

Contents

<i>Preface</i>		<i>xv</i>
0.1	For the Reader	xv
0.2	For the Expert	xviii
0.3	Background and Conventions	xix
0.4**	The Goals of This Book	xx

PART I

Preliminaries

1	Just Enough Category Theory to Be Dangerous	3
1.1	Categories and Functors	5
1.2	Universal Properties Determine an Object up to Unique Isomorphism	9
1.3	Limits and Colimits	16
1.4	Adjoint	20
1.5	An Introduction to Abelian Categories	23
1.6*	Spectral Sequences	32
2	Sheaves	41
2.1	Motivating Example: The Sheaf of Smooth Functions	41
2.2	Definition of Sheaf and Presheaf	42
2.3	Morphisms of Presheaves and Sheaves	47
2.4	Properties Determined at the Level of Stalks, and Sheafification	50
2.5	Recovering Sheaves from a “Sheaf on a Base”	53
2.6	Sheaves of Abelian Groups, and $\mathcal{O}_X$ -Modules, Form Abelian Categories	56
2.7	The Inverse Image Sheaf	58

PART II

Schemes

3	Toward Affine Schemes: The Underlying Set, and Topological Space	65
3.1	Toward Schemes	65
3.2	The Underlying Set of an Affine Scheme	67
3.3	Visualizing Schemes: Generic Points	76
3.4	The Underlying Topological Space of an Affine Scheme	77
3.5	A Base of the Zariski Topology on $\text{Spec } A$: Distinguished Open Sets	79
3.6	Topological (and Noetherian) Properties	80
3.7	The Function $I(\cdot)$, Taking Subsets of $\text{Spec } A$ to Ideals of A	87

4	The Structure Sheaf, and the Definition of Schemes in General	89
4.1	The Structure Sheaf of an Affine Scheme	89
4.2	Visualizing Schemes: Nilpotents	92
4.3	Definition of Schemes	95
4.4	Three Examples	97
4.5	Projective Schemes, and the Proj Construction	102
5	Some Properties of Schemes	109
5.1	Topological Properties	109
5.2	Reducedness and Integrality	110
5.3	The Affine Communication Lemma, and Properties of Schemes That Can Be Checked “Affine-Locally”	112
5.4	Normality and Factoriality	115
6	Rings Are to Modules as Schemes Are to . . .	121
6.1	Quasicoherent Sheaves	121
6.2	Characterizing Quasicoherence Using the Distinguished Affine Base	123
6.3	Quasicoherent Sheaves Form an Abelian Category	127
6.4	Finite Type Quasicoherent, Finitely Presented, and Coherent Sheaves	128
6.5	Algebraic Interlude: The Jordan–Hölder Package	130
6.6	Visualizing Schemes: Associated Points and Zerodivisors	133
6.7**	Coherent Modules over Non-Noetherian Rings	143

PART III

Morphisms of Schemes

7	Morphisms of Schemes	147
7.1	Motivations for the “Right” Definition of Morphism of Schemes	147
7.2	Morphisms of Ringed Spaces	148
7.3	From Locally Ringed Spaces to Morphisms of Schemes	150
7.4	Maps of Graded Rings and Maps of Projective Schemes	156
7.5	Rational Maps from Reduced Schemes	157
7.6*	Representable Functors and Group Schemes	163
7.7**	The Grassmannian: First Construction	166
8	Useful Classes of Morphisms of Schemes	169
8.1	“Reasonable” Classes of Morphisms (Such as Open Embeddings)	169
8.2	Another Algebraic Interlude: Lying Over and Nakayama	171
8.3	A Gazillion Finiteness Conditions on Morphisms	174
8.4	Images of Morphisms: Chevalley’s Theorem and Elimination Theory	181

9	Closed Embeddings and Related Notions	187
9.1	Closed Embeddings and Closed Subschemes	187
9.2	Locally Closed Embeddings and Locally Closed Subschemes	189
9.3	Important Examples from Projective Geometry	191
9.4	The (Closed Sub)scheme-Theoretic Image	196
9.5	Slicing by Effective Cartier Divisors, Regular Sequences and Regular Embeddings	199
10	Fibered Products of Schemes, and Base Change	205
10.1	They Exist	205
10.2	Computing Fibered Products in Practice	210
10.3	Interpretations: Pulling Back Families, and Fibers of Morphisms	213
10.4	Properties Preserved by Base Change	217
10.5*	Properties Not Preserved by Base Change, and How to Fix Them	219
10.6	Products of Projective Schemes: The Segre Embedding	226
10.7	Normalization	228
11	Separated and Proper Morphisms, and (Finally!) Varieties	233
11.1	Fun with Diagonal Morphisms, and Quasiseparatedness Made Easy	233
11.2	Separatedness, and Varieties	235
11.3	The Locus where Two Morphisms from X to Y Agree, and the “Reduced-to-Separated” Theorem	241
11.4	Proper Morphisms	243

PART IV

“Geometric” Properties of Schemes

12	Dimension	251
12.1	Dimension and Codimension	251
12.2	Dimension, Transcendence Degree, and Noether Normalization	254
12.3	Krull’s Theorems	261
12.4	Dimensions of Fibers of Morphisms of Varieties	266
13	Regularity and Smoothness	271
13.1	The Zariski Tangent Space	271
13.2	Regularity, and Smoothness over a Field	275
13.3	Examples	280
13.4	Bertini’s Theorem	283
13.5	Discrete Valuation Rings, and Algebraic Hartogs’s Lemma	286
13.6	Smooth (and Étale) Morphisms: First Definition	292
13.7*	Valuative Criteria for Separatedness and Properness	294
13.8*	More Sophisticated Facts about Regular Local Rings	298
13.9*	Filtered Rings and Modules, and the Artin-Rees Lemma	299

PART V

Quasicoherent Sheaves on Schemes, and Their Uses

14	More on Quasicoherent and Coherent Sheaves	303
14.1	Vector Bundles “=” Locally Free Sheaves	303
14.2	Locally Free Sheaves on Schemes in Particular	307
14.3	More Pleasant Properties of Finite Type and Coherent Sheaves	311
14.4	Pushforwards of Quasicoherent Sheaves	315
14.5	Pullbacks of Quasicoherent Sheaves: Three Different Perspectives	316
14.6	The Quasicoherent Sheaf Corresponding to a Graded Module	322
15	Line Bundles, Maps to Projective Space, and Divisors	325
15.1	Some Line Bundles on Projective Space	325
15.2	Line Bundles and Maps to Projective Space	328
15.3	The Curve-to-Projective Extension Theorem	335
15.4	Hard but Important: Line Bundles and Weil Divisors	337
15.5	The Payoff: Many Fun Examples	343
15.6	Effective Cartier Divisors “=” Invertible Ideal Sheaves	346
15.7	The Graded Module Corresponding to a Quasicoherent Sheaf	348
16	Maps to Projective Space, and Properties of Line Bundles	351
16.1	Globally Generated Quasicoherent Sheaves	351
16.2	Ample and Very Ample Line Bundles	353
16.3	Applications to Curves	357
16.4*	The Grassmannian as a Moduli Space	362
17	Projective Morphisms, and Relative Versions of Spec and Proj	367
17.1	Relative Spec of a (Quasicoherent) Sheaf of Algebras	367
17.2	Relative Proj of a (Quasicoherent) Sheaf of Graded Algebras	370
17.3	Projective Morphisms	373
18	Čech Cohomology of Quasicoherent Sheaves	379
18.1	(Desired) Properties of Cohomology	379
18.2	Definitions and Proofs of Key Properties	383
18.3	Cohomology of Line Bundles on Projective Space	387
18.4	Riemann–Roch, and Arithmetic Genus	390
18.5	A First Glimpse of Serre Duality	395
18.6	Hilbert Functions, Hilbert Polynomials, and Genus	398
18.7	Higher Pushforward (or Direct Image) Sheaves	403
18.8*	Serre’s Characterizations of Ampleness and Affineness	407
18.9*	From Projective to Proper Hypotheses: Chow’s Lemma and Grothendieck’s Coherence Theorem	409

19	Application: Curves	413
19.1	A Criterion for a Morphism to Be a Closed Embedding	413
19.2	A Series of Crucial Tools	415
19.3	Curves of Genus 0	418
19.4	Classical Geometry Arising from Curves of Positive Genus	418
19.5	Hyperelliptic Curves	421
19.6	Curves of Genus 2	424
19.7	Curves of Genus 3	424
19.8	Curves of Genus 4 and 5	426
19.9	Curves of Genus 1	428
19.10	Elliptic Curves Are Group Varieties	434
19.11	Counterexamples and Pathologies Using Elliptic Curves	439
20*	Application: A Glimpse of Intersection Theory	443
20.1	Intersecting n Line Bundles with an n -Dimensional Variety	443
20.2	Intersection Theory on a Surface	447
20.3	The Grothendieck Group of Coherent Sheaves, and an Algebraic Version of Homology	452
20.4**	The Nakai–Moishezon and Kleiman Criteria for Ampleness	454
21	Differentials	459
21.1	Motivation and Game Plan	459
21.2	Definitions and First Properties	460
21.3	Examples	472
21.4	The Riemann–Hurwitz Formula	476
21.5	Understanding Smooth Varieties Using Their Cotangent Bundles	481
21.6	Generic Smoothness, and Consequences	485
21.7	Unramified Morphisms	489
22*	Blowing Up	491
22.1	Motivating Example: Blowing Up the Origin in the Plane	491
22.2	Blowing Up, by Universal Property	492
22.3	The Blow-up Exists, and Is Projective	497
22.4	Examples and Computations	501

PART VI

More Cohomological Tools

23	Derived Functors	511
23.1	The Tor Functors	511
23.2	Derived Functors in General	514
23.3	Derived Functors and Spectral Sequences	517

23.4	Derived Functor Cohomology of $\mathcal{O}$ -Modules	523
23.5	Čech Cohomology and Derived Functor Cohomology Agree	525
24	Flatness	531
24.1	Easier Facts	532
24.2	Flatness through Tor	537
24.3	Ideal-Theoretic Criteria for Flatness	540
24.4**	Aside: The Koszul Complex and the Hilbert Syzygy Theorem	546
24.5	Topological Implications of Flatness	548
24.6	Local Criteria for Flatness	552
24.7	Flatness Implies Constant Euler Characteristic	557
24.8	Smooth and Étale Morphisms, and Flatness	560
25	Cohomology and Base Change Theorems	567
25.1	Statements and Applications	567
25.2	Proofs of Cohomology and Base Change Theorems	572
25.3	Applying Cohomology and Base Change to Moduli Problems	576
26	Depth and Cohen–Macaulayness	581
26.1	Depth	581
26.2	Cohen–Macaulay Rings and Schemes	584
26.3	Serre’s $R_1 + S_2$ Criterion for Normality	587
27	The Twenty-Seven Lines on a Cubic Surface	589
27.1	Preliminary Facts	590
27.2	Every Smooth Cubic Surface (over $\bar{k}$) Contains 27 Lines	591
27.3	Every Smooth Cubic Surface (over $\bar{k}$) is a Blown-Up Plane	595
28	Power Series and the Theorem on Formal Functions	599
28.1	Algebraic Preliminaries	599
28.2	Types of Singularities	603
28.3	The Theorem on Formal Functions	604
28.4	Zariski’s Connectedness Lemma and Stein Factorization	606
28.5	Zariski’s Main Theorem	608
28.6	Castelnuovo’s Criterion for Contracting (-1) -Curves	612
28.7**	Proof of the Theorem on Formal Functions 28.3.2	615
29*	Proof of Serre Duality	619
29.1	Desiderata	619
29.2	Ext Groups and Ext Sheaves for $\mathcal{O}$ -Modules	624
29.3	Serre Duality for Projective k -Schemes	626
29.4	The Adjunction Formula for the ω_X , and $\omega_X = \mathcal{K}_X$	629

<i>Bibliography</i>	635
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<i>Index</i>	641
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Chapter 1

Just Enough Category Theory to Be Dangerous

Was mich nicht umbringt, macht mich stärker. That which does not kill me, makes me stronger.

—F. Nietzsche [N, aphorism number 8]

Before we get to any interesting geometry, we need to develop a language to discuss things cleanly and effectively. This is best done in the language of categories. There is not much to know about categories to get started; it is just a very useful language. Like all mathematical languages, category theory comes with an embedded logic, which allows us to abstract intuitions in settings we know well to far more general situations.

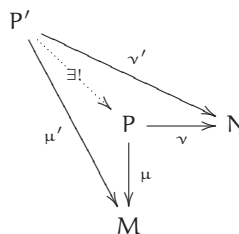
Our motivation is as follows. We will be creating some new mathematical objects (such as schemes, and certain kinds of sheaves), and we expect them to act like objects we have seen before. We could try to nail down precisely what we mean by “act like,” and what minimal set of things we have to check in order to verify that they act the way we expect. Fortunately, we don’t have to—other people have done this before us, by defining key notions, such as *abelian categories*, which behave like modules over a ring.

Our general approach will be as follows. I will try to tell you what you need to know, and no more. (This I promise: If I use the word “topoi,” you can shoot me.) I will begin by telling you things you already know, and describing what is essential about the examples, in a way that we can abstract a more general definition. We will then see this definition in less familiar settings, and get comfortable with using it to solve problems and prove theorems.

1.0.1. Example: product. For example, we will define the notion of *product* of schemes. We could just give a definition of product, but then you should want to know why this precise definition deserves the name of “product.” As a motivation, we revisit the notion of product in a situation we know well: (the category of) sets. One way to define the product of sets U and V is as the set of ordered pairs $\{(u, v) : u \in U, v \in V\}$. But someone from a different mathematical culture might reasonably define it as the set of symbols $\{u^v : u \in U, v \in V\}$. These notions are “obviously the same.” Better: There is “an obvious bijection between the two.”

This can be made precise by giving a better definition of product, in terms of a *universal property*. Given two sets M and N , a product is a set P , along with maps $\mu : P \rightarrow M$ and $\nu : P \rightarrow N$, such that for any set P' with maps $\mu' : P' \rightarrow M$ and $\nu' : P' \rightarrow N$, these maps factor *uniquely* through P :

(1.0.1.1)



(The symbol $\exists$ means “there exists,” and the symbol $!$ means “unique.”) Thus a **product** is a *diagram*

$$\begin{array}{ccc} P & \xrightarrow{\nu} & N \\ \mu \downarrow & & \\ M & & \end{array},$$

and not just a set P , although the maps μ and ν are often left implicit.

This definition agrees with the traditional definition, with one twist: there isn't just a single product; but any two products come with a *unique* isomorphism between them. In other words, the product is unique up to unique isomorphism. Here is why: If you have a product

$$\begin{array}{ccc} P_1 & \xrightarrow{\nu_1} & N \\ \mu_1 \downarrow & & \\ M & & \end{array}$$

and I have a product

$$\begin{array}{ccc} P_2 & \xrightarrow{\nu_2} & N \\ \mu_2 \downarrow & & \\ M & & \end{array}$$

then by the universal property of my product (letting (P_2, μ_2, ν_2) play the role of (P, μ, ν) and (P_1, μ_1, ν_1) play the role of (P', μ', ν') in (1.0.1.1)), there is a unique map $f: P_1 \rightarrow P_2$ making the appropriate diagram commute (i.e., $\mu_1 = \mu_2 \circ f$ and $\nu_1 = \nu_2 \circ f$). Similarly, by the universal property of your product, there is a unique map $g: P_2 \rightarrow P_1$ making the appropriate diagram commute. Now consider the universal property of my product, this time letting (P_2, μ_2, ν_2) play the role of both (P, μ, ν) and (P', μ', ν') in (1.0.1.1). There is a unique map $h: P_2 \rightarrow P_2$ such that

$$\begin{array}{ccccc} P_2 & & & & \\ & \searrow \nu_2 & & \searrow \nu_2 & \\ & & P_2 & \xrightarrow{\nu_2} & N \\ & \searrow h & \downarrow \mu_2 & & \\ & & M & & \end{array}$$

commutes. However, I can name two such maps: the identity map id_{P_2} and $f \circ g$. Thus $f \circ g = \text{id}_{P_2}$. Similarly, $g \circ f = \text{id}_{P_1}$. Thus the maps f and g arising from the universal property are bijections. In short, there is a unique bijection between P_1 and P_2 preserving the “product structure” (the maps to M and N). This gives us the right to name any such product $M \times N$, since any two such products are uniquely identified.

This definition has the advantage that it works in many circumstances, and once we define categories, we will soon see that the above argument applies verbatim in any category to show that products, if they exist, are unique up to unique isomorphism. Even if you haven't seen the definition of category before, you can verify that this agrees with your notion of product in some category that you have seen before (such as the category of vector spaces, or the category of manifolds).

This is handy even in cases that you understand. For example, one way of defining the product of two manifolds M and N is to cut them both up into charts, then take products of charts, then glue them together. But if I cut up the manifolds M and N in one way, and you cut them up in another, how do we know our resulting product manifolds are the “same”? We could wave our hands, or make an annoying argument about refining covers, but instead, we should just show

that they are “categorical products” and hence canonically the “same” (i.e., isomorphic). We will formalize this argument in §1.2.

Another set of notions we will abstract is that of categories that “behave like modules.” We will want to define kernels and cokernels for new notions, and we should make sure that these notions behave the way we expect them to. This leads us to the definition of *abelian categories*, first defined by Grothendieck in his Tôhoku paper [Gr1].

In this chapter, we will give an informal introduction to these and related notions, in the hope of our developing just enough familiarity to comfortably use them in practice.

1.1 Categories and Functors

The introduction of the digit 0 or the group concept was general nonsense too, and mathematics was more or less stagnating for thousands of years because nobody was around to take such childish steps.

—A. Grothendieck, [BroP, pp. 4–5]

Before functoriality, people lived in caves.

—B. Conrad

We begin with an informal definition of categories and functors.

1.1.1. Categories.

A **category** consists of a collection of **objects**, and for each pair of objects, a set of **morphisms** (or **arrows**) between them. (For experts: technically, this is the definition of a *locally small category*. In the correct definition, the morphisms need only form a class, not necessarily a set, but see Caution 0.3.1.) Morphisms are often informally called **maps**. The collection of objects of a category $\mathcal{C}$ is often denoted by $\text{obj}(\mathcal{C})$, but we will usually denote the collection also by $\mathcal{C}$. If $A, B \in \mathcal{C}$, then the set of morphisms from A to B is denoted by $\text{Mor}(A, B)$. A morphism is often written $f: A \rightarrow B$, and A is said to be the **source** of f , and B the **target** of f . (Of course, $\text{Mor}(A, B)$ is taken to be disjoint from $\text{Mor}(A', B')$ unless $A = A'$ and $B = B'$.)

Morphisms compose as expected: there is a composition $\text{Mor}(B, C) \times \text{Mor}(A, B) \rightarrow \text{Mor}(A, C)$, and if $f \in \text{Mor}(A, B)$ and $g \in \text{Mor}(B, C)$, then their composition is denoted by $g \circ f$. Composition is associative: if $f \in \text{Mor}(A, B)$, $g \in \text{Mor}(B, C)$, and $h \in \text{Mor}(C, D)$, then $h \circ (g \circ f) = (h \circ g) \circ f$. For each object $A \in \mathcal{C}$, there is always an **identity morphism** $\text{id}_A: A \rightarrow A$, such that when you (left- or right-) compose a morphism with the identity, you get the same morphism. More precisely, for any morphisms $f: A \rightarrow B$ and $g: B \rightarrow C$, $\text{id}_B \circ f = f$ and $g \circ \text{id}_B = g$. (If you wish, you may check that “identity morphisms are unique”: there is only one morphism deserving the name id_A .) This ends the definition of a category.

