the first section of [Apian 1541], *Instrumentum hoc primi mobilis componere*.

[22] [Maurolico 1558] (on which see [Moscheo 1992] on editorial issues) includes Latin editions of Theodosius's *Spherics*, Menelaus's *Spherics*, Autolycus's *Spherics*, Theodosius's *De habilitationibus*, and Euclid's *Phenomena* as well as several small trigonometric tables (sine, *tabula fecunda, tabula benefica*, and declinations and ascensions) and a *Compendium mathematicae*. On Maurolico's sources for his edition of Menelaus, see [Taha/Pinel 1997] or [Taha/Pinel 2001]. See also [Napoli 1876] for an edition of Maurolico's *Geometricarum quaestionum*. [Rose 1975, 159–184] is a good account of Maurolico's life and work.

[23] [Maurolico 1558, f. 58].

[24] *De sphaera* is the first of a number of short treatises in *Opuscula mathematica*, [Maurolico 1575]; the definitions of *umbra versa* and *umbra recta* may be found on page 13. For Maurolico's attack on Copernicus, see [Rosen 1957].

[25] [Maurolico 1558, f. 60], *Demonstratio tabulae beneficae*.

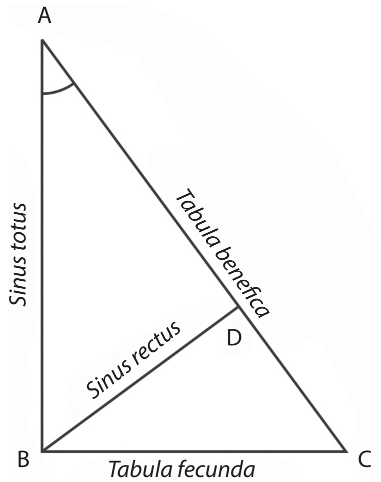


Figure 1.7
Maurolico's trigonometric functions.

Maurolico's new table introduces what today we call the secant.[26] A short table at the end of the book[27] gives secant values, with $R = 100{,}000$, for integer arguments from 1° to 89°. Rheticus, of course, had already published tables of all six trigonometric functions seven years earlier in his *Canon doctrinae triangulorum*. But he had used his own unique terms and definitions, which make no appearance in Maurolico's work.[28] Instead, consider Maurolico's figure 1.7: within right triangle ABC, segment BD is perpendicular to AC. Set AB equal to $R = 100{,}000$. Then, given $\measuredangle A$ at the top of the diagram, we may find BD from a table of sines, BC from the *tabula fecunda*, and AC from the *tabula benefica*.

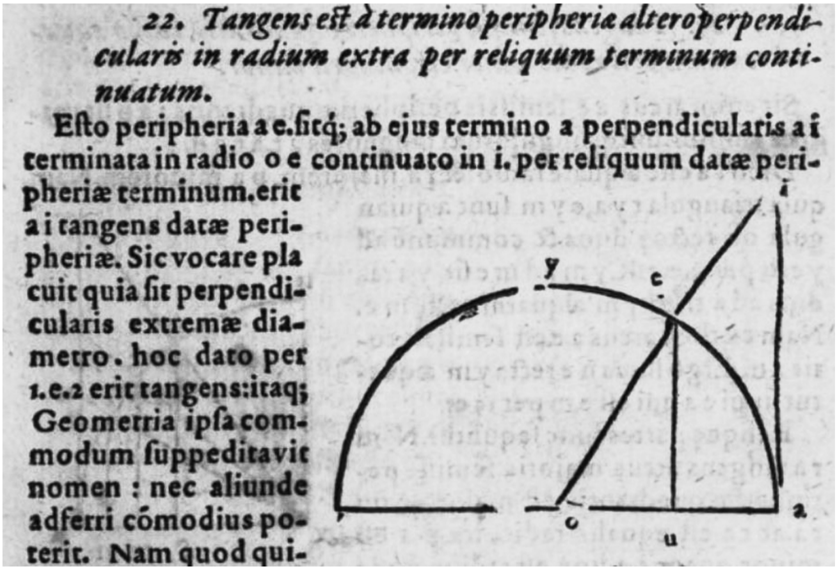
It took another quarter century for the *tabula fecunda* and *tabula benefica* to take on their modern names of tangent and secant in Danish scholar Thomas Fincke's (1561–1656) *Geometriae rotundi* ("Geometry of Circles and Spheres").[29] Still a 22-year-old student in 1583 at its publication, Fincke switched to the study of medicine that same year. Over the course of his very long career, he held professorial positions in medicine, rhetoric, and mathematics and held a number of senior administrative posts (including rector and

[26] Copernicus composed a table of secants by hand, but it was never published. See [Glowatzki/Göttsche 1990, 190–192]. For an analysis of Maurolico's table, see [Van Brummelen/Byrne, forthcoming].

[27] Folio 66. As we shall see later, a controversy arose over whether Maurolico's table owed an unpaid debt to Rheticus.

[28] Here we differ from von Braunmuhl's opinion that Maurolico was following Rheticus; see [von Braunmühl 1900/1903, I, 150–151].

[29] [Fincke 1583]. De Morgan first makes this identification in [De Morgan 1846]. See [Schönbeck 2004] for a detailed account of Fincke's life and a summary of the *Geometriae rotundi*.


22. Tangens eſt à termino peripheriæ altero perpendicularis in radium extra per reliquum terminum continuatum.

Eſto peripheria a e. ſitq; ab ejus termino a perpendicularis a i terminata in radio o e continuato in i. per reliquum datæ peripheriæ terminum. erit a i tangens datæ peripheriæ. Sic vocare placuit quia ſit perpendicularis extremæ diametro. hoc dato per 1.e.2 erit tangens: itaq; Geometria ipſa commodum ſuppeditavit nomen : nec aliunde adferri cōmodius poterit. Nam quod qui-

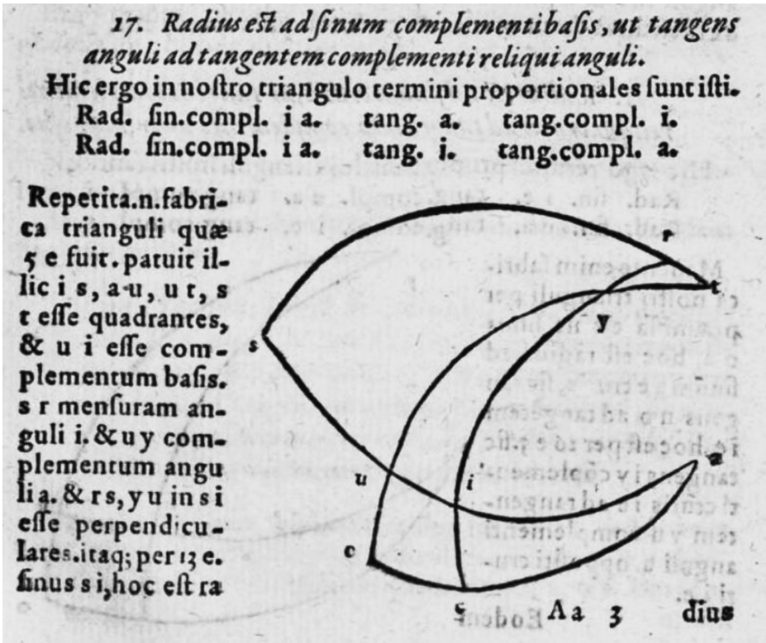
Figure 1.8
Defining the tangent in Fincke's *Geometria rotundi*.

dean of the medical school for over half a century) at the University of Copenhagen. But the *Geometriae rotundi* remains his most enduring legacy. Inspired by Peter Ramus's 1569 *Geometria*, in a way the book is a step back to an older time, with its emphasis on the ancient spherical Menelaus's theorem.[30] However, it was found to be extremely clear and readable, and it was spoken of highly for several decades.

One of the *Geometriae rotundi*'s most lasting contributions was its creative use of language to simplify the presentation. Among his innovations were the inventions of the names "*tangens*" and "*secans*" for the tangent and secant functions respectively. In Proposition V.22 (figure 1.8), Fincke takes a semicircle of given radius, draws a vertical tangent from its rightmost point, and extends a diagonal at a given angle from center O until it touches the tangent line at I. Then the length of AI, naturally, is the "tangent" of that angle. A few propositions later (V.27), Fincke calls OI the secant since it crosses the circle's edge.[31]

The new names were instantly popular among Fincke's colleagues; they are found already three years later in Christoph Clavius's 1586 edition of The-

[30] See [Van Brummelen 2009, 56–61].
[31] [Fincke 1583, 73–74, 76].


17. Radius eſt ad ſinum complementi baſis, ut tangens anguli ad tangentem complementi reliqui anguli.
Hic ergo in noſtro triangulo termini proportionales ſunt iſti.
Rad. ſin.compl. i a. tang. a. tang.compl. i.
Rad. ſin.compl. i a. tang. i. tang.compl. a.

Repetita.n.fabrica trianguli quæ 5 e fuit. patuit illic i s, a u, u t, s t eſſe quadrantes, & u i eſſe complementum baſis. s r menſuram anguli i. & u y complementum anguli a. & r s, y u in s i eſſe perpendiculares. itaq; per 13 e. ſinus s i, hoc eſt ra dius

Figure 1.9
Fincke's expression of the relation cos $c = \cot A \cot B$ for a right-angled spherical triangle, *Geometria rotundi* XIV.17. Book XIV contains the earliest appearances of the abbreviations "sin," "tang," and "sec"; the first two can be seen here.

odosius's *Spherics*[32] as well as in Antonio Magini's 1592 *De planis triangulis* (which also contains the terms *tangens secunda* and *secans secunda* for cotangent and cosecant, paralleling the earlier usages of *sinus primus* and *sinus secundus* for sine and cosine), among other works.[33] The abbreviations for the words varied from one author to the next; well into the seventeenth century they had not yet become standardized. François Viète himself objected to the new terms, arguing that they could be too easily confused with other ways that the terms are used in geometry.[34] But in this case at least, Viète's opinion did not transform into practice.

[32] [Clavius 1586]. In addition to Theodosius's *Spherics*, the book contains tables of tangents and secants (in which the name *benefica* also appears) and trigonometric treatises by Clavius himself.

[33] [Magini 1592]. [Cajori 1928–1929, vol. 2, 150–151] also refers to the use of these terms by Brahe, Lansberg, Blundeville, and Pitiscus.

[34] [Viète 1593, the third folio numbered 38] ("Immo vero artem confundunt, cum his vocibus necessae habeat uti Geometra abs relatione"); see also [Cajori 1928–1929, vol. 2, 150].

Book XIV, concluding Fincke's *Geometriae rotundi* with some spherical trigonometric results, contains a significant notational development. Perhaps due to the length of text that would otherwise be required to state these theorems, Fincke abbreviates the trigonometric functions in ways that we recognize today. Here we find for the first time "sin." for sine; "tan." and "tang." for tangent; "sec." for secant; and "sin. comp." or "sin. compl." for cosine (and similarly for cotangent and cosecant). In figure 1.9, for instance, we see Fincke's expression of the relation "*R* is to Cos *ia* as Tan *a* is to Cot *i*" in the right-angled spherical triangle at the bottom of the diagram, equivalent to our $\cos c = \cot A \cot B$.

Trigonometric Tables Evolving

Until machines took over the world of computation, numerical tables were how trigonometry was used in the sciences, surveying, and navigation. Hipparchus's invention of the trigonometric table to convert geometric statements into quantitative results was to extend far beyond his predictions of eclipses. In turn, the need for easily computed, yet accurate tables was the motive behind many of the theorems that are now taught in school. The basic formulas of plane trigonometry—for instance, the sine and cosine sum and difference laws and the half-angle formulas—were invented to simplify computations of tables.[35] And as we just saw, the tangent and the secant functions were introduced in Europe not as functions but as tables (the *tabula fecunda* and *tabula benefica*).

The late sixteenth century saw a spectacular rise in the production of trigonometric tables in terms of both the industry required to generate them and the quality of the results.[36] Almost every author participated in the table-making process (see figure 1.10); composing a table was a major part of what it meant to be a practitioner of the doctrine of triangles. Dealing with fractional quantities outside of the astronomers' traditional sexagesimal (base 60) arithmetic was not in the standard toolbox until late in the sixteenth century; table makers usually got around this problem by using a base circle radius equal to some large power of ten.[37] Then, they could represent Sines, Cosines, and so on as large whole numbers.

[35] See [Van Brummelen 2009, 41–46, 70–77] for descriptions of trigonometric tables in ancient Greece and in multiple places elsewhere in the book for discussions of tables in medieval cultures.

[36] See [Glowatzki/Göttsche 1990] for a study of Regiomontanus's trigonometric tables and those of his successors.

[37] At least one astronomer of the fifteenth century (Giovanni Bianchini) took some early steps toward decimal fractional notation, including the invention of the decimal point, which we shall describe shortly.

Author	Work	sin	tan	sec	*R*	Step size	Worst case error
Regiomontanus	*Tabulae directionum* (1490)	✓	✓		60,000 (sine) 100,000 (tangent)	1′ (sine) 1° (tangent)	4th of 7 decimal places
Apian	*Introductio geographica* (1541)	✓			100,000	1′	
Regiomontanus	*Tractatus Georgii Peurbachii…* (1541)	✓			6,000,000; 10,000,000	1′	
Copernicus	*De lateribus triangulorum* (1542)	✓			10,000,000	1′	
Rheticus	*Canon doctrinae triangulorum* (1551)	✓	✓	✓	10,000,000	10′	5th of 10 decimal places
Reinhold	*Tabularum directionum* (1554)		✓		10,000,000	1′ (10″ after 89°)	4th of 12 decimal places (for 89°59′:5th of 11 places)
Maurolico	*Theodosii sphaericorum* (1558)	✓	✓	✓	100,000	1°	7th of 7 decimal places (for 89°59′: 6th of 9 places)
Viète	*Canon mathematicus seu ad triangula* (1579)	✓	✓	✓	100,000.000	1′	9th of 9 decimal places
Bressieu	*Metrices astronomicae* (1581)	✓	✓	✓	60 (three sexagesimal places)	1°	3rd of 4 sexagesimal places
Fincke	*Geometriae rotundi* (1583)	✓	✓	✓	10,000,000	1′	5th of 11 decimal places
Rheticus/Otho	*Opus palatinum* (1596)	✓	✓	✓	10,000,000,000	10″	7th of 15 decimal places (for 89°59′:9th of 14 places)
Pitiscus	*Trigonometriae* (1600)	✓	✓	✓	100,000	1′	5th of 9 decimal places
Van Roomen	*Canon triangulorum sphaericorum* (1607)	✓	✓	✓	1,000,000,000	10′	6th of 12 decimal places
Pitiscus (Rheticus)	*Thesaurus mathematicus* (1613)	✓			1,000,000,000,000,000	10″	

Figure 1.10
Trigonometric tables from Regiomontanus to the eve of logarithms.

A quick examination of figure 1.10 reveals several noteworthy facts. Firstly, it took almost no time for the tangent and the secant functions, under various names, to be accepted and tabulated along with the sine.[38] Secondly, the increments between the arguments became smaller and smaller, achieving more accuracy at the cost of increased labor; the standard increment soon became 1′ or even smaller. Finally, often unaware of it, all authors struggled with the entries of a trigonometric table that are most difficult to compute accurately: namely, values for the tangent and secant where the argument approaches 90°. These values were often calculated by dividing by a very small quantity such as the cosine of an angle near 90°.[39] Small rounding errors in

[38] In spherical trigonometry the function arcsin (sin x sin y). had currency through the sixteenth century and was often tabulated; see [Van Brummelen 2009, 263] on Regiomontanus's table and [Glowatski/Göttsche 1990, 197–207] for a summary.
[39] See [Pritchard, forthcoming].

the cosine values were thus magnified and became much larger errors in the tangent and secant values.[40]

Several sixteenth-century European authors discussed their methods for computing sines.[41] Usually their methods did not go much beyond what one finds already in the chord table in Ptolemy's *Almagest* along with those developed in early Islam and transmitted to Europe through al-Andalus. A typical early sixteenth-century text is Regiomontanus's *Compositio tabularum sinuum rectorum*, published 65 years after his death in 1541.[42] Regiomontanus begins this work simply by stating that one can find the Sine of the complement of an arc whose Sine is known, using the Pythagorean Theorem:

$$\sin(90° - \theta) = \sqrt{R^2 - \mathrm{Sin}^2\theta}. \tag{1.1}$$

He then determines the Sines of the *kardajas*, namely, the multiples of 15°, which can be obtained from the Sines of 30°, 45°, and 60°, a simple geometric argument deriving the Sine of 15°, and (1.1).[43] This results in a small table of sines, listed in the order of their computation rather than in increasing order, with $R = 600{,}000{,}000$:

Arcus	*Sinus*
90	600000000
30	300000000
60	519615242
45	424264069
15	155291427
75	579555496

[40] We have already discussed this problem with respect to Rheticus's tables in the *Opus palatinum*, their identification by Adriaan van Roomen, and the repairs to the table made by Pitiscus; see [Van Brummelen 2009, 280–282]. For the secant function, the alternative method $\sec^2\theta = 1 + \tan^2\theta$ was much less prone to error (assuming one has an accurate tangent table) and used occasionally; see [Van Brummelen/Byrne, forthcoming].

