

CONTENTS

List of Examples xiii

Preface xvii

1	Introduction	1
2	Linear Spaces and Transformations	3
2.1	Properties of Linear Spaces	3
2.2	Vector Spaces	4
2.3	Linear Transformations	4
3	Differentiable Manifolds	7
3.1	Charts and Coordinates	7
3.2	Definition	10
3.3	Local Coordinate Changes	11
3.4	Functions on Manifolds	13
3.5	Orientable Manifolds	13
4	Vectors and One-Forms	15
4.1	Vectors	15
4.1.1	Vectors as Tangents to Curves	16
4.1.2	Bases and Coordinates	17

4.1.3	Vector Field	19
4.1.4	Transformations	20
4.2	One-Forms	23
4.2.1	Duality	23
4.2.2	Bases	24
4.2.3	Transformations	26
4.3	Alternative Perspective	28
4.4	Lie Bracket	29
5	Tensors	33
5.1	Definition	33
5.2	Operations on Tensors	35
5.2.1	Addition	35
5.2.2	Tensor Product.....	35
5.2.3	Contraction	36
5.2.4	Transpose of (2,0) and (0,2) Tensors	38
5.2.5	Transpose of a (1,1) Tensor	40
5.3	Transformations	41
5.3.1	Tetrad Formalism.....	43
5.3.2	Pseudotensors.....	46
5.4	Kronecker or Identity Tensor	46
5.5	Logarithms and Exponentials of (1,1) Tensors	46
5.6	Tensor Densities and Capacities	47
5.6.1	Pseudotensor Densities and Capacities	48
5.7	Levi-Civita Density and Capacity	48
5.8	Determinant of Rank-2 Tensors	50
5.9	Inverse of Rank-2 Tensors	50
5.10	Metric Tensor	51
5.10.1	Formulation	51
5.10.2	Geometrical Meaning.....	55
5.10.3	Norm of Vectors and One-Forms.....	56
5.10.4	Metric in Tetrads	56
5.11	Adjoint of a (1,1) Tensor	57

5.12	Tensor Densities and Capacities Revisited	58
5.13	Levi-Civita Pseudotensor	58
5.14	Kronecker Determinants	60
5.15	Rotations	63
5.15.1	Euler Angles	64
5.15.2	Rodrigues's Formula	66
6	Maps between Manifolds	71
6.1	Maps	71
6.2	Maps between Manifolds of Different Dimensions	72
6.2.1	Pullback	72
6.2.2	Pushforward	74
6.3	Maps between Manifolds of the Same Dimensions	76
7	Differentiation on Manifolds	85
7.1	Covariant Derivative	85
7.1.1	Formulation	85
7.1.2	Transformation of Connection Coefficients	89
7.1.3	Divergence	91
7.1.4	Parallel Transport	91
7.1.5	Torsion and Curvature Tensors	94
7.1.6	Bianchi Identities	97
7.1.7	Torsion-Free Connection	97
7.1.8	Covariant Derivative of the Metric Tensor	98
7.1.9	Mixed Covariant Derivative in Tetrad Basis	101
7.1.10	Spin Connection	102
7.1.11	Contracted Bianchi Identities	102
7.1.12	Covariant Derivative of Tensor Densities and Capacities	107
7.1.13	Nonmetricity	107
7.2	Euler Derivative	112
7.3	Lie Derivative	114
7.3.1	Lie Derivative of Vectors	114
7.3.2	Geometrical Interpretation	117

7.3.3	Autonomous Lie Derivative	118
7.3.4	Lie Derivative of One-Forms	118
7.3.5	Lie Derivative of (p, q) Tensors	119
7.3.6	Lie Derivative of Functions	120
7.3.7	Lie Derivative of Metric Tensors	120
7.3.8	Lie Derivative of Levi-Civita Tensor	123
8	Differential Forms	127
8.1	Definition	127
8.2	Operations on Forms	130
8.2.1	Addition	130
8.2.2	Exterior Product	130
8.2.3	Interior Product	133
8.3	k-Vectors	133
8.4	Hodge Dual	134
8.5	Volumes	135
8.5.1	Properties	136
8.6	Surfaces	137
8.7	Exterior Derivative	139
8.7.1	Coordinate-Free Definition	140
8.7.2	Exact Forms	142
8.7.3	Commutativity with Pullback and Pushforward	144
8.8	Lie Derivative of a Form	148
8.9	Vector- and Tensor-Valued Forms	150
8.9.1	Transformations of Tensor-Valued Forms	151
8.9.2	Operations on Tensor-Valued Forms	159
8.9.3	Connection One-Forms	161
8.9.4	Torsion Two-Forms	162
8.9.5	Exterior Covariant Derivative	164
8.9.6	Covariant Lie Derivative	167
8.9.7	Curvature Two-Forms	169
8.9.8	Commutator of Covariant Lie and Exterior Covariant Derivatives	171