We have a notion of **isomorphism** between two objects of a category (a morphism $f: A \rightarrow B$ such that there exists some—necessarily unique—morphism $g: B \rightarrow A$, where $f \circ g$ and $g \circ f$ are the identities on B and A respectively).

1.1.2. Example. The prototypical example to keep in mind is the category of sets, denoted by *Sets*. The objects are sets, and the morphisms are maps of sets. (Because Russell’s paradox shows that there is no set of all sets, we did not say earlier that there is a set of all objects. But as stated in §0.3, we are deliberately omitting all set-theoretic issues.)

1.1.3. Example. Another good example is the category Vec_k of vector spaces over a given field k . The objects are k -vector spaces, and the morphisms are linear transformations. (What are the isomorphisms?)

1.1.A. UNIMPORTANT EXERCISE. A category in which each morphism is an isomorphism is called a **groupoid**. (This notion is not important in what we will discuss. The point of this

exercise is to give you some practice with categories, by relating them to an object you know well.)

- (a) A perverse definition of a *group* is: a groupoid with one object. Make sense of this. (Similarly, in case you care, a perverse definition of a **monoid** is: a category with one object.)
- (b) Describe a groupoid that is not a group.

1.1.B. EXERCISE. If A is an object in a category $\mathcal{C}$, show that the invertible elements of $\text{Mor}(A, A)$ form a group (called the **automorphism group of A** , denoted by $\text{Aut}(A)$). What are the automorphism groups of the objects in Examples 1.1.2 and 1.1.3? Show that two isomorphic objects have isomorphic automorphism groups. (For readers with a topological background: if X is a topological space, then the fundamental groupoid is the category where the objects are points of X , and the morphisms $x \rightarrow y$ are paths from x to y , up to homotopy. Then the automorphism group of x_0 is the (pointed) fundamental group $\pi_1(X, x_0)$. In the case where X is connected, and $\pi_1(X)$ is not abelian, this illustrates the fact that for a connected groupoid—whose definition you can guess—the automorphism groups of the objects are all isomorphic, but not canonically isomorphic.)

1.1.4. Example: Abelian groups. The abelian groups, along with group homomorphisms, form a category *Ab*.

1.1.5. Important Example: Modules over a ring. If A is a ring, then the A -modules form a category Mod_A . (This category has additional structure; it will be the prototypical example of an *abelian category*; see §1.5.) Taking $A = k$, we obtain Example 1.1.3; taking $A = \mathbb{Z}$, we obtain Example 1.1.4.

1.1.6. Example: Rings. There is a category *Rings*, where the objects are rings, and the morphisms are maps of rings in the usual sense (maps of sets that respect addition and multiplication, and that send 1 to 1 by our conventions; see §0.3).

1.1.7. Example: Topological spaces. The topological spaces, along with continuous maps, form a category *Top*. The isomorphisms are homeomorphisms.

In all of the above examples, the objects of the categories were in obvious ways sets with additional structure (a **concrete category**, although we won't use this terminology). This needn't be the case, as the next example shows.

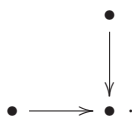
1.1.8. Example: Partially ordered sets. A **partially ordered set** (or **poset**) is a set S along with a binary relation $\geq$ on S satisfying:

- (i) $x \geq x$ (reflexivity),
- (ii) $x \geq y$ and $y \geq z$ imply $x \geq z$ (transitivity), and
- (iii) if $x \geq y$ and $y \geq x$ then $x = y$ (antisymmetry).

A partially ordered set $(S, \geq)$ can be interpreted as a category whose objects are the elements of S , and with a single morphism from x to y if and only if $x \geq y$ (and no morphism otherwise).

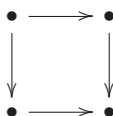
A trivial example is $(S, \geq)$ where $x \geq y$ if and only if $x = y$. Another example is

(1.1.8.1)



Here there are three objects. The identity morphisms are omitted for convenience, and the two non-identity morphisms are depicted. A third example is

(1.1.8.2)



Here the “obvious” morphisms are again omitted: the identity morphisms, and the morphism from the upper left to the lower right. Similarly,

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

depicts a partially ordered set, where, again, only the “generating morphisms” are depicted.

1.1.9. Example: *The category of subsets of a set, and the category of open subsets of a topological space.* If X is a set, then the subsets form a partially ordered set, where arrows are given by inclusion. (Be careful: you may be expecting the arrows to go the other way, because of Example 1.1.8.) Informally, if $U \subset V$, then we have exactly one morphism $U \rightarrow V$ in the category (and otherwise none). Similarly, if X is a topological space, then the *open* sets form a partially ordered set, where the maps are given by inclusions.

1.1.10. Definition. A **subcategory** $\mathcal{A}$ of a category $\mathcal{B}$ has as its objects some of the objects of $\mathcal{B}$, and some of the morphisms of $\mathcal{B}$, such that the objects of $\mathcal{A}$ include the sources and targets of the morphisms of $\mathcal{A}$, and the morphisms of $\mathcal{A}$ include the identity morphisms of the objects of $\mathcal{A}$, and are preserved by composition. (For example, (1.1.8.1) is in an obvious way a subcategory of (1.1.8.2). Also, we have an obvious “inclusion” $i: \mathcal{A} \rightarrow \mathcal{B}$, which will soon be an example of a functor.)

1.1.11. Functors.

A **covariant functor** F from a category $\mathcal{A}$ to a category $\mathcal{B}$, denoted by $F: \mathcal{A} \rightarrow \mathcal{B}$, is the following data. It is a map of objects $F: \text{obj}(\mathcal{A}) \rightarrow \text{obj}(\mathcal{B})$, and for each $A_1, A_2 \in \mathcal{A}$, and morphism $m: A_1 \rightarrow A_2$, a morphism $F(m): F(A_1) \rightarrow F(A_2)$ in $\mathcal{B}$. We require that F preserve identity morphisms (for $A \in \mathcal{A}$, $F(\text{id}_A) = \text{id}_{F(A)}$), and that F preserve composition ($F(m_2 \circ m_1) = F(m_2) \circ F(m_1)$). (You may wish to verify that covariant functors send isomorphisms to isomorphisms.) A trivial example is the **identity functor** $\text{id}: \mathcal{A} \rightarrow \mathcal{A}$, whose definition you can guess. Here are some less trivial examples.

1.1.12. Example: *A forgetful functor.* Consider the functor from the category of vector spaces (over a field k) Vec_k to *Sets* that associates to each vector space its underlying set. The functor sends a linear transformation to its underlying map of sets. This is an example of a **forgetful functor**, where some additional structure is forgotten. Another example of a forgetful functor is $\text{Mod}_A \rightarrow \text{Ab}$ from A -modules to abelian groups, which remembers only the abelian group structure of the A -module.

1.1.13. Topological examples. Examples of covariant functors include the fundamental group functor π_1 , which sends a topological space X with choice of a point $x_0 \in X$ to a group $\pi_1(X, x_0)$ (what are the objects and morphisms of the source category?), and the i th homology functor $\text{Top} \rightarrow \text{Ab}$, which sends a topological space X to its i th homology group $H_i(X, \mathbb{Z})$. The covariance corresponds to the fact that a (continuous) morphism of pointed topological spaces $\phi: X \rightarrow Y$ with $\phi(x_0) = y_0$ induces a map of fundamental groups $\pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$, and similarly for homology groups.

1.1.14. Example. Suppose A is an object in a category $\mathcal{C}$. Then there is a functor $h^A: \mathcal{C} \rightarrow \text{Sets}$ sending $B \in \mathcal{C}$ to $\text{Mor}(A, B)$ and sending $f: B_1 \rightarrow B_2$ to $\text{Mor}(A, B_1) \rightarrow \text{Mor}(A, B_2)$, described by

$$[g: A \rightarrow B_1] \longmapsto [f \circ g: A \rightarrow B_1 \rightarrow B_2].$$

This seemingly silly functor ends up surprisingly being an important concept.

1.1.15. Definitions. If $F: \mathcal{A} \rightarrow \mathcal{B}$ and $G: \mathcal{B} \rightarrow \mathcal{C}$ are covariant functors, then we define a functor $G \circ F: \mathcal{A} \rightarrow \mathcal{C}$ (the **composition** of G and F) in the obvious way. Composition of functors is associative in an evident sense.

A covariant functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is **faithful** if for all $A, A' \in \mathcal{A}$, the map $\text{Mor}_{\mathcal{A}}(A, A') \rightarrow \text{Mor}_{\mathcal{B}}(F(A), F(A'))$ is injective, and **full** if it is surjective. A functor that is full and faithful is **fully faithful**. (For various philosophical reasons, the notion of “full” functor on its own is unimportant; “fully faithful” is the useful notion.) A subcategory $i: \mathcal{A} \rightarrow \mathcal{B}$ is a **full subcategory** if i is full.

(Inclusions are always faithful, so there is no need for the phrase “faithful subcategory.”) Thus a subcategory $\mathcal{A}'$ of $\mathcal{A}$ is full if and only if for all $A, B \in \text{obj}(\mathcal{A}')$, $\text{Mor}_{\mathcal{A}'}(A, B) = \text{Mor}_{\mathcal{A}}(A, B)$. For example, the forgetful functor $\text{Vec}_k \rightarrow \text{Sets}$ is faithful, but not full; and if A is a ring, the category of finitely generated A -modules is a full subcategory of the category Mod_A of A -modules.

1.1.16. Definition. A **contravariant functor** is defined in the same way as a covariant functor, except the arrows switch directions: in the above language, $F(A_1 \rightarrow A_2)$ is now an arrow from $F(A_2)$ to $F(A_1)$. (Thus $F(m_2 \circ m_1) = F(m_1) \circ F(m_2)$, not $F(m_2) \circ F(m_1)$.)

It is wise to state whether a functor is covariant or contravariant, unless the context makes it very clear. If it is not stated (and the context does not make it clear), the functor is often assumed to be covariant.

Sometimes people describe a contravariant functor $\mathcal{C} \rightarrow \mathcal{D}$ as a covariant functor $\mathcal{C}^{\text{opp}} \rightarrow \mathcal{D}$, where $\mathcal{C}^{\text{opp}}$ is the same category as $\mathcal{C}$ except that the arrows go in the opposite direction. Here $\mathcal{C}^{\text{opp}}$ is said to be the **opposite category** to $\mathcal{C}$.

One can define fullness, etc. for contravariant functors, and you should do so.

1.1.17. Linear algebra example. If Vec_k is the category of k -vector spaces (introduced in Example 1.1.3), then taking duals gives a contravariant functor $(\cdot)^\vee: \text{Vec}_k \rightarrow \text{Vec}_k$. Indeed, to each linear transformation $f: V \rightarrow W$, we have a dual transformation $f^\vee: W^\vee \rightarrow V^\vee$, and $(f \circ g)^\vee = g^\vee \circ f^\vee$.

1.1.18. Topological example (cf. Example 1.1.13) *for those who have seen cohomology.* The i th cohomology functor $H^i(\cdot, \mathbb{Z}): \text{Top} \rightarrow \text{Ab}$ is a contravariant functor.

1.1.19. Example. There is a contravariant functor $\text{Top} \rightarrow \text{Rings}$ taking a topological space X to the ring of real-valued continuous functions on X . A morphism of topological spaces $X \rightarrow Y$ (a continuous map) induces the pullback map from functions on Y to functions on X .

1.1.20. Example (the functor of points; cf. Example 1.1.14). Suppose A is an object of a category $\mathcal{C}$. Then there is a contravariant functor $h_A: \mathcal{C} \rightarrow \text{Sets}$ sending $B \in \mathcal{C}$ to $\text{Mor}(B, A)$, and sending the morphism $f: B_1 \rightarrow B_2$ to the morphism $\text{Mor}(B_2, A) \rightarrow \text{Mor}(B_1, A)$ via

$$[g: B_2 \rightarrow A] \longmapsto [g \circ f: B_1 \rightarrow B_2 \rightarrow A].$$

This example initially looks weird and different, but Examples 1.1.17 and 1.1.19 may be interpreted as special cases; do you see how? What is A in each case? This functor might reasonably be called the *functor of maps* (to A), but is actually known as the **functor of points**. We will meet this functor again in §1.2.11 and (in the category of schemes) in Definition 7.3.10.

1.1.21.* Natural transformations (and natural isomorphisms) of covariant functors, and equivalences of categories.

(This notion won’t come up in an essential way until at least Chapter 7, so you shouldn’t read this section until then.) Suppose F and G are two covariant functors from $\mathcal{A}$ to $\mathcal{B}$. A **natural transformation of covariant functors** $F \rightarrow G$ is the data of a morphism $m_A: F(A) \rightarrow G(A)$ for each $A \in \mathcal{A}$ such that for each $f: A \rightarrow A'$ in $\mathcal{A}$, the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ m_A \downarrow & & \downarrow m_{A'} \\ G(A) & \xrightarrow{G(f)} & G(A') \end{array}$$

commutes. A **natural isomorphism** of functors is a natural transformation such that each m_A is an isomorphism. (We make analogous definitions when F and G are both contravariant.)

The data of functors $F: \mathcal{A} \rightarrow \mathcal{B}$ and $F': \mathcal{B} \rightarrow \mathcal{A}$ such that $F \circ F'$ is naturally isomorphic to the identity functor $\text{id}_{\mathcal{B}}$ on $\mathcal{B}$ and $F' \circ F$ is naturally isomorphic to $\text{id}_{\mathcal{A}}$ is said to be an **equivalence of categories**. The right notion of when two categories are “essentially the same” is not *isomorphism*

(a functor giving bijections of objects and morphisms) but *equivalence*. Exercises 1.1.C and 1.1.D might give you some vague sense of this. Later exercises (for example, to show that “rings” and “affine schemes” are essentially the same once arrows are reversed, Exercise 7.3.E) may help, too.

Two examples might make this strange concept more comprehensible. The double dual of a finite-dimensional vector space V is *not* V , but we learn early to say that it is canonically isomorphic to V . We can make that precise as follows. Let $f.d.Vec_k$ be the category of finite-dimensional vector spaces over k . Note that this category contains oodles of vector spaces of each dimension.

1.1.C. EXERCISE. Let $(\cdot)^{\vee\vee} : f.d.Vec_k \rightarrow f.d.Vec_k$ be the double dual functor from the category of finite-dimensional vector spaces over k to itself. Show that $(\cdot)^{\vee\vee}$ is naturally isomorphic to the identity functor on $f.d.Vec_k$. (Without the finite-dimensionality hypothesis, we only get a natural transformation of functors from id to $(\cdot)^{\vee\vee}$.)

Let $\mathcal{V}$ be the category whose objects are the k -vector spaces k^n for each $n \geq 0$ (there is one vector space for each n), and whose morphisms are linear transformations. The objects of $\mathcal{V}$ can be thought of as vector spaces with bases, and the morphisms as matrices. There is an obvious functor $\mathcal{V} \rightarrow f.d.Vec_k$, as each k^n is a finite-dimensional vector space.

1.1.D. EXERCISE. Show that $\mathcal{V} \rightarrow f.d.Vec_k$ gives an equivalence of categories, by describing an “inverse” functor. (Recall that we are being cavalier about set-theoretic assumptions—see Caution 0.3.1—so feel free to simultaneously choose bases for each vector space in $f.d.Vec_k$. To make this precise, you will need to use Gödel-Bernays set theory or else replace $f.d.Vec_k$ with a very similar small category, but we won’t worry about this.)

1.1.22. *Aside for experts.*** Your argument for Exercise 1.1.D will show that (modulo set-theoretic issues) this definition of equivalence of categories is the same as another one commonly given: a covariant functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is an equivalence of categories if it is fully faithful and every object of $\mathcal{B}$ is isomorphic to an object of the form $F(A)$ for some $A \in \mathcal{A}$ (F is **essentially surjective**, a term we will not need).

1.2 Universal Properties Determine an Object up to Unique Isomorphism

Given some category that we come up with, we often will have ways of producing new objects from old. In good circumstances, definitions will be usefully made using the notion of a *universal property*. Informally, we wish that there were an object with some property. We first show that if it exists, then it is essentially unique, or more precisely, is unique up to unique isomorphism. Then we go about constructing an example of such an object to show existence.

Explicit constructions are sometimes easier to work with than universal properties, but with a little practice, universal properties are useful in proving things quickly and slickly. Indeed, when learning the subject, people often find explicit constructions more appealing, and use them more often in proofs, but as they become more experienced, they find universal property arguments more elegant and insightful.

1.2.1. Products were defined by a universal property. We have seen one important example of a universal property argument already in §1.0.1: products. You should go back and verify that our discussion there gives a notion of product in any category, and shows that products, *if they exist*, are unique up to unique isomorphism.

1.2.2. Initial, final, and zero objects. Here are some simple but useful concepts that will give you practice with universal property arguments. An object of a category $\mathcal{C}$ is an **initial object** if it has precisely one map to every object. It is a **final object** if it has precisely one map from every object. It is a **zero object** if it is both an initial object and a final object.

1.2.A. EXERCISE. Show that any two initial objects are uniquely isomorphic. Show that any two final objects are uniquely isomorphic.

In other words, *if* an initial object exists, it is unique up to unique isomorphism, and similarly for final objects. This (partially) justifies the phrase “*the* initial object” rather than “*an* initial object,” and similarly for “*the* final object” and “*the* zero object.” (Convention: We often say “*the*,” not “*a*,” for anything defined up to unique isomorphism.)

1.2.B. EXERCISE. What are the initial and final objects in *Sets*, *Rings*, and *Top* (if they exist)? How about in the two examples of §1.1.9?

1.2.3. Localization of rings and modules. Another important example of a definition by universal property is the notion of *localization* of a ring. We first review a constructive definition, and then reinterpret the notion in terms of universal property. A **multiplicative subset** S of a ring A is a subset closed under multiplication containing 1. We define a ring $S^{-1}A$. The elements of $S^{-1}A$ are of the form a/s where $a \in A$ and $s \in S$, and where $a_1/s_1 = a_2/s_2$ if (and only if) *for some* $s \in S$, $s(s_2a_1 - s_1a_2) = 0$. We define $(a_1/s_1) + (a_2/s_2) = (s_2a_1 + s_1a_2)/(s_1s_2)$, and $(a_1/s_1) \times (a_2/s_2) = (a_1a_2)/(s_1s_2)$. (If you wish, you may check that this equality of fractions really is an equivalence relation and the two binary operations on fractions are well-defined on equivalence classes and make $S^{-1}A$ into a ring.) We have a canonical ring map

$$(1.2.3.1) \quad A \longrightarrow S^{-1}A$$

given by $a \mapsto a/1$. Note that if $0 \in S$, $S^{-1}A$ is the 0-ring.

There are two particularly important flavors of multiplicative subsets. The first is $\{1, f, f^2, \dots\}$, where $f \in A$. This localization is denoted by A_f . (Can you describe an isomorphism $A_f \xrightarrow{\sim} A[t]/(tf - 1)$?) The second is $A \setminus \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. This localization $S^{-1}A$ is denoted by $A_{\mathfrak{p}}$. (Notational warning: If $\mathfrak{p}$ is a prime ideal, then $A_{\mathfrak{p}}$ means you’re allowed to divide by elements not in $\mathfrak{p}$. However, if $f \in A$, A_f means you’re allowed to divide by f . This can be confusing. For example, if (f) is a prime ideal, then $A_f \neq A_{(f)}$.)