[41] Occasionally they also discussed the computation of tangents and secants but usually only briefly and simply.

[42] Published as an appendix to [Peurbach 1541]; [Glowatzki/Göttsche 1990, 11–24] contains a reproduction of the manuscript and a translation to German. This is not the earliest sixteenth-century publication describing the calculation of a sine table; Peter Apian's *Introductio geographica* (1533) contains both a sine table (reprinted a year later in his *Instrumentum sinuum seu primi mobilis*) and a description. See [Folkerts 1997, 225–226] for a brief account. The *Instrumentum sinuum seu primi mobilis* also contains a small table of arc sines, the earliest such table of which I am aware with clearly trigonometric intent. An early description of the construction of a sine table, using similar methods and almost contemporaneous with Regiomontanus, may be found in Oronce Fine's 1542 *De sinibus*; see [Ross 1977].

[43] The *kardajas*, from the Persian for "sections," are found in medieval India, Islam, and Europe. For a modern account of this and the following proposition, see [Zeller 1944, 33–34].

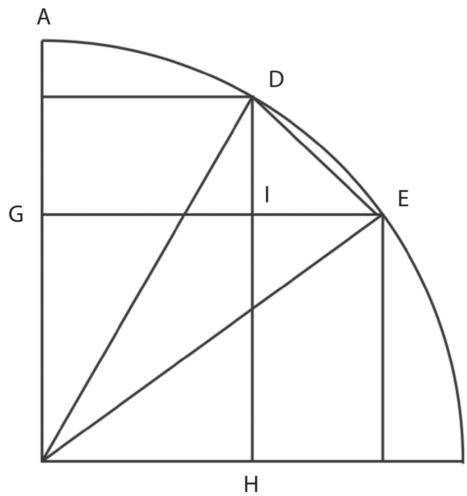


Figure 1.11
Regiomontanus's calculation of the length of a side of a regular 15-gon.

Proposition 3 gives

$$\frac{\frac{1}{2}R}{\operatorname{Sin}\theta} = \frac{\operatorname{Sin}\theta}{\operatorname{Vers} 2\theta}, \tag{1.2}$$

an equivalent to the Sine half-angle formula. This gives Regiomontanus all the tools he needs to find the Sines of all the multiples of 3°45′, which he promptly does, in a table similar in form to the above. Proposition 4 uses constructions of the regular pentagon and decagon inscribed in a circle, as Ptolemy and many others had done, to determine the values of a couple of more difficult Sines.[44] For instance, since the side of the inscribed pentagon is equal to the chord of a 72° arc, half of the side of the pentagon is the Sine of 36°. Once these values are known, proposition 5 allows Regiomontanus to find the length of a side of a regular 15-gon inscribed in a circle, as follows (figure 1.11): in a quadrant of radius R, let $AD = 30°$ and $AE = 54°$. Then $EI = EG - IG = \operatorname{Sin} 54° - \operatorname{Sin} 30° = \operatorname{Sin}(90° - 36°) - \operatorname{Sin} 30°$ and $ID = DH - HI = \operatorname{Sin} 60° - \operatorname{Sin} 36°$, so we can calculate a value for $ED = \sqrt{EI^2 + ID^2}$. But $\widehat{ED} = 24°$ is one side of the regular 15-gon, so $\frac{1}{2}ED = \operatorname{Sin} 12°$. Now that we have a value for Sin 12°, apply the half-angle formula four times to get Sin 45′. Once Regiomontanus has this value under his belt, he needs only time, patience, and the Sine sum and difference laws to find the Sines of all multiples of 45′.

But all of this work is only a precursor to the most challenging problem in the calculation of Sine tables, namely that of finding the Sine values of multiples not of 45′ but of 1° (or 1′). The only Sines that can be found using

[44] See [Van Brummelen 2009, 72–74].

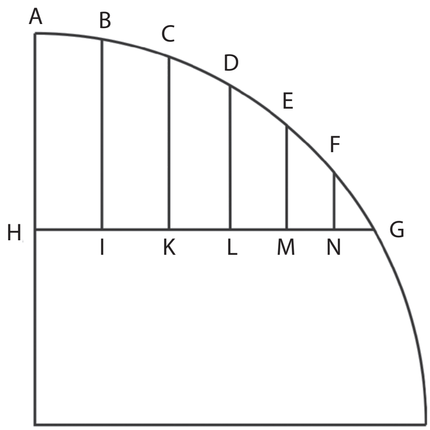


Figure 1.12
Regiomontanus's method to calculate Sin 1°.

geometry alone are those whose arcs can be written in the form $3m/2^{n\circ}$ for whole numbers *m* and *n*. To go beyond this set, mathematicians since Ptolemy had had to find a way somehow to break the bounds of the methods available to geometry. Regiomontanus proceeds as follows. Within the quadrant, cut six arcs of ¼° each: *AB*, *BC*, . . . , *FG* (figure 1.12); and drop a perpendicular from *G* onto *AH*. Then drop perpendiculars from *B*, *C*, *D*, *E*, and *F* to *HG*. *HI*, *HK*, . . . are then the Sines of the successive multiples of ¼°, up to $HG = \text{Sin}1\frac{1^\circ}{2}$. By a lemma (omitted here, although one can see it is true by inspection), Regiomontanus knows that $HI > IK > \ldots > NG$. Since he already knows from his table calculations that $HL = \text{Sin}\frac{3^\circ}{4} = 7{,}853{,}773$ (in a circle of radius 600,000,000), he determines

$$\text{Sin }1^\circ = HM = HL + LM < \frac{4}{3}HL = \frac{4}{3}\text{Sin}\frac{3}{4}^\circ = \frac{4}{3}(7853773) \\ = 10{,}471{,}697. \tag{1.3}$$

Similarly, knowing also that $HG = \text{Sin }1\frac{1^\circ}{2} = 15{,}706{,}169$, he finds

$$\text{Sin }1^\circ = HM = HL + LM > HL + \frac{1}{3}LG = \\ \text{Sin}\frac{3}{4}^\circ + \frac{1}{3}\left(\text{Sin }1\frac{1}{2}^\circ - \text{Sin}\frac{3}{4}^\circ\right) = 10{,}471{,}238. \tag{1.4}$$

The result is a narrow interval containing Sin 1°:

$$10{,}471{,}238 < \text{Sin }1^\circ < 10{,}471{,}697. \tag{1.5}$$

From here Regiomontanus uses his half-angle formula to obtain[45]

$$5{,}235{,}818 < \mathrm{Sin}\frac{1^\circ}{2} < 5{,}236{,}044. \tag{1.6}$$

Since he wishes to compute a sine table with $R = 6{,}000{,}000$ rather than 600,000,000, Regiomontanus divides by 100, leaving

$$52{,}358 < \mathrm{Sin}\frac{1^\circ}{2} < 52{,}360, \tag{1.7}$$

from which he concludes that $\mathrm{Sin}\frac{1^\circ}{2} = 52{,}359$. Armed with this approximation, the half-angle formula, the Sine sum and difference laws, and a lot of patience, he is able to fill in the Sines of all the multiples of ¼°.[46]

This technique is an enhancement on the approach used by Ptolemy in the *Almagest*, but it is essentially the same idea. Various eastern Arabic enhancements of Ptolemy's procedure from the tenth and eleventh centuries had generated similar results.[47] Curiously, only a few decades before Regiomontanus wrote this treatise but far to the East in Samarqand, Jāmshīd al-Kāshī had overturned the rules of this problem by introducing algebra and an iterative procedure that allows the determination of Sin 1° to as many places as one has the patience to calculate. However, his solution was not to find its way to Europe.[48] Even more curiously and much closer to Regiomontanus's home, his older colleague Giovanni Bianchini had done something similar, also with a method capable of generating arbitrary levels of precision, and we know that Regiomontanus became aware of it at some point.[49] However, there is no trace of anything new on this topic in this work.

The divide over terminology that we saw in the previous section was about to make a reappearance in the context of tables. Rheticus's new structure and his tables for all six trigonometric functions appeared only a decade after the publication of Regiomontanus's book, in the 1551 *Canon doctrinae triangulorum*. While this latter work eventually became very difficult to find, clearly the word about it spread through the mathematical community; his name is mentioned frequently in the late sixteenth century in conjunction with the new trigonometric functions well before his massive *Opus palatinum*,

[45] These two values are in error in the last two places, but this is about to become irrelevant.
[46] Regiomontanus goes on to describe how to enhance the process to work one's way down to Sin 1′, which would allow him to build a table with an increment of 1′, but he does not provide the calculations.
[47] See [Van Brummelen 2009, 140–145].
[48] See [Van Brummelen 2009, 146–149].
[49] See [Gerl 1989, 265–268]. A marginal note by Regiomontanus in the margin of the manuscript Cracow BJ 558 (f.22v) states that Bianchini's method is superior to Ptolemy's.

a full treatment of his trigonometry with gigantic tables, was published in 1596. In fact, although Maurolico published his table of secants under a different name (*tabula benefica*) imitating the style of Regiomontanus in 1558, Thomas Fincke asserted in his 1583 *Geometriae rotundi* that Maurolico had simply taken over Rheticus's secant table. Magini, in his 1592 *De planis triangulis*, defended Maurolico, arguing that he had worked independently of Rheticus.

The question may be resolved by a closer inspection of Maurolico's table, which gives the secant for $R = 100{,}000$ and for every degree up to 89°.[50] Since the secant grows without bound as the argument approaches 90°, the last few values in any secant table are difficult to compute and are highly sensitive to rounding errors. For instance, the correct value of Sec 89° is 5,729,869. Maurolico's value is 5,729,868 while Rheticus's is 5,729,838.[51] Another example: immediately below Maurolico's table, he gives a few values of Sec θ for arguments greater than 89°, one of which (89°30′) has the same argument as an entry in Rheticus's table. The correct value of Sec 89°30′ is 11,459,301; Maurolico's is 11,459,309; Rheticus's value is 11,459,348. In both cases (and in others) Maurolico's value is much more accurate than Rheticus's. Therefore, he did not appropriate Rheticus's table.[52]

François Viète dealt with the problem of finding Sine values for arguments where geometry alone does not suffice, both early and late in his career. In his 1579 *Canon mathematicus seu ad triangula*, he determines sin 1′ as follows.[53] Beginning with $\sin 30° = 0.5$, he applies the sine half-angle formula (in the form $\sin^2(\theta/2) = \frac{1}{2}\text{vers}\,\theta$) 11 times in a row. In his last two iterations he finds

$$\sin^2\left(\frac{450'}{256}\right) = 0.000000261455205834 \text{ and}$$

$$\sin^2\left(\frac{225'}{256}\right) = 0.000000065363805733. \tag{1.8}$$

[50] [Von Braunmühl 1900/1903, vol. 1, 151–152] reports on the controversy and mentions a table of secants by Maurolico with arguments up to 45°; this table is mentioned by several later writers, apparently taking their information from von Braunmühl. The manuscript in fact does contain a secant table as described by von Braunmühl but in two columns, the first of which ends at 45°. Perhaps von Braunmühl did not notice the second column and thus did not have the opportunity to compare the values in the two secant tables for arguments near 90°.

[51] This entry cannot be a typographical error since Rheticus's interpolation column confirms this value. Since Rheticus's value for R is larger, it contains two more decimal places, suppressed here; likewise for the entry for sec 89°30′.

[52] For a full analysis and the background to the controversy, see [Van Brummelen/Byrne, forthcoming].

[53] See [Viète 1579, 62–67]. For the reader's ease, we have converted Viète's calculations to a base circle of $R = 1$.

From these values Viète derives two estimates for sin 1′ as follows:

$$\sin 1' > \sqrt{\left(\frac{256}{450}\right)^2 \cdot 0.000000261455205834} = 0.0002908881959 \tag{1.9}$$

and

$$\sin 1' < \sqrt{\left(\frac{256}{225}\right)^2 \cdot 0.000000065363805733} = 0.0002908882056. \tag{1.10}$$

The former comes from the assertion that $\frac{\sin\left(\frac{450'}{256}\right)}{\sin 1'} < \frac{450/256}{1}$; the latter comes from $\frac{\sin 1'}{\sin\left(\frac{225'}{256}\right)} < \frac{1}{225/256}$. As impressive as these calculations are, this inequality—the heart of Viète's method—goes all the way back to Ptolemy's *Almagest*. Now, since $\frac{225'}{256}$ is closer to 1′ than $\frac{450'}{256}$ is, Viète proposes (but does not carry out in the text) that the final value for Sin 1′ should be a weighted average favoring (1.10) over (1.9). This would result in Sin 1′ ≈ 0.0002908882042, a value that is completely accurate except for the last decimal place. Decades later, Viète would invent (but not carry out) a method that applies algebra to the problem in the spirit of al-Kāshī; we shall examine it later in this chapter.

Also in the *Canon mathematicus*, we find a very large and rather odd table, the *Canonion triangulorum laterum rationalium*.[54] Within it, Viète provides 45 pages of over 1,400 Pythagorean triples, scaled so that one of the three sides of the triangle is exactly equal to 100,000. These triples are ordered sequentially so that they can be used as a trigonometric table. Their values can be quite complicated. For instance, the first entry is

$$\frac{19{,}988{,}480{,}000}{49{,}942{,}416{,}589} \text{ and } 99{,}999\frac{49{,}942{,}376{,}589}{49{,}942{,}416{,}589};$$

and in fact, the square root of the sum of the squares of these two numbers is precisely 100,000. Viète himself states at the end of the *Canon mathematicus* that the *Canonion* "is of very little use."[55] One wonders, then, why he put so much effort into it. Perhaps he was concerned about issues of roundoff error in conventional tables, or he wished not to stray from the realm of pure geometry into approximation, or he thought of this work more as number theory

[54] [Viète 1579], pages numbered separately as pp. 1–45. See also [Tanner 1977] for offshoots of this work by Torporley and Harriot, [Hutton 1811b, 5–6], [Zeller 1944, 73–74], and [Roegel 2011h] for a reconstruction of Viète's table.
[55] [Viète 1579, 75].

than as support for astronomy. We shall encounter this "rational trigonometry" again in chapter 5.

Before we move on, it is also worth mentioning an unusual small treatise by Nicolaus Raymarus Ursus (1551–1600), a German astronomer known primarily for his rivalry with Tycho Brahe over priority to the geoheliocentric system for the motions of the planets. The work in which he propounded this model, his 1588 *Fundamentum astronomicum*,[56] also contains some computational mathematics, including discussions of the computation of sine tables. Here he refers, not entirely clearly, to a method developed by his teacher Joost Bürgi involving finite differences, which we shall discuss later.[57] The method Ursus describes for finding sin 1′ is similar to those we have seen before. However, once he has it, he uses the identity

$$2\sin(A-x)\cos x - \sin A = \sin(A-2x) \tag{1.11}$$

cleverly to fill in the remaining entries: starting with $A=90°$ and $x=1'$ and the knowledge of sin 90° and sin 89°59′, he uses it to calculate sin 89°58′; and by decreasing A again and again by one minute, he is able to calculate the sines of 89°57′, 89°56′, and so forth.[58] We shall see identities used in this way again, in chapter 3.

Meanwhile, Rheticus had died in 1574, but the massive tables of the *Opus palatinum* were finally published in 1596 by Valentin Otho. We have already described these tables elsewhere.[59] The 700-page tables, the largest ever compiled up to that time, contain all six of the standard trigonometric functions. Computed for every 10″ of arc to ten decimal places, they constitute one of the most intensive computational efforts in human history. However, the methods Rheticus used, although inventive, did not extend beyond the approximation methods we have seen in this section. In fact, in figure 1.10 we see that Rheticus encountered the same difficulties with numerically sensitive trigonometric values that plagued almost all of his colleagues. The errors in Rheticus's tables were noticed by Romanus[60] and repaired by Pitiscus in 1607. Six years later Pitiscus would release *Thesaurus mathematicus*, an even more precise set of tables based on some of Rheticus's unpublished calculations.[61]

56 [Ursus 1588]. On sine tables, see especially the second of the seven chapters.
57 See [Delambre 1821, vol. 1, 289–291, 299–301].
58 See an account in [Delambre 1821, vol. 1, 306–307].
59 See [Van Brummelen 2009, 273–282]. Since then a recomputation of the entire set of tables has appeared ([Roegel 2011e]).
60 See [Bockstaele 1992] for a Latin edition of the passage and a modern account of Romanus's criticism.
61 See the description in [Van Brummelen 2009, 281–282]. Since then [Roegel 2011c] has given a recomputation.