8.9.9	Bianchi Identities Revisited	172
8.9.10	Nonmetricity Revisited	173
8.10	Integration of Forms	174
8.10.1	Line Integrals	174
8.10.2	Surface Integrals	175
8.10.3	Volume Integrals	177
8.11	Stokes's Theorem	178
8.11.1	Fundamental Theorem of Calculus	178
8.11.2	Green's Theorem	179
8.11.3	Gauss's Theorem	180
8.11.4	Stokes's Theorem	180
8.11.5	Variational Principles	180
8.11.6	Noether's Theorem	181
8.11.7	Applications	183

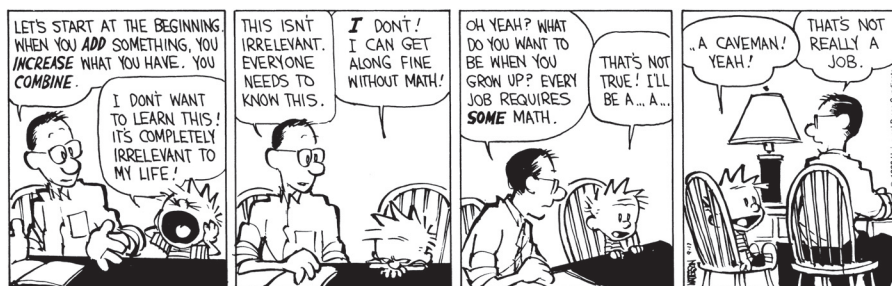
Glossary	211
Bibliography	219
Author Index	223
General Index	225

Chapter 1

INTRODUCTION

You have to spend some energy and effort to see the beauty of math.

—MARYAM MIRZAKHANI



Unfortunately for Calvin, every job requires *some* math. Monday, September 17, 1990, Calvin and Hobbes. © Watterson. Reprinted with permission of ANDREWS MCMEEL SYNDICATION. All rights reserved.

In the late nineteenth century, Woldemar Voigt introduced the word “tensor” in the context of elasticity theory, a name that reflects its mechanical origins (Maugin, 2013). At that time, the mathematical world was striving toward generalizations, and physical interest in the theory of elasticity was driven by the now outdated notion of ether. The appearance of tensors in the mathematical language came from these developments.

Tensors have become a fundamental tool for science and engineering. They are widely used to describe physical quantities and their relationships in multidimensional spaces. Tensors are essential for studying fluid dynamics, continuum mechanics, electromagnetism, and general relativity. They enable precise descriptions of physical properties that have multiple dimensions and magnitudes.

In recent years, tensors have become increasingly important for data analytics and machine learning due to their ability to represent and analyze complex data and relationships in multidimensional spaces. Tensors serve as data structures in deep learning and machine learning algorithms, allowing for the representation and processing of data

in neural networks and enabling the modeling of complex relationships and patterns in data.

The geometrical description of tensors used in this book provides an elegant and powerful tool for comprehending their behavior, relationships, and applications in various fields. It aids in linking abstract mathematical concepts to physical phenomena, enabling researchers, scientists, and engineers to understand and utilize tensors in their respective domains.

This book provides the mathematical background for geometrical descriptions of tensors and their calculus. Its approach is inspired by various sources, including Schutz (1980), Carroll (2004), and Misner et al. (2017). For precise mathematical definitions of the concepts, the reader is directed toward more mathematical textbooks, such as Bishop & Goldberg (1980), Dubrovin et al. (1985), Lovelock & Rund (1989), Marsden & Hughes (1983), Martin (2002), and Stone & Goldbart (2009).

The main objective is to expand \mathbb{R}^n , *Euclidean space*, into a *manifold*, a curved space that behaves like \mathbb{R}^n locally. The sphere and torus are examples of manifolds. Manifolds are invoked for several reasons. Deformed bodies can adopt forms that cannot be described within \mathbb{R}^n . Certain mathematical spaces—such as *Hamiltonian mechanics' phase space*, which is spanned by the components of coordinates and momenta, or *spacetime* in the theory of *general relativity*—are not \mathbb{R}^n .

The book is organized as follows. Chapter 2 discusses *linear spaces* and *coordinate transformations*, while subsequent chapters explore *tensors*, which can be viewed as *linear operators* acting on *vectors* and their *duals*. Chapter 3 is devoted to manifolds and their coordinate descriptions. Chapter 4 extends the concept of vectors in \mathbb{R}^n to manifolds with curvilinear coordinates, introducing the dual entities of vectors: one-forms. The focus is on coordinate systems, including their *bases* and *components* in terms of these bases. Locally, any differentiable manifold of n dimensions behaves like \mathbb{R}^n , so attention is paid to the neighborhoods of points of the manifold. In chapter 5, tensors are introduced and viewed as *multilinear maps* from vectors and one-forms to numbers, \mathbb{R} . Chapter 6 introduces maps of tensors between manifolds, and differentiation on manifolds is discussed in chapter 7. Chapter 8 introduces exterior *differential forms* and the *exterior derivative*, which acts on k -forms to produce $(k + 1)$ -forms. The gradient is an example of the exterior derivative, and other familiar operations, such as the divergence and curl in \mathbb{R}^3 , can be viewed as particular cases of exterior differentiation. The *fundamental theorem of exterior calculus* generalizes vector-calculus integral theorems, such as Gauss's and Stokes's theorems. The book contains over eighty examples from continuum mechanics, sourced from the book by Tromp (2025), and general relativity, which are used to illustrate the theory.