Warning: Sometimes localization is first introduced in the special case where A is an integral domain and $0 \notin S$. In that case, $A \hookrightarrow S^{-1}A$, but this isn’t always true, as shown by the following exercise. (But we will see that noninjective localizations needn’t be pathological, and we can sometimes understand them geometrically; see Exercise 3.2.L.)

1.2.C. EXERCISE. Show that $A \rightarrow S^{-1}A$ is injective if and only if S contains no zerodivisors. (A **zerodivisor** of a ring A is an element a such that there is a nonzero element b with $ab = 0$. The other elements of A are called **non-zerodivisors**. For example, an invertible element is never a zerodivisor. Counterintuitively, 0 is a zerodivisor in every ring but the 0-ring. More generally, if M is an A -module, then $a \in A$ is a **zerodivisor for** M if there is a nonzero $m \in M$ with $am = 0$. The other elements of A are called **non-zerodivisors for** M . Equivalently, and *very* usefully, $a \in A$ is a non-zerodivisor for M if and only if $\times a : M \rightarrow M$ is an injection, or equivalently in the language of §1.4.4, if

$$0 \longrightarrow M \xrightarrow{\times a} M$$

is exact.)

If A is an integral domain and $S = A \setminus \{0\}$, then $S^{-1}A$ is called the **fraction field** of A , which we denote by $K(A)$. The previous exercise shows that A is a subring of its fraction field $K(A)$. We now return to the case where A is a general (commutative) ring.

1.2.D. EXERCISE. Verify that $A \rightarrow S^{-1}A$ satisfies the following universal property: $S^{-1}A$ is initial among A -algebras B where every element of S is sent to an invertible element in B . (Recall: The data of “an A -algebra B ” and “a ring map $A \rightarrow B$ ” are the same.) Translation: Any map $A \rightarrow B$ where every element of S is sent to an invertible element factors uniquely through $A \rightarrow S^{-1}A$. Another translation: a ring map out of $S^{-1}A$ is the same thing as a ring map from A that sends every element of S to an invertible element. Furthermore, an $S^{-1}A$ -module is the same thing as an A -module for which $s \times \cdot : M \rightarrow M$ is an A -module isomorphism for all $s \in S$.

In fact, it is cleaner to *define* $A \rightarrow S^{-1}A$ by the universal property, and to show that it exists, and to use the universal property to check various properties $S^{-1}A$ has. Let's get some practice with this by *defining* localizations of modules by the universal property. Suppose M is an A -module. We define the A -module map $\phi: M \rightarrow S^{-1}M$ as being initial among A -module maps $M \rightarrow N$ such that elements of S are invertible in N ($s \times \cdot: N \rightarrow N$ is an isomorphism for all $s \in S$). More precisely, any such map $\alpha: M \rightarrow N$ factors uniquely through ϕ :

$$\begin{array}{ccc} M & \xrightarrow{\phi} & S^{-1}M \\ & \searrow \alpha & \downarrow \exists! \\ & & N \end{array} .$$

(Translation: $M \rightarrow S^{-1}M$ is universal (initial) among A -module maps from M to modules that are actually $S^{-1}A$ -modules. Can you make this precise by defining clearly the objects and morphisms in this category?)

Notice: (i) This determines $\phi: M \rightarrow S^{-1}M$ up to unique isomorphism (you should think through what this means); (ii) we are defining not only $S^{-1}M$, but also the map ϕ at the same time; and (iii) essentially by definition the A -module structure on $S^{-1}M$ extends to an $S^{-1}A$ -module structure.

1.2.E. EXERCISE. Show that $\phi: M \rightarrow S^{-1}M$ exists, by constructing something satisfying the universal property. Hint: Define elements of $S^{-1}M$ to be of the form m/s where $m \in M$ and $s \in S$, and $m_1/s_1 = m_2/s_2$ if and only if for some $s \in S$, $s(s_2m_1 - s_1m_2) = 0$. Define the additive structure by $(m_1/s_1) + (m_2/s_2) = (s_2m_1 + s_1m_2)/(s_1s_2)$, and the $S^{-1}A$ -module structure (and hence the A -module structure) as given by $(a_1/s_1) \cdot (m_2/s_2) = (a_1m_2)/(s_1s_2)$.

1.2.F. EXERCISE.

- Show that localization commutes with finite products, or equivalently, with finite direct sums. In other words, if $M_1, \dots, M_n$ are A -modules, describe an isomorphism (of A -modules, and of $S^{-1}A$ -modules) $S^{-1}(M_1 \times \dots \times M_n) \rightarrow S^{-1}M_1 \times \dots \times S^{-1}M_n$.
- Show that localization commutes with *arbitrary* direct sums.
- Show that “localization does not necessarily commute with infinite products”: the obvious map $S^{-1}(\prod_i M_i) \rightarrow \prod_i S^{-1}M_i$ induced by the universal property of localization is not always an isomorphism. (Hint: $(1, 1/2, 1/3, 1/4, \dots) \in \mathbb{Q} \times \mathbb{Q} \times \dots$.)

1.2.4. Remark. Localization does not always commute with Hom ; see Example 1.5.10. But Exercise 1.5.H will show that in good situations (if the first argument of Hom is *finitely presented*), localization *does* commute with Hom .

1.2.5. Tensor products. Another important example of a universal property construction is the notion of a **tensor product** of A -modules:

$$\begin{array}{ccc} \otimes_A: & \text{obj}(\text{Mod}_A) \times \text{obj}(\text{Mod}_A) & \longrightarrow \text{obj}(\text{Mod}_A) \\ & (M, N) & \longmapsto M \otimes_A N \end{array}$$

The subscript A is often suppressed when it is clear from context. The tensor product is often defined as follows. Suppose you have two A -modules M and N . Then elements of the tensor product $M \otimes_A N$ are finite A -linear combinations of symbols $m \otimes n$ ($m \in M$, $n \in N$), subject to relations $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$, $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$, $a(m \otimes n) = (am) \otimes n = m \otimes (an)$ (where $a \in A$, $m_1, m_2 \in M$, $n_1, n_2 \in N$). More formally, $M \otimes_A N$ is the free A -module generated by $M \times N$, quotiented by the submodule generated by $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$, $(m, n_1 + n_2) - (m, n_1) - (m, n_2)$, $a(m, n) - (am, n)$, and $a(m, n) - (m, an)$ for $a \in A$, $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$. The image of (m, n) in this quotient is $m \otimes n$.

If A is a field k , we recover the tensor product of vector spaces.

1.2.G. EXERCISE (IF YOU HAVEN'T SEEN TENSOR PRODUCTS BEFORE). Show that $\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12) \cong \mathbb{Z}/(2)$. (This exercise is intended to give some hands-on practice with tensor products.)

1.2.H. IMPORTANT EXERCISE: RIGHT-EXACTNESS OF $(\cdot) \otimes_A N$. Show that $(\cdot) \otimes_A N$ gives a covariant functor $Mod_A \rightarrow Mod_A$. Show that $(\cdot) \otimes_A N$ is a **right-exact functor**, i.e., if

$$M' \rightarrow M \rightarrow M'' \rightarrow 0$$

is an exact sequence of A -modules (which means $f: M \rightarrow M''$ is surjective, and M' surjects onto the kernel of f ; see §1.4.4), then the induced sequence

$$M' \otimes_A N \rightarrow M \otimes_A N \rightarrow M'' \otimes_A N \rightarrow 0$$

is also exact. This exercise is repeated in Exercise 1.5.G, but you may get a lot out of doing it now. (You will be reminded of the definition of right-exactness in §1.5.6.)

In contrast, you can quickly check that the tensor product is not left-exact: tensor the exact sequence of $\mathbb{Z}$ -modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/(2) \longrightarrow 0$$

with $\mathbb{Z}/(2)$.

The constructive definition of $\otimes$ is a weird definition, and really the “wrong” definition. To motivate a better one: notice that there is a natural A -bilinear map $M \times N \rightarrow M \otimes_A N$. (If $M, N, P \in Mod_A$, a map $f: M \times N \rightarrow P$ is **A -bilinear** if $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$, $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$, and $f(am, n) = f(m, an) = af(m, n)$.) Any A -bilinear map $M \times N \rightarrow P$ factors through the tensor product uniquely: $M \times N \rightarrow M \otimes_A N \rightarrow P$. (Think this through!)

We can take this as the *definition* of the tensor product as follows. It is an A -module T along with an A -bilinear map $t: M \times N \rightarrow T$, such that given any A -bilinear map $t': M \times N \rightarrow T'$, there is a unique A -linear map $f: T \rightarrow T'$ such that $t' = f \circ t$.

$$\begin{array}{ccc} M \times N & \xrightarrow{t} & T \\ & \searrow t' & \swarrow \exists! f \\ & T' & \end{array}$$

1.2.I. EXERCISE. Show that $(T, t: M \times N \rightarrow T)$ is unique up to unique isomorphism. Hint: First figure out what “unique up to unique isomorphism” means for such pairs, using a category of pairs (T, t) . Then follow the analogous argument for the product.

In short, given M and N , there is an A -bilinear map $t: M \times N \rightarrow M \otimes_A N$, unique up to unique isomorphism, defined by the following universal property: for any A -bilinear map $t': M \times N \rightarrow T'$ there is a unique A -linear map $f: M \otimes_A N \rightarrow T'$ such that $t' = f \circ t$.

As with all universal property arguments, this argument shows uniqueness *assuming existence*. To show existence, we need an explicit construction.

1.2.J. EXERCISE. Show that the construction of §1.2.5 satisfies the universal property of tensor product.

The three exercises below are useful facts about tensor products with which you should be familiar.

1.2.K. IMPORTANT EXERCISE.

- If M is an A -module and $A \rightarrow B$ is a morphism of rings, give $B \otimes_A M$ the structure of a B -module (this is part of the exercise). Show that this describes a functor $Mod_A \rightarrow Mod_B$.
- (tensor product of rings)** If further $A \rightarrow C$ is another morphism of rings, show that $B \otimes_A C$ has a natural structure of a ring. Hint: Multiplication will be given by $(b_1 \otimes c_1)(b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2)$. (Exercise 1.2.U will interpret this construction as a fibered coproduct.)

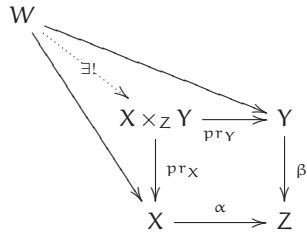
1.2.L. IMPORTANT EXERCISE. If S is a multiplicative subset of A and M is an A -module, describe a natural isomorphism $(S^{-1}A) \otimes_A M \xrightarrow{\sim} S^{-1}M$ (as $S^{-1}A$ -modules *and* as A -modules).

1.2.M. EXERCISE ($\otimes$ COMMUTES WITH $\oplus$). Show that tensor products commute with arbitrary direct sums: if M and $\{N_i\}_{i \in I}$ are all A -modules, describe an isomorphism

$$M \otimes (\oplus_{i \in I} N_i) \xrightarrow{\sim} \oplus_{i \in I} (M \otimes N_i).$$

1.2.6. Essential Example: Fibered products. Suppose we have morphisms $\alpha: X \rightarrow Z$ and $\beta: Y \rightarrow Z$ (in *any* category). Then the **fibered product** (or *fibred product*) is an object $X \times_Z Y$ along with morphisms $\text{pr}_X: X \times_Z Y \rightarrow X$ and $\text{pr}_Y: X \times_Z Y \rightarrow Y$, where the two compositions $\alpha \circ \text{pr}_X, \beta \circ \text{pr}_Y: X \times_Z Y \rightarrow Z$ agree, such that given any object W with maps to X and Y (whose compositions to Z agree), these maps factor through some unique $W \rightarrow X \times_Z Y$:

(1.2.6.1)

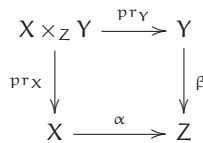


(Warning: The definition of the fibered product depends on α and β , even though they are omitted from the notation $X \times_Z Y$.)

By the usual universal property argument, if it exists, it is unique up to unique isomorphism. (You should think this through until it is clear to you.) Thus the use of the phrase “the fibered product” (rather than “a fibered product”) is reasonable, and we should reasonably be allowed to give it the name $X \times_Z Y$. We know what maps to it are: they are precisely maps to X and maps to Y that agree as maps to Z .

1.2.7. Definition. As an example, if $\pi: X \rightarrow Y$ is a morphism, and the fibered product $X \times_Y X$ exists, then this determines a **diagonal morphism** $\delta_\pi: X \rightarrow X \times_Y X$. The diagonal morphism will turn out to be a very useful notion.

Depending on your religion, the diagram



is called a **fibered/pullback/Cartesian diagram/square** (six possibilities—and even more are possible if you prefer “fibred” to “fibered”).

The right way to interpret the notion of fibered product is first to think about what it means in the category of sets.

1.2.N. EXERCISE (FIBERED PRODUCTS OF SETS). Show that in *Sets*,

$$X \times_Z Y = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}.$$

More precisely, show that the right side, equipped with its evident maps to X and Y , satisfies the universal property of the fibered product. (This will help you build intuition for fibered products.)

1.2.O. EXERCISE. If X is a topological space, show that fibered products always exist in the category of open sets of X , by describing what a fibered product is. (Hint: It has a one-word description.)

1.2.P. EXERCISE. If Z is the final object in a category $\mathcal{C}$, and $X, Y \in \mathcal{C}$, show that “ $X \times_Z Y = X \times Y$ ”: “the” fibered product over Z is uniquely isomorphic to “the” product. Assume all relevant (fibered) products exist. (This is an exercise about unwinding the definition.)

1.2.Q. USEFUL EXERCISE: TOWERS OF CARTESIAN DIAGRAMS ARE CARTESIAN DIAGRAMS. If the two squares in the following commutative diagram are Cartesian diagrams, show that the “outside rectangle” (involving U, V, Y , and Z) is also a Cartesian diagram.

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

1.2.R. EXERCISE. Given morphisms $X_1 \rightarrow Y, X_2 \rightarrow Y$, and $Y \rightarrow Z$, show that there is a natural morphism $X_1 \times_Y X_2 \rightarrow X_1 \times_Z X_2$, assuming that both fibered products exist. (This is trivial once you figure out what it is saying. The point of this exercise is to see why it is trivial.)

1.2.S. IMPORTANT EXERCISE: THE DIAGONAL-BASE-CHANGE DIAGRAM. Suppose we are given morphisms $X_1, X_2 \rightarrow Y$ and $Y \rightarrow Z$. Show that the following diagram is a Cartesian square.

$$\begin{array}{ccc} X_1 \times_Y X_2 & \longrightarrow & X_1 \times_Z X_2 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \times_Z Y \end{array}$$

Assume all relevant (fibered) products exist. (If this exercise is too hard now, you can try it again at Exercise 1.3.B.) You will appreciate how useful this diagram is when you repeatedly use the diagonal morphism in proofs and constructions.

If you liked this problem, you may enjoy Exercise 11.1.C.

1.2.8. Coproducts. Define **coproduct** in a category by reversing all the arrows in the definition of product. Define **fibered coproduct** in a category by reversing all the arrows in the definition of fibered product. Coproduct is denoted by $\coprod$.

1.2.T. EXERCISE. Show that coproduct for *Sets* is disjoint union. This is why we use the notation $\coprod$ for disjoint union.

1.2.U. EXERCISE. Suppose $A \rightarrow B$ and $A \rightarrow C$ are two ring morphisms, so in particular B and C are A -modules. Recall (Exercise 1.2.K) that $B \otimes_A C$ has a ring structure. Show that there is a natural morphism $B \rightarrow B \otimes_A C$ given by $b \mapsto b \otimes 1$. (This is not necessarily an inclusion; see Exercise 1.2.G.) Similarly, there is a natural morphism $C \rightarrow B \otimes_A C$. Show that this gives a fibered coproduct on rings, i.e., that

$$\begin{array}{ccc} B \otimes_A C & \longleftarrow & C \\ \uparrow & & \uparrow \\ B & \longleftarrow & A \end{array}$$

satisfies the universal property of fibered coproduct.

1.2.9. Monomorphisms and epimorphisms.

1.2.10. Definition. A morphism $\pi: X \rightarrow Y$ is a **monomorphism** if any two morphisms $\mu_1: Z \rightarrow X$ and $\mu_2: Z \rightarrow X$ such that $\pi \circ \mu_1 = \pi \circ \mu_2$ must satisfy $\mu_1 = \mu_2$. In other words, there is at most one

way of filling in the dotted arrow so that the diagram

$$\begin{array}{ccc} & Z & \\ \leq 1 \downarrow & \searrow & \\ X & \xrightarrow{\pi} & Y \end{array}$$

commutes—for any object Z , the natural map $\text{Mor}(Z, X) \rightarrow \text{Mor}(Z, Y)$ is an injection. Intuitively, it is the categorical version of an injective map, and indeed this notion generalizes the familiar notion of injective maps of sets. (The reason we don't use the word "injective" is that in some contexts, "injective" will have an intuitive meaning that may not agree with "monomorphism." One example: in the category of divisible groups, the map $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$ is a monomorphism but not injective. This is also the case with "epimorphism"—to be defined shortly—in contrast with "surjective.")

1.2.V. EXERCISE. Show that the composition of two monomorphisms is a monomorphism.

1.2.W. EXERCISE. Prove that a morphism $\pi: X \rightarrow Y$ is a monomorphism if and only if the fibered product $X \times_Y X$ exists, and the induced diagonal morphism $\delta_\pi: X \rightarrow X \times_Y X$ (Definition 1.2.7) is an isomorphism. We may then take this as the definition of monomorphism. (Monomorphisms aren't central to future discussions, although they will come up again. This exercise is just good practice.)

1.2.X. EASY EXERCISE. We use the notation of Exercise 1.2.R. Show that if $Y \rightarrow Z$ is a monomorphism, then the morphism $X_1 \times_Y X_2 \rightarrow X_1 \times_Z X_2$ you described in Exercise 1.2.R is an isomorphism. (Hint: For any object V , give a natural bijection between maps from V to the first and maps from V to the second. It is also possible to use the Diagonal-Base-Change diagram, Exercise 1.2.S.)

The notion of **epimorphism** is "dual" to the definition of monomorphism, where all the arrows are reversed. This concept will not be central for us, although it turns up in the definition of an abelian category. Intuitively, it is the categorical version of a surjective map. (But be careful when working with categories of objects that are sets with additional structure, as epimorphisms need not be surjective. Example: In the category *Rings*, $\mathbb{Z} \rightarrow \mathbb{Q}$ is an epimorphism, but obviously not surjective.)

1.2.11. Representable functors and Yoneda's Lemma. Much of our discussion about universal properties can be cleanly expressed in terms of representable functors, under the rubric of "Yoneda's Lemma." Yoneda's lemma is an easy fact stated in a complicated way. Informally speaking, you can essentially recover an object in a category by knowing the maps into it. For example, we have seen that the data of maps to $X \times Y$ are naturally (canonically) the data of maps to X and to Y . Indeed, we have now taken this as the *definition* of $X \times Y$.