Algebraic Gems by Viète

A tantalizing hint suggests that Rheticus was dissatisfied with existing methods for the construction of sine tables; he may have been aware that the $3m/2^{n\circ}$ barrier could be broken by solving an appropriate cubic equation as al-Kāshī had done (unbeknownst to Rheticus) just over a century earlier. Rheticus visited Gerolamo Cardano in 1545, the year Cardano published his solution to the cubic in his *Ars Magna*, "hoping it would be of some use to me in grappling with the science of triangles."[62] But he was sent away empty handed, and the *Opus palatinum* contains no hint of the use of a cubic equation. Its accomplishment, then, owes as much to industry as it does to creativity.

On the other hand, François Viète managed to make the transition to the algebraic problem, showed how to solve the relevant equations, and described how they could be used to generate sine tables—but he seems never to have implemented the solution. His methods appear in *Ad angularium sectionum analyticen*, published by Alexander Anderson in 1615 more than a decade after Viète's death.[63] The key to the solution comes early in this work where Viète determines recurrence relations for sin $n\theta$ and cos $n\theta$.

Text 1.3
Viète, Finding a Recurrence Relation for sin $n\theta$
(from *Ad angularium sectionum analyticen*)

Theorem IIII: If beginning as a point on the circumference of a circle any number of equal segments are laid off and straight lines are drawn [from the beginning point] to the individual points marking the segments, as the shortest is to the one next to it, so any of the others above the shortest will be [to] the sum of the two nearest to it.

[A geometric proof follows.]

(After Theorem VII:) Cut the circumference of a circle into a number of equal parts beginning at any assumed point and from it draw straight lines to the ends of the equal arcs. Let the shortest of these lines be Z and the next shortest B. Hence, from Theorem IIII, the first is to the second as the second is to the sum of the first and the third. The third, therefore, will be $(B^2 - Z^2)/Z$. By the same method used in the preceding [theorem],

the fourth will be $\dfrac{B^3 - 2Z^2B}{Z^2}$

[62] [Danielson 2006, 121].

[63] See [Viète 1615]; it also appears as "Theoremata ad sectiones angulares" in [Viète 1646]. See [Viète (Witmer) 1983, 418–450] for a translation.

the fifth will be $\dfrac{B^4 - 3Z^2B^2 + Z^4}{Z^3}$

. . .

the tenth will be $\dfrac{B^9 - 8Z^2 + 21Z^4B^5 - 20Z^6B^3 + 5Z^8B}{Z^8}$.[64]

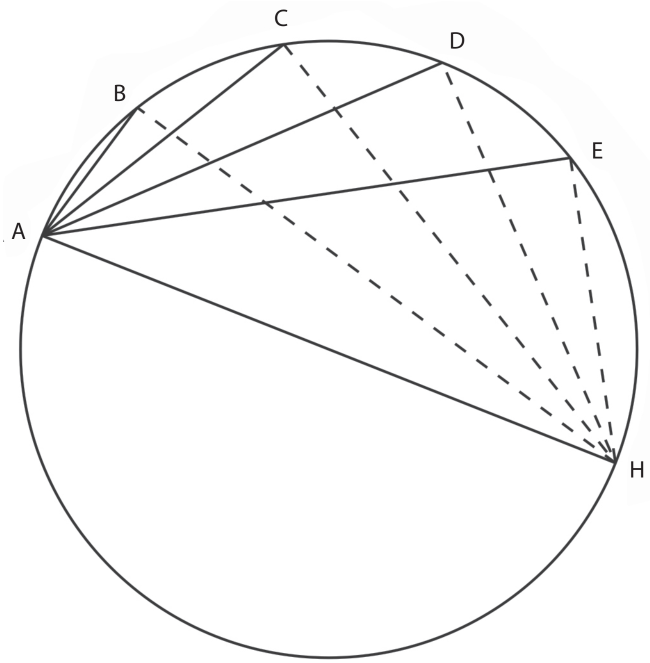


Figure 1.13
Viète's diagram for the sine multiple-angle recurrence relation (simplified). The dashed lines are used in our explanation but do not appear in Viète's figure.

Explanation: (See figure 1.13.) First, we must understand Viète's notation. Arcs $\widehat{AB}, \widehat{BC}, \widehat{CD}$, and $\widehat{DE}$ are all equal; it is understood that AH has been divided into arbitrarily many arcs. AH is a diameter, which implies that the triangles ABH, ACH, and so forth are all right angled. Let θ be the angles $\angle AHB$, $\angle BHC$, $\angle CHD$, and $\angle DHE$; by *Elements* III.20, they are equal to half the posited arcs $\widehat{AB}, \widehat{BC}$, and so on. Then (assuming we are in a unit circle)

[64] [Viète (Witmer) 1983, 426, 435–436]. Viète's algebraic notation in the original differs somewhat from Witmer's transcription; for instance, $(B^2-Z^2)/Z$ is rendered as $\dfrac{\begin{array}{c}\text{Bq.}\\ -\text{Zq.}\end{array}}{\text{Z}}$.

chord $Z = AB$ is equal to 2 sin θ while chord $B = AC$ is equal to 2 sin 2θ. Viète asserts that

$$\frac{Z}{B} = \frac{D}{C+E}, \tag{1.12}$$

where D is the second-longest chord in the diagram, C is the third longest, and E is the longest. In modern notation, this turns out to be equivalent to the recurrence relation

$$\frac{\sin\theta}{\sin 2\theta} = \frac{\sin(n-1)\,\theta}{\sin(n-2)\,\theta + \sin n\theta}. \tag{1.13}$$

Viète also determines a recurrence relation for cosines:

$$\frac{1}{2\cos\theta} = \frac{\cos(n-1)\theta}{\cos(n-2)\theta + \cos n\theta}. \tag{1.14}$$

By increasing n successively by one and solving for sin $n\theta$ each time, Viète is able to generate formulas for sin $n\theta$ for any n, including an equivalent to the sine triple-angle formula used by al-Kāshī.[65]

Viète compiles a table of the coefficients in the formulas for cos $n\theta$, going as far as $n = 21$.[66] Clearly, this would have been virtually impossible without his symbolic notation.

Was Viète simply showing off by deriving higher and higher multiple-angle formulas in this way? Perhaps. Certainly, he could hardly have illustrated more effectively the power of combining symbolic algebra with trigonometry; higher-order formulas beyond the triple-angle formula had not been discovered anywhere else, even in the Islamic world. But there was more to it than demonstrating his prowess. He reveals at least part of his intent at the end of *Ad angularium sectionum analyticen*: to find a precise value for sin 1′ in order to construct a table of sines. He begins with a value for sin 18°, which is a value that one can compute using geometric theorems. From it, Viète applies his sine quintuple-angle formula, generating sin 3°36′. This requires solving a quintic equation, which Viète does not explain how to do; however,

[65] It came to light in the nineteenth century that Joost Bürgi had followed a similar algebraic path; see [Wolf 1872–1876, 7–28; 1890, vol. 1, 169–175] and [von Braunmühl 1900/1903, vol. 1, 205–208] for accounts and [Roegel 2010a, 5–7] for a discussion of his sine table. Unfortunately, Bürgi's failure to publish rendered his work a dead end.

[66] Viète also derives equivalents to multiple-angle sine and cosine formulas up to $n = 5$ in Propositions 48–51 of his *Ad logisticem speciosam notae priores*, published in 1631 with notes by Jean de Beaugrand; it is the second treatise in [Viète (van Schooten) 1646]. For an English translation see [Viète (Witmer) 1983, 72–74]; for a French translation see [Ritter 1868, 245–276]. Witmer remarks (pp. 6–7) that Viète comes close to, but does not quite arrive at, general expressions for cos $n\theta$ and sin $n\theta$.

in another work he had shown how to approximate solutions to polynomial equations.[67] Likewise, using the sine triple-angle formula (and solving a cubic), we may move from sin 60° to sin 20°. Trisect again to get sin 6°40′; then bisect to get sin 3°20′. Apply the sine difference law to 3°36′ and 3°20′ to get sin 16′; finally, bisect four times, and we have sin 1′.[68] Viète never did implement this method, but three decades later Henry Briggs would exploit it in the construction of massive trigonometric tables in his *Trigonometria Britannica*.

We are not yet finished with Viète's algebra. Before applying his multiple-angle formulas to sine tables in the *Ad angularium*, Viète shows how one may work sometimes in the other direction using trigonometry to solve problems in algebra. His most spectacular example is his 1595 *Ad problema quod omnibus mathematicis totius orbis construendum proposuit Adrianus Romanus*.[69] This dramatic story begins two years earlier. In 1593 Romanus had proposed to the world an apparently unsolvable problem, to find roots of the 45th-degree equation

$$\begin{aligned}&45x - 3795x^3 + 95634x^5 - 1138500x^7 + 7811375x^9 - 34512075x^{11}\\ &\quad + 105306075x^{13} - 232676280x^{15} + 384942375x^{17} - 488494125x^{19}\\ &\quad + 483841800x^{21} - 378658800x^{23} + 236030652x^{25} - 117679100x^{27}\\ &\quad + 46955700x^{29} - 14945040x^{31} + 3764565x^{33} - 740259x^{35} + 111150x^{37}\\ &\quad -12300x^{39} + 945x^{41} - 41x^{43} + x^{45} = K. \qquad (1.15)\end{aligned}$$

A quick examination reveals that this is no ordinary 45th-degree polynomial; for instance, all the powers of x are odd. However, at first glance it is a mystery how, when presented this problem by a Dutch ambassador through the king of France, Viète was able to come up with one solution almost immediately, and 22 others by the next day.

[67] *De numerosa potestatum purarum* [Viète 1600]; also available in [Viète 1646, 163–228]. The method for the extraction of roots is based on finding an initial approximation a to the solution x of the polynomial, substituting $a + b$ for x in the polynomial, and applying the binomial theorem to expand the result. See also [Goldstine 1977, 66–68].

[68] [Viète 1615, 47]; an English translation is in [Viète (Witmer) 1983, 450].

[69] [Viète 1595]; also available in [Viète 1646, 305–324]. Our account is based on [Viète (Witmer) 1983, 445n46], a translation of [Viète 1595, folio 12]. Viète deals with these issues in other treatises as well, including *De aequationum recognitione* and *Supplementum geometriae*, both available in [Viète 1646]. Viète's calculus of triangles, appearing also in *Ad logisticen speciosam notae priores* and *Zeteticorum*, has drawn attention; some of its calculations are isomorphic to the use of arithmetic with complex numbers, although [Glushkov 1977] is careful to point out the danger of such "unhistorical analysis"; see also [Itard 1968], [Bekken 2001], and [Reich 1973, chapter 3]. Also, [Bachmakova/Slavutin 1977] argue that Viète's calculations with triangles are dedicated to the solution of indeterminate equations.

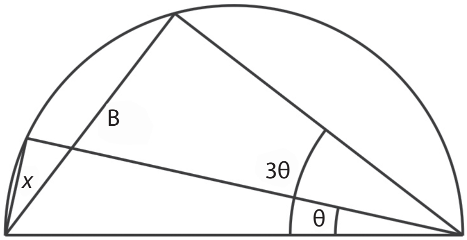


Figure 1.14
Viète's solution of the irreducible cubic equation.

We illustrate with a (thankfully) simpler case, an example of the first "Theoremation" of *Ad problema*: the equation $3x - x^3 = 1$, an example of the irreducible (sometimes called "depressed") cubic $ax - x^3 = b$ that Scipione del Ferro, Tartaglia, and Gerolamo Cardano had solved several decades earlier. Viète recognizes that the form of this cubic equation is related to the sine triple-angle formula that he expresses as $3R^2x - x^3 = R^2B$, where R is the base circle radius, x is the chord subtending angle θ in figure 1.14, and B is the chord subtending 3θ. If we are in a unit circle, then we may verify that $B = 2 \sin 3\theta$ and $x = 2 \sin \theta$. For our example we have $B = 1$. This implies that $\sin 3\theta = \frac{1}{2}$. Thus $3\theta = 30°$ or $150°$, so $\theta = 10°$ or $50°$. Hence $x = 2 \sin 10° = 0.347296$ or $x = 2 \sin 50° = 1.53208$, and Viète has found two of the three roots of the cubic equation. (Since Viète can consider only angles between 0 and 180° he cannot find the third root, which is negative.)

This remarkable use of trigonometry to solve the irreducible cubic can be extended to certain polynomials of higher powers using higher multiple-angle formulas, thereby extending beyond Cardano's solutions of the cubic and quartic equations. Of course, bringing in a sine table to solve a polynomial alters the problem by expanding the set of tools permitted to generate a solution. Nevertheless, it is ingenious and, within its parameters, successful. One can see now how Viète upheld the honor of French mathematics by solving the 45th-degree polynomial so quickly: he recognized that it is the result of two angle trisections and a quintisection ($3 \times 3 \times 5$). He was able to generate only 23 of the 45 solutions for the same reason that we generated only two of the three solutions in our cubic; the other solutions are negative.[70]

Through this tour de force, Viète had clearly demonstrated the power of the new algebra. He ends the treatise, and we end our treatment of Viète's contributions to trigonometry, as follows: "Embrace the new, lovers of knowledge; farewell, and consult the just and the good."[71]

[70] [Hollingdale 1984, 135–136] contains an account of how Viète might have gone about solving Romanus's equation.
[71] [Viète 1595, unnumbered folio after folio 13].

New Theorems, Plane and Spherical

Complete solutions to all conceivable triangles, both plane and spherical, had existed in Europe since Regiomontanus's *De triangulis omnimodis*, which remained the dominant textbook for most of the sixteenth century. One might wonder, then, what there was left to do. But Regiomontanus's book was written before advances in the mid-sixteenth century made possible certain ways to streamline the theory. Primary among these was the advent of the new functions, especially the tangent and the secant. Regiomontanus, restricted to the sine, cosine (expressed as the sine of the complement of the angle), and the versed sine, naturally approached solutions of triangles with only these three functions in mind. As the tangent and secant (and their complements) gradually established themselves as members of an expanded set of primitive functions, new and more attractive options for solving triangles became readily available.

Today, the most well known of the new sixteenth-century formulas is the planar Law of Tangents,[72]

$$\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}. \tag{1.16}$$

Most modern sources assign the first European appearance of this formula to Thomas Fincke in proposition X.15 of his 1583 *Geometriae rotundi*.[73] He introduces the law to solve triangles where two sides and the included angle are known. His first example illustrates how it works. Let $a=21$, $b=13$, and $\angle C=67°22'49''$; then $\frac{1}{2}(a+b)=17$ and $\frac{1}{2}(a-b)=4$.[74] We also know that $\frac{1}{2}(A+B)=\frac{1}{2}(180°-C)=56°18'35''$, so by the Law of Tangents, $\frac{1}{2}(A-B)=19°26'24''$. Finally, A and B may be found as the sum and difference of $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$ respectively, namely, 75°45′ and 36°52′11″.

Many other authors picked up the Law of Tangents shortly after its appearance in Fincke's book.[75] We find it used for the same purpose in, for instance, Christoph Clavius's 1586 *Triangula rectilinea,*[76] Philip van Lansberge's 1591 *Triangulorum geometriae*,[77] and Viète's 1593 *Variorum de*

[72] The theorem was known in medieval Islam, but (as far we know) it was not transmitted to Europe.

[73] [Fincke 1583, 292–293].

[74] Fincke expresses the left side of the Law of Tangents as $\frac{1}{2}(a-b)/\frac{1}{2}(a+b)$, which simplifies the calculations slightly.

[75] See [Tropfke 1903, vol. 2, 238] for a short discussion.

[76] In an appendix to his edition of Theodosius's *Spherics* [Clavius 1586, 328–329].

[77] [Van Lansberge 1591, 162].

rebus mathematicis responsorum, liber VIII.[78] So the theorem was integrated quickly into the standard corpus of plane trigonometry and has remained there ever since.

It comes as a mild surprise that the Law of Tangents does not appear directly in Viète's earlier *Canon mathematicus seu ad triangula* (1579), for that work is full of new identities, most of which have fallen out of common use today.[79] Some of the more interesting of Viète's new theorems are equivalents in his notation to

$$\tan(45° + \theta/2) = 2\tan\theta + \tan(45° - \theta/2) \tag{1.17}$$

and

$$\sec\theta = \frac{1}{2}\tan(45° + \theta/2) + \frac{1}{2}\tan(45° - \theta/2). \tag{1.18}$$

The first allows a tangent table to be computed quickly (using only additions) once the entries up to an argument of 45° have been found; the second allows the easy completion of a secant table once a tangent table has been completed. Others of Viète's theorems include

$$\cot\frac{\alpha+\beta}{2} = -\frac{\sin\alpha - \sin\beta}{\cos\alpha - \cos\beta} \tag{1.19}$$

and

$$\frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}} = \frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta}, \tag{1.20}$$

with the latter being related to the Law of Tangents. As part of his work on solving planar oblique triangles, Viète also presents the sine and cosine *difference-to-product* identities,[80]

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \tag{1.21}$$

and

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}. \tag{1.22}$$

[78] [Viète 1593, 32].
[79] See [Delambre 1821, vol. 2, 19] on the identities useful for computing tables. For a survey of the new identities in the *Canon mathematicus seu ad triangula*, see [Ritter 1895, 48–53].
[80] There are corresponding formulas for the sums of sines and cosines.