AUTHOR INDEX

- Acharya, A. 206
Adkins, C. J. 195
Amari, S. 145
Appel, W. 134
- Bilby, B. A. 145
Bishop, R. L. 2
Borgnakke, C. 195
Bullough, B. R. 145
- Callen, H. B. 195
Carroll, S. M. 2, 17, 101, 102, 134, 167, 206
Cartan, E. 104
Cosserat, E. 188
Cosserat, F. 188
- Dahlen, F. A. 65, 81
deWit, R. 205
do Carmo, M. P. 7, 11, 127
Dubrovin, B. A. 2
Dudek, M. 206
- Edelen, D. G. B. 146, 205
Edmonds, A. R. 64
Einstein, A. 104
- Flanders, H. 127
Fomenko, A. T. 2
Frankel, T. 59, 133, 135, 137, 148, 178
- Garecki, J. 206
Gazonas, G. 9
Goldbart, P. 2
- Goldberg, S. I. 2
Gonzalez, D. 206
Gupta, A. 108, 174
- Hehl, F. W. 104, 105, 107, 147, 151, 173, 174, 206, 207, 209
Hencky, H. 82
Hilbert, D. 206
Holländer, E. F. 145
Hopkins, D. 9
Hughes, J. R. 2, 79, 123, 148
- Ito, H. 204
- Jackson, J.D. 108
Jaumann, G. 113
- Kerlick, G. D. 104, 174, 206, 207
Kibble, T. W. B. 104
Kleinert, H. 146
Kondo, K. 145, 204
Kosevich, A. M. 145
- Lagoudas, D. C. 146, 205
Liberati, S. 107, 174
Lovelock, D. 2, 42, 71, 97
- Malvern, L. 38, 53, 188, 194
Marsden, J. E. 2, 79, 123, 148
Martin, D. 2, 7
Maugin, G. A. 1, 188
McCrea, J. D. 107, 151, 173, 174, 206, 209
Metrikine, A. V. 188
- Mielke, E. W. 107, 173, 174, 206, 209
Misner, C. 2, 23, 28, 29, 34, 184
Montesinos, M. 206
Munkres, J. R. 7, 16
Mura, T. 145
- Nabarro, F. R. N. 145
Ne'eman, Y. 107, 173, 174, 206, 209
Nester, J. M. 104, 174, 206, 207
Noether, E. 181
Noll, W. 145
Novikov, S. P. 2
Nowacki, W. 188
Nye, J. F. 145
- Obukhov, Y. N. 147
Oshima, N. 145
- Penrose, R. 96
Powers, B. 9
- Randono, A. 206
Rodrigues, O. 67
Romero, R. 206
Roychowdhury, A. 108, 174
Ruggiero, M. L. 147
Rund, H. 2, 42, 71, 97
- Sakata, S. 145
Schey, H. M. 142
Schutz, B. 2, 16, 19, 27, 42, 114, 178
Sciama, D. W. 104
Sedov, L. I. 9

224 Author Index

- | | | |
|------------------------|---|--------------------------------------|
| Shimbo, M. 145 | Tartaglia, A. 147 | Vitagliano, V. 107, 174 |
| Smith, E. 145 | Thorne, K. 2, 23, 28, 29, 34, 184 | von der Heyde, P. 104, 174, 206, 207 |
| Sonntag, R. E. 195 | Torres del Castillo, G. F. 7 | |
| Sotiriou, T. 107, 174 | Toupin, R. A. 82, 188 | Wald, R. M. 85, 95, 103 |
| Spivak, M. 7, 114, 178 | Trautman, A. 104, 105, 136, 137, 174, 206 | Weile, D. 9 |
| Stone, M. 2 | Tromp, J. 2, 65, 81, 147, 174, 188, 205 | Weinberg, S. 104, 105 |
| | Truesdell, C. A. 82, 188 | Wheeler, J. 2, 23, 28, 29, 34, 184 |
| Takeo, M. 204 | | |
| Tarantola, A. 58 | Uehara, T. 145 | Zaremba, S. 113 |