Recall Example 1.1.20. Suppose A is an object of category $\mathcal{C}$. For any object $C \in \mathcal{C}$, we have a set of morphisms $\text{Mor}(C, A)$. If we have a morphism $f: B \rightarrow C$, we get a map of sets

$$(1.2.11.1) \quad \text{Mor}(C, A) \longrightarrow \text{Mor}(B, A)$$

by composition: given a map from C to A , we get a map from B to A by precomposing with $f: B \rightarrow C$. Hence this gives a contravariant functor $h_A: \mathcal{C} \rightarrow \text{Sets}$. Yoneda's Lemma states that the functor h_A determines A up to unique isomorphism. More precisely:

1.2.Y. IMPORTANT EXERCISE THAT YOU SHOULD DO ONCE IN YOUR LIFE (YONEDA'S LEMMA).

(a) Suppose you have two objects A and A' in a category $\mathcal{C}$, and maps

$$(1.2.11.2) \quad i_C: \text{Mor}(C, A) \longrightarrow \text{Mor}(C, A')$$

that commute with the maps (1.2.11.1). Show that the i_C (as C ranges over the objects of $\mathcal{C}$) are induced from a unique morphism $g: A \rightarrow A'$. More precisely, show that there is a unique morphism $g: A \rightarrow A'$ such that for all $C \in \mathcal{C}$, i_C is $u \mapsto g \circ u$.

- (b) If furthermore the ι_C are all bijections, show that the resulting g is an isomorphism. (Hint for both: This is much easier than it looks. This statement is so general that there are really only a couple of things that you could possibly try. For example, if you're hoping to find a morphism $A \rightarrow A'$, where will you find it? Well, you are looking for an element $\text{Mor}(A, A')$. So just plug $C = A$ into (1.2.11.2), and see where the identity goes.)

There is an analogous statement with the arrows reversed, where instead of maps into A , you think of maps *from* A . The role of the contravariant functor h_A of Example 1.1.20 is played by the covariant functor h^A of Example 1.1.14. Because the proof is the same (with the arrows reversed), you needn't think it through.

The phrase “Yoneda’s Lemma” properly refers to a more general statement. Although it looks more complicated, it is no harder to prove.

1.2.Z.* EXERCISE.

- (a) Suppose A and B are objects in a category $\mathcal{C}$. Give a bijection between the natural transformations $h^A \rightarrow h^B$ of covariant functors $\mathcal{C} \rightarrow \text{Sets}$ (see Example 1.1.14 for the definition) and the morphisms $B \rightarrow A$.

- (b) State and prove the corresponding fact for contravariant functors h_A (see Example 1.1.20).

Remark: A contravariant functor F from $\mathcal{C}$ to Sets is said to be **representable** if there is a natural isomorphism

$$\xi: F \xrightarrow{\sim} h_A.$$

Thus the representing object A is determined up to unique isomorphism by the pair (F, ξ) . There is a similar definition for covariant functors. (We will revisit this in §7.6, and this problem will appear again as Exercise 7.6.C. The element $\xi^{-1}(\text{id}_A) \in F(A)$ is often called the “universal object”; do you see why?)

- (c) **Yoneda’s Lemma.** Suppose F is a covariant functor $\mathcal{C} \rightarrow \text{Sets}$, and $A \in \mathcal{C}$. Give a bijection between the natural transformations $h^A \rightarrow F$ and $F(A)$. (The corresponding fact for contravariant functors is essentially Exercise 10.1.B.)

In fancy terms, Yoneda’s lemma states the following. Given a category $\mathcal{C}$, we can produce a new category, called the **functor category** of $\mathcal{C}$, where the objects are contravariant functors $\mathcal{C} \rightarrow \text{Sets}$, and the morphisms are natural transformations of such functors. We have a functor (which we can usefully call h) from $\mathcal{C}$ to its functor category, which sends A to h_A . Yoneda’s Lemma states that this is a fully faithful functor, called the *Yoneda embedding*. (Fully faithful functors were defined in §1.1.15.)

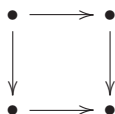
1.2.12. Joke. The Yoda embedding, contravariant it is.

1.3 Limits and Colimits

Two important definitions, those of limits and colimits, are determined by universal properties. They generalize a number of familiar constructions. I will give the definitions first, and then show you why they are familiar. For example, fractions will be motivating examples of colimits (Exercise 1.3.D(a)), and the p -adic integers (Example 1.3.4) will be motivating examples of limits.

1.3.1. Limits. We say that a category is a **small category** if the objects form a set and the morphisms form a set. (This is a technical condition intended only for experts.) Suppose $\mathcal{I}$ is any small category, and $\mathcal{C}$ is any category. Then a functor $F: \mathcal{I} \rightarrow \mathcal{C}$ (i.e., with an object $A_i \in \mathcal{C}$ for each element $i \in \mathcal{I}$, and appropriate commuting morphisms dictated by $\mathcal{I}$) is said to be a **diagram indexed by $\mathcal{I}$** . We call $\mathcal{I}$ an **index category**. Our index categories will usually be partially

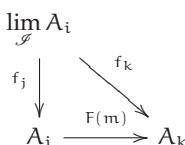
ordered sets (Example 1.1.8), in which, in particular, there is at most one morphism between any two objects. (But other examples are sometimes useful.) For example, if $\square$ is the category



and $\mathcal{A}$ is a category, then a functor $\square \rightarrow \mathcal{A}$ is precisely the data of a commuting square in $\mathcal{A}$.

Then the **limit of the diagram** is an object $\lim_{\mathcal{I}} A_i$ (or $\varprojlim_{\mathcal{I}} A_i$) of $\mathcal{C}$ along with morphisms $f_j: \lim_{\mathcal{I}} A_i \rightarrow A_j$ for each $j \in \mathcal{I}$, such that if $m: j \rightarrow k$ is a morphism in $\mathcal{I}$, then

(1.3.1.1)

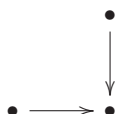


commutes, and the object and the maps to each A_i are universal (final) with respect to this property. More precisely, given any other object W along with maps $g_i: W \rightarrow A_i$ commuting with the $F(m)$ (if $m: j \rightarrow k$ is a morphism in $\mathcal{I}$, then $g_k = F(m) \circ g_j$), there is a unique map

$$g: W \rightarrow \lim_{\mathcal{I}} A_i$$

so that $g_i = f_i \circ g$ for all i . (In some cases, the limit is sometimes called the **inverse limit** or **projective limit**. We won't use this language.) By the usual universal property argument, if the limit exists, it is unique up to unique isomorphism.

1.3.2. Examples: Products. For example, if $\mathcal{I}$ is the partially ordered set



we obtain the fibered product.

If $\mathcal{I}$ is

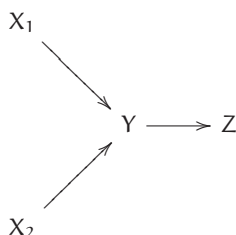


we obtain the product.

If $\mathcal{I}$ is a set (i.e., its only morphisms are the identity maps), then the limit is called the **product** of the A_i , and is denoted by $\prod_i A_i$. The special case where $\mathcal{I}$ has two elements is the example of the previous paragraph.

1.3.A. EXERCISE (REALITY CHECK). Suppose that the partially ordered set $\mathcal{I}$ has an initial object e . Show that the limit of any diagram indexed by $\mathcal{I}$ exists.

1.3.B. EXERCISE: THE DIAGONAL-BASE-CHANGE DIAGRAM, AGAIN. Solve 1.2.S again by identifying both $X_1 \times_Y X_2$ and $Y \times_{(Y \times_Z Y)} (X_1 \times_Z X_2)$ as the limit of the diagram



1.3.3. Example: Formal power series. For a ring A , the **(formal) power series**, $A[[x]]$, are often described informally (and somewhat unnaturally) as being the ring

$$A[[x]] = \{a_0 + a_1x + a_2x^2 + \cdots\}$$

(where $a_i \in A$, and the ring operations are the “obvious” ones). It is an example of a limit in the category of rings:

$$\begin{array}{c} A[[x]] \\ \swarrow \quad \searrow \quad \searrow \quad \searrow \\ \cdots \longrightarrow A[x]/(x^3) \longrightarrow A[x]/(x^2) \longrightarrow A[x]/(x). \end{array}$$

The universal property of limits yields a natural ring morphism $A[x] \rightarrow A[[x]]$. If $A = \mathbb{R}$ or $\mathbb{C}$, this map factors through the ring of *convergent power series*.

1.3.4. Example: The p -adic integers. For a prime number p , the **p -adic integers** (or more informally, **p -adics**), $\mathbb{Z}_p$, are often described informally (and somewhat unnaturally) as being of the form

$$a_0 + a_1p + a_2p^2 + \cdots$$

(where $0 \leq a_i < p$). They are an example of a limit in the category of rings:

$$\begin{array}{c} \mathbb{Z}_p \\ \swarrow \quad \searrow \quad \searrow \quad \searrow \\ \cdots \longrightarrow \mathbb{Z}/(p^3) \longrightarrow \mathbb{Z}/(p^2) \longrightarrow \mathbb{Z}/(p). \end{array}$$

(Warning: $\mathbb{Z}_p$ is sometimes used to denote the integers modulo p , but $\mathbb{Z}/(p)$ or $\mathbb{Z}/p\mathbb{Z}$ is better to use for this, to avoid confusion. Worse: by §1.2.3, $\mathbb{Z}_p$ also denotes those rationals whose denominators are a power of p . Hopefully the meaning of $\mathbb{Z}_p$ will be clear from the context.)

The similarity of Examples 1.3.3 and 1.3.4 is no coincidence. Formal power series and the p -adic integers are examples of *completions*, the topic of Chapter 28.

Limits do not always exist for any index category $\mathcal{I}$. However, you can often easily check that limits exist if the objects of your category can be interpreted as sets with additional structure, and arbitrary products exist (respecting the set-like structure).

1.3.C. IMPORTANT EXERCISE. Show that in the category *Sets*,

$$\left\{ (a_i)_{i \in \mathcal{I}} \in \prod_i A_i : f(m)(a_j) = a_k \text{ for all } m \in \text{Mor}_{\mathcal{I}}(j, k) \subset \text{Mor}(\mathcal{I}) \right\},$$

along with the obvious projection maps to each A_i , is the limit $\lim_{\mathcal{I}} A_i$.

This clearly also works in the category Mod_A of A -modules (in particular, Vec_k and Ab), as well as *Rings*.

From this point of view, $2 + 3p + 2p^2 + \cdots \in \mathbb{Z}_p$ can be understood as the sequence $(2, 2 + 3p, 2 + 3p + 2p^2, \dots)$.

1.3.5. Colimits. More immediately relevant for us will be the dual (arrow-reversed version) of the notion of limit (or inverse limit). We just flip the arrows f_i in (1.3.1.1), and get the notion of a **colimit**, which is denoted by $\text{colim}_{\mathcal{I}} A_i$ (or $\varinjlim_{\mathcal{I}} A_i$). (You should draw the corresponding diagram.) Again, if it exists, it is unique up to unique isomorphism. (In some cases, the colimit is sometimes called the **direct limit**, **inductive limit**, or **injective limit**. We won’t use this language. I prefer using limit/colimit in analogy with kernel/cokernel and product/coproduct. This is more than analogy, as kernels and products may be interpreted as limits, and similarly with cokernels

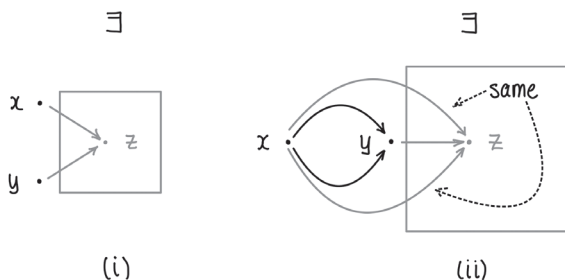


Figure 1.1 A filtered category (pictorial definition).

and coproducts. Also, I remember that kernels “map to,” and cokernels are “mapped to,” which reminds me that a limit maps *to* all the objects in the big commutative diagram indexed by $\mathcal{I}$; and a colimit has a map *from* all the objects.)

1.3.6. Joke. A comathematician is a device for turning cotheorems into ffee.

Even though we have just flipped the arrows, colimits behave quite differently from limits.

1.3.7. Example. The abelian group $5^{-\infty}\mathbb{Z}$ of rational numbers whose denominators are powers of 5 is a colimit $\text{colim}_{i \in \mathbb{Z}^+} 5^{-i}\mathbb{Z}$. More precisely, $5^{-\infty}\mathbb{Z}$ is the colimit of the diagram

$$\mathbb{Z} \longrightarrow 5^{-1}\mathbb{Z} \longrightarrow 5^{-2}\mathbb{Z} \longrightarrow \dots$$

in the category of abelian groups.

The colimit over an index *set* I is called the **coproduct**, denoted by $\coprod_i A_i$, and is the dual (arrow-reversed) notion to the product.

1.3.D. EXERCISE.

- Interpret the statement “ $\mathbb{Q} = \text{colim } \frac{1}{n}\mathbb{Z}$.”
- Interpret the union of some subsets of a given set as a colimit. (Dually, the intersection can be interpreted as a limit.) The objects of the category in question are the subsets of the given set.

Colimits do not always exist, but there are two useful large classes of examples for which they do.

1.3.8. Definition. A nonempty partially ordered set $(S, \geq)$ is **filtered** (or is said to be a **filtered set**) if for each $x, y \in S$, there is a z such that $x \geq z$ and $y \geq z$. More generally (see Figure 1.1), a nonempty category $\mathcal{I}$ is **filtered** if:

- for each $x, y \in \mathcal{I}$, there is a $z \in \mathcal{I}$ and arrows $x \rightarrow z$ and $y \rightarrow z$, and
- for every two arrows $u: x \rightarrow y$ and $v: x \rightarrow y$, there is an arrow $w: y \rightarrow z$ such that $w \circ u = w \circ v$.

(Other terminologies are also commonly used, such as “directed partially ordered set” and “filtered index category,” respectively.)

1.3.E. EXERCISE. Suppose $\mathcal{I}$ is filtered. (We will almost exclusively use the case where $\mathcal{I}$ is a filtered set.) Recall the symbol $\coprod$ for disjoint union of sets. Show that any diagram in *Sets* indexed by $\mathcal{I}$ has the following, with the obvious maps to it, as a colimit:

$$\left\{ (a_i, i) \in \coprod_{i \in \mathcal{I}} A_i \right\} / \left((a_i, i) \sim (a_j, j) \text{ if and only if there are } f: A_i \rightarrow A_k \text{ and } g: A_j \rightarrow A_k \text{ in the diagram for which } f(a_i) = g(a_j) \text{ in } A_k \right)$$

(You will see that the “filtered” hypothesis is there to ensure that $\sim$ is an equivalence relation.)

For example, in Example 1.3.7, each element of the colimit is an element of something upstairs, but you can't say in advance what it is an element of. For instance, $17/125$ is an element of $5^{-3}\mathbb{Z}$ (or $5^{-4}\mathbb{Z}$, or later ones), but not $5^{-2}\mathbb{Z}$.

This idea applies to many categories whose objects can be interpreted as sets with additional structure (such as abelian groups, A -modules, groups, etc.). For example, the colimit $\text{colim } M_i$ in the category of A -modules Mod_A can be described as follows. The set underlying $\text{colim } M_i$ is defined as in Exercise 1.3.E. To add the elements $m_i \in M_i$ and $m_j \in M_j$, choose an $\ell \in \mathcal{J}$ with arrows $u: i \rightarrow \ell$ and $v: j \rightarrow \ell$, and then define the sum of m_i and m_j to be $F(u)(m_i) + F(v)(m_j) \in M_\ell$. The element $m_i \in M_i$ is 0 if and only if there is some arrow $u: i \rightarrow k$ for which $F(u)(m_i) = 0$, i.e., if and only if it becomes 0 “later in the diagram.” Last, multiplication by an element of A is defined in the obvious way.

1.3.F. EXERCISE. Verify that the A -module described above is indeed the colimit. (Make sure you verify that addition is well-defined, i.e., is independent of the choice of representatives m_i and m_j , the choice of ℓ , and the choice of arrows u and v . Similarly, make sure that scalar multiplication is well-defined.)

1.3.G. USEFUL EXERCISE (LOCALIZATION AS A COLIMIT). Generalize Exercise 1.3.D(a) to interpret localization of an integral domain as a colimit over a filtered set: suppose S is a multiplicative set of A , and interpret $S^{-1}A = \text{colim}_s \frac{1}{s}A$ where the colimit is over $s \in S$, and in the category of A -modules. (Aside: Can you make some version of this work even if A isn't an integral domain, e.g., $S^{-1}A = \text{colim } A_s$? This will work in the category of A -algebras.)

A variant of this construction works without the filtered condition if you have another means of “connecting elements in different objects of your diagram.” For example:

1.3.H. EXERCISE: COLIMITS OF A -MODULES WITHOUT THE FILTERED CONDITION. Suppose you are given a diagram of A -modules indexed by $\mathcal{J}: \mathcal{J} \rightarrow \text{Mod}_A$, where we let $M_i := F(i)$. Show that the colimit is $\bigoplus_{i \in \mathcal{J}} M_i$ modulo the relations $m_i - F(n)(m_i)$ for every $n: i \rightarrow j$ in $\mathcal{J}$ (i.e., for every arrow in the diagram). (Somewhat more precisely: “modulo” means “quotiented by the submodule generated by.”)

1.3.9. Summary. One useful thing to informally keep in mind is the following. In a category where the objects are “set-like,” an element of a limit can be thought of as a family of elements of each object in the diagram that are “compatible” (Exercise 1.3.C). And an element of a colimit can be thought of as (“has a representative that is”) an element of a single object in the diagram (Exercise 1.3.E). Even though the definitions of limit and colimit are the same, just with arrows reversed, these interpretations are quite different.

1.3.10. Small remark. In fact, colimits exist in the category of sets for all reasonable (“small”) index categories (see for example [E, Thm. A6.1]), but that won't matter to us.

1.3.11. Joke. What do you call someone who reads a paper on category theory? Answer: A coauthor.

1.4 Adjoints

We next come to a very useful notion closely related to universal properties. Just as a universal property “essentially” (up to unique isomorphism) determines an object in a category (assuming such an object exists), “adjoints” essentially determine a functor (again, assuming it exists). Two *covariant* functors $F: \mathcal{A} \rightarrow \mathcal{B}$ and $G: \mathcal{B} \rightarrow \mathcal{A}$ are **adjoint** if there is a natural bijection for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$(1.4.0.1) \quad \tau_{AB}: \text{Mor}_{\mathcal{B}}(F(A), B) \rightarrow \text{Mor}_{\mathcal{A}}(A, G(B)).$$

We say that (F, G) form an **adjoint pair**, and that F is **left-adjoint** to G (and G is **right-adjoint** to F). We say F is a **left adjoint** (and G is a **right adjoint**). By “natural” we mean the following. For all

$f: A \rightarrow A'$ in $\mathcal{A}$, we require

$$(1.4.0.2) \quad \begin{array}{ccc} \text{Mor}_{\mathcal{B}}(F(A'), B) & \xrightarrow{Ff^*} & \text{Mor}_{\mathcal{B}}(F(A), B) \\ \downarrow \tau_{A'B} & & \downarrow \tau_{AB} \\ \text{Mor}_{\mathcal{A}}(A', G(B)) & \xrightarrow{f^*} & \text{Mor}_{\mathcal{A}}(A, G(B)) \end{array}$$

to commute, and for all $g: B \rightarrow B'$ in $\mathcal{B}$ we want a similar commutative diagram to commute. (Here f^* is the map induced by $f: A \rightarrow A'$, and Ff^* is the map induced by $Ff: F(A) \rightarrow F(A')$.)