These two equations are close cousins of the *product-to-difference* (or just *product*) identities

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \tag{1.23}$$

and

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)], \tag{1.24}$$

which were of considerable interest. They were studied intensely, first by Johann Werner in the early sixteenth century and then in the 1580s by Nicolai Ursus and the group led by Tycho Brahe.[81] Their attraction lay in the fact that they could be used to transform the need to multiply two trigonometric quantities, a tedious process common in spherical trigonometry and astronomy, into the much easier task of adding or subtracting—essentially the same benefit that would be associated later with logarithms. This became known as *prosthaphairesis*; we already discussed its history in the previous volume.[82]

Spherical trigonometry also saw its share of new theorems; in fact, the subject underwent a metamorphosis during the sixteenth century. We begin where the theory itself begins, with right-angled triangles. The modern treatment reduces to these ten identities:

$\sin b = \tan a \cot A$	$\sin a = \sin A \sin c$
$\cos c = \cot A \cot B$	$\cos A = \sin B \cos a$
$\sin a = \cot B \tan b$	$\cos B = \cos b \sin A$
$\cos A = \tan b \cot c$	$\sin b = \sin c \sin B$
$\cos B = \cot c \tan a$	$\cos c = \cos a \cos b$

Many of these results had been known already to ancient and medieval astronomers, especially those in the right column consisting entirely of sines and cosines. In various forms, some of them may be found buried in texts as old as Ptolemy's *Almagest*, embedded in the language of chords and often presented within solutions to problems in spherical astronomy. The second and third identities on the right are known as Geber's theorem, named after the twelfth-century Andalusian astronomer. But neither the ancient Greek nor the medieval eastern Islamic astronomers dealt solely with the triangle as the fundamental figure of spherical trigonometry; the Greeks worked with Menelaus's theorem, and in eastern Islam after the tenth century the emphasis

[81] We do not suggest that the later interest in these formulas came from Viète.
[82] [Van Brummelen 2009, 264–265].

was on the Rule of Four Quantities.[83] Our ordered list of identities would not have been familiar to either culture.

The idea of gathering the ten fundamental identities into a unified whole is first hinted at by Georg Rheticus in a six-page dialogue at the end of his 1551 *Canon doctrinae triangulorum*.[84] Explicitly rejecting both Ptolemy and Geber, Rheticus claims to have a new approach to spherical trigonometry that requires knowledge of only ten identities applied to a right triangle. One can hardly imagine what else he may have meant, other than these. But in this dialogue, he does not elaborate or even state what they are.

Rheticus's comprehensive theory of spherical trigonometry would not appear until 22 years after his death in the 1596 *Opus palatinum* with Valentin Otho. In the meantime, several authors had beaten him to publication. The first was François Viète in his 1579 *Canon mathematicus seu ad triangula*. Viète lists all ten of the basic identities in a table as follows:[85]

	Totus	**Sinus**	**Sinus**	**Sinus**		**Totus**	**Fecundus**	**Fecundus**	**Sinus**
I	C	A B	A	C B	VI	C	A C	$\not{B}$	C B
II	C	A B	B	A C	VII	C	C B	$\not{A}$	A C
III	C	$\not{A}\,\not{C}$	A	$\not{B}$	VIII	C	$\not{A}\,\not{B}$	C B	$\not{B}$
IIII	C	$\not{C}\,\not{B}$	B	$\not{A}$	IX	C	$\not{A}\,\not{B}$	A C	$\not{A}$
V	C	$\not{A}\,\not{C}$	$\not{C}\,\not{B}$	$\not{A}\,\not{B}$	X	C	$\not{B}$	$\not{A}$	$\not{A}\,\not{B}$

The table may be read as follows. Under "Totus" the C represents the sine of the right angle at C, in other words, the radius of the base circle. "Sinus" represents the sine; "fecundus" represents the tangent. A pair of letters represents the side we would represent by the missing letter (i.e., AB represents *c*). A strikethrough represents the complementary function of that quantity. Each row expresses an equality of ratios. Thus the first row represents $\frac{\sin 90°}{\sin c} = \frac{\sin A}{\sin a}$ or $\sin a = \sin A \sin c$. The other rows give the remaining nine identities; for instance, rows III and IIII are Geber's Theorem, and row V is the spherical Pythagorean Theorem. As we shall see, the ten identities exhibit an extraordinary structure when arranged appropriately, but Viète's arrangement does not reflect this structure. Viète proceeds to rearrange the identities in various ways corresponding to his version of Rheticus's scheme for

[83] In two nested right-angled triangles sharing the angles on the bases, the ratio of the sines of the altitudes is equal to the ratio of the sines of the hypotenuses.

[84] [Rheticus 1551, third and fourth pages of the dialogue].

[85] [Viète 1579, 36–37].

grouping right-angled triangles in three species. This results in another 50 mathematically trivial variations of the ten identities. He does not prove any of them; his interest here (and elsewhere in the *Canon*) is to present the theorems compactly and systematically so that the reader may apply them easily to any triangle problem—provided that Viète's unique notation is mastered.

A couple of pages later, Viète presents another table of 60 identities.[86] The first ten are as follows:

	Sinus	**Sinus**	**Sinus**	**Sinus**		**Sinus**	**Faecundus**	**Faecundus**	**Sinus**
I	B	A̸	A̸ C̸	A̸ B̸	VI	A	B̸	A C	A B
II	A	B̸	C̸ B̸	A̸ B̸	VII	B	A̸	C B	A B
III	C̸ B̸	A̸	A B	A C	VIII	A	C B	A̸ B̸	A̸ C̸
IIII	A̸ C̸	B̸	A B	C B	IX	B	A C	A̸ B̸	C̸ B̸
V	A	C B	B	A C	X	C̸ B̸	A̸	B̸	A̸ C̸

The notation is identical to the preceding table, so for instance, the first identity should be read as $\sin B/\cos A = \cos b/\cos c$. Each of these ten identities may be derived by solving for the same term in two of the original ten theorems and setting them equal to each other; for example, this one may be found by solving for $\cos a$ in $\cos A = \sin B \cos a$ and $\cos c = \cos a \cos b$. Hence these new results are not particularly interesting here. But Viète's thoroughness occasionally leads him to stumble upon theorems that had had currency in medieval Islam; for instance, the third identity is $\cos a/\cos b = \sin c/\sin b$, which had appeared three centuries earlier in Naṣīr al-Dīn al-Ṭūsī's thirteenth-century *Treatise on the Quadrilateral*.[87]

Viète seemed to realize that such a surfeit of formulas could be confusing to the reader. Later, in his 1593 *Variorum de rebus mathematicis responsorum*, following the textbook writers of the previous decade, he selected and reported on the identities most useful for solving triangles according to which of the triangle's elements are known and which are to be found.[88] As we shall see, the theory was streamlined between 1580 and 1609; Simon Stevin has been credited with the conclusion that the original ten identities are sufficient for all right triangles in his 1608 book *Driehouckhandel*.[89]

[86] [Viète 1579, 40–41]
[87] See [Van Brummelen 2009, 190].
[88] [Viète 1593, folios 32–35].
[89] Within [Stevin 1608a]; a Latin version may be found at the beginning of the first volume of [Stevin 1608b]; the credit is given in [von Braunmühl 1900/1903, vol. 1, 227]. Here and elsewhere, he and some other writers sometimes refer to six rather than ten identities; this reflects

As for oblique spherical triangles, many authors continued to treat them simply by dropping a perpendicular from one of the vertices onto the opposite side and working with the resulting pair of right triangles, an approach that would later pay dividends in the age of logarithms. But others treated oblique triangles directly. The two fundamental results are the Law of Sines,

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}; \tag{1.25}$$

and the Law of Cosines,

$$\cos c=\cos a \cos b+\sin a \sin b \cos C. \tag{1.26}$$

Both had been stated and proved already in Regiomontanus's *De triangulis omnimodis*.[90] However, Regiomontanus's expression of the Law of Cosines is in a form that might not be recognized immediately today. It refers not to cosines but rather to versed sines:

$$\frac{\text{vers } C}{\text{vers } c-\text{vers}(a-b)}=\frac{1}{\sin a \sin b}. \tag{1.27}$$

The Law of Cosines refers to all three sides of the triangle but only one angle. There is another spherical Law of Cosines, this one referring to three angles and one side:

$$\cos C=-\cos A \cos B+\sin A \sin B \cos c. \tag{1.28}$$

This theorem did not appear in Regiomontanus or anywhere else for some time; it is stated for the first time in print (but not proven), again in a form that applies the versed sine rather than the cosine, in IV.16 of Phillipp van Lansberge's 1591 *Triangulorum geometricae*.[91] It seems that it was known earlier to Brahe[92] and possibly others. In both van Lansberge's book and in its next appearance in Viète's 1593 *Variorum de rebus mathematicis responsorum* (the latter using cosines rather than versed sines), it is placed in direct

the fact that four of the identities are identical to four others, up to switching the *A*s with the *B*s and the *a*s with the *b*s.

[90] The Law of Sines is theorem IV.17, [Regiomontanus (Hughes) 1967, 225–227]; the Law of Cosines is theorem V.2, [Regiomontanus (Hughes) 1967, 271–275].

[91] [Van Lansberge 1591, 196–197]. In later editions it appears as IV.17. Lansberge claims the theorem as his own and inserts a proof, on which we shall comment shortly, in the second edition, [van Lansberge 1631, 158–161].

[92] [Von Braunmühl 1900, vol. 1, 181] notes its appearance in one of Brahe's unpublished manuscripts.

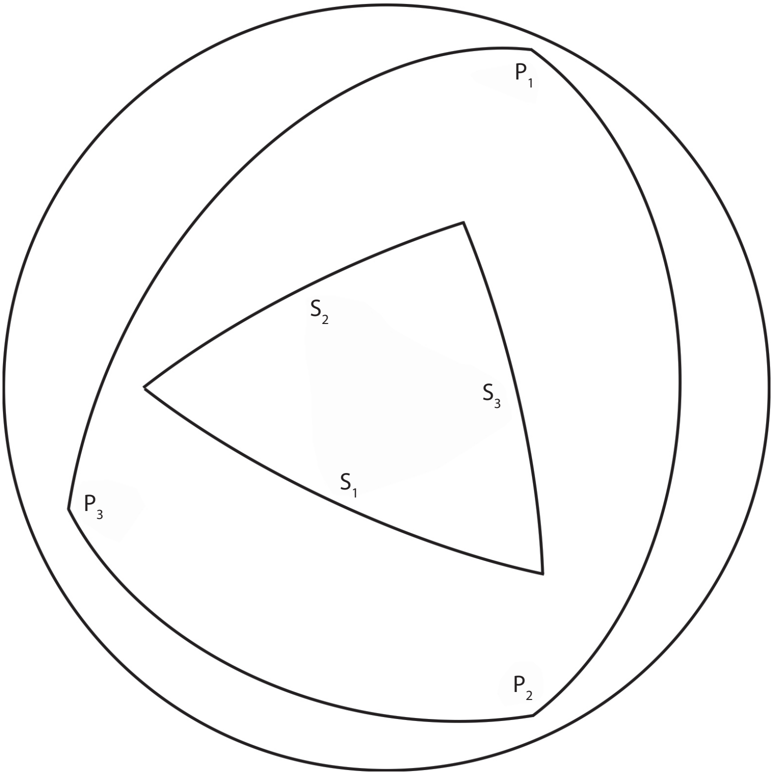


Figure 1.15
The construction of the polar triangle. For each side S_n of the original triangle, draw the pole P_n on the side of S_n that contains the interior of the triangle; then join the P_ns.

parallel with the Law of Cosines.[93] The earliest proof may be found a couple of years later, in Pitiscus's 1595 edition of the *Trigonometriae*.[94]

The correspondence between the two Laws of Cosines is no coincidence; they are linked by a duality relation. If one considers each side of a given spherical triangle to be an equator and draws the pole of that equator on the side of the triangle's interior, then joins the three poles, the resulting *polar triangle* has some remarkable properties (figure 1.15). In particular, the polar triangle of the polar triangle is the original triangle, the sides of the polar triangle are the supplements of the angles of the original, and the angles of the polar triangle are the supplements of the sides of the original. Applying this

[93] [Viète 1593, 36].
[94] [Zeller 1944, 103].

latter statement to the Law of Cosines immediately gives the Law of Cosines for Angles and vice versa.

The polar triangle had been discovered centuries earlier by astronomer Abū Naṣr Manṣūr ibn ʿIrāq around the turn of the millennium,[95] but it (along with most other trigonometric innovations from eastern Islam) does not seem to have found its way to Europe. The story of its rediscovery is more complicated. We find something like the polar triangle first in the creative hands of François Viète in his 1593 *Variorum de rebus mathematicis responsorum* where he refers somewhat obscurely to sides and angles of triangles being reciprocal.[96] Later in the same chapter Viète constructs diagrams of triangles with great circles connecting all six poles of the three sides of the original triangle; the polar triangle is one of the triangles in these figures.[97] Later in the same text, Viète gives a series of eight theorems about spherical triangles that happen to be aligned in four pairs, a theorem along with its dual result through the polar triangle. This has been taken as evidence that Viète was in fact using polar triangles as a device to convert theorems to their dual partners. In any case, Viète's presentation is sufficiently vague that it appears not to have spread far; until Viète's work was reexamined much later, credit went instead to Willebrord Snell.[98] The expression of polar triangles in the latter's 1627 *Doctrinae triangulorum canonicae* is certainly much clearer and gives a good sense as to how they can be used.

Text 1.4
Snell on Reciprocal Triangles
(from *Doctrinae triangulorum canonicae*)

Book III: PROPOSITION 8: If from the three given angles of the triangles [taken as] poles, great circles are described, the sides and angles of the triangle will be expressed, [and] the remaining sides and angles are first found reciprocally.[99]

[95] See [Van Brummelen 2009, 184–185].

[96] The tenth statement on spherical triangles in [Viète 1593, folio 41], which reads: "Si sub apicibus singulia propositi Tripleuri sphaerici describantur maximi circuli, Tripleurum ita descriptum Tripleuri primum propositi lateribus et angulis est reciprocum."

[97] [Viète 1593, folios 42–45]. See an exposition of one of the examples of this section in [Zeller 1944, 83–84].

[98] [Delambre 1819, 478–479] argues that Viète's words are not sufficiently clear to be certain that he was referring specifically to the polar triangle; [Ritter 1895, 56] disagrees. [Von Braunmühl 1898] and [1900/1903, vol. 1, 182–183] pay special attention to the problem, noting the sequence of theorems in polar pairs as evidence. [Tropfke 1923, vol. 5, 125] notes that Viète's sparse presentation likely led to the public credit passing to Snell.

[99] [Snell 1627, 120].

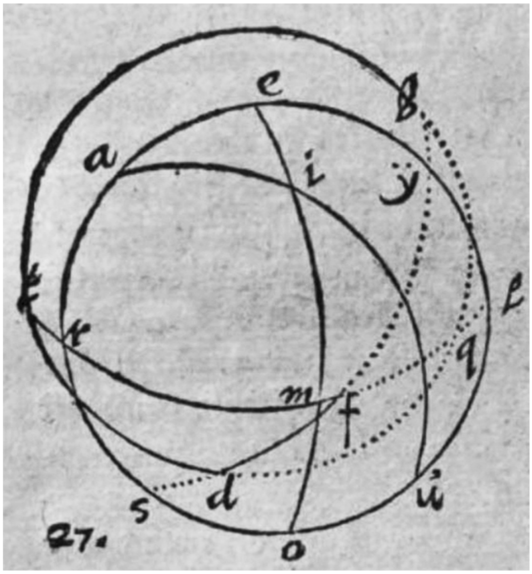


Figure 1.16
The polar triangle in Snell's *Doctrinae triangulorum canonicae*.

Explanation: (See figure 1.16) The diagram represents a sphere; the original triangle is *aei*. Snell instructs us to draw the equator *sdy* with pole *a*; equator *rfl* with pole *e*; and equator *tdqb* with pole *i*. Snell's construction is a little different than how it is usually done today; it begins by considering the poles of the original triangle and constructs equators rather than the other way around. The relation between spherical triangles and their polar duals implies that there is no difference in the final result as long as one selects the correct triangle among those formed by the intersections of the three equators.