GENERAL INDEX

- absolute temperature, 194
accidentals, 123
action, 180, 201, 202, 206; Hilbert, 206;
 metric-affine, 206; Palatini, 206
adiabatic process, 195, 198
adjoint, 57
adjoint operator, 75
affine connection one-forms, 206
Almansi strain, 82
alternating symbol, 48
Ampère's law, 109
angular momentum three-form, 188
angular velocity; material, 192
anholonomic, 31
anholonomicity, 30
antisymmetric, 39
antisymmetric tensor, 39
associative, 3
atlas, 7
autonomous Lie derivative, 118
- basis, 17; anholonomic, 31; holonomic, 30,
 94, 95; linearly independent, 17
basis vector, 17
beach ball, 40
beach ball representation, 39
Bianchi identities, 97, 147, 172–174;
 contracted, 102–105; post-Riemannian,
 174
bijective, 11, 73
bulk modulus, 69
Burgers vector, 147, 204, 205
- caloric equation of state, 196
canonical energy-momentum current, 208
capacity; Levi-Civita, 48; tensor, 47, 58
Cartan homotopy formula, 149, 167
Cartan magic formula, 149, 167
Cartan tensor, 104, 105
Cauchy stress; weighted, 83
Cauchy stress tensor, 38
Cauchy stress two-forms, 152, 159
Cauchy-Green tensor, 81
chart, 7, 77, 78
Christoffel symbol; first kind, 100;
 Lagrangian, 101; second kind, 100
circulation tube, 132, 158
classical vector calculus, 142
Clausius–Duhem inequality, 195
closed form, 139, 182
co-Einstein tensor, 207
co-Ricci tensor, 207
co-Weyl vector, 208
coframe, 43, 44, 130; orthogonal, 57
commutative, 3
commutator, 29
comoving coordinates, 9, 78
comoving volume, 183
compact, 179
compatibility, 32
compatibility conditions, 32
compliance tensor, 69
congruence, 19, 114
conjugate pair, 156
conjugate thermodynamic parameters, 196
- conjugate variables, 196
connection; spin, 102, 167; torsion-free, 97
connection coefficients, 88; frames, 101;
 Lagrangian, 100; nonmetricity, 107;
 torsion-free, 100; transformation, 89
connection one-forms, 161; coordinate,
 166; Eulerian, 166; Lagrangian, 145,
 166; tetrad, 166
conservation law, 182
conservation of angular momentum, 188
conservation of energy, 190; with spin, 192
conservation of linear momentum, 186
conservation of mass, 186
continuity equation, 186; Maxwell's
 equations, 110
continuous symmetry, 181
continuum mechanics, 18, 21, 22, 27, 38,
 42, 43, 45, 53, 59, 68, 77, 81–83, 90, 95,
 100, 108, 112, 119, 121, 124, 145, 151,
 156, 159, 166, 168, 183, 186, 188, 190,
 192, 194, 195, 201, 202, 204
contortion tensor, 99, 108
contracted Bianchi identities, 102, 103
contraction, 36, 159; tensor-valued form,
 159
contravariant, 34, 36
convected coordinates, 78, 115
coordinate connection one-forms, 166
coordinate system, 7; convected, 115
coordinates, 7; comoving, 9, 78; convected,
 78; Eulerian, 9; Lagrangian, 9;
 referential, 77; spatial, 9