1.4.A. EXERCISE. Write down what this diagram should be.

1.4.B. EXERCISE. Show that the map τ_{AB} (1.4.0.1) has the following properties. For each A there is a map $\eta_A: A \rightarrow GF(A)$ so that for any $g: F(A) \rightarrow B$, the corresponding $\tau_{AB}(g): A \rightarrow G(B)$ is given by the composition

$$A \xrightarrow{\eta_A} GF(A) \xrightarrow{Gg} G(B).$$

Similarly, there is a map $\epsilon_B: FG(B) \rightarrow B$ for each B so that for any $f: A \rightarrow G(B)$, the corresponding map $\tau_{AB}^{-1}(f): F(A) \rightarrow B$ is given by the composition

$$F(A) \xrightarrow{Ff} FG(B) \xrightarrow{\epsilon_B} B.$$

Here is a key example of an adjoint pair.

1.4.C. EXERCISE. Suppose M, N , and P are A -modules (where A is a ring). Describe a bijection $\text{Hom}_A(M \otimes_A N, P) \leftrightarrow \text{Hom}_A(M, \text{Hom}_A(N, P))$. (Hint: Try to use the universal property of $\otimes$.)

1.4.D. EXERCISE (TENSOR-HOM ADJUNCTION). Show that $(\cdot) \otimes_A N$ and $\text{Hom}_A(N, \cdot)$ are adjoint functors.

1.4.E. EXERCISE. Suppose $B \rightarrow A$ is a morphism of rings. If M is an A -module, you can create a B -module M_B by considering it as a B -module. This gives a functor $\cdot_B: \text{Mod}_A \rightarrow \text{Mod}_B$. Show that this functor is right-adjoint to $\cdot \otimes_B A$. In other words, describe a bijection

$$\text{Hom}_A(N \otimes_B A, M) \cong \text{Hom}_B(N, M_B)$$

functorial in both arguments. (This adjoint pair is very important.)

1.4.1.* Fancier remarks we won't use. You can check that the left adjoint determines the right adjoint up to natural isomorphism, and vice versa. The maps η_A and ϵ_B of Exercise 1.4.B are called the *unit* and *counit* of the adjunction. This leads to a different characterization of adjunction. Suppose functors $F: \mathcal{A} \rightarrow \mathcal{B}$ and $G: \mathcal{B} \rightarrow \mathcal{A}$ are given, along with natural transformations $\eta: \text{id}_{\mathcal{A}} \rightarrow GF$ and $\epsilon: FG \rightarrow \text{id}_{\mathcal{B}}$ with the property that $G\epsilon \circ \eta G = \text{id}_G$ (for each $B \in \mathcal{B}$, the composition of $\eta_{G(B)}: G(B) \rightarrow GFG(B)$ and $G(\epsilon_B): GFG(B) \rightarrow G(B)$ is the identity) and $\epsilon F \circ \eta = \text{id}_F$. Then you can check that F is left-adjoint to G . These facts aren't hard to check, so if you want to use them, you should verify everything for yourself.

1.4.2. Examples from other fields. For those familiar with representation theory: Frobenius reciprocity may be understood in terms of adjoints. Suppose V is a finite-dimensional representation of a finite group G , and W is a representation of a subgroup $H < G$. Then induction and restriction are an adjoint pair $(\text{Ind}_H^G, \text{Res}_H^G)$ between the category of G -modules and the category of H -modules.

Topologists' favorite adjoint pair may be the suspension functor and the loop space functor.

1.4.3. Example: Groupification of abelian semigroups. Here is another motivating example: getting an abelian group from an abelian semigroup. (An **abelian semigroup** is just like an

abelian group, except we don't require an identity or an inverse. Morphisms of abelian semigroups are maps of sets preserving the binary operation. One example is the nonnegative integers $\mathbb{Z}^{\geq 0} = \{0, 1, 2, \dots\}$ under addition. Another is the positive integers $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ under addition. Yet another is the positive integers $\mathbb{Z}^+$ under multiplication. You may enjoy groupifying all three.) From an abelian semigroup, you can create an abelian group. In our examples, from the nonnegative integers under addition ($\mathbb{Z}^{\geq 0}, +$), we create the integers ($\mathbb{Z}, +$), and from the positive integers under multiplication ($\mathbb{Z}^+, \times$), we create the positive rationals ($\mathbb{Q}^+, \times$). Here is a formalization of that notion. A **groupification** of an abelian semigroup S is a map of abelian semigroups $\pi: S \rightarrow G$ such that G is an abelian group, and any map of abelian semigroups from S to an abelian group G' factors *uniquely* through G :

$$\begin{array}{ccc} S & \xrightarrow{\pi} & G \\ & \searrow & \downarrow \exists! \\ & & G' \end{array}$$

(Perhaps “abelian groupification” would be more precise than “groupification.”)

1.4.F. EXERCISE (AN ABELIAN GROUP IS GROUPIFIED BY ITSELF). Show that if an abelian semigroup is *already* a group then the identity morphism is the groupification. (More correct: the identity morphism is *a* groupification.) Note that you don't need to construct groupification (or even know that it exists in general) to solve this exercise.

1.4.G. EXERCISE. Construct the “groupification functor” H from the category of nonempty abelian semigroups to the category of abelian groups. (One possible construction: given an abelian semigroup S , the elements of its groupification $H(S)$ are ordered pairs $(a, b) \in S \times S$, which you may think of as $a - b$, with the equivalence that $(a, b) \sim (c, d)$ if $a + d + e = b + c + e$ for some $e \in S$. Describe addition in this group, and show that it satisfies the properties of an abelian group. Describe the abelian semigroup map $S \rightarrow H(S)$.) Let F be the forgetful functor from the category of abelian groups Ab to the category of abelian semigroups. Show that H is left-adjoint to F .

(Here is the general idea for experts: We have a full subcategory of a category. We want to “project” from the category to the subcategory. We have

$$\text{Mor}_{\text{category}}(S, H) = \text{Mor}_{\text{subcategory}}(G, H)$$

automatically; thus we are describing the left adjoint to the forgetful functor. How the argument worked: we constructed something which was in the smaller category, which automatically satisfies the universal property.)

1.4.H. EXERCISE (CF. EXERCISE 1.4.E). The purpose of this exercise is to give you more practice with “left adjoints of forgetful functors,” the means by which we get abelian groups from abelian semigroups, and sheaves from presheaves. Suppose A is a ring, and S is a multiplicative subset. Then $S^{-1}A$ -modules are a full subcategory (§1.1.15) of the category of A -modules (via the obvious inclusion $\text{Mod}_{S^{-1}A} \hookrightarrow \text{Mod}_A$). Then $\text{Mod}_A \rightarrow \text{Mod}_{S^{-1}A}$ can be interpreted as an adjoint to the forgetful functor $\text{Mod}_{S^{-1}A} \rightarrow \text{Mod}_A$. State and prove the correct statements.

(Here is the larger story. Every $S^{-1}A$ -module is an A -module, and this is an injective map, so we have a forgetful functor $F: \text{Mod}_{S^{-1}A} \rightarrow \text{Mod}_A$. In fact this is a fully faithful functor: it is injective on objects, and the morphisms between any two $S^{-1}A$ -modules *as A -modules* are just the same when they are considered as $S^{-1}A$ -modules. Then there is a functor $G: \text{Mod}_A \rightarrow \text{Mod}_{S^{-1}A}$, which might reasonably be called “localization with respect to S ,” which is left-adjoint to the forgetful functor. Translation: If M is an A -module, and N is an $S^{-1}A$ -module, then $\text{Mor}(GM, N)$ (morphisms as $S^{-1}A$ -modules, which are the same as morphisms as A -modules) are in natural bijection with $\text{Mor}(M, FN)$ (morphisms as A -modules).)

situation	category $\mathcal{A}$	category $\mathcal{B}$	left adjoint $F: \mathcal{A} \rightarrow \mathcal{B}$	right adjoint $G: \mathcal{B} \rightarrow \mathcal{A}$
A-modules (Ex. 1.4.D)	Mod_A	Mod_A	$(\cdot) \otimes_A N$	$Hom_A(N, \cdot)$
ring maps $B \rightarrow A$ (Ex. 1.4.E)	Mod_B	Mod_A	$(\cdot) \otimes_B A$ (extension of scalars)	$M \mapsto M_B$ (restriction of scalars)
(pre)sheaves on a topological space X (Ex. 2.4.K)	presheaves on X	sheaves on X	sheafification	forgetful
(semi)groups (§1.4.3)	semigroups	groups	groupification	forgetful
sheaves, $\pi: X \rightarrow Y$ (Ex. 2.7.B)	sheaves on Y	sheaves on X	π^{-1}	π_*
sheaves of abelian groups or $\mathcal{O}$ -modules, open embeddings $\pi: U \hookrightarrow Y$ (Ex. 23.4.G)	sheaves on U	sheaves on Y	$\pi_!$	π^{-1}
quasicoherent sheaves, $\pi: X \rightarrow Y$ (Prop. 14.5.7)	$QCoh_Y$	$QCoh_X$	π^*	π_*
ring maps $B \rightarrow A$ (Ex. 17.1.J)	Mod_A	Mod_B	$M \mapsto M_B$ (restriction of scalars)	$N \mapsto$ $Hom_B(A, N)$
quasicoherent sheaves, affine $\pi: X \rightarrow Y$ (Ex. 17.1.K(b))	$QCoh_X$	$QCoh_Y$	π_*	$\pi^{! ?}$

Table 1.1 Some important adjoint pairs.

Table 1.1 gives most of the adjoints that will come up for us. Other examples will also come up, such as the adjoint pair $(\sim, \Gamma_\bullet)$ between graded modules over a graded ring, and quasicoherent sheaves on the corresponding projective scheme (§15.7).

1.4.4. *Last comments only for people who have seen adjoints before.* If (F, G) is an adjoint pair of functors, then F commutes with colimits, and G commutes with limits. Also, limits commute with limits and colimits commute with colimits. We will prove these facts (and a little more) in §1.5.14.

1.5 An Introduction to Abelian Categories

Ton papier sur l'Algèbre homologique a été lu soigneusement, et a converti tout le monde (même Dieudonné, qui semble complètement fonctorisé!) à ton point de vue.

Your paper on homological algebra was read carefully and converted everyone (even Dieudonné, who seems to be completely functorised!) to your point of view.

—J.-P. Serre, letter to A. Grothendieck, July 13, 1955 [GrS, pp. 17–18]

Since learning linear algebra, you have been familiar with the notions and behaviors of kernels, cokernels, etc. Later in your life you saw them in the category of abelian groups, and later still in the category of A -modules.

We will soon define some new categories (certain sheaves) that will have familiar-looking behavior, reminiscent of that of modules over a ring. The notions of kernels, cokernels, images,

and more will make sense, and they will behave “the way we expect” from our experience with modules. This can be made precise through the notion of an *abelian category*. Abelian categories are the right general setting in which one can do “homological algebra,” in which notions of kernel, cokernel, and so on are used, and one can work with complexes and exact sequences.

We will see enough to motivate the definitions that we will see in general: monomorphism (and subobject), epimorphism, kernel, cokernel, and image. But in this book we will avoid showing that they behave “the way we expect” in a general abelian category because the examples we will see are directly interpretable in terms of modules over rings. In particular, it is not worth memorizing the definition of abelian category.

Two central examples of an abelian category are the category Ab of abelian groups, and the category Mod_A of A -modules. The first is a special case of the second (just take $A = \mathbb{Z}$). As we give the definitions, you should verify that Mod_A is an abelian category.

We first define the notion of *additive category*. We will use it only as a stepping stone to the notion of an abelian category. Two examples you can keep in mind while reading the definition: the category of free A -modules (where A is a ring), and real (or complex) Banach spaces.

1.5.1. Definition. A category $\mathcal{C}$ is said to be **additive** if it satisfies the following properties.

- Ad1. For each $A, B \in \mathcal{C}$, $\text{Mor}(A, B)$ is an abelian group, such that composition of morphisms distributes over addition. (You should think about what this means—it translates to two distinct statements.)
- Ad2. $\mathcal{C}$ has a zero object, denoted by 0 . (This is an object that is simultaneously an initial object and a final object, Definition 1.2.2.)
- Ad3. It has products of two objects (a product $A \times B$ for any pair of objects), and hence, by induction, products of any finite number of objects.

In an additive category, the morphisms are often called **homomorphisms**, and Mor is denoted by Hom . In fact, the notation Hom is a good indication that you’re working in an additive category. A functor between additive categories preserving the additive structure of Hom is called an **additive functor**.

1.5.2. Remarks. It is a consequence of the definition of additive category that finite products are also finite coproducts (i.e., sums)—the details don’t matter to us. The symbol $\oplus$ is used for this notion. Also, it is quick to show that additive functors send zero objects to zero objects (show that Z is a 0-object if and only if $\text{id}_Z = 0_Z$; additive functors preserve both id and 0), and preserve products.

One motivation for the name 0-object is that the 0-morphism in the abelian group $\text{Hom}(A, B)$ is the composition $A \rightarrow 0 \rightarrow B$. (We also remark that the notion of 0-morphism thus makes sense in any category with a 0-object.)

(A cleaner axiomatization of additive categories that makes clear that the abelian group structure of $\text{Mor}(A, B)$ is intrinsic to the category itself is the following [Lur, pp. 21–22]. A0. $\mathcal{C}$ has a zero object. A1. $\mathcal{C}$ has products of any two objects, and coproducts of any two objects. By the universal property of product and coproduct, we have natural morphisms $\phi_{AB} : A \coprod B \rightarrow A \times B$. A2. ϕ_{AB} is an isomorphism. This allows us to define a binary operation on $\text{Mor}(A, B)$, with $f + g$ (for $f, g \in \text{Mor}(A, B)$) defined by the composition

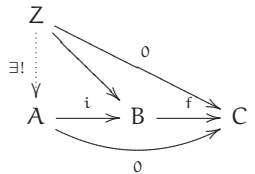
$$A \xrightarrow{(f, g)} B \times B \xrightarrow{\phi_{BB}^{-1}} B \coprod B \longrightarrow B,$$

where the last map is the “codiagonal” defined by the universal property of coproduct. A little work shows that this endows $\text{Mor}(A, B)$ with the structure of a **commutative monoid**, i.e., an abelian semigroup with identity. The identity is the composition $A \rightarrow 0 \rightarrow B$. A3. *This commutative monoid $\text{Mor}(A, B)$ is an abelian group.*

1.5.3. The category of A -modules Mod_A is clearly an additive category, but it has even more structure. We now formalize some essential aspects of this structure in the notion of *abelian category*.

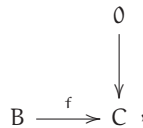
1.5.4. Definition. Let $\mathcal{C}$ be a category with a 0-object (and thus 0-morphisms). A **kernel** of a morphism $f: B \rightarrow C$ is *defined to be* a map $i: A \rightarrow B$ such that $f \circ i = 0$, and that is universal with respect to this property. Diagrammatically:

(1.5.4.1)



(Note that the kernel is not just an object; it is a morphism of an object to B . In practice, the term is often applied to just the object, and the intended interpretation is clear from the context.) Hence it is unique up to unique isomorphism by universal property nonsense. The kernel is written $\ker f \rightarrow B$. A **cokernel** (denoted by $\text{coker } f$) is defined dually by reversing the arrows—do this yourself. The kernel of $f: B \rightarrow C$ is the limit (§1.3) of the diagram

(1.5.4.2)



and similarly the cokernel is a colimit (see (2.6.0.1)).

If $i: A \rightarrow B$ is a monomorphism, then we say that A is a **subobject** of B , where the map i is implicit. There is also the notion of **quotient object**, defined dually to subobject.

An **abelian category** is an additive category satisfying three additional properties.

- (1) Every map has a kernel and cokernel.
- (2) Every monomorphism is the kernel of its cokernel.
- (3) Every epimorphism is the cokernel of its kernel.

It is a nonobvious (and imprecisely stated) fact that every property you want to be true about kernels, cokernels, etc. follows from these three. (Warning: In part of the literature, additional hypotheses are imposed as part of the definition.)

The **image** of a morphism $f: A \rightarrow B$ is defined as $\text{im}(f) = \ker(\text{coker } f)$ whenever it exists (e.g., in every abelian category). The morphism $f: A \rightarrow B$ factors uniquely through $\text{im } f \rightarrow B$ whenever $\text{im } f$ exists, and $A \rightarrow \text{im } f$ is an epimorphism and a cokernel of $\ker f \rightarrow A$ in every abelian category. The reader may want to verify this as a (hard!) exercise.

The cokernel of a monomorphism is called the **quotient**. The quotient of a monomorphism $A \rightarrow B$ is often denoted by B/A (with the map from B implicit).

We will leave the foundations of abelian categories untouched. The key thing to remember is that if you understand kernels, cokernels, images and so on in the category of modules over a given ring, you can manipulate objects in any abelian category. This is made precise by the Freyd–Mitchell Embedding Theorem (Remark 1.5.5).

However, the abelian categories we will come across will obviously be related to modules, and our intuition will clearly carry over, so we needn't invoke a theorem whose proof we haven't read. For example, we will show that sheaves of abelian groups on a topological space X form an abelian category (§2.6), and the interpretation in terms of “compatible germs” will connect notions of kernels, cokernels etc. of sheaves of abelian groups to the corresponding notions of abelian groups.

1.5.5. Small remark on chasing diagrams. It is useful to prove facts (and solve exercises) about abelian categories by chasing elements. Unfortunately, some commonly used abelian categories,

such as the category of complexes (to be defined in Exercise 1.5.C), do not have “elements”—they are not naturally “sets with additional structure” in any obvious way. Nonetheless, proof by element-chasing can be justified by the Freyd–Mitchell Embedding Theorem: If $\mathcal{C}$ is an abelian category whose objects form a set, then there is a ring A and an exact, fully faithful functor from $\mathcal{C}$ into Mod_A , which embeds $\mathcal{C}$ as a full subcategory. (Unfortunately, the ring A need not be commutative.) A proof is sketched in [Weib, §1.6], and references to a complete proof are given there. A proof is also given in [KS2, §9.6]. The upshot is that to prove something about a diagram in some abelian category, we may assume that it is a diagram of modules over some ring, and we may then “diagram-chase” elements. Moreover, any fact about kernels, cokernels, and so on that holds in Mod_A holds in any abelian category.

If invoking a theorem whose proof you haven’t read bothers you, a short alternative is Mac Lane’s “elementary rules for chasing diagrams,” [Mac, Thm. 3, p. 204]; [Mac, Lem. 4, p. 205] gives a proof of the Five Lemma (Exercise 1.6.6) as an example.

But in any case, do what you need to do to put your mind at ease, so you can move forward. Do as little as your conscience will allow.

1.5.6. Complexes, exactness, and homology.

(In this entire discussion, we assume we are working in an abelian category.) We say a sequence

$$(1.5.6.1) \quad \cdots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \cdots$$

is a **complex at B** if $g \circ f = 0$, and is **exact at B** if $\ker g = \text{im } f$. (More specifically, g has a kernel that is an image of f . Exactness at B implies being a complex at B —do you see why?) A sequence is a **complex** (resp., **exact**) if it is a complex (resp., exact) at each (internal) term. A **short exact sequence** is an exact sequence with five terms, the first and last of which are zeros—in other words, an exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

For example, $0 \longrightarrow A \longrightarrow 0$ is exact if and only if $A = 0$;

$$0 \longrightarrow A \xrightarrow{f} B$$

is exact if and only if f is a monomorphism (with a similar statement for $A \xrightarrow{f} B \longrightarrow 0$);

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

is exact if and only if f is an isomorphism; and

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if and only if f is a kernel of g (with a similar statement for $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$). To show some of these facts it may be helpful to prove that (1.5.6.1) is exact at B if and only if the cokernel of f is a cokernel of the kernel of g .