There is one other candidate for the discovery of the polar triangle in Europe. As noted above, the Law of Cosines for Angles is stated in Philip van Lansberge's 1591 *Triangulorum geometricae*. It seems a natural inference that he might have used polar triangles to arrive at the statement of this theorem.[100] In his second edition, published four years after Snell's book in 1631, van Lansberge inserts a proof based on the idea of the polar triangle, introducing it as follows: "the second part of the [Law of Cosines for Angles], which we have the right to claim that we were the first to discover, is proved in the same way as the first, if first we describe a new triangle by means of the poles of the sides of the given triangle."[101] This is a claim for the discovery of the Law

[100] The suggestion is made in [von Braunmühl 1900/1903, vol. 1, 192–193].
[101] [Van Lansberge 1631, 158]. The description and diagram in [Zeller 1944, 97] are from the 1631 edition and are not found in the 1591 first edition.

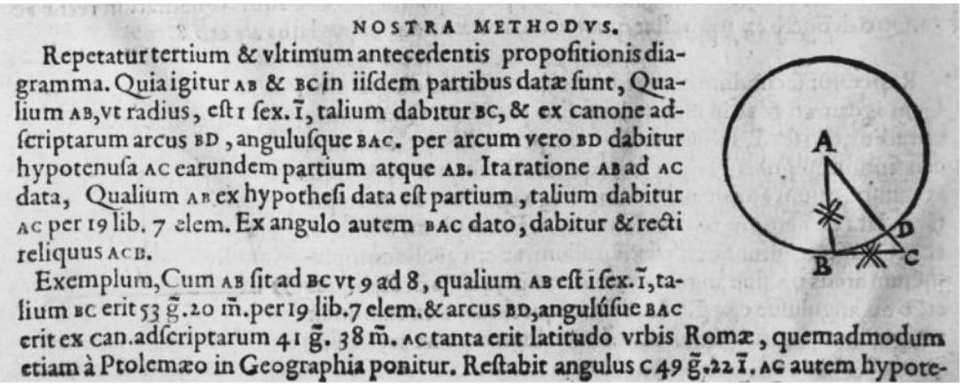
of Cosines for Angles but not quite for polar triangles. Nevertheless Simon Stevin credits van Lansberge in his 1608 *Hypomnemata mathematica* (the Latin version of his *Dreihouckhandel*)[102] and provides essentially the same proof of the complementarity of sides and angles. Perhaps van Lansberge had circulated his ideas privately.

Consolidating the Solutions of Triangles

François Viète's 1579 *Canon mathematicus* seems to have triggered a period of about three decades of textbook writing. There were enough new trigonometric functions, theorems, and approaches to solving triangles that a book to replace Regiomontanus's universal triangle solver *De triangulis omnimodis* was sorely needed, and a number of authors attempted to fill the gap. Neither Viète's notation nor the structure of his 1579 *Canon mathematicus* conformed to Regiomontanus's style, which most of his contemporaries were used to reading. Thus, while clearly most mathematicians read Viète and profited by his work, many continued to approach trigonometry within Regiomontanus's tradition (soon to be augmented by Fincke's 1583 introduction of the "tangent" and "secant"). Perhaps the earliest of these textbooks was Maurice Bressieu's 1581 *Metrices astronomicae,*[103] written and titled to position the science of triangles as a computational foundation for astronomy. Bressieu presents various different kinds of triangles and in each case outlines how to solve it, often presenting alternate methods he credits to Ptolemy and sometimes to Regiomontanus; following this he provides a numerical example. Figure 1.17 shows his solution to a plane right triangle where the two sides adjoining the right angle are known and the beginning of a numerical example.[104] Note the hash marks drawn on the given segments; Bressieu seems to have been the first of a number of authors to indicate the givens in the diagram in this way.[105]

One of the most influential of the early texts, Thomas Fincke's 1583 *Geometriae rotundi* appeared two years later. The book itself was not especially innovative mathematically, relying especially on Menelaus's theorem for its spherical results (although, as we saw, it does contain the first appearance of the planar Law of Tangents). However, it came recommended by Clavius, Pitiscus, and Napier for its exceptional clarity. Fincke's presentation (Book X for plane triangles, Book XIV for spherical) is organized around theorems

[102] [Stevin 1608b, vol. 1, 223–224].
[103] Little has been written about Bressieu; see [de Merez 1880] for a short biography.
[104] For readers attempting to translate the Latin, the "canone adscriptarum" refers to a tangent table.
[105] [Zeller 1944, 87].



NOSTRA METHODVS.

Repetatur tertium & vltimum antecedentis propositionis diagramma. Quia igitur AB & BC in iisdem partibus datæ sunt, Qualium AB, vt radius, est 1 sex. 1̄, talium dabitur BC, & ex canone adscriptarum arcus BD, angulusque BAC. per arcum vero BD dabitur hypotenusa AC earundem partium atque AB. Ita ratione AB ad AC data, Qualium AB ex hypothesi data est partium, talium dabitur AC per 19 lib. 7 elem. Ex angulo autem BAC dato, dabitur & recti reliquus ACB.

Exemplum, Cum AB sit ad BC vt 9 ad 8, qualium AB est 1 sex. 1̄, talium BC erit 53 g̃. 20 m̃. per 19 lib. 7 elem. & arcus BD, angulúsue BAC erit ex can. adscriptarum 41 g̃. 38 m̃. AC tanta erit latitudo vrbis Romæ, quemadmodum etiam à Ptolemæo in Geographia ponitur. Restabit angulus C 49 g̃. 22 1̄. AC autem hypote-

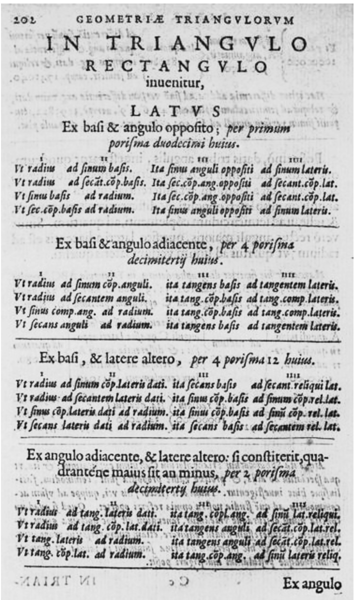
Figure 1.17
The beginning of Maurice Bressieu's solution of a right-angled triangle. The word "adscriptum" refers to his version of a tangent.

rather than triangles: that is, he presents a theorem and afterward describes how it may be used to solve a certain kind of triangle rather than the other way around. Christoph Clavius's 1586 text[106] similarly emphasizes theorems and proofs, interspersing them with "problems" that demonstrate how to use the theorems to solve certain triangles. Pitiscus's famous 1595 *Trigonometriae*[107] follows Regiomontanus's model in *De triangulis omnimodis*: he states all the theorems first and then uses them to solve various kinds of triangles. For spherical triangles he begins with four results, calling them "axioms": the Rule of Four Quantities, the Law of Tangents, the Law of Sines, and the Law of Cosines.

However, perhaps driven by the increasing use of trigonometry in applications such as surveying, navigation, and science, some texts started to emphasize an algorithmic approach based on the presentation of triangles rather than theorems: if the triangle has such and such a property, then follow this path; if it does not, then the triangle does not exist; and so forth. Some of the books we have just mentioned had an inkling of such schemes in short indexes that list the various types of triangles in sequence and indicate where one should go in the text to solve them. The index in Phillip van Lansberge's

106 Published as a supplement to his edition of Theodosius's *Sphaerica*; see [Clavius 1586].

107 Pitiscus's *Trigonometriae* first appeared at the end of Scultetus's *Sphaericorum* in 1595 and was published separately in a revised edition five years later [Pitiscus 1600]. For Handson's translation, see [Pitiscus (Handson) 1614]; the frontispiece is reproduced in the preface of this book. For a summary of the various editions and translations of the *Trigonometriae* and Pitiscus's other works, see [Archibald 1949a]. See also [Delambre 1821, vol. 2, 28–35]; [Gravelaar 1898] in Dutch, mostly on the computation of tables; [Hellmann 1997] for some discussion of the mathematics; and [Miura 1986] on the applications.



202 GEOMETRIÆ TRIANGVLORVM

IN TRIANGVLO
RECTANGVLO
inuenitur,

LATVS

Ex basi & angulo opposito; *per primum porisma duodecimi huius.*

I	II	III	IIII
Vt radius	*ad sinum basis.*	*Ita sinus anguli oppositi*	*ad sinum lateris.*
Vt radius	*ad secãt.cõp.basis.*	*Ita sec.cõp.ang.oppositi*	*ad secant.cõp.lat.*
Vt sinus basis	*ad radium.*	*Ita sec.cõp.ang.oppositi*	*ad secant.cõp.lat.*
Vt sec.cõp.basis	*ad radium.*	*Ita sinus anguli oppositi*	*ad sinum lateris.*

Ex basi & angulo adiacente, *per 4 porisma decimitertij huius.*

I	II	III	IIII
Vt radius	*ad sinum cõp.anguli.*	*ita tangens basis*	*ad tangentem lateris.*
Vt radius	*ad secantem anguli.*	*ita tang.cõp.basis*	*ad tang.comp.lateris.*
Vt sinus comp.ang.	*ad radium.*	*ita tang.cõp.basis*	*ad tang.comp.lateris.*
Vt secans anguli	*ad radium.*	*ita tangens basis*	*ad tangentem lateris.*

Ex basi, & latere altero, *per 4 porisma 12 huius.*

I	II	III	IIII
Vt radius	*ad sinum cõp.lateris dati.*	*ita secans basis*	*ad secant.reliqui lat.*
Vt radius	*ad secantem lateris dati.*	*ita sinus cõp.basis*	*ad sinum cõp.rel.lat.*
Vt sinus cõp.lateris dati	*ad radium.*	*ita sinu cõp.basis*	*ad sinũ cõp. rel. lat.*
Vt secans lateris dati	*ad radium.*	*ita secans basis*	*ad secantem rel. lat.*

Ex angulo adiacente, & latere altero: si constiterit, quadranténe maius sit an minus, *per 2 porisma decimitertij huius.*

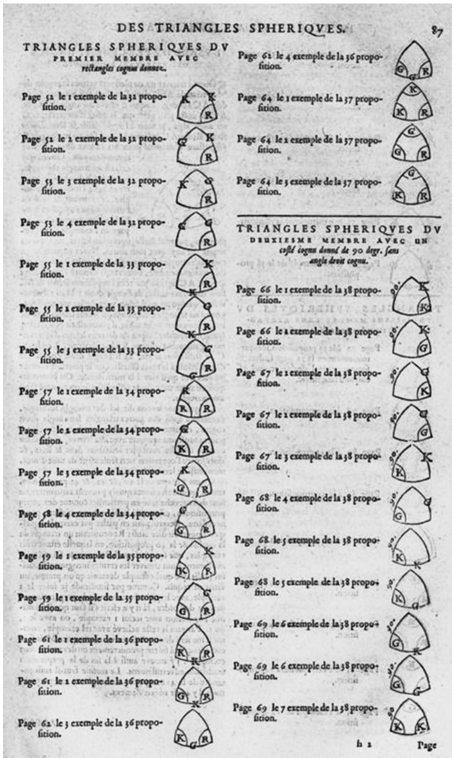
I	II	III	IIII
Vt radius	*ad tang. lateris dati.*	*ita tang. cõpl.ang.*	*ad sinũ lat.reliqui.*
Vt radius	*ad tang. cõp.lat.dati.*	*ita tangens anguli*	*ad secãt.cõp.lat.rel.*
Vt tang. lateris	*ad radium.*	*ita tangens anguli*	*ad secãt.cõp.lat.rel.*
Vt tang. cõp.lat.	*ad radium.*	*ita tang.cõp.ang.*	*ad sinũ lateris reliq.*

Ex angulo

Figure 1.18
A page from van Lansberge's 1591 *Triangulorum geometriae*, classifying methods to solve right-angled spherical triangles.

1591 *Triangulorum geometriae* is elaborate; figure 1.18, for instance, shows the first of three pages of his index for right-angled spherical triangles, grouping the various identities according to what element is sought and what elements are known.[108] Antonio Magini's 1609 *Primum mobile* goes further with similar classifications, grouping different types of spherical triangle in a 16-page-long scheme[109] and elsewhere providing grids showing which problem in his treatise solves which type of triangle.[110] Simon Stevin's *Driehouckhandel* (*Trigonometry*), published as part of his 1608 *Wisconstighe Ghedachtenissen* (*Mathematical Memoirs*),[111] divides the discussions of both planar and spherical triangles into three distinct parts: (a) preliminary theorems, (b) identities, and (c) solutions of triangles. This structure endured for hundreds of years; it is found in Todhunter's *Spherical Trigonometry*, the dominant textbook of the late nineteenth and early twentieth centuries.[112]

[108] [Van Lansberge 1591, 202].
[109] [Magini 1609, folios 38–45].
[110] [Magini 1609, folios 47, 68]. Several other grids in this work explain how to handle certain cases of problem.
[111] [Stevin 1608a]. The book, written in Dutch, was translated several times. See, for instance, the Latin edition by Snell [Stevin 1608b], and a French translation with supplements by Albert Girard [Stevin (Girard) 1634]. A selection from the treatise appears in Struik's *The Principal Works of Simon Stevin* [Struik 1958, vol. IIB, 757–761].
[112] The original edition is [Todhunter 1859]; it was revised and expanded in [Todhunter/Leathem 1901].



DES TRIANGLES SPHERIQVES. 87

TRIANGLES SPHERIQVES DV PREMIER MEMBRE AVEC rectangles cognus donnez.

Page 52 le 1 exemple de la 32 proposition.
Page 52 le 2 exemple de la 32 proposition.
Page 53 le 3 exemple de la 32 proposition.
Page 53 le 4 exemple de la 32 proposition.
Page 55 le 1 exemple de la 33 proposition.
Page 55 le 2 exemple de la 33 proposition.
Page 55 le 3 exemple de la 33 proposition.
Page 57 le 1 exemple de la 34 proposition.
Page 57 le 2 exemple de la 34 proposition.
Page 57 le 3 exemple de la 34 proposition.
Page 58 le 4 exemple de la 34 proposition.
Page 59 le 1 exemple de la 35 proposition.
Page 59 le 1 exemple de la 35 proposition.
Page 61 le 1 exemple de la 36 proposition.
Page 61 le 2 exemple de la 36 proposition.
Page 62 le 3 exemple de la 36 proposition.
Page 62 le 4 exemple de la 36 proposition.
Page 64 le 1 exemple de la 37 proposition.
Page 64 le 2 exemple de la 37 proposition.
Page 64 le 3 exemple de la 37 proposition.

TRIANGLES SPHERIQVES DV DEUXIESME MEMBRE AVEC UN costé cognu donné de 90 degr. sans angle droit cognu.

Page 66 le 1 exemple de la 38 proposition.
Page 66 le 2 exemple de la 38 proposition.
Page 67 le 2 exemple de la 38 proposition.
Page 67 le 2 exemple de la 38 proposition.
Page 67 le 3 exemple de la 38 proposition.
Page 68 le 4 exemple de la 38 proposition.
Page 68 le 5 exemple de la 38 proposition.
Page 68 le 5 exemple de la 38 proposition.
Page 69 le 6 exemple de la 38 proposition.
Page 69 le 6 exemple de la 38 proposition.
Page 69 le 7 exemple de la 38 proposition.

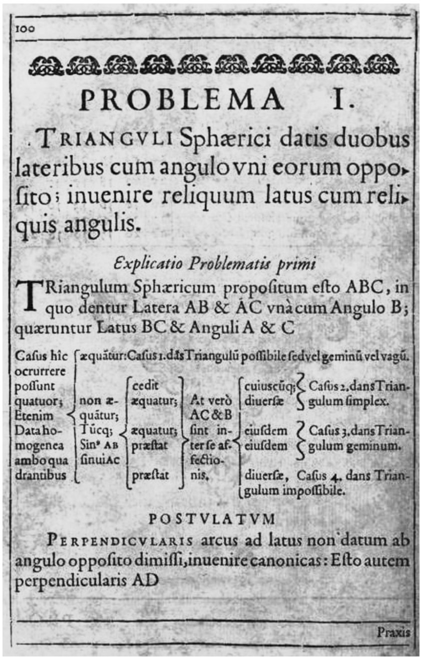
h 2 Page

Figure 1.19
A page from Albert Girard's 1634 edition of Stevin's trigonometry, showing part of the classification of spherical triangles.

Figure 1.19 shows part of the index from Albert Girard's French edition of the *Driehouckhandel*, illustrating the classification of spherical triangles (including a special category of quadrantal triangles).[113]

But when it came to algorithmic thinking, no one went further than Adrianus Romanus in his 1609 *Canon triangulorum sphaericorum*. Other than its tables and a section describing how to compute them, the entire book is a 270-page-long detailed algorithm for solving spherical triangles with dozens of examples. Book II begins with a detailed nine-page classification of triangles into various genera, followed by 40 pages of examples and diagrams of each genus. The remaining 200 pages are divided into six problems: the first dealing with triangles where two sides are given as well as one of the angles not included between the given sides, the second dealing with two given angles and one of the sides not included between the given angles, and so on. In each case Romanus provides an algorithm for solving the triangle and for handling the various cases that arise. At the bottom of figure 1.20,

[113] [Stevin 1634, vol. 2, 87]. A quadrantal triangle has a side (not an angle) equal to 90°.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PROBLEMA I.