226 General Index

- corotation, 112
- corotational material derivative, 112; Zaremba-Jaumann, 113
- cosmological constant, 105
- Cosserat micropolar medium, 188
- cotangent bundle, 23
- cotangent space, 23
- couple stress, 189; three-forms, 203
- couple-stress two-forms, 189
- covariant, 34, 36
- covariant derivative, 85; mixed, 102; modified, 105; tensor capacity, 107; tensor density, 107; tetrad, 101; torsion-free, 97
- covariant Lie derivative, 167; Eulerian, 168; Lagrangian, 168
- covariant vector, 27
- covector, 27
- Cramer's rule, 62
- cross product, 49, 60, 134
- cubic symmetry, 68
- curl, 141
- current; dilation, 209; shear, 209; spin, 208; strain, 209
- curvature, 34, 36, 108; Gaussian curvature, 96; intrinsic, 205; Ricci, 103
- curvature tensor, 34, 36, 37, 41, 94–96
- curvature two-forms, 169, 170
- curve, 16
- dark energy and cosmological constant, 105
- defects, 32, 107, 145, 205; interstitial, 109; point, 109; substitutional, 109; vacancy, 109
- deformation; elastic, 196, 198; plastic, 196, 198
- deformation gradient, 22
- deformation rate, 90
- deformation tensor; Green, 53; right Cauchy–Green, 53
- deformation-rate tensor, 91, 121, 123
- density; Levi-Civita, 48; tensor, 47, 58
- derivative; corotational material, 112; covariant, 85; Euler, 112; exterior, 24, 139; exterior covariant, 164; Lie, 114; material, 112; substantial, 112
- determinant, 46, 47, 50
- deviatoric strain, 69
- diffeomorphism, 8, 72, 182
- differentiable manifold, 7, 8, 15
- differential, 24
- differential forms, 127
- differential geometry, 15
- dilation current, 209
- directional derivative, 92
- disclination, 32, 96, 145–147, 205; twist, 147; wedge, 147
- disclination two-forms, 147, 204, 205
- disformation tensor, 108
- dislocation, 32, 96, 145–147, 204; edge, 147; screw, 147
- dislocation line, 147
- dislocation two-forms, 147, 204, 205
- distance, 51
- distributive, 4
- divergence, 91, 141
- dot product, 37, 55
- double-dot product, 38, 55, 66, 159
- Dreibein, 43
- dual, 23, 29
- dual space, 23
- dual vector, 27
- duality, 23, 29
- duality condition, 24
- duality product, 24, 41
- dummy index, 35
- dynamic equations, 105; Einstein general relativity, 106; Einstein-Cartan general relativity, 106
- earthquake, 39
- edge dislocation, 147
- Einstein general relativity, 104, 174; dynamic equations, 106; field equations, 104, 210
- Einstein summation convention, 17
- Einstein tensor, 103, 207
- Einstein-Cartan general relativity, 104, 147, 174; dynamic equations, 106; field equations, 105, 209
- Einstein-Hilbert action, 206
- elastic deformation, 196, 198
- elastic tensor, 67; isotropic, 68
- electric charge density, 109
- electric displacement, 111
- electric field, 109
- endomorphism, 58
- entropy, 194
- equation of state, 195; caloric, 196
- Euclidean space, 7
- Euler angles, 64
- Euler derivative, 112, 113, 149
- Euler-Lagrange equations, 181
- Eulerian coordinates, 9, 77
- Eulerian strain, 82
- event, 10, 29
- exact form, 139, 142, 205
- explosion, 40
- exterior covariant derivative, 164; Eulerian, 166; Lagrangian, 166; mixed, 166
- exterior derivative, 24, 139; covariant, 164
- exterior differential forms, 127
- exterior product, 130, 159; Grassmann product, 131; wedge product, 131
- Faraday tensor, 143
- Faraday's law, 109
- field equations, 104, 206; Einstein general relativity, 104, 210; Einstein-Cartan general relativity, 105, 209; metric-affine gravity, 208
- first law of thermodynamics, 190; with spin, 192
- first Piola-Kirchhoff stress, 83; two-forms, 154
- first postulate, 55
- flat b , 123
- flow of matter, 21
- fluid, 69
- flux, 133, 150
- focal hemisphere, 39
- form; closed, 139, 182; connection, 161; exact, 139, 142, 205; integration, 174; Lie derivative, 148; operations, 130; stress, 151; tensor-valued, 147, 150; vector-valued, 147, 150
- form-versions of Maxwell's equations, 143
- four-velocity, 29, 45, 54
- frame, 43; inertial, 106; nonspinning, 106