If you would like practice in playing with these notions before thinking about homology, you can prove the Snake Lemma (stated in Example 1.6.5, with a stronger version in Exercise 1.6.B), or the Five Lemma (stated in Example 1.6.6, with a stronger version in Exercise 1.6.C). (I would do this in the category of A -modules, but see [KS2, Lem. 12.1.1, Lem. 8.3.13] for proofs in general.)

If (1.5.6.1) is a complex at B , then its **homology at B** (often denoted by H) is $\ker g / \text{im } f$. (More precisely, there is some monomorphism $\text{im } f \hookrightarrow \ker g$, and H is the cokernel of this monomorphism.) Therefore, (1.5.6.1) is exact at B if and only if its homology at B is 0. We say that elements of $\ker g$ (assuming the objects of the category are sets with some additional structure) are the **cycles**, and elements of $\text{im } f$ are the **boundaries** (so homology is “cycles mod boundaries”). If the complex is indexed in decreasing order, the indices are often written as subscripts, and H_i is the

homology at $A_{i+1} \rightarrow A_i \rightarrow A_{i-1}$. If the complex is indexed in increasing order, the indices are often written as superscripts, and the homology H^i at $A^{i-1} \rightarrow A^i \rightarrow A^{i+1}$ is often called **cohomology**.

An exact sequence

$$(1.5.6.2) \quad A^\bullet: \quad \dots \longrightarrow A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} \dots$$

can be “factored” into short exact sequences

$$0 \longrightarrow \ker f^i \longrightarrow A^i \longrightarrow \ker f^{i+1} \longrightarrow 0,$$

which is helpful in proving facts about long exact sequences by reducing them to facts about short exact sequences.

More generally, if (1.5.6.2) is assumed only to be a complex, then it can be “factored” into short exact sequences.

$$(1.5.6.3) \quad \begin{aligned} 0 &\longrightarrow \ker f^i \longrightarrow A^i \longrightarrow \operatorname{im} f^i \longrightarrow 0, \\ 0 &\longrightarrow \operatorname{im} f^{i-1} \longrightarrow \ker f^i \longrightarrow H^i(A^\bullet) \longrightarrow 0 \end{aligned}$$

1.5.A. EXERCISE. Describe exact sequences

$$(1.5.6.4) \quad \begin{aligned} 0 &\longrightarrow \operatorname{im} f^i \longrightarrow A^{i+1} \longrightarrow \operatorname{coker} f^i \longrightarrow 0, \\ 0 &\longrightarrow H^i(A^\bullet) \longrightarrow \operatorname{coker} f^{i-1} \longrightarrow \operatorname{im} f^i \longrightarrow 0 \end{aligned}$$

(These are somehow dual to (1.5.6.3). In fact, in some mirror universe this might have been given as the standard definition of homology.) Assume the category is that of modules over a fixed ring for convenience, but be aware that the result is true for any abelian category.

1.5.B. EXERCISE AND IMPORTANT DEFINITION. Suppose

$$0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} 0$$

is a complex of finite-dimensional k -vector spaces (often called $A^\bullet$ for short). Define $h^i(A^\bullet) := \dim H^i(A^\bullet)$. Show that $\sum (-1)^i \dim A^i = \sum (-1)^i h^i(A^\bullet)$. In particular, if $A^\bullet$ is exact, then $\sum (-1)^i \dim A^i = 0$. (If you haven’t dealt much with cohomology, this will give you some practice.)

1.5.C. IMPORTANT EXERCISE. Suppose $\mathcal{C}$ is an abelian category. Define the **category $\operatorname{Com}_{\mathcal{C}}$ of complexes** as follows. The objects are infinite complexes

$$A^\bullet: \quad \dots \longrightarrow A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} \dots$$

in $\mathcal{C}$, and the morphisms $A^\bullet \rightarrow B^\bullet$ are commuting diagrams

$$(1.5.6.5) \quad \begin{array}{ccccccc} \dots & \longrightarrow & A^{i-1} & \xrightarrow{f^{i-1}} & A^i & \xrightarrow{f^i} & A^{i+1} \xrightarrow{f^{i+1}} \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & B^{i-1} & \xrightarrow{g^{i-1}} & B^i & \xrightarrow{g^i} & B^{i+1} \xrightarrow{g^{i+1}} \dots \end{array}$$

Show that $\operatorname{Com}_{\mathcal{C}}$ is an abelian category.

Feel free to deal with the special case of modules over a fixed ring. (Remark for experts: Essentially the same argument shows that $\mathcal{C}^{\mathcal{J}}$ is an abelian category for any small category $\mathcal{J}$ and any abelian category $\mathcal{C}$. This immediately implies that the category of presheaves on a topological space X with values in an abelian category $\mathcal{C}$ is automatically an abelian category; cf. §2.3.5.)

1.5.D. IMPORTANT EXERCISE. Show that (1.5.6.5) induces a map of homology $H^i(A^\bullet) \rightarrow H^i(B^\bullet)$. Show furthermore that H^i is a covariant functor $Com_{\mathcal{C}} \rightarrow \mathcal{C}$. (Again, feel free to deal with the special case Mod_A .)

1.5.7. Homotopic maps induce the same maps on homology. We say two maps of complexes $f: C^\bullet \rightarrow D^\bullet$ and $g: C^\bullet \rightarrow D^\bullet$ are **homotopic** if there is a sequence of maps $w: C^i \rightarrow D^{i-1}$ such that $f - g = dw + wd$.

1.5.E. EXERCISE. Show that two homotopic maps give the same map on homology.

1.5.8. Theorem (long exact sequences) — *A short exact sequence of complexes*

$$\begin{array}{ccccccc}
 0^\bullet & : & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
 A^\bullet & : & \cdots & \longrightarrow & A^{i-1} & \xrightarrow{f^{i-1}} & A^i & \xrightarrow{f^i} & A^{i+1} & \xrightarrow{f^{i+1}} & \cdots \\
 \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
 B^\bullet & : & \cdots & \longrightarrow & B^{i-1} & \xrightarrow{g^{i-1}} & B^i & \xrightarrow{g^i} & B^{i+1} & \xrightarrow{g^{i+1}} & \cdots \\
 \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
 C^\bullet & : & \cdots & \longrightarrow & C^{i-1} & \xrightarrow{h^{i-1}} & C^i & \xrightarrow{h^i} & C^{i+1} & \xrightarrow{h^{i+1}} & \cdots \\
 \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
 0^\bullet & : & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
 \end{array}$$

induces a long exact sequence in cohomology

$$\begin{array}{ccccccc}
 & & & & \cdots & \longrightarrow & H^{i-1}(C^\bullet) & \longrightarrow \\
 & & & & & & & \\
 H^i(A^\bullet) & \longrightarrow & H^i(B^\bullet) & \longrightarrow & H^i(C^\bullet) & \longrightarrow & & \\
 & & & & & & & \\
 H^{i+1}(A^\bullet) & \longrightarrow & \cdots & & & & &
 \end{array}$$

(This requires a definition of the **connecting homomorphism** $H^{i-1}(C^\bullet) \rightarrow H^i(A^\bullet)$, which is “natural” in an appropriate sense.) In the category of modules over a ring, Theorem 1.5.8 will come out of our discussion of spectral sequences—see Exercise 1.6.F—but this is a somewhat perverse way of proving it. For a proof in general, see [KS2, Theorem 12.3.3]. You may want to prove it yourself, by first proving a weaker version of the Snake Lemma (Example 1.6.5), where in the hypotheses (1.6.5.1), the 0’s in the bottom left and top right are removed, and in the conclusion (1.6.5.2), the first and last 0’s are removed.

1.5.9. Exactness of functors. If $F: \mathcal{A} \rightarrow \mathcal{B}$ is an additive covariant functor from one abelian category to another, we say that F is **right-exact** if the exactness of

$$A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

in $\mathcal{A}$ implies that

$$F(A') \longrightarrow F(A) \longrightarrow F(A'') \longrightarrow 0$$

is also exact. Dually, we say that F is **left-exact** if the exactness of

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \quad \text{implies}$$

$$0 \longrightarrow F(A') \longrightarrow F(A) \longrightarrow F(A'') \quad \text{is exact.}$$

An additive contravariant functor is **left-exact** if the exactness of

$$A' \longrightarrow A \longrightarrow A'' \longrightarrow 0 \quad \text{implies}$$

$$0 \longrightarrow F(A'') \longrightarrow F(A) \longrightarrow F(A') \quad \text{is exact.}$$

The reader should be able to deduce what it means for a contravariant functor to be **right-exact**.

An additive covariant or contravariant functor is **exact** if it is both left-exact and right-exact.

1.5.F. EXERCISE. Suppose F is an exact functor. Show that applying F to an exact sequence preserves exactness. For example, if F is covariant, and $A' \rightarrow A \rightarrow A''$ is exact, then $FA' \rightarrow FA \rightarrow FA''$ is exact. (This will be generalized in Exercise 1.5.I(c).)

1.5.G. EXERCISE. Suppose A is a ring, $S \subset A$ is a multiplicative subset, and M is an A -module.

- Show that localization of A -modules $Mod_A \rightarrow Mod_{S^{-1}A}$ is an exact covariant functor.
- Show that $(\cdot) \otimes_A M$ is a right-exact covariant functor $Mod_A \rightarrow Mod_A$. (This is a repeat of Exercise 1.2.H.)
- Show that $\text{Hom}(M, \cdot)$ is a left-exact covariant functor $Mod_A \rightarrow Mod_A$. If $\mathcal{C}$ is any abelian category, and $C \in \mathcal{C}$, show that $\text{Hom}(C, \cdot)$ is a left-exact covariant functor $\mathcal{C} \rightarrow Ab$.
- Show that $\text{Hom}(\cdot, M)$ is a left-exact contravariant functor $Mod_A \rightarrow Mod_A$. If $\mathcal{C}$ is any abelian category, and $C \in \mathcal{C}$, show that $\text{Hom}(\cdot, C)$ is a left-exact contravariant functor $\mathcal{C} \rightarrow Ab$.

1.5.H. EXERCISE. Suppose M is a **finitely presented A -module**: M has a finite number of generators, and with these generators it has a finite number of relations; or, equivalently, and usefully fits in an exact sequence

$$(1.5.9.1) \quad A^{\oplus q} \longrightarrow A^{\oplus p} \longrightarrow M \longrightarrow 0.$$

Use (1.5.9.1) and the left-exactness of Hom to describe an isomorphism

$$S^{-1} \text{Hom}_A(M, N) \xleftarrow{\sim} \text{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

(You might be able to interpret this in light of a variant of Exercise 1.5.I below, for left-exact contravariant functors rather than right-exact covariant functors.)

1.5.10. Example: *Hom doesn't always commute with localization.* In the language of Exercise 1.5.H, take $A = N = \mathbb{Z}$, $M = \mathbb{Q}$, and $S = \mathbb{Z} \setminus \{0\}$.

1.5.11.* Two useful facts in homological algebra.

We now come to two (sets of) facts I wish I had learned as a child, as they would have saved me lots of grief. They encapsulate what is best and worst of abstract nonsense. The statements are so general as to be nonintuitive. The proofs are very short. They generalize some specific behavior that is easy to prove on an ad hoc basis. Once they are second nature to you, many subtle facts will become obvious to you as special cases. And you will see that they will get used (implicitly or explicitly) repeatedly.

1.5.12.* Interaction of homology and (right/left-)exact functors.

You might wait to prove this until you learn about cohomology in Chapter 18, when it will first be used in a serious way.

1.5.I. IMPORTANT EXERCISE (THE FHHF THEOREM). This result can take you far, and perhaps for that reason it has sometimes been called the Fernbahnhof (FernbaHnHoF) Theorem, notably in [Vak1, Exer. 1.5.I]. Suppose $F: \mathcal{A} \rightarrow \mathcal{B}$ is a covariant functor of abelian categories, and $C^\bullet$ is a complex in $\mathcal{A}$.

- (F right-exact yields $FH^\bullet \longrightarrow H^\bullet F$) If F is right-exact, describe a natural morphism $FH^\bullet \rightarrow H^\bullet F$. (More precisely, for each i , the left side is F applied to the cohomology at piece i of $C^\bullet$, while the right side is the cohomology at piece i of $FC^\bullet$.)

- (b) (F left-exact yields $FH^\bullet \longleftarrow H^\bullet F$) If F is left-exact, describe a natural morphism $H^\bullet F \rightarrow FH^\bullet$.
- (c) (F exact yields $FH^\bullet \xrightarrow{\sim} H^\bullet F$) If F is exact, show that the morphisms of (a) and (b) are inverses and thus isomorphisms.

Hint for (a): Use $C^i \xrightarrow{d^i} C^{i+1} \longrightarrow \text{coker } d^i \longrightarrow 0$ to give an isomorphism $F \text{coker } d^i \xrightarrow{\sim} \text{coker } Fd^i$. Then use the first line of (1.5.6.4) to give an epimorphism $F \text{im } d^i \twoheadrightarrow \text{im } Fd^i$. Then use the second line of (1.5.6.4) to give the desired map $FH^i C^\bullet \rightarrow H^i F C^\bullet$. While you are at it, you may as well describe a map for the fourth member of the quartet $\{\text{coker}, \text{im}, H, \ker\}$: $F \ker d^i \rightarrow \ker Fd^i$.

1.5.13. If this makes your head spin, you may prefer to think of it in the following specific case, where both $\mathcal{A}$ and $\mathcal{B}$ are the category of A -modules, and F is $(\cdot) \otimes_A N$ for some fixed A -module N . Your argument in this case will translate without change to yield a solution to Exercise 1.5.I(a) and (c) in general. If $\otimes N$ is exact, then N is called a **flat** A -module. (The notion of flatness will turn out to be very important, and is discussed in detail in Chapter 24.)

For example, localization is exact (Exercise 1.5.G(a)), so $S^{-1}A$ is a *flat* A -algebra for all multiplicative sets S . Thus taking cohomology of a complex of A -modules commutes with localization—something you could verify directly.

1.5.14. *Interaction of adjoints, (co)limits, and (left- and right-) exactness.*

A surprising number of arguments boil down to the following statement:

Limits commute with limits and right adjoints. In particular, in an abelian category, because kernels are limits, both limits and right adjoints are left-exact.

And to its dual:

Colimits commute with colimits and left adjoints. In particular, because cokernels are colimits, both colimits and left adjoints are right-exact.

These statements were promised in §1.4.4, and will be proved below. The latter has a useful extension:

In Mod_A , colimits over filtered index categories are exact. “Filtered” was defined in §1.3.8.

1.5.15. Caution.** It is not true that in abelian categories in general, colimits over filtered index categories are exact. (Grothendieck realized the desirability of such colimits being exact, and formalized this as his “AB5” axiom; see, for example, [Stacks, tag 079A].) Here is a counterexample. Because the axioms of abelian categories are self-dual, it suffices to give an example in which a *cofiltered limit* fails to be exact (where **cofiltered** has the obvious dual definition to *filtered*), and we do this. Fix a prime p . In the category Ab of abelian groups, for each positive integer n , we have an exact sequence $\mathbb{Z} \rightarrow \mathbb{Z}/(p^n) \rightarrow 0$. Taking the limit over all n in the obvious way, we obtain $\mathbb{Z} \rightarrow \mathbb{Z}_p \rightarrow 0$, which is certainly not exact.)

Unimportant Remark 1.5.18 will dash another hope you may have.

1.5.16. If you want to use these statements (for example, later in this book), you will have to prove them. Let’s now make them precise.

1.5.J. EXERCISE (KERNELS COMMUTE WITH LIMITS). Suppose $\mathcal{C}$ is an abelian category, and $\alpha: \mathcal{I} \rightarrow \mathcal{C}$ and $\beta: \mathcal{I} \rightarrow \mathcal{C}$ are two diagrams in $\mathcal{C}$ indexed by $\mathcal{I}$. For convenience, let $A_i = \alpha(i)$ and $B_i = \beta(i)$ be the objects in those two diagrams. Let $h_i: A_i \rightarrow B_i$ be maps commuting with the maps in the diagrams. (Translation: h is a natural transformation of functors $\alpha \rightarrow \beta$; see §1.1.21.) Then the $\ker h_i$ form another diagram in $\mathcal{C}$ indexed by $\mathcal{I}$. Describe a canonical isomorphism $\lim \ker h_i \xrightarrow{\sim} \ker(\lim A_i \rightarrow \lim B_i)$, assuming the limits exist.

Implicit in the previous exercise is the idea that limits should somehow be understood as functors.

1.5.K. EXERCISE. Make sense of the statement that “limits commute with limits” in a general category, and prove it. (Hint: Recall that kernels are limits. The previous exercise should be a corollary of this one.)

1.5.17. Proposition (right adjoints commute with limits) — Suppose $(F: \mathcal{C} \rightarrow \mathcal{D}, G: \mathcal{D} \rightarrow \mathcal{C})$ is a pair of adjoint functors. If $A = \lim A_i$ is a limit in $\mathcal{D}$ of a diagram indexed by $\mathcal{I}$, then $GA = \lim GA_i$ (with the corresponding maps $GA \rightarrow GA_i$) is a limit in $\mathcal{C}$.

Proof. We must show that $GA \rightarrow GA_i$ satisfies the universal property of limits. Suppose we have maps $W \rightarrow GA_i$ commuting with the maps of $\mathcal{I}$. We wish to show that there exists a unique $W \rightarrow GA$ extending the $W \rightarrow GA_i$. By adjointness of F and G , we can restate this as: Suppose we have maps $FW \rightarrow A_i$ commuting with the maps of $\mathcal{I}$. We wish to show that there exists a unique $FW \rightarrow A$ extending the $FW \rightarrow A_i$. But this is precisely the universal property of the limit. $\square$

Of course, the dual statements to Exercise 1.5.K and Proposition 1.5.17 hold by the dual arguments.

If F and G are additive functors between abelian categories, and (F, G) is an adjoint pair, then (as kernels are limits and cokernels are colimits) G is left-exact and F is right-exact.

1.5.L. EXERCISE. Show that in Mod_A , colimits over filtered index categories are exact. (Your argument will apply without change to any abelian category whose objects can be interpreted as “sets with additional structure.”) Right-exactness follows from the above discussion, so the issue is left-exactness. (Possible hint: After you show that localization is exact, Exercise 1.5.G(a), or stalkification is exact, Exercise 2.6.E, in a hands-on way, you will be easily able to prove this. Conversely, if you do this exercise, those two will be easy.)

1.5.M. EXERCISE. Show that filtered colimits commute with homology in Mod_A . Hint: Use the FHHF Theorem (Exercise 1.5.I), and the previous exercise.

In light of Exercise 1.5.M, you may want to think about how limits (and colimits) commute with homology in general, and which way maps go. The statement of the FHHF Theorem should suggest the answer. (Are limits analogous to left-exact functors, or right-exact functors?) We won’t directly use this insight, but see §18.1 (vii) for an example.

Just as colimits are exact (not just right-exact) in especially good circumstances, limits are exact (not just left-exact), too. The following will be used twice in Chapter 28.