. TRIANGVLI Sphærici datis duobus lateribus cum angulo vni eorum oppoſito; inuenire reliquum latus cum reliquis angulis.

Explicatio Problematis primi

TRiangulum Sphæricum propoſitum eſto ABC, in quo dentur Latera AB & AC vnà cum Angulo B; quæruntur Latus BC & Anguli A & C

Caſus hîc ocrurrere poſſunt quatuor; Etenim Data homogenea ambo quadrantibus
æquãtur: Caſus 1. dãs Triangulũ poſſibile ſed vel geminũ vel vagũ.
non æquãtur; Tũcq; Sin⁹ AB ſinui AC
cedit æquatur; æquatur; præſtat præſtat
At verò AC & B ſint inter ſe affectionis,
cuiuscũq; diuerſæ — Caſus 2. dans Triangulum ſimplex.
eiuſdem eiuſdem — Caſus 3. dans Triangulum geminum.
diuerſæ, Caſus 4. dans Triangulum impoſſibile.

POSTVLATVM

PERPENDICVLARIS arcus ad latus non datum ab angulo oppoſito dimiſſi, inuenire canonicas: Eſto autem perpendicularis AD

Praxis

Figure 1.20
A page from Adrianus Romanus's 1609 *Triangulorum sphaericorum*.

the beginning of his algorithm for the first problem, Romanus solves it (as many others did) by dropping a perpendicular from a vertex to the opposite side, thereby dividing it into two right triangles. He then applies (but does not prove) the right-angled triangle identities to the two right triangles.[114]

One of the most eccentric, yet remarkable methods ever developed to solve spherical triangles appears in Christoph Clavius's lengthy treatise, the *Astrolabium*.[115] This mostly astronomical treatise works extensively with the technique for spherical geometry known as the ***analemma***.[116] Dating back to ancient Greece, the analemma deals with a problem in spherical geometry by rotating one or more circles on the sphere into the plane of a particular great circle, thereby reducing it to a problem in plane geometry.

The central topic of the *Astrolabium* is ***stereographic projection***, which maps a sphere onto the plane through its equator as follows. In figure 1.21

[114] [Romanus 1609, 100]. We should mention Nathaniel Torporley's bizarre 1602 *Diclides Coelometricae*, which we saw before. Its unique approach reduces the six cases of right spherical triangles to two, but its obscurity renders it close to impenetrable. See [Delambre 1821, vol. 2, 37–40], [von Braunmühl 1900/1903, vol. 1, 183–186], [Zeller 1944, 106–107], and [Silverberg 2009].

[115] [Clavius 1593]. For an account of Clavius's interactions with Ptolemy's and Copernicus's cosmological theories, see [Lattis 1994]. For a survey of Clavius's mathematics, see [Naux 1983].

[116] See [Van Brummelen 2009, 66–67].

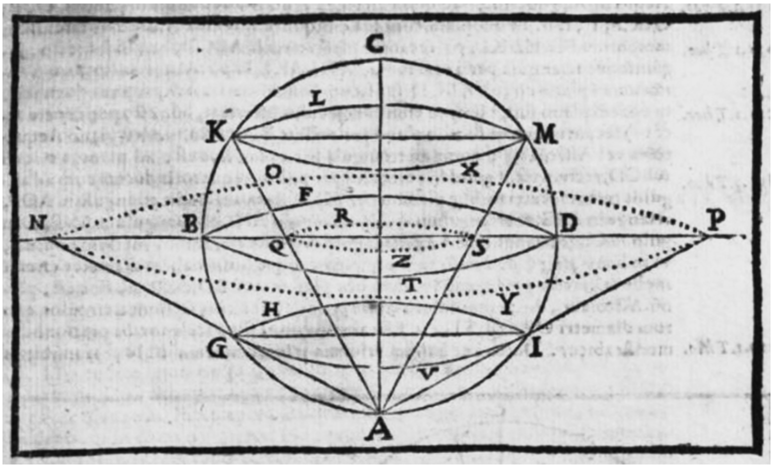


Figure 1.21
From Clavius's *Astrolabium*, illustrating stereographic projection.

the south pole is *A*, and a plane is drawn through equator *BFDT*. For any point *M* on the sphere, a line is drawn from *M* to *A*. Point *S*, where the line crosses the plane through the equator, is considered to be the projection of *M* onto the plane.[117] Stereographic projection has two advantages: circles on the sphere map to circles or lines on the plane and the angle between two great circles is mapped to the same angle on the plane. The ancient astronomical instrument, the astrolabe, is simply a physical realization of a stereographic projection of the celestial sphere.

Clavius's approach to solving spherical triangles begins by positioning the sphere so that some side of the triangle is placed along the equator (called by later authors the *primitive circle*) or so that one vertex is at the north pole. Using the given quantities, as much as possible of the projected triangle is drawn on the primitive circle. Once this is done, the remaining elements are constructed geometrically, if possible. Sometimes other great circles are rotated onto the plane, taking the place of the primitive circle. Once the projected triangle has been drawn, the sought angles and sides are measured with a ruler or protractor. Finally, these data are used as inputs into a mathematical process that reconstructs the values of the sought elements of the original spherical triangle.[118]

[117] Points on the sphere below the equator are mapped to points on the plane outside the equator; for instance, *G* maps to *N*.

[118] Clavius's methods, as well as related work by Dutch mathematician Adrian Metius (1571–1635) in Book V of *De astrolabio catholico* (1633), are described in [Haller 1899].

This method, ingenious as it is, was not seen as very practical even by some of its adherents; the famous instrument maker Benjamin Martin, introducing the subject in his 1736 *Young Trigonometer's Compleat Guide*, states that "this way is (generally speaking) more artful than useful"; but he goes on to say that "by a little use, [it] is very practicable and easy."[119] It had currency in some textbooks until as late as the nineteenth century, appearing alongside more conventional solutions as a legitimate alternative.[120]

Widening Applications

Through the fifteenth century and into the sixteenth, trigonometry had been a handmaid to astronomy; Regiomontanus himself called it "the foot of the ladder to the stars."[121] In medieval Islam, spherical trigonometry had come to be applied to finding distances and directions on the surface of the earth, originally through the determination of the direction of Mecca. But even these calculations had taken place on the celestial rather than the terrestrial sphere. This makes the sixteenth century one of the most remarkable periods in the history of mathematics, for it was during the latter part of this century that trigonometry started to become genuinely applicable to the physical world: not just for determining distances and directions in the heavens but also on the earth and sea. Raphe Handson's 1614 translation of Pitiscus's *Trigonometriae* presents a transformed view of trigonometry, liberated from its servanthood to astronomy by linking to many other earthly activities:

> All arts are in themselves so infinite, that the life of man is first consumed before he comes to know; yet, the pleasure is such (especially in the mathematics) that the more a man understandeth, the less he thinks to know; as still covetous of more, and never satisfied. And amongst all the sciences mathematical, this trigonometry, or dimension of triangles, is copious in the contemplation of it, and more profitable in the practice: For thereby all heights, depths, distances, questions of the map, globe, sphere, or astrolabe, may be more truly supputated [calculated], than by any instrument whatsoever; besides

[119] [Martin 1736, vol. 2, 150]. See pp. 150–160 for his treatment of the subject and [Van Brummelen 2013, 133–139] for a modern mathematical explanation based on Martin's text. See also [von Braunmühl 1900/1903, vol. 1, 189–191], who expresses admiration but also reserves doubts about its efficacy.

[120] See for instance [Wilson 1720] and [Keith 1826]. Other graphical methods were invented to solve spherical triangles, and interest continued (especially in educational circles) as late as the 1950s, at which point interest in spherical trigonometry itself faded away. [Bradley 1920] contains a useful bibliography of references up to that date.

[121] [Regiomontanus (Hughes) 1967, 28–29].

> the infinite use thereof in geometry, astronomy, cosmography, etc. Wherefore I have adventured thereon, as a subject, which generally in its own nature carrieth much reputation amongst the sincere lovers of those sciences.[122]

Modern students may dispute Handson's characterization of the pleasure of the subject but perhaps not its practical value.

The most obvious places for trigonometry to spread its wings were still with mathematics—in particular, to measurement within geometry, for which there was a healthy tradition dating back to ancient times. From the sixteenth century onward, a number of authors were interested in questions of goniometry and cyclometry. These related subjects dealt with measurements of various lengths, angles, and areas of certain geometric figures, especially regular polygons and circles. Trigonometry can of course be applied to such questions, but it can also benefit from such study. For instance, the study of the lengths of regular polygons is related to the determination of the sines of small arcs (such as sin 1°, which is half the length of a side of a 180-gon inscribed in a unit circle). Cyclometry in particular is intimately related to approximations for π. It was at this time that Adrianus Romanus, Ludolph van Ceulen, and Philip van Lansberge derived their values of π accurate to 16, 35, and 28 digits respectively.[123]

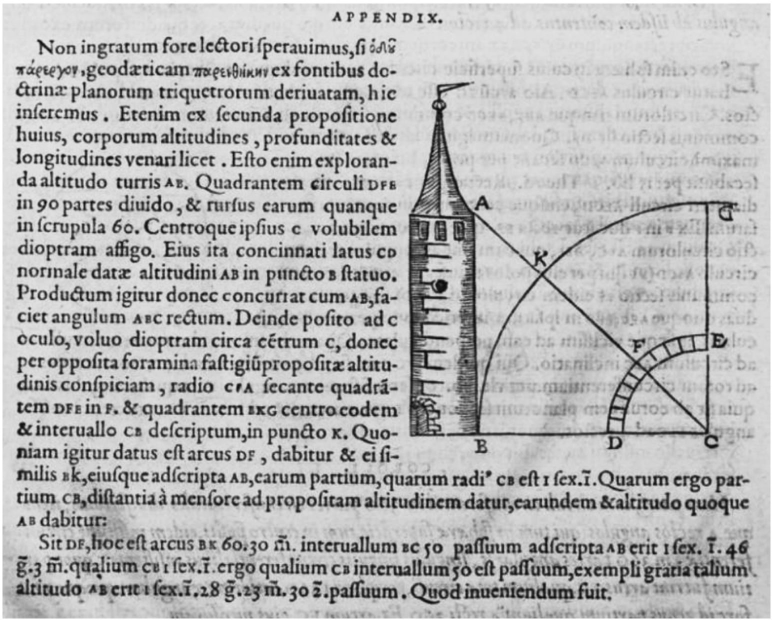
Genuine applications of trigonometry outside of mathematics were more difficult to find at first. There was of course no end to the uses of trigonometry in astronomy: they had been present since the birth of the subject, especially models of the motions of the planets, spherical astronomy, and solar timekeeping. However, earthly applications were much rarer.[124] From the thirteenth century, the genre of "practical geometry" had dealt with questions related to altimetry, stereometry, and mensuration. This subject was defined by its interaction with the physical world and often involved the use of measurements made by instruments. Its audience consisted of surveyors, architects, cartographers, observational astronomers, navigators, the military, and artists, among others.[125] A few of these treatises made some small use of trig-

[122] [Pitiscus (Handson) 1614, beginning of the dedicatory epistle].

[123] Romanus's text is the incomplete *Ideae mathematicae pars prima* [Romanus 1593]; van Ceulen's is *Arithmetische en Geometrische Fondamenten* [van Ceulen 1615] (which contains 33 digits; his final value appears in [Snell 1621]); van Lansberge's is the *Cycometriae novae* [van Lansberge 1616].

[124] One must not forget the determination of the *qibla* in medieval Islam; but even here, correct solutions relied primarily on spherical astronomy. Some other uses of trigonometry in geography in Islam do exist; see [Van Brummelen 2009, 215–217].

[125] The literature on practical geometry and its history is too extensive to be described exhaustively here; we refer only to a few texts. See [Victor 1979] for a description of its origins in medieval Europe; [Busard 1998, 7–12] for a survey of practical geometry to the mid-sixteenth century; [Taylor 1954] for a history of practical mathematics in England from 1485 to 1715;



APPENDIX.

Non ingratum fore lectori ſperauimus, ſi ὁδοῦ πάρεργον, geodæticam παρενθήκην ex fontibus doctrinæ planorum triquetrorum deriuatam, hic inſeremus. Etenim ex ſecunda propoſitione huius, corporum altitudines, profunditates & longitudines venari licet. Eſto enim exploranda altitudo turris AB. Quadrantem circuli DFE in 90 partes diuido, & rurſus earum quanque in ſcrupula 60. Centroque ipſius C volubilem dioptram affigo. Eius ita concinnati latus CD normale datæ altitudini AB in puncto B ſtatuo. Productum igitur donec concurrat cum AB, faciet angulum ABC rectum. Deinde poſito ad C oculo, voluo dioptram circa cẽtrum C, donec per oppoſita foramina faſtigiũ propoſitæ altitudinis conſpiciam, radio CFA ſecante quadrãtem DFE in F. & quadrantem BKG centro eodem & interuallo CB deſcriptum, in puncto K. Quoniam igitur datus eſt arcus DF, dabitur & ei ſimilis BK, eiuſque adſcripta AB, earum partium, quarum radi⁹ CB eſt 1 ſex. ī. Quarum ergo partium CB, diſtantia à menſore ad propoſitam altitudinem datur, earundem & altitudo quoque AB dabitur:

Sit DF, hoc eſt arcus BK 60. 30 m̃. interuallum BC 50 paſſuum adſcripta AB erit 1 ſex. ī. 46 g̃. 3 m̃. qualium CB 1 ſex. ī. ergo qualium CB interuallum 50 eſt paſſuum, exempli gratia talium altitudo AB erit 1 ſex. ī. 28 g̃. 23 m̃. 30 2̃. paſſuum. Quod inueniendum fuit.

Figure 1.22
Finding the altitude of a tower in Bressieu's *Metrices astronomicae*.

onometry, but most were devoid of it, sticking to basic geometric tools like similar triangles and the Pythagorean theorem.[126]

Prior to 1580, texts devoted to trigonometry had stayed within the confines of mathematics and astronomy. This changed dramatically with the consolidation movement of the 1580s and beyond. What one might consider to be the first practical "story problem" in a trigonometry textbook appears at the end of the chapter on planar trigonometry in Maurice Bressieu's 1581 *Metrices astronomicae*.[127]

Bressieu seems hesitant to introduce the world of practice into his trigonometry, separating the problem from his main text and introducing it with the phrase, "Hoping it will not be unwelcome to the reader." His goal is to find the height of a tower (figure 1.22) where the distance *BC* from the base is given and the angle of altitude from the observer at *C* of the top of the tower is measured. This elementary problem had been solved previously in practical geometry textbooks but not with trigonometry. Bressieu replaces the

and several of Jim Bennett's publications, especially [Bennett 1998], a survey of the relation between instruments and practical geometry.

126 See [Van Brummelen 2009, 224–230, 239–240], where the works of Abraham bar Hiyya, Fibonacci, and John of Murs are considered.

127 [Bressieu 1581, 49].

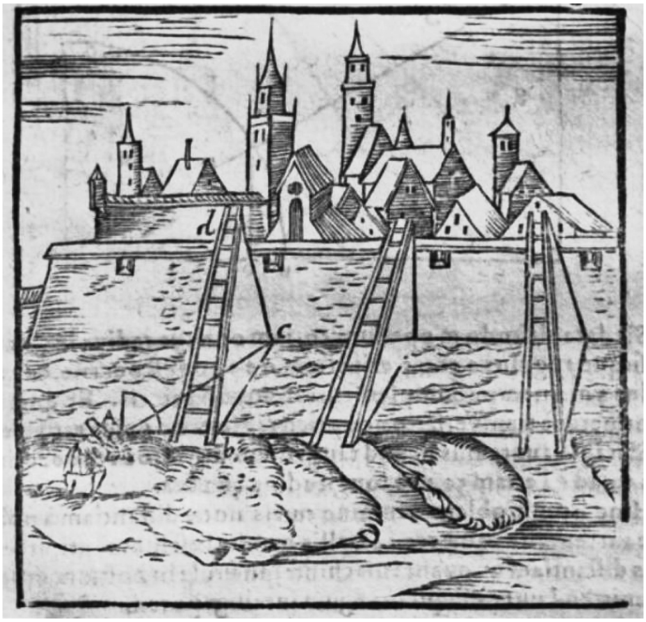


Figure 1.23 From Book XI of Fincke's *Geometriae rotundi*.

shadow square and similar triangles with an angle measurement and a tangent table, eventually finding the height as an equivalent to 50 tan θ, where θ is the altitude. In the example calculation Bressieu has an angle of elevation of 60.5° and the distance to the tower of 50 paces, which makes the height of the tower an impressive 88.5 paces.