- frames; inertial; nonspinning, 106;
orthogonal, 57
- Frank vector, 147, 204, 205
- Frobenius inner product, 38
- Frobenius norm, 38, 47
- fundamental theorem of calculus, 178
- fundamental theorem of exterior calculus,
179
- Galilean reference frame, 9
- Galilean spacetime, 95
- gas, 69
- Gauss's theorem, 180
- Gauss's law; electric field, 109; magnetic
field, 109
- general relativity, 7, 10, 36, 37, 41, 43, 44,
52, 54, 63, 94, 104, 105, 108, 137, 173,
206; Einstein, 174; Einstein field
equations, 104; Einstein-Cartan, 174;
Einstein-Cartan field equations, 105;
instein-Cartan, 147; metric-affine, 174;
metric-affine field equations, 208
- generalized Kronecker delta symbol, 60
- generalized Stokes's theorem, 178
- geodesic, 93
- geodesic equation, 93
- geometry; metric-affine, 206;
Riemann-Cartan, 206; Riemannian, 206
- Gibbs relation, 196
- gradient, 141; deformation, 22
- gravitation; Einstein, 174; Einstein-Cartan,
107, 174; metric-affine, 107, 174
- Green deformation tensor, 53
- Green's Theorem, 179
- Hamilton's principle, 201
- Hausdorff manifold, 11
- heat, 190
- heat flux, 191
- heat flux two-form, 194
- helicity, 96
- Helmholtz free energy, 195
- Hencky strain, 47, 82
- Hilbert action, 206
- Hodge dual, 134
- Hodge star, 134
- holonomic, 31
- holonomic basis, 94, 95
- holonomicity, 30
- holonomy, 96
- homeomorphism, 72
- Hooke's law, 38, 67, 159; form version,
159
- hypermomentum current, 208
- hypocenter, 39
- identity tensor, 46
- implosion, 40
- incompatibility, 32
- incompatibility tensor, 32
- incompatible motion, 145
- incompressibility, 69
- induction equation, 110, 111
- inertia density tensor, 193
- inertial frame, 106
- inertial frames; nonspinning, 106
- inertial reference frame, 9
- infinitesimal strain tensor, 38, 68, 82
- injective, 8, 73
- inner product, 37, 52
- integral; line, 174, 175; surface, 175, 176;
volume, 177
- integral curve, 19
- interior product, 133
- internal energy density, 190
- internal heating, 191
- internal state variables, 196
- interstitial defect, 109
- intrinsic curvature, 205
- intrinsic spin, 188, 192
- intrinsic torsion, 204, 205
- inverse of rank-2 tensors, 50, 63
- inverse problems, 58
- irreversible process, 195, 196
- isotropic symmetry, 68
- Jacobian, 13
- Jacobian of the motion, 79, 124
- k -form, 127, 129, 130, 133
- k -vector, 133
- kinetic energy; rotational, 192
- kinetic energy density, 190
- Kirchhoff stress, 83
- Kronecker delta symbol, 20; generalized,
60
- Kronecker determinants, 60
- Kronecker tensor, 46
- Lagrangian connection coefficients, 100
- Lagrangian coordinates, 9, 77
- Lagrangian density, 180, 206; four-form,
202; matter, 208; three-form, 201
- Lagrangian strain, 82
- Legendre transformation, 196
- Leibniz's rule, 87, 183
- Levi-Civita capacity, 48, 58
- Levi-Civita density, 48, 58
- Levi-Civita pseudotensor, 48, 58, 59
- Levi-Civita symbol, 48
- Lie bracket, 29
- Lie derivative, 112, 114, 115, 149;
autonomous, 118; covariant, 167;
covariant, 167; Euler derivative, 113;
form, 148; function, 120; general tensor,
119; geometrical interpretation, 117;
Levi-Civita tensor, 123; metric tensor,
120; one-form, 118; vectors, 114
- Lie dragging, 117
- line integral, 175
- line of nodes, 64
- linear momentum three-form, 186
- linear momentum two-form, 186
- linear space, 3; external operation, 4;
distributivity, 4; internal operation, 3;
associativity, 3; commutativity, 3; linear
transformation, 3; null element, 4;
opposite element, 4; properties, 3;
tensor, 3; vector, 7; vector space, 4; zero
element, 4
- linear transformation, 3–5; linear space, 3
- linear vector dipole; compensated, 40
- linearity, 86
- logarithmic strain, 82
- Lorentz factor, 55
- Lorentz force, 111
- Maclaurin series, 47
- macroscopic Maxwell's equations, 111
- magnetic diffusivity, 111
- magnetic field, 109