1.5.N. EXERCISE. Suppose

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & A_{n+1} & \longrightarrow & B_{n+1} & \longrightarrow & C_{n+1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A_0 & \longrightarrow & B_0 & \longrightarrow & C_0 \longrightarrow 0
 \end{array}$$

is an inverse system of exact sequences of modules over a ring, such that the maps $A_{n+1} \rightarrow A_n$ are surjective. (We say: “transition maps of the left term are surjective.”) Show that the limit

$$(1.5.17.1) \quad 0 \longrightarrow \lim A_n \longrightarrow \lim B_n \longrightarrow \lim C_n \longrightarrow 0$$

is also exact. (You will need to define the maps in (1.5.17.1).)

1.5.18. Unimportant remark. Based on these ideas, you may suspect that right-exact functors always commute with colimits. The fact that the tensor product commutes with infinite direct sums (Exercise 1.2.M) may reinforce this idea. Unfortunately, it is not true—“double dual” $\cdot^{\vee\vee} : \text{Vec}_k \rightarrow \text{Vec}_k$ is covariant and right exact (in fact, exact), but does not commute with infinite direct sums, as $\bigoplus_{i=1}^{\infty} (k^{\vee\vee})$ is not isomorphic to $(\bigoplus_{i=1}^{\infty} k)^{\vee\vee}$.

1.5.19.* Dreaming of derived functors. When you see a left-exact functor, you should always dream that you are seeing the end of a long exact sequence. If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence in abelian category $\mathcal{A}$, and $F: \mathcal{A} \rightarrow \mathcal{B}$ is a left-exact functor, then

$$0 \longrightarrow FM' \longrightarrow FM \longrightarrow FM''$$

is exact, and you should always dream that it continues in some natural way. For example, the next term should depend only on M' —call it R^1FM' —and if it is zero, then $FM \rightarrow FM''$ is an epimorphism. This remark holds true for left-exact and contravariant functors too. In good cases, such a continuation exists, and is incredibly useful. We will discuss this in Chapter 23.

1.6* Spectral Sequences

Je suis quelque peu affolé par ce déluge de cohomologie, mais j'ai courageusement tenu le coup. Ta suite spectrale me paraît raisonnable (je croyais, sur un cas particulier, l'avoir mise en défaut, mais je m'étais trompé, et cela marche au contraire admirablement bien).

I am a bit panic-stricken by this flood of cohomology, but have borne up courageously. Your spectral sequence seems reasonable to me (I thought I had shown that it was wrong in a special case, but I was mistaken, on the contrary it works remarkably well).

—J.-P. Serre, letter to A. Grothendieck, March 14, 1956 [GrS, p. 38]

Spectral sequences are a powerful bookkeeping tool for proving things involving complicated commutative diagrams. They were introduced by Leray in the 1940s at the same time as he introduced sheaves. They have a reputation for being abstruse and difficult. It has been suggested that the name ‘spectral’ was given because, like specters, spectral sequences are terrifying, evil, and dangerous. I have heard no one disagree with this interpretation, which is perhaps not surprising since I just made it up.

Nonetheless, the goal of this section is to tell you enough that you can use spectral sequences without hesitation or fear, and why you shouldn't be frightened when they come up in a seminar. What is perhaps different in this presentation is that we will use spectral sequences to prove things that you may have already seen, and that you can prove easily in other ways. This will allow you to get some hands-on experience in how to use them. We will also see them only in the special case of double complexes (the version by far the most often used in algebraic geometry), and not in the general form usually presented (filtered complexes, exact couples, etc.). See [Weib, Ch. 5] for more detailed information if you wish.

You should *not* read this section when you are reading the rest of Chapter 1. Instead, you should read it just before you need it for the first time. When you finally *do* read this section, you *must* do the exercises up to Exercise 1.6.F.

(continued...)

Index

Page numbers in boldface indicate definitions.

- (-1) -curve, **448**, 506, 590, 595, 597, 599, 606, 612, 614, *see also* Castelnuovo's Criterion; blowing down
- $A(I)_\bullet$ (Rees algebra), **299**
- $\hat{A}$ (completion), **599**
- A -bilinear, **12**
- C^{reg} , **435**
- $\mathcal{F} \otimes \mathcal{G}$ for quasicohherent sheaves, **125**
- $\mathcal{F}(U)$ (section of sheaf $\mathcal{F}$ over U), **43**
- $\mathcal{F}(m)$ ("twist" of $\mathcal{F}$, $= \mathcal{F} \otimes \mathcal{O}(m)$), **328**
- $\mathcal{F}|_U$ (restriction of sheaf to open subset), **45**
- $\mathcal{F}^{\text{sh}}$ (sheafification of presheaf $\mathcal{F}$), **52**
- $\mathcal{F}^\vee$ (dual of $\mathcal{O}_X$ -module $\mathcal{F}$), **48**
- $\mathcal{F}_p$ (stalk of (pre)sheaf $\mathcal{F}$), **44**
- $\mathcal{F}_{\text{lf}}$ (torsion-free quotient of $\mathcal{F}$), **313**
- $\mathcal{F}_{\text{tors}}$ (torsion subsheaf of $\mathcal{F}$), **313**
- $f|_U = \text{res}_{V|U}(f)$ (restriction of f to U), **43**
- $\mathcal{G}|_X$ (pullback of $\mathcal{G}$ to X), **317**
- $\sqrt{I}$ (radical of ideal I), **78**
- i_p (inclusion of point p), **46**
- $i_{p,*}$ (pushforward along i_p), **45**
- $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F})$ (intersection of $\mathcal{L}_1, \dots, \mathcal{L}_n$ with $\mathcal{F}$), **443**
- $|\mathcal{L}|$ (complete linear series), **329**
- $(M : x)$, **537**
- $M(I)_\bullet$ ($A(I)_\bullet$ -module), **299**
- $M(m)_\bullet$ ("shift" or "twist" of $M_\bullet$ by m), **327**
- $M_\bullet$ ($\mathbb{Z}$ -graded module), **322**
- M_{tors} (torsion submodule of M), **123**
- $M_\bullet$ (quasicohherent sheaf corresponding to graded module $M_\bullet$), **322**
- in terms of "compatible germs", **322**
- is an exact functor, **322**
- may not be an isomorphism, **322**
- $\tilde{M}$ (quasicohherent sheaf corresponding to a module M), **91**
- $[p]$ (point of $\text{Spec } A$ corresponding to p), **67**
- $p_a(X)$ (arithmetic genus), **392**
- $P_g(X)$ (geometric genus), **482**
- $S^{-1}A$ (localization of ring), **10**
- $S^{-1}M$ (localization of module), **11**
- $\underline{S}$ (constant sheaf), **45**
- $s \otimes t$ (tensor of sections), **125**, **305**
- Y^{red} (reduction of a scheme), **199**, **232**
- Δ (diagonal locally closed subscheme of $X \times_Y X$), **236**
- $\Gamma(U, \mathcal{F})$ (section of sheaf $\mathcal{F}$ over U), **43**
- Γ adjoint to Spec , **152**
- $\Gamma_\bullet$ (graded analog of Γ), **348**, **349**
- $\Omega_{A/B}$ (module of Kähler differentials), **460**, **466**
- $\Omega_{X/Y}$ (relative cotangent sheaf), **468**
- $\Omega_{X/Y}^i$ (sheaf of relative i -forms), **481**
- $\chi(X, \mathcal{F})$ (Euler characteristic), **327**, **390**
- $\coprod$ (coproduct), **14**, **19**, **96**
- γ_π (graph morphism), **237**
- $\hookrightarrow$, **170**, **187**, **190**, *see* (open, closed, locally closed) embedding
- λ -invariant, **430**
- μ_n (group scheme), **166**
- ν_d (Veronese embedding), **194**
- ω (canonical/dualizing sheaf), **396**, **415**, **416**, **418**, **423**–**425**, **481**, **619**–**621**, **621**, **622**–**633**
- $\oplus$ (direct sum, in an additive category), **24**
- $\otimes$ (tensor product), **11**
- $\otimes_{\mathcal{O}_X}$ (tensor product of $\mathcal{O}_X$ -modules), **58**
- $\boxtimes$ (box-times), **331**, **331**, **353**, **354**, **449**, **571**
- ϕ_q^p, ϕ_Z^p (cohomology and base change map), **567**, *see also* push-pull map
- π -ample, *see* ampleness, with respect to π (relatively ample)
- π -very ample, *see* very ampleness, with respect to π (relatively very ample)
- π^*s (pullback of section s), **319**
- π^* (pullback of $\mathcal{O}$ -module/quasicohherent sheaf), **148**, **316**, **318**, **319**, *see also* pullback; projection formula; push-pull map
- $\pi^{1?}$ (occasional right adjoint to π_*), **369**, **369**, **370**, **407**, **626**–**628**
- computable affine-locally, **369**, **370**
- for $\mathcal{O}$ -modules and closed embeddings, **630**
- π^{-1} , **59**, *see* inverse image sheaf
- π_* (pushforward sheaf), **46**, *see* sheaf, pushforward
- $\pi_!$, **61**, **524**, *see* extension by zero
- $\rho(X)$ (Picard number), **395**, **395**, **451**, **484**
- finiteness of, **614**
- $\bigwedge^\bullet M, \bigwedge^\bullet \mathcal{F}$ (exterior algebra), **310**, **310**, **311**, **311**, **364**, **365**, **481**, **506**, **546**
- A^1 (affine line), **67**–**70**
- A^n (affine n -space), **71**
- A_X^n , **210**
- A -scheme, **114**
- $A'_d(X)$ (variant of Chow group), **453**
- map to homology, **454**
- A_n -singularities, **502**, **503**, **603**
- Ab (category of abelian groups), **6**
- is an abelian category, **24**
- Ab_X (category of sheaves of abelian groups on X), **48**, **48**, **49**, **57**, **58**
- has enough injectives, **523**
- is an abelian category, **57**
- abelian category, **5**, **6**, **15**, **23**–**25**, **25**, **26**–**30**, **32**, **49**, **56**, **121**, **127**, **130**, **131**, **144**, **303**, **305**, **306**, **514**–**516**, **520**, **523**, **552**, *see also* Ab ; Ab_X ; Coh_X ; $\text{Com-}\mathcal{C}$; complex; $f.d.\text{Vec}_K$; $\text{Mod } A$; $\text{Mod } \mathcal{O}_X$; objects, in an abelian category; QCoh_X ; category; Vec_K ; enough injectives; enough projectives; FHHF Theorem; Jordan–Hölder package; spectral sequence
- e.g., coherent A -modules, **128**
- e.g., presheaves of abelian groups on X , **49**
- abelian cone associated to coherent sheaf, **369**
- abelian group scheme, **165**
- abelian semigroup, *see* semigroup (abelian)
- abelian varieties, **246**, **246**, **248**, **434**
- are abelian, **248**
- absolute Frobenius, *see* Frobenius
- abstract nonsense, **3**
- action of group variety/scheme, **166**, **343**, **365**
- transitive, **487**
- acyclic object, **518**, **519**, **523**–**525**, **624**, **626**, *see also* resolution (complexes)
- additive category, **24**, **24**, **25**, **48**, **49**, **56**
- additive functor, **24**, **24**, **28**, **31**, **516**, **517**, **519**, **526**, **552**
- ADE singularities, **503**, **603**
- $\text{adj}(M)$ (adjugate matrix), **172**, **174**, **231**
- adjoints, **20**
- (co)unit, **21**
- left, **20**
- are right-exact, **30**
- commute with colimits, **23**
- of forgetful functors, **22**
- right, **20**
- are left-exact, **30**
- commute with limits, **23**, **31**
- preserves injectives, **526**
- table of, **23**

- ul style="list-style-type: none; padding-left: 0;">
- adjunction formula, 590, 633
 - for a curve on a surface, **448**
 - for the canonical bundle $\mathcal{K}$, **481**, 482, 590
 - for the dualizing sheaf ω , 629, 632, **632**
- Affine Communication Lemma, 91, 112, **113**, 115, 121, 129, 130, 150, 169, 171, 175, 176, 178, 180, 181, 344
- affine cone, 103, 108, 195, **195**, 255, 257, 263, 279, 323, 345, 498, 585
 - relative version, 373
- affine cover cohomology vanishing, 380, **380**, 381, 386, 406, 408
- affine line, 67, 68, **68–70**, 188
 - functorial characterization, 163
- affine morphisms, **175**
 - $R^{i>0}\pi_*\mathcal{F} = 0$ for all quasicoherent $\mathcal{F}$, 405
 - π_* exact (for quasicoherent sheaves), 316, 607
 - π_* preserves cohomology of quasicoherent sheaves, 380
- affine-local on target, 176
- are quasicoherent and quasiseparated, 176
- are separated, 236
- as *Spec*, 368
 - e.g., closed embeddings, 187
 - e.g., finite morphisms, 176
- preserved by base change, 218
- preserved by composition, 175
- reasonable class, 218
- affine n -space, **71**, 72, 75, 82
- (over a field) is regular, 283
 - (over any base) is flat, 533
 - coordinate functions, **71**
 - functorial characterization, 164
 - minus origin, or with doubled origin, *see* recurring
 - (counter)examples
 - over a scheme X ($\mathbb{A}_X^n$), **210**
- affine open subscheme/subset, 95, **96**
- affine plane, 70, 73, 74, 76, 77, 94, 98, 100, 243, 285, 491, 502
- affine schemes/varieties, *see* schemes; varieties
- affine-diagonal morphisms, 239
- affine-local properties, 112, **113**
- implied by stalk-local, 113, 169
 - on the source, 171
 - on the target, 169, **171**
- Alexandrov space, 525
- algebraic equivalence, **558**
- implies numerical equivalence, 559
- algebraic field extension, *see* field extension
- algebraic group, 246, **246**, 247
- Algebraic Hartogs's Lemma, 101, 102, 115, 117, 290, **290**, 291, 292, 308, 337, 339, 345, 419, 481, 595
- algebraic space, 333
- Alteration Theorem, 503
- ample cone, 394, 451, **454**, 456
 - = interior of nef cone, 456
 - is open, 454, 456
- ampleness, 351, 353, 451, *see also*
 - Kleiman's criterion; Kodaira Embedding Theorem;
 - Nakai–Moishezon criterion;
 - Serre's cohomological criterion
- $\boxtimes$ ample is ample, 354
 - $\otimes$ ample is ample, 354
 - $\otimes$ base-point-free is ample, 354
 - $\otimes$ nef is ample, 456
 - = positive degree for smooth projective integral curves, 417
 - better-behaved than very ampleness, 354
 - cohomological criterion, 354
 - implies nef, 395, 456
 - in the absolute setting, **356**
 - properties of, 357
 - many characterizations of, 353
 - of vector bundle, 377
 - open condition in families, 354
 - over A , *see* ampleness, with respect to π
 - preserved by finite pullback, 354, 376, 393, 407
 - with respect to π (relatively ample), **353**, 357, **376**
 - affine-local on the target, 377
- analytification, 155, **155**, 156, 239, 246, 293, 307, 347, 360, 392, 395, 401, 439, 447, 471, 479, 483, *see also* varieties, complex
- of a morphism, **155**
- André-Quillen homology, 463, 464
- $\text{Ann}_A \mathfrak{m}$ (annihilator ideal of $\mathfrak{m}$), **135**
- annihilator ideal, 134, **135**, 137, 138, 490, 541
- anticanonical bundle, 596, *see also* canonical, bundle
- arithmetic genus, *see* genus, arithmetic
- arrow (category), *see* morphism (category)
- Artin-Rees Lemma, 271, 286, 299, **299**, 300, 602
- Artin-Schreier cover, 479
- Artinian, *see also* finite length;
 - Jordan–Hölder package; length
- local ring, **133**, 254, 293
- ring, **133**
- scheme, **133**, 293
- ascending chain condition, **85**, 86
- $\text{Ass}_A M$ (associated points/primes of M), 134, **136**
- assassin, **136**
- associated graded, 273, 453, 613
- associated points, 94, 133, 134, **135**, **136**, 141, 142, 198, 200, 225, 243, 344, 346, 347, 360, 362, 393, 399, 400, 444, 445, 447, 453, 472, 494, 535, 543, 550, 581, *see also* Associated-to-Separated Theorem
- and depth, 581
 - and flatness, 535
 - behavior in exact sequences, 137, 138
 - commutes with localization, 134, 140
 - finitely many, 134
 - of coherent sheaf, **135**
 - of integral schemes, 142
 - of locally Noetherian scheme, **141**
- zerodivisors must vanish at one, 134, 137
- associated prime ideals, 133, 134, **136**, 138, 140, 262, 291, 535, 581–583, *see also* associated points; primary ideals
- and depth, 583
- annihilator of some element of M , 134, 137
- e.g., minimal prime ideals, 139
- localization at, 136
- of a ring, 136, **136**
- zerodivisors = element of associated prime, 137
- Associated-to-Separated Theorem, **243**
- Asymptotic Riemann–Roch, 446, **446**, 448, 455
- Atiyah flop, **505**
- Atiyah, M., xix
- Auslander–Buchsbaum Theorem, **298**, 299, 348, 447, 581, 588
- automorphism group
- infinitesimal, 484
 - of $\mathbb{P}_k^1$, 343, 423, 424, 429–431, 437, 485
 - of $\mathbb{P}_k^n$, 331, 343, 425, 485, 544
 - of a curve, 423, 424, 426, 428, 437, 480, 481
 - of a scheme, 151
 - of an object in a category, 6
- Axiom of Choice, xx, 512, 514, *see also* Zorn's Lemma
- $\text{Bl}_X Y$ (blow-up of Y along X), **492**, **493**
- base (of a topology), **53**, *see also* presheaf, on a base; sheaf, on a base
- base (target of a morphism), **213**
- base change, **213**
 - base ring, **105**
 - base scheme, **151**, **213**
- base change diagram/square, **213**, *see also* Cartesian diagram/square; fibered diagram/square; pullback, diagram/square
- base field extension, **212**
- base identity axiom/base gluability axiom, **54**, 89–91, *see also* gluability axiom; identity axiom
- base locus, **329**, 504, 505, *see also* scheme-theoretic, base locus
- base point, **329**, 416, 456, 612
- base-point-freeness, **329**, **329**, 351, *see also* global generation
- $\otimes$ ample is ample, 354
 - $\otimes$ very ample is very ample, 353
 - and maps to projective space, 329
- base-point-free locus, **329**
- implied by very ampleness, 353
- independent of base field, **387**
- preserved by $\otimes$, *see* global generation
- with respect to π (relatively base-point-free), 357, **376**
- Bertini's Theorem, 271, 281, 283, **283**, 284, 401, 403, 425, 489, 504, 590,