Bressieu's tentative foray into practical geometry seems not to have had the feared effect of deterring readers, for several texts over the next decade traversed similar ground but with much more commitment. Only two years later, Thomas Fincke's *Geometriae rotundi* devoted the entire 11th book (out of 14) to problems involving altitudes and distances.[128] Its mathematics is straightforward; it consists of applying similar triangles and (often) the tangent function to measurements obtained with a quadrant and other simple instruments to determine various heights, distances, and lengths. Fincke's text and images would have appealed to surveyors, the military, and architects (see figure 1.23). Pitiscus's *Trigonometriae* goes even further; it includes chapters on geodesy, altimetry, geography, gnomometry (sundials), and astronomy and in a later edition another chapter on architecture (especially military). These applications take up over half of his text (aside from the tables).[129]

[128] [Fincke 1583, 296–322].
[129] [Miura 1986] contains a brief account of the applications chapters in Pitiscus's *Trigonometriae*.

Trigonometry and practical geometry truly came together in a meaningful way with Christopher Clavius's revolutionary new *Geometria practica* in 1604. Near the beginning of this book Clavius presents a full summary of the solutions of planar triangles via trigonometry, although with no proofs as befitted a practical work.[130] Armed with these new tools, he goes on to solve the usual surveying problems (heights of castles, etc.), but using plane trigonometry rather than the usual tools of practical geometry.

Text 1.5
Clavius on a Problem in Surveying
(from *Geometria practica*)

On the distance along the ground, whether it is accessible or inaccessible, by means of quadrant measurements at two stations in the same plane, when at its endpoint some perpendicular altitude is erected, even if [the base] is not seen at its lowest extreme. And here we determine the height.

Let the distance, or the sought length be *AB*, in plane *CB*, and erected at the endpoint *B* is some perpendicular altitude *BG*, although the endpoint *B* is not visible. Let the height of the measurer, from the eye to the feet, be *DA*. . . . Extend through *D* a parallel *EF* to *CB*, starting in the first station *D* and ending in the second station *E*, the furthest point; and line *DE*, the distance between the stations, is known by an ordinary measurement. Then, guided by the side of the quadrant *HK* that has the sights, . . . set the sights so that the peak *G* may be seen, dropping perpendicular *HI*. And . . . angle *GDF* in minutes, equal to arc *IL*, may be seen on the quadrant, clearly the complement of arc *IK*. For when thread *HI* is perpendicular to line *DF*, angle *GDF*, the complement of angle *DHI*, clearly will be equal to angle *IHL*, which is itself the complement of angle *DHI*. And we will call this angle *GDF* the angle of observation. In the same way angle *GEF* is observed at the second station, by rays from the eye, through the quadrant's sights to the peak at *G*. Taking *EM* equal to *DN*, erect perpendiculars *M<H>* and *NO*. . . . Therefore, if we set *EM* and *DN* as the *sinus toti*, *MH* and *NO* will be tangents of the angles of observation at *E* and *D*. Also draw *DQ* parallel to *EG*, crossing *NO* at *P*. Angle *NDP* is equal to angle *E*. Therefore, the two angles *N* and *D* in triangle *NDP* are equal to two angles *M* and *E*, . . . and sides *DN* and *EM*, which are adjacent, are equal. Sides *NP* and *MH* will be equal, so *OP* will be the difference between the tangents of the angles of observation. Because of this, as *OP* is to *PN*, so is *GQ* to *QF*. And as *GQ* is to *QF*, so is *ED* to *DF*. . . . Hence if [the following] is done:

As OP, the difference between the tangents of the angles of observation is to PN (or HM), tangent of the smaller [angle], so is ED, the distance between the noted stations in a common measure to the other, that is, to DF,

[130] [Clavius 1604, 45–52].

[it] produces the sought distance, *DF* or *AB*, the same measure of the distance to the station; and if it is added to the distance *ED* between the stations, we will also learn the distance *EF*, or *CB*, to the furthest station.[131]

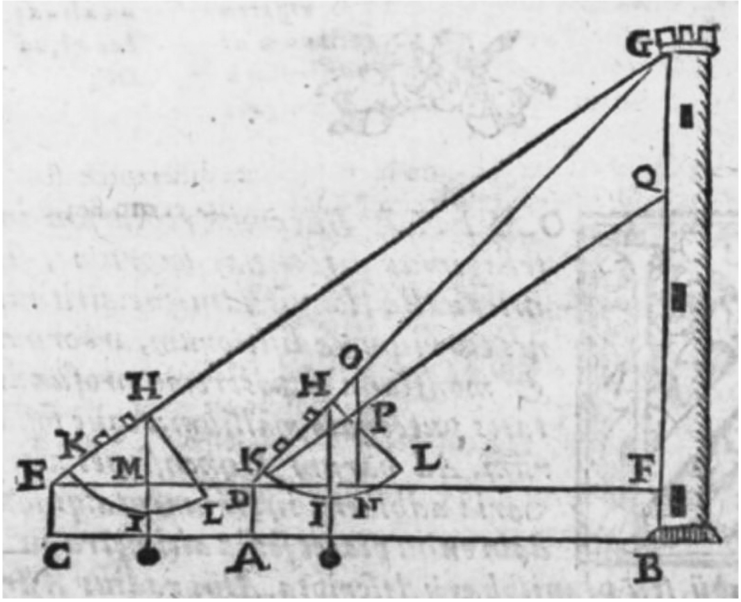


Figure 1.24
Finding the altitude of a tower if the base is inaccessible, from Clavius's 1604 *Geometria practica*.

Explanation: (See figure 1.24.) The goal is to determine the distance to a tower when its base *F* is inaccessible or hidden from view. Observers at two stations in a direct line from the tower, at *D* and *E*, measure the altitude of the pinnacle of the tower *G* (the "angles of observation" $\theta_1 = \angle GDF$ and $\theta_2 = \angle GEF$) with their quadrants. Slide ΔEMH to the right so that $\angle E$ is at *D*, defining *N* and *P*; extend *DP* to *Q*. Then $ON = DN \tan\ \theta_1$ and $MH = NP = DN \tan\ \theta_2$. So $OP = ON - NP = DN(\tan\ \theta_1 - \tan\ \theta_2)$, and hence $\dfrac{OP}{NP} = \dfrac{\tan\theta_1 - \tan\theta_2}{\tan\theta_1}$. But $\dfrac{OP}{NP} = \dfrac{GQ}{QF} = \dfrac{ED}{DF}$; and *ED* is the measured distance while *DF* is the sought distance from the first station to the base of the tower. So $DF = ED \cdot \dfrac{\tan\theta_1}{\tan\theta_1 - \tan\theta_2}$.

[131] [Clavius 1604, 54–55].

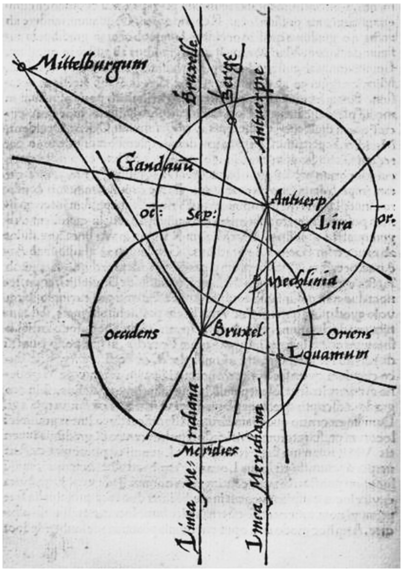


Figure 1.25
Gemma Frisius on surveying, from the 1540 edition of Apian's *Cosmographia*.

The merger of trigonometry with practical needs in geodesy and altimetry provided practitioners with much more powerful and precise mathematical tools. However, without corresponding improvements in the instruments used to measure distances and angles, the extra precision would be superfluous. The new methods were not adopted very widely at their outset. The power of geometry had been revealed to surveyors as early as 1533, with Gemma Frisius's introduction of the notion of triangulation on the surface of the earth (figure 1.25).[132] Although his techniques had required angle measurements, they had not employed trigonometry. Various instruments were invented for use in surveying through the sixteenth century, including a device called a "trigonometer," which formed with its arms a triangle similar to the triangle being measured on the ground. However, only the simple theodolite, measuring azimuths but not altitudes, seems to have gained much traction in practice. It would not be until the first half of the seventeenth century that the power of geometry in general and trigonometry in particular would become generally accepted in surveying practice.[133] This late adoption may have been aided at least in part by the wave of surveying applications in the trigonometry

[132] This appears first in Gemma Frisius's 1533 edition of Peter Apian's *Cosmographia* [Apian 1533a]. See analyses of Gemma Frisius's, Brahe's, and Snell's approaches to triangulation in [Haasbroeck 1968]. On triangulation in Gemma Frisius's work, see also [Taylor 1927] and [Pogo 1935]; the latter contains a facsimile edition.
[133] [Bennett 1991b], especially pp. 348–354.

Figure 1.26
Frontispiece of Aaron Rathborne's 1616 *The Surveyor.*

textbooks but may have had more to do with logarithms, which we shall see in chapter 2. A notable step forward was Aaron Rathborne's 1616 *The Surveyor* (figure 1.26), which introduces trigonometry in certain contexts and even mentions Pitiscus and Napier in one of the earliest references to logarithms outside of mathematics and astronomy.[134] Rathborne was a member of the peculiarly English trade of "mathematical practitioner." These men earned their living, at least in part, through tutoring mathematics useful for purposes such as engineering and gunnery rather than the higher pursuits of natural philosophy.[135]

[134] [Rathborne 1616, 142].

[135] Much has been written about the culture of the English mathematical practitioners. For a start on the literature, see [Taylor 1954] and [Taylor 1966]. A more recent account, arguing (in part) that the upper classes were not entirely separate from the trade, is [Feingold 1984]. See also [Bennett 1982], [Bennett 1991a], [Johnston 1994], [Neal 1999], [Hackmann 2000], and [Cormack 2006], among others.

It is thus no surprise that England took the lead in the integration of trigonometry with navigation. This had not yet begun in the mid-sixteenth century, with European trigonometry still in its infancy and still firmly attached to astronomy; as Leonard Digges in his 1553 *Prognostication* had lamented, "but those who have tried [to introduce trigonometry] know how far this passes the capacity of the common man."[136] However, in 1581, surely before he had seen the flood of trigonometry textbooks that was just starting to appear, naval officer William Borough advocated using trigonometric tables to calculate the sun's azimuth, referring to Regiomontanus and the tables of Copernicus, Reinhold, and Rheticus, although apparently he had not seen Viète's *Canon mathematicus.*[137]

Trigonometry was circulating in England, but it did not really enter into English publications until its use in navigation became clearer near the end of the century as awareness of the practical value of the subject was growing.[138] An appendix to Thomas Blundeville's popular 1594 *Exercises* dedicated to astronomy, geography, and navigation,[139] larger than the rest of the book, contained the first trigonometric tables published in England (explicitly borrowed from Clavius). Blundeville illustrated the use of these tables to solve astronomical problems important for navigation and printed them in a compact size helpful for use at sea.[140]

Two major navigational books, both published in 1614, solidified the union of trigonometry with navigation. The first was a partial translation of Pitiscus's *Trigonometriae* by Ralph Handson, a friend of Aaron Rathborne and a student of Henry Briggs, of whom we shall say more in chapter 2. Handson added a section on navigation, "wherein is manifested, the disagreement betwixt the ordinarie sea-Chart, and the globe, and the agreement betwixt the globe, and a true sea-chart: made after Mercator's way, or Mr. Edw. Wright's projection: whereby the excellency of the art of triangles will be the more perspicuous."[141] (We shall discuss this projection shortly.) Among Handson's

[136] Quoted in [Taylor 1954, 52].

[137] Quoted in [Taylor 1957, 211]. Borough speaks of completing for himself the second half of Rheticus's *Canon doctrinae triangulorum* (Rheticus had calculated the table for arguments up to 45°), either unaware that one may simply read the columns backward to generate the entries for arguments greater than 45° (although Rheticus had provided arguments working backward on the right side of his table for this purpose) or hoping to provide tables with arguments up to 90° for easier use in "Navigation and Cosmographie."

[138] John Blagrave's *The Mathematical Jewel* [Blagrave 1585], a description of a new mathematical instrument, contains definitions of trigonometric functions; however, he solves triangles not with the functions but with his new instrument.

[139] [Blundeville 1594]; see also the facsimile edition [Blundeville 1971].

[140] [Waters 1958, 355–356].

[141] [Pitiscus (Handson) 1614, nautical section, 1].

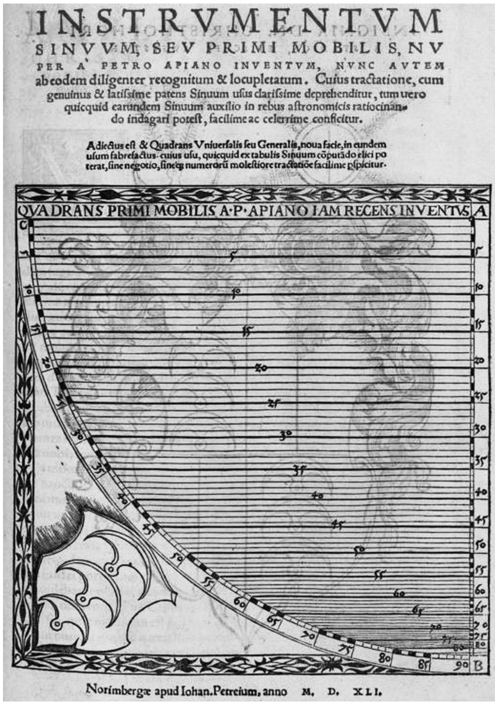


Figure 1.27
The title page of Peter Apian's 1541 *Instrumentum sinuum, seu primi mobilis.*

contributions was the "mid-latitude formula," which allowed sailors to determine, from the longitudes and latitudes of two places, their bearing and distance from one another. Ease of calculation, important for navigators, was important to Handson; he emphasizes the benefits of prosthaphairesis to convert multiplications to additions in the same year that his Scottish colleague John Napier was to render it obsolete.[142] Handson's book was aided into publication by his colleague John Tapp, who the same year published a new edition of Robert Norman's *The Newe Attractive* and William Borough's *Discourse on the Variation of the Cumpas*, to which he appended a set of navigational techniques for use with trigonometric tables.[143] Tapp's intent was to promote the "arithmetical sailing," that is, trigonometric methods with tables. The computational barriers to these methods, not inconsiderable in practice, were to become much more benign before the year was out.

However, in the meantime, the need to calculate—especially multiplication and division with trigonometric quantities—was a near-fatal disadvantage; while seamen might have been capable of the task, it was cumbersome

[142] For an account see [Waters 1958, 393–399].
[143] [Norman 1614]. See [Waters 1958, 559–562] for an account of Tapp's trigonometric navigation.

when required on a regular basis and, more seriously, prone to error. The alternative to calculation with trigonometric tables was the use of mathematical instruments, which worked much more quickly and easily, and the loss of precision caused by the use of a physical device was insignificant for navigation. Several such instruments had existed for centuries; see for instance the sine quadrant on the title page of Peter Apian's 1541 *Instrumentum sinuum seu primi mobilis* (figure 1.27). However, the needs of tradesmen and navigators in the context of the new practical mathematics seems to have brought instruments freshly into the discussion; in 1598 Thomas Hood and Galileo independently invented "sectors" with multiple uses that were predecessors to the slide rule.[144]

However, the sector that really made arithmetical navigation accessible was invented by Edmund Gunter around 1606. A young recent graduate of Oxford, Gunter would become associated with Henry Briggs and Edward Wright at Gresham College several years later. His fame rests on his instruments, especially the sector and a quadrant also named for him. Indeed, his connection with instruments and hence the class of mathematical practitioners seems at least once to have decreased his reputation. John Aubrey recounted his interview with Henry Savile for the first Savilian chair of geometry at Oxford:

> [Gunter] came and brought with him his sector and quadrant, and fell to resolving of triangles and doing a great many fine things. Said the grave knight, "Do you call this reading of Geometry? This is showing of tricks, man!" and so dismissed him with scorn, and sent for Briggs from Cambridge.[145]

It took Gunter until two years before his death to publish a book on his invention, the *De sector et radio* (1624), but his work had circulated widely in manuscript long before that.[146] Likely inspired by Hood's device, Gunter's sector is a simpler instrument honed for the purpose of calculation (figure 1.28).

[144] The origins of the sector are not entirely clear; see [Williams/Tomash 2003] for a survey and a description of the other lines on the instrument. On Hood and his sector, see [Johnston 1991] and [Taylor 2013]. On Galileo's sector see [Galileo 1978]. [Drake 1977] demonstrates that Hood and Galileo worked independently. For the contribution of Antwerp mathematician Michiel Coignet, see [Meskens 1997].