- magnetization field, 111
magnetizing field, 111
manifold, 7; chart, 7; coordinates, 7;
differentiable, 7, 15; Hausdorff, 11;
orientable, 13; atlas, 13; chart, 13; circle,
13; Jacobian, 13; Klein bottle, 13;
Möbius strip, 13; tangent space, 13;
referential, 77; spacetime, 104; spatial,
77, 78; surface; orientable, 13
map, 8, 71; between manifolds, 72;
bijection, 71; bijective, 11; injective, 8,
71; one-to-one, 8; surjective, 71
maps between manifolds, 76
mass flux two-form, 186
mass three-form, 186
material angular velocity, 192
material defects, 32
material derivative, 112, 120
material point, 77
material velocity, 18, 21, 90, 95, 112, 121,
124
matter Lagrangian density, 208
Maxwell's equations, 108; form version,
144; in matter, 111; in vacuum, 110;
macroscopic, 111
Mercator series, 47
metric signature, 54
metric tensor, 51, 63; covariant derivative,
98; inverse, 53, 63; Riemannian, 52
metric-affine action, 206
metric-affine geometry, 206
metric-affine gravitation theory, 107, 108,
174; field equations, 208
metrical energy-momentum current, 208
micropolar medium, 188
Minkowski metric, 54
Minkowski spacetime, 54, 167
mixed covariant derivative, 102
mixed exterior covariant derivative, 166
modified covariant derivative, 105
modified torsion tensor, 104, 105
modulus; bulk, 69; shear, 69
moment tensor, 39, 40
monifold; referential, 154
monoclinic symmetry, 68
motion, 12; incompatible, 145; Jacobian,
79
Nanson's relation, 81
natural \natural , 123
no-more continuum, 145
nodal-plane ambiguity, 40
Noether current, 182, 183
nondegenerate, 52
nonholonomic, 31
nonmetricity, 107, 108, 173, 174
nonmetricity one-forms, 173
nonmetricity tensor, 107
nonspinning frame, 106
norm, 56; Frobenius, 47
normal, 137
normal strain two-vector, 158
normal traction two-forms, 153
normal vector, 137
notation, 15, 37, 47
objective rate, 114
Ohm's law, 110
one-form, 23; basis, 24; connection, 145,
161; Eulerian, 166; Lagrangian, 166;
field, 23; nonmetricity, 173; norm, 56;
spin connection, 167; strain, 158
one-forms; affine connection, 206; spin
connection, 167
one-to-one, 8, 73
open set, 8
orientability, 13
orientable manifold, 13
oriented, 179
oriented surface, 59
orthogonal group, 64; special, 64
orthogonal transformations, 64
orthogonality, 56
orthotropic symmetry, 68
outer product, 35; tensor product, 35
Palatini action, 206
Palatini tensor, 207
Palatini torsion tensor, 104
parallel transport, 91, 92, 114; analytic
formulation, 92; geometrical
interpretation, 92
particle rotation, 199
permeability, 110
permittivity, 109
Piola transformation, 82
Piola-Kirchhoff stress; first, 83; second, 83;
two-forms, 154
plastic deformation, 196, 198
plasticity, 198
Poincaré's lemma, 139, 182
point defects, 107–109
polarization field, 111
post-Riemannian Bianchi identities,
174
product; cross, 134; dot, 37; double-
dot, 38, 159; duality, 24; exterior, 130,
131, 159; inner, 37; interior, 133; outer,
35; wedge, 128, 130, 131
product space, 16
proper time, 29
pseudoform, 133, 135
pseudoscalar, 60
pseudotensor, 46; Levi-Civita, 59
pseudotensor capacity, 48
pseudotensor density, 48
pull down operator, 133
pullback, 72, 144, 159, 178
pushforward, 74, 75, 144
Pythagorean theorem, 56
radiation pattern, 39
reference frame; Galilean, 9; inertial, 9
referential coordinates, 77
referential manifold, 77, 154
referential state, 77
relative tensors, 47
relativity, 54
reversible process, 195, 196
Reynolds transport theorem, 183
Ricci curvature, 103
Ricci identity, 95
Ricci scalar, 103, 206
Ricci tensor, 37, 103; symmetry, 106
Riemann tensor, 35, 95, 103
Riemann-Cartan geometry, 206
Riemannian geometry, 206
Riemannian manifold; differentiation, 99;
Levi-Civita pseudotensor, 60; metric
tensor, 51, 55, 60
Riemannian metric; definition, 52
right Cauchy-Green tensor, 53

- rigid rotation, 114
 rigid translation, 114
 rigidity, 69
 Rodrigues formula, 66, 67
 rotation, 46, 63, 202; angle, 65; axis, 65;
 particle, 199
 rotation tensor, 63
 rotational kinetic energy, 192
- scalar, 34, 35, 38; field, 35
 screw dislocation, 147
 second law of thermodynamics, 194, 195
 second Piola-Kirchhoff stress, 82, 83;
 two-forms, 155
 second Piola-Kirchhoff stress two-forms,
 156
 second postulate, 54
 seismology, 38–40, 68
 semicolon notation, 86
 sharp \sharp , 123
 shear current, 209
 shear modulus, 69
 smooth, 11
 source; dip-slip, 40; explosion, 40; eyeball,
 40; fried-egg, 40; implosion, 40;
 strike-slip, 40; thrust, 40
 source mechanism, 39; double-couple, 39;
 non-double-couple, 39
 spacetime, 7; Galilean, 95; general
 relativity, 10; Minkowski, 167
 spacetime manifold, 10, 29, 104
 spatial coordinates, 9
 spatial manifold, 77, 78
 spatial point, 77
 special orthogonal group, 64
 special relativity, 54; first postulate, 55;
 second postulate, 54
 specific entropy density, 194
 speed of light, 110
 spin, 192
 spin angular momentum, 188
 spin angular momentum three-forms, 192
 spin connection, 102, 167
 spin connection one-forms, 167
 spin current, 208
 spin tensor, 105
 Stokes's theorem, 178, 180; classical, 180
- strain, 156, 158; Almansi, 82; deviator, 69;
 deviatoric, 69; Eulerian, 82; form
 version, 156; Hencky, 47, 82;
 Lagrangian, 82; logarithmic, 82;
 two-vectors, 158
 strain current, 209
 strain deviator, 69
 strain energy density, 68
 strain one-forms, 158
 strain tensor, 158; infinitesimal, 38, 68
 strain two-vector, 157, 159; normal, 158
 stress, 38; couple, 189; first
 Piola-Kirchhoff, 83; Kirchhoff, 83;
 second Piola-Kirchhoff, 82; three-form,
 203; vector-valued two-form, 151
 stress tensor; Cauchy, 38; weighted
 Cauchy, 83
 stress-energy tensor, 104
 structure coefficients, 30, 31, 43
 substitutional defect, 109
 summation convention, 17
 surface, 137
 surface integral, 175, 176
 surface one-form, 59
 surface torque, 188
 surface two-form, 137
 surjective, 71, 73
 symmetric tensor, 39
 symmetry; cubic, 68; isotropic, 68;
 monoclinic, 68; orthotropic, 68;
 transversely isotropic, 68; triclinic, 68
 symmetry classes, 68
- tangent bundle, 16
 tangent space, 15, 77
 Taylor series, 47
 tensor, 33; addition, 35; adjoint, 57;
 alternating, 58; antisymmetric, 39;
 capacity, 47, 58; Cartan, 104;
 co-Einstein, 207; contortion, 99, 108;
 contraction, 36; curvature, 94–96;
 deformation-rate, 91; density, 47, 58;
 determinant, 50; disformation, 108;
 Einstein, 103, 207; elastic, 67;
 exponential, 46; field, 35; identity, 46;
 incompatibility, 32; inertia density, 193;
 inverse, 50, 63; Kronecker, 46;
- Levi-Civita, 58; logarithm, 46; metric,
 51; modified torsion, 104; moment, 39;
 nonmetricity, 107; operations, 35;
 Palatini, 207; Palatini torsion, 104;
 Ricci, 103; Riemann, 95, 103; rotation,
 63; skewsymmetric, 133; strain, 158;
 symmetric, 38; torsion, 94, 96; trace, 37;
 traction, 153; transformation, 41;
 transpose, 38, 40; two-point, 83;
 valence, 34; vorticity, 91
 tensor capacity, 47, 58; covariant
 derivative, 107; weight, 47
 tensor density, 47, 58; covariant derivative,
 107; weight, 47
 tensor field, 35; divergence of, 91
 tensor product, 35; outer product, 35
 tensor-valued form, 96, 147, 150
 tensors; relative, 47
 tetrad, 43, 101, 102, 106, 130; covariant
 derivative, 101
 tetrad connection one-forms, 166
 thermodynamic mechanical parameter,
 196
 thermodynamics, 190
 three-form, 129, 132; angular momentum,
 188; canonical energy-momentum
 current, 208; disclination, 205;
 hypermomentum current, 208; internal
 energy, 190; internal heating, 191;
 kinetic energy density, 190; kinetic
 energy density energy, 192; linear
 momentum, 186; mass, 186; spin
 angular momentum, 192; spin current,
 208; strain current, 209; volume, 135;
 volume force, 186; volume torque, 188
- time derivative; Euler, 112
 time dilation, 55
 torque; surface, 188; volume, 188
 torque two-forms, 192
 torsion, 108; intrinsic, 204, 205
 torsion form, 104
 torsion tensor, 94, 96; modified, 105
 torsion two-forms, 162, 163
 torsion-free connection, 97
 total angular velocity, 192
 total rotation, 202
 trace, 37

230 **General Index**

- traction, 153; two-forms, 153
- traction tensor, 153
- traction two-form, 159; normal, 153, 154;
 - shear, 153, 154
- transpose, 38, 40; (1,1) tensor, 40
- transversely isotropic symmetry, 68
- triclinic symmetry, 68
- true strain, 82
- twist disclination, 147
- two-form, 128; Cauchy stress, 151, 152, 154, 157, 159, 187; couple-stress, 189; curvature, 169, 170; disclination, 147, 204, 205; dislocation, 147, 204, 205; first Piola-Kirchhoff stress, 154; heat flux, 191, 194; linear momentum, 186; mass flux, 186; Piola-Kirchhoff stress, 154; second Piola-Kirchhoff stress, 155, 156; surface, 137; tensor-valued, 96; torque, 188, 192; torsion, 162, 163; traction, 153, 154, 159; vector-valued, 96, 150
- two-point tensor, 83, 154
- two-vector; strain, 157, 159
- unit vector, 56
- universal electric constant, 109
- universal magnetic constant, 110
- vacancy defect, 109
- valence, 34
- variational principle, 180
- vector, 15, 16; basis, 17; field, 19; length, 56; norm, 56; normal, 137; unit, 56
- vector calculus, 142
- vector field, 19; constancy, 92
- vector space, 4, 17; dimension, 5; basis, 5; linear space, 4
- vector-valued form, 96, 147, 150
- vector-valued two-form, 150
- velocity, 183
- Vielbein, 43
- Vierbein, 43, 45
- Volterra cut-and-weld protocols, 147
- volume, 135
- volume element, 135
- volume force three-forms, 186
- volume form, 59, 124, 135; properties, 137
- volume integral, 177
- volume torque, 188
- volume transport, 60
- vorticity, 90, 192
- vorticity tensor, 91
- wedge disclination, 147
- wedge product, 128, 130; exterior product, 131; Grassmann product, 131
- weighted Cauchy stress tensor, 83
- Weyl vector, 208
- worldline, 29
- Zaremba-Jaumann rate, 113