- 591, *see also* Kleiman–Bertini Theorem
- improved characteristic 0 version, **488**
- with weaker hypotheses, 285
- Betti numbers, 484, 568
- Bézout’s Theorem, 192, 194, 229, 282, 331, **400**, 401, **401**, 402, 426, 427, 434, 443, 585, 595
- for plane curves, **401**, 419, 420, 448
- bilinear form (index of), **451**
- bilinear map, **12**
- birational, 484
 - (rational) map, 159, **159**, 161, 228, 360, 503–505, 595, 615
 - factorization of, 506
 - invariant, 481, 615
 - e.g., geometric genus, 482
 - e.g., $h^0(X, \Omega_{X/k}^1) = h^{1,0}$, 481
 - e.g., j th plurigenus, 482
 - e.g., Kodaira dimension, 482
 - model, 615
 - model of integral curves, 357
 - morphism, 159, **159**, 228, 231, 358, 410, 492, 595, 607, 608, 611, 612, 614
 - Zariski’s Main Theorem version, **607**, 609
 - varieties/schemes, 159, **159**, 358, 359, 449, 482, 486, 491, 506, 615
 - may not be isomorphic, 344
 - often means isomorphic dense open subschemes, 159
- birational transform, *see* proper transform
- blow-up, 216, 273, 491, **493**, *see also* $\text{Bl}_X Y$; $E_X Y$; exceptional divisor; Hironaka’s Theorem; proper transform; resolution (singularities); total transform; universal property; Weak Factorization Theorem
- and line bundles, 506
- and the canonical bundle, 506
- commutes with flat base change, 537, 603
- computable locally, 493, 495, 497
- does not commute with general base change, 537
- exists, 497
- is projective, 497
- locally principal closed subscheme, 494
- not flat, 536
- of a nonreduced subscheme, 503
- of affine space, 369, 492
- of an effective Cartier divisor, 493
- of regular embeddings, 499
- of smooth in smooth is smooth, 500
- of the plane, 216, 344, 448, 449, 491, 501, 589, 594–597
 - as graph of rational map, 243
- preserves irreducibility and reducedness, 494
- Proj* description, 497
- to resolve base locus, 504
- to resolve singularities, 502, 503
- Blow-Up Closure Lemma, 493, 494, 496, **496**, 502, 504
- blowing down, 506, *see also* Weak Factorization Theorem
 - (-1) -curve, 612, *see also* Castelnuovo’s Criterion
- Borel–Moore homology, 454
- boundary (homological algebra), **26**
- box-times, *see* $\boxtimes$
- branch
 - divisor, **477**, 478
 - locus, **477**, **489**
 - points, 421, **421**, 422, 430, **478**, 480
- branched cover, 177, 421, 422, 425, 429, 439, 473, 476, 479, 531, *see also* double cover
- $c_1(L) \cup$, **453**, *see also* Chern class
- Calabi–Yau variety, 266, 482, **482**, 483, *see also* trichotomy
- Cancellation Theorem, 170, 233, 234, **234**, 237
 - for affine morphisms, 539
 - for closed embeddings, 353
 - for projective morphisms (“unreasonable” class), **375**, 607, 613, 627
 - for proper morphisms, 244, 246, 607, 611
 - for quasicompact morphisms, 175, 378, 379
 - for reasonable classes of morphisms, **234**
 - for separated morphisms, 379
 - in *Sets*, 234
- canonical
 - bundle $\mathcal{K}_X$, 396, **481**, 506, 590, 596
 - is Serre-dualizing, 396, 428, 473, 480–485, 619, 629
 - curve, **425**, 427, 483
 - embedding $|\mathcal{K}|$, 424, 425, **425**, 426, 427, 434
 - map $|\mathcal{K}|$, **423**, 424–426
 - sheaf ω_X , 380, 395, 427, **619**, **621**
- Caroll, L., 77
- Cartan–Eilenberg resolution, 519, 520, 522, **522**
- Cartesian diagram/square, **13**, *see also* base change diagram/square; fibered diagram/square; pullback, diagram/square
- Cartier divisors, xix, **348**, *see also* effective Cartier divisors
- Castelnuovo’s Criterion, 498, 595, 599, 606, 612, **612**, 614
- category, **5**, *see also* Ab ; Ab_X ; abelian category; additive category; Coh_X ; $\text{Com}_{\mathcal{C}}$; $f.d.\text{Vec}_k$; equivalence; *Groups*; Mod_A ; $\text{Mod}_{\mathcal{O}_X}$; opposite; QCoh_X ; *Rings*; *Sets*; Sets_X ; Sch ; Sch_S ; subcategory; *Top*; Vec_k
- concrete, **6**
- filtered index, 19
- index, **16**
- locally small, **5**
- of k -schemes/ A -schemes/ S -schemes, *see* Sch_k / Sch_A / Sch_S
- of complexes, *see* $\text{Com}_{\mathcal{C}}$
- of open subsets, 7
- of schemes, *see* Sch
- of subsets of a set, 7
- of vector bundles, 309
- small, 9, 16, **16**, 20, 27
- catenary rings, **258**, 587
- Cayley–Bacharach Theorem, 419
- Čech cohomology, xix, 233, 379, 383, 384, **384**, 385, 516, 526, 528–530, *see also* cohomology
- = derived functor cohomology, 511, 523, 525
- and change of base field, 386
- behaves well in flat families, 557
- commutes with filtered colimits, 380
- commutes with flat base change, 536, 567
- dimensional vanishing, **380**, 387, 507, 546, 630
- more generally, 387
- relative, 406
- of line bundles on projective space, 387
- pullback, **380**
- Čech complex, 384, **384**, 385, 386, 388, 389, 404, 422, 528, 529, 557, 572, 573, 622
- Čech cover, 526, 529
- Čech resolution, **385**, 557
- change of base, *see* base, change
- Chasles’ Theorem, 419
- Chern class, 392, 446, 453, 454
- Chevalley’s Theorem, 181–183, **183**, 184, 185, 196, 216, 217, 266, 446, 487, 549, 550
- for finitely presented morphisms, 183, 217
- for locally finitely presented morphisms, **217**
- Chinese Remainder Theorem, 93, 133, 211
 - as geometric fact, 101
- Chow group, 454
- Chow’s Lemma, 409, 410, **410**, 411, 444
- other versions, 412
- $\text{Cl}(\mathbb{A}_k^n) = 0$, 341
- $\text{Cl}(X)$ (class group), **341**
- class (in set theory), 5
- class group, 119, 341, **341**, 342, 344, 345, *see also* excision exact sequence
- and unique factorization, 345, 346
- in number theory, **307**, 342, 345
- closed embeddings, 177, 187, **187**
- affine-local on target, 187, 227
- are affine, 187
- are finite, 187
- are monomorphisms, 189, 212
- are proper, 244
- are separated, 240
- base change by, 210
- in projective space, 191
- intersections of, 210
- preserved by base change, 210, 217
- preserved by composition, 187
- reasonable class, 217
- closed immersions, *see* closed embeddings

- closed map, **177**, **185**, 243, 412, *see also*
 - universally closed morphisms
 - not preserved by base change, 244
- closed points, 72, **82**, **109**, 472, *see also*
 - degree, of closed point; Nullstellensatz
 - dense for finitely generated k -algebras, 82
 - dense for varieties, 109
 - may not be dense, 83
 - of $\mathbb{P}_k^n$, 100, 101, 107, 193
 - of $\text{Spec } \mathbb{Q}(t) \otimes_{\mathbb{Q}} \mathbb{C}$, 212
 - of $\text{Spec } A$ correspond to maximal ideals, 82
 - of affine n -space, 82
 - of locally finite type k -schemes, 114
 - quasicompact schemes have one, 109
 - schemes need not have one, 109, 334
- closed subscheme exact sequence, **188**, 309, 320, 321, 323, 346, 347, 391, 399, 408, 536, 579, 613, 632
- closed subscheme-theoretic image, *see* scheme-theoretic, image
- closed subschemes, 93, 96, 104, 107, 170, 182, 187, **187**, *see also*
 - effective Cartier divisors; closed embeddings
- = quasicohherent sheaves of ideals, 187
- arbitrary intersections of, 188
- cut out by function s , **189**
- cut out by section s of locally free sheaf, **308**
- finite union of, 188
- intersections of, 210
- locally principal, 199, **199**, 495, 496, 504, 555, *see also* divisor, locally principal
- codimension ≤ 1 , 262
- pull back, 218
- pull back, 210
- closure, *see* scheme-theoretic, closure; Zariski, closure
- cocycle condition, **56**, **99**, **304**
- codim (codimension), **253**
- codimension
 - at most difference of dimensions, 253
 - inequality for fibers, 267
 - of a point, **253**
 - of a prime ideal, **253**
 - of an irreducible subset, **253**
 - of regular embedding, **202**
 - pathologies, 253, 265
 - sometimes difference of dimensions, 258
- cofiltered, **30**
- cofinal, 599, **599**, 602, 610, 613, 616
- Cohen Structure Theorem, 600
- Cohen–Macaulayness, 134, 258, 298, 299, 396, 447, 581, 584, **584**, 585–588, 620, 623, 627–629, 632, 633, *see also* Miracle Flatness; Serre duality
- and adding a variable, 587
- e.g., all dimension 0 Noetherian schemes, 584, 585
- e.g., all dimension 1 Noetherian schemes without embedded points, 584
- e.g., all dimension 2 normal varieties, 588
- e.g., Gorenstein, 633
- e.g., regular embeddings in smooth k -schemes, 585
- e.g., regular local rings, 581, 585
- fancy properties, 586
- implies catenary, 587
- implies no embedded points, 584, 585
- open condition, 586
- preserved by localization, 584, 586
- slicing criterion, 584
- stalk-local, 584
- Coherence Theorem, *see* Grothendieck’s Coherence Theorem
- coherent modules, **128**, 130, 143, 144, *see also* coherent sheaves
 - form an abelian category, 144
 - Homs are coherent, 144
 - over non-Noetherian rings, 143
- coherent sheaves, 108, **130**, 303, *see also*
 - coherent modules; finite type, quasicohherent sheaves; finitely presented, sheaves; Grothendieck’s Coherence Theorem
- abelian cone of, **369**
- affine-local nature, 130
- dual, 312
- Euler characteristic, **390**
- flat = locally free, 542
- higher pushforward, *see* sheaf, higher pushforward
- non-Noetherian context, 143
- not good notion in smooth geometry, 130
- on X form an abelian category Coh_X , 130, 144, 306
- pushforward, *see* sheaf, pushforward
- cohomology, xix, 8, **27**, *see also* Čech
 - cohomology; cup product; derived functor cohomology; homology; long exact sequence
 - Čech = derived functor, 511, 523, 525
 - of a double complex, **33**
- Cohomology and Base Change Theorem, 316, 403, 405, 569, **569**, 570–576, 579, *see also* push-pull map
- Coh_X (category of coherent sheaves on X), **130**
- coker (cokernel), **25**
- $\text{coker}_{\text{pre}}$ (cokernel presheaf), 49
- cokernel, **25**, *see also* universal property
- presheaf, **49**
- sheaf, **56**
- colim (colimit), **18**
- colimits, **18**, *see also* universal property
 - are right-exact, 30
 - commute with colimits, 23, 30
- commute with left-adjoints, 23, 30
- e.g., cokernels, 30
- e.g., localization, 20
- e.g., stalks, 44
- filtered, 19
 - commute with homology in Mod_A , 31, 541
 - in Mod_A are exact, 30, 31, 60
 - not always exact, 30
 - in Mod_A (non-filtered case), 20
 - may not commute with right-exact functors, 32
- collapse (of spectral sequence), **35**
- comathematician, 19
- $\text{Com}_{\mathcal{C}}$ (category of complexes in $\mathcal{C}$), **27**
 - is abelian category, 27
- commutative monoid, **24**
- compatible germs, 25, **50**, 51–54, 60, 107, 124, 188, 322, 333
- complete k -scheme, **244**, *see also* proper morphisms
- complete intersections, **203**, 403, 427, 434, 482, *see also* local complete intersections
 - in affine space
 - are Cohen–Macaulay, 585
 - implies no embedded points, 585
 - in projective space, 403, 426, 427, 434, 482, 483, 632
 - positive-dimensional ones are connected, 403
- complete linear series, *see* linear series
- complete with respect to an ideal I , **600**
- completion, 18, 143, 273, 544, 565, 599, **599**, 600, 601, 603, 605, 606, 615
- and Cohen–Macaulayness, 587
- commutes with finite direct sums, 602
- is flat, 601, 603
- not always exact, 601
- of A along I , **599**
- preserves exact sequences of finitely generated modules, 601, 602
- complex (in abelian category), **26**
 - double complex, 32, **33**
 - (hyper)cohomology of, **33**
 - cohomology, 33
 - first quadrant, **34**
 - total complex of, **33**
- exactness, 26, **26**
 - (for sheaves) is stalk-local, 127
 - of sequence of sheaves is stalk-local, 57
- factorization into short exact sequences, 27
- homotopic maps give same map on homology, **28**, 512
- complex geometry, *see* analytification; manifolds, complex; varieties, complex
- complex manifolds, *see* manifolds, complex
- complex varieties, *see* varieties, complex
- component
 - connected, 84, **84**, 85, 109, 113, 222, 305, *see also*

- idempotent-connectedness package
- are closed, 84
- may not be open, 84
- irreducible, **84**, **84**, 85, 88, **109**, 225
- = minimal primes for Spec A , 88
- are closed, 84
- Noetherian spaces have finite #, 84
- composition series, **131**, *see also*
 - Jordan–Hölder package;
 - modules
- composition-of-functors spectral sequence, *see* Grothendieck
- composition-of-functors spectral sequence
- compositum of subfields, **255**
- cone, *see also* abelian cone; affine cone;
 - ample cone; conormal cone;
 - effective cone; mapping cone;
 - nef cone; normal cone;
 - projective cone; tangent cone
- over smooth quadric surface, 92, 102, 116, 118, 161, 227, 256, 273, 279, 281, 290, 345, 346, 505, *see also* recurring (counter)examples; quadric surface
- not factorial, 273
- over twisted cubic/ rational normal curve, 81, 196, 255
- traditional, i.e. $x^2 + y^2 = z^2$, 106, 195, 196, 229, 273, 290, 340, 344, 346, 426, 438, 439, 502, 503
- conic (curve in $\mathbb{P}^2$), **192**
 - $\cong \mathbb{P}^1$ if regular with a rational point, 161, 418
 - classification of, 193
 - universal, 393
- coniveau filtration on $K(X)$, **453**
- connected, **80**, *see also* component;
 - fibers;
 - idempotent-connectedness package
 - schemes, **109**
- connecting homomorphism, **28**
- conormal
 - bundle, 464, 491
 - cone, 491
 - exact sequence, 463, 464, **470**
 - affine version, **464**
 - for smooth varieties, **471**
 - left-exact in good circumstances, 464, 470, 471, 565
- module, **464**
- sheaf, 464, **465**, 491
 - of regular embedding is locally free, 465
- constant (pre)sheaf, **45**
- constant rank implies locally free on reduced (for finite type quasicohherent sheaves), 314, 361, 542, 558
- constructible subsets, **182**, *see also* Chevalley’s Theorem
 - are finite disjoint unions of locally closed subsets, 182
 - in more general situations, 217
- contravariant functor, *see* functor
- coordinates, **67**
 - on affine space, e.g., x_i , **71**
 - projective x_i , 100, 101, **107**, 475
- coproduct, **14**, **19**
 - of schemes (is disjoint union), **153**, 205
 - of sets, 14
- corank, **275**, *see also* Jacobian corank
 - function; semicontinuity
- cotangent, *see also* tangent
 - bundle, 396
 - complex, 463, 565
 - exact sequence, 463, **470**, 563
 - affine version, **462**, 463
 - left-exact in good circumstances, 463, 470, 477, 565
 - map, 274
 - sheaf, 459, **468**, *see also* tangent sheaf
 - relative, **468**
- space, 42, **271**
 - computed by Jacobian at rational points, 274
 - computed by Jacobian at separable closed points, 462, 472
 - is fiber of Ω , 462
- vector, **271**, 459, **459**
 - = differential, 271
 - relative, **459**
- counterexamples, *see* recurring (counter)examples
- covariant functor, *see* functor
- cover
 - by open subfunctors, **209**
 - open, **41**
- covering space, 292, 560
- Cremona transformation, **161**, 243, 336, 505, 597
- cross-ratio, 430, **430**
 - and the j -invariant, 430
 - classifies 4 points on $\mathbb{P}^1$, 430
- cubic, 229, 420, 427, 431–437, 440, 441, 504, 544, 546
 - curves, 504, *see also* cuspidal cubic curve; nodal cubic curve; twisted cubic curve
 - hypersurface, **192**
 - plane curves, 281, 425
 - surface, xvii, xviii, 261, 426, 427, 504, 505, 589–596, *see also* del Pezzo surfaces; Fermat cubic surface
 - is blown-up plane, 595
 - singular, 591
- cup product, 620, 623, 629
- Curve-to-Projective Extension
 - Theorem, 160, 161, 335, **335**, 336, 337, 358–360, 431, 433
 - necessity of hypotheses, 335
- curves, **192**, **252**, *see also* (-1) -curve;
 - canonical, curve; elliptic curves; genus; Fermat curve; hyperelliptic curves; Klein quartic curve; $\mathcal{M}_g$; rational normal curve; Tate curve; twisted cubic curve
 - curve singularities, 603
 - exists regular projective birational model, 357
- genus, 413
 - arithmetic = topological, 392
 - geometric = topological, 479
- genus 0, 402, 418–420, 423, 424, 428, 479, 485, 571, 604
 - are conics, 418
- genus 1, 402, 416, 420, 421, 423, 424, 428, 429, 436–438, 440, 546, *see also* elliptic curves
 - classified by j -invariant, 437
 - not classified by j -invariant, 437
- genus 2, 423, 424, 479
 - hyperelliptic in precisely one way, **423**
- genus 3, 392, 402, 423–427, 479, 604
 - not all hyperelliptic, 423
- genus 4, 403, 426, 427, 434
- genus 5, 426, 427
- genus 6, 402, 427, 604
- moduli space $\mathcal{M}_g$, 532
- non-rational, 162
- nonhyperelliptic, 423, 425–427, 434
- of every genus, 422
- plane, 281, 401, 419, 425, 426, 438, 448, 461, 496, 502, 546, 589, 603, 604, *see also* Bézout’s Theorem
- proper implies projective, 417
- trigonal, **425**, 427
- various categories are equivalent, 359
- cuspidal, 229, 281, 413, 489, 503, **603**, 604, *see also* normalization, of cusp
- cuspidal cubic curve, 229, 281, 489, 502
 - normalization, 550
- cycle (homological algebra), **26**
- $d: \mathcal{O}_X \rightarrow \Omega_{X/Y}$, **468**
- $d: A \rightarrow \Omega_{A/B}$, **460**
- $D_+(f)$ (projective distinguished open set), 80, **106**, 327
- $D(f)$ (distinguished open set), *see* open set, distinguished
- D_n -singularities, **503**, **603**
- δ_π (diagonal morphism), **13**, **233**
- δ -functors, **516**, 625, 628, 629
 - covariant/contravariant, **517**
 - e.g., derived functors, 516
 - morphism of, **517**
 - universal, 516, **517**, 625, 626, 628
- δ -invariant, **604**
- d -uple embedding $\vee_d$, *see* Veronese embedding, **194**
- Dedekind domain, 118, 119, 230, 258, **289**, 307
 - unique factorization of ideals, 131, 289
- deformation theory, 274, 484, 532, 541, 589, 592
- deformation to the normal cone, 508, **508**, 543, *see also* normal cone
- $\deg \mathcal{F} = \deg_{\mathbb{C}} \mathcal{F}$ (degree of coherent sheaf on curve), **392**
 - when $\mathcal{F}$ is line bundle, **391**
- degenerate linear series, **329**, *see also* linear series
- degree
 - d map, **331**
 - algebraic = analytic for maps of curves, 360

- degree (cont.)
 - in projective space, **400**, 531
 - additive for unions, 402
 - constant in flat families, 557
 - of curve, 331
 - topological definition, 401
 - of closed point of finite type
 - k -scheme, **115