[145] [Aubrey 1982, 117]; see [Higton 2001] for a discussion of the context of the issue. Even today the attitude persists; the *Dictionary of Scientific Biography* entry says that "the tools he provided were of immense value long afterward," but that his contributions were "essentially of a practical nature," and that he was merely a "competent but unoriginal mathematician" [Pepper 1972, 593].

[146] [Gunter 1624]; see [Higton 2013] on the illustrations and diagrams in this work. On Gunter's sector and other contributions to navigation, see [Waters 1958, 358–392] and [Cotter 1981].

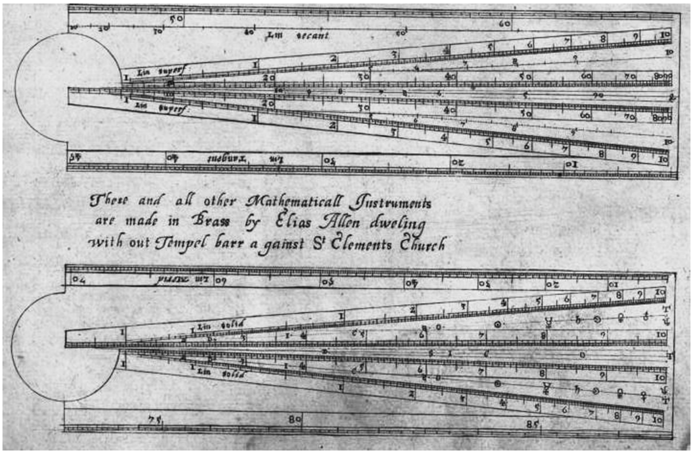


Figure 1.28
Gunter's sector, from his 1636 *Description and Use of the Sector, Crosse-Staffe and Other Instruments.*

It has two arms fixed with a hinge at one end and various scales marked on both sides of each arm. Scales for the sine, tangent, and secant allow the device to solve any triangle, plane or spherical. For instance, the sine scale is marked so that the distance of any point from the hinge corresponds to the sine of the angle indicated at that point. The arms open outward, and with a pair of compasses the user is able to form similar triangles that correspond to various ratios such as those that arise in the solutions of right-angled spherical triangles.

Text 1.6
Gunter on Solving a Right-Angled Spherical Triangle with His Sector
(from *De Sectore et Radio*)

In a rectangle triangle: To find a side by knowing the base, and the angle opposite to the required side.

As the Radius
is to the sine of the base;
So the sine of the opposite angle
to the sine of the side required.

As in the rectangle *ACB*, having the base *AB*, the place of the Sun 30° from the equinoctial point, and the angle *BAC* of 23°30′ the greatest declination, if it were required to find the side *BC* the declination of the Sun.

Take either the lateral sine of 23°30′ and make it a parallel radius; so the parallel sine of 30° taken and measured in the side of the Sector; shall give the side required 11°30′. Or take the sine of 30° and make it a parallel radius; so the parallel sine of 23°30′ taken and measured in the lateral sines, shall be 11°30′ as before.[147]

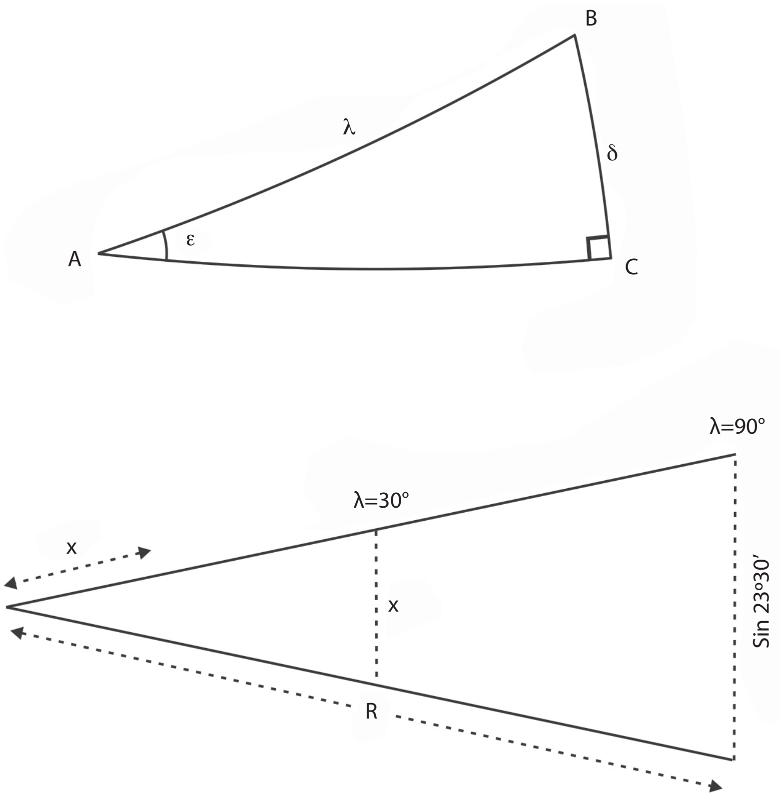


Figure 1.29
Finding the declination using Gunter's sector.

Explanation: (See figure 1.29) By "base," Gunter means the hypotenuse of the spherical triangle. Gunter's first example is a standard astronomical problem: find the sun's declination δ from its ecliptic longitude λ. The solution is $\sin\delta = \sin\lambda \sin\epsilon$ (where $\epsilon = 23°30'$ is the obliquity of the ecliptic), equivalent to the modern formula $\sin a = \sin A \sin c$ for a right triangle. But Gunter expresses it as $\frac{\text{Sin}\,90°}{\text{Sin}\,\epsilon} = \frac{\text{Sin}\,\lambda}{\text{Sin}\,\delta}$ for good reason.

The "line of sines" is the unequally marked scale near the middle in figure 1.28, ending at 90°, displayed on both arms of the sector. Set a compass

[147] [Gunter 1624, 76].

along the line of sines so that the distance between the two tips is equal to Sin 23°30′, that is, the distance along the line of sines from zero to 23°30′. Move the compass to the end of the sector, and spread the sector's arms so that their ends touch both ends of the compass. Without changing the angle of the pivot, move the compass inward (narrowing the gap between its tips) so that its two ends touch the two locations on the sector corresponding to 30°. We now have two similar triangles; from them, we have $\sin 90° / \sin 23°30' = \sin 30° / x$, where x is the new distance between the compass tips. From Gunter's ratio above, we know that x is equal to $\sin \delta$. Move the compass so that one of its tips is at the pivot. On the line of sines, the point corresponding to the other tip (11°30′) is δ.[148]

One of the scales on Gunter's instrument is entitled "meridional parts," and therein lies the final episode of this chapter. The shortest voyage between two ports is of course the great circle arc between them. However, traveling along this course is difficult because one's bearing changes continuously and so frequent course corrections are required. A simpler choice (although a slightly longer journey) is to travel along a path with a constant bearing, called a *loxodrome* or *rhumb line*. It would be helpful for a navigator to have in his possession a map with the property that a straight line on the map corresponds to a rhumb line on the ocean. A straight line drawn on the map, say, at a 45° angle upward and to the right, would follow a northeast bearing at every point. Pedro Nuñez had discovered the difference between great circles and rhumb lines in 1533. The first to construct a map with the desired property (in 1569) was none other than Gerard Mercator, a former pupil of Gemma Frisius.[149] For a map to achieve the required property, it turns out that the lines corresponding to latitude circles must be spaced not at equal intervals but at ever greater distances from each other as one moves from the equator to a pole (figure 1.30). Although this notion is at the heart of the Mercator projection,

[148] The reader may object that the arms cannot be spread far enough apart to fit R (the length of the sector) between the two 23°30′ indicators. However, elsewhere in the treatise Gunter explains how one may scale quantities up and down using linear scales (the "line of lines," marked from zero to ten printed on the other side of the sector. Using similar triangles as above, one may use compass distances of $R/10$ and $\sin 30°/10$. On the "line of sines" side of the sector, separate the 23°30′ indicators by $R/10$. Then insert the compass points separated by $\sin 30°/10$ at the appropriate place on the line of sines to find δ as before.

[149] [Mercator 1961]. The literature on Mercator is enormous; we point out only a few recent items. [Crane 2002] and [Taylor 2004] are two of the most recent biographies while [Monmonier 2004] is a social history of the projection, including the early history but also the modern debate with the alternative Peters projection. [D'Hollander 2005] deals with the projection itself. [Delevsky 1942] also considers the possible sources of Mercator's ideas. There also have been more than a few volumes of collected papers over the past two decades on Mercator and his historical context.

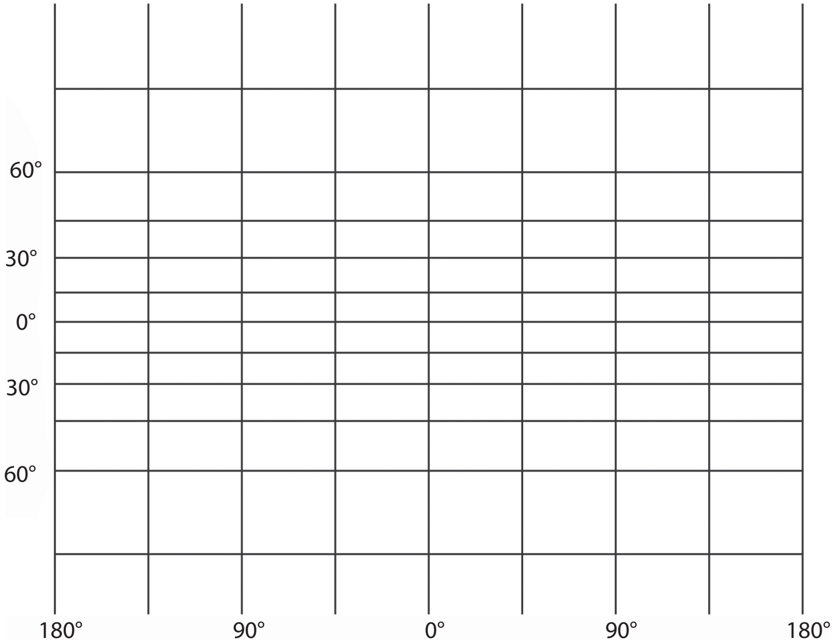


Figure 1.30
Longitude and latitude lines in Mercator's projection.

Mercator's latitude circles on his own map were not very accurately placed, and it is unclear what process he invoked to place them.

The first published solution to the problem of the spacing of the latitude lines is due to Edward Wright in his 1599 *Certaine Errors in Navigation* (figure 1.31).[150] Also known for his translation of Napier's *Mirifici logarithmorum canonis descriptio* in 1618 (published the year after Napier died), Wright also collaborated with Henry Briggs for many years; Wright, Briggs, and Edmund Gunter were together at Gresham College in 1615. Wright's interest in logarithms and mathematical instruments was practical—for the service of navi-

[150] [Wright 1599], although some of his table of meridional parts had previously appeared multiple times, first in [Blundeville 1594]. Indeed, the whole book nearly appeared under someone else's name before Wright was compelled to publish; clearly the demand for Wright's ideas was strong. In addition to various scholarly treatments (including within some of the books on Mercator we mentioned previously), the method has been described in several popular articles; see for instance [Rickey/Tuchinsky 1980], [Fernández Garcia/Jiménez Alcón/Muñoz Prieto 2001], and [Maor 2002, 174–177]. On Wright and his work, see [Parsons/Morris 1939] and [Waters 1958, especially pp. 219–229].

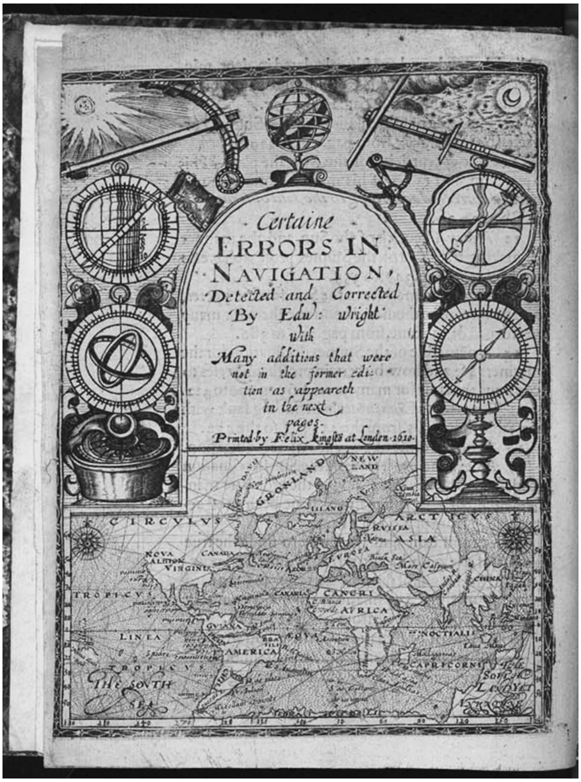


Figure 1.31
The frontispiece of the second edition (1610) of Edward Wright's *Certaine Errors in Navigation.*

gation. His *Certaine Errors* became a landmark, if one may put it that way, for finding one's way at sea.

The idea behind Wright's solution to the problem of the spacing of the latitude circles is straightforward. The latitude circles in figure 1.30 are all drawn as if they have the same length, but on the globe their lengths vary in proportion to cos ϕ, where ϕ is the latitude. Therefore, horizontal distances (longitudes) have been stretched relative to the equator by a factor of the reciprocal of cos ϕ, that is, sec ϕ. To preserve bearings, the vertical distances ΔA (where A is the northward distance on the map from the equator to the latitude line corresponding to ϕ) must be stretched by the same ratio. So, at latitude ϕ, ΔA should be proportional to sec $\phi \cdot \Delta A$ (at latitude 0°), but since there is no stretching at the equator, at latitude 0° ΔA is equal to $\Delta\phi$. Hence $\Delta A = k \sec\phi \cdot \Delta\phi$.

The modern calculus student will notice immediately that this is the same as $A(\phi) = k\int_0^{\phi} \sec\varphi \, d\varphi$, but at Wright's time calculus was still many decades away. So, to construct his table of meridional parts, Wright was forced into an onerous calculation (which he described as "an easy way laid open"):

> For . . . by perpetuall addition of the secantes answerable to the latitudes of each point or parallel unto the summe compounded of all the former secants, beginning with the secans of the first parallel's latitude, and thereto adding the secans of the second parallel's latitude, and to the summe of both of these adjoyning the secans of the third parallel's latitude, and so forth in all the rest, we may make a table which shall shew the sections and points of latitude in the meridians of the nautical planisphere: by which sections, the parallels are to be drawne.

Effectively, then, Wright uses a Riemann sum to compute $A(\phi)$. He chooses $\Delta\phi = 1'$ but helpfully refers readers to Rheticus's *Opus palatinum* should someone wish to take on the thankless task of improving the accuracy of the calculation by decreasing $\Delta\phi$ to $10''$.[151]

Wright was not the only English navigator working on the problem of meridional parts. John Dee (1527–1609), a friend of Mercator and Nuñez and a student of Gemma Frisius, had produced tables that predated Mercator's 1569 map, although the method he used to calculate them is unknown.[152] Later, Dee's colleague Thomas Harriot (1560–1621) (who himself served on an ocean-going expedition to Virginia with Sir Walter Raleigh in the 1580s) would also venture in this direction. Harriot's highly innovative work in mathematics and science never saw a printing press during his life. Today it exists only in manuscripts, notes, and modern scholarly editions. In mathematics he is known especially for his contribution to the theory of equations; closer to our interests here, he also was the first to state the area of a spherical triangle (although he did not prove his result). Harriot constructed tables of meridional parts in the 1580s or 1590s, not long after his voyage with Raleigh. He revisited the topic late in his life and in 1614 constructed a large table of meridional parts with the aid of finite difference interpolation.[153] We shall discuss this topic in chapter 2.

[151] [Wright 1599, chapter entitled "Faults in the common sea chart," from the 17th to the 19th page].

[152] The literature devoted to Dee is extensive, but not much attention has been paid to his interest in navigation; see [Taylor 1955], [Taylor 1957, 195–207], [Alexander 2005], and [Baldwin 2006]. For his table of meridional parts see [Taylor 1963, 415–433].

[153] The history of Harriot's contribution to the problem of meridional parts has been controversial. See [Taylor/Sadler 1953], [George 1956], [Lohne 1965/66], [Pepper 1967a], [Pepper 1967b], [George 1968], and especially [Pepper 1968] and [Pepper 1976]. [Taylor/Sadler 1953] and [Pepper 1967b, 23–25] reveal that Harriot somehow knew some formula for meridional parts.

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Arabic names are alphabetized after the initial "al-" if it appears. Thus, for instance, "al-Bīrūnī" appears in the Bs. Historical works are listed under their author's name, if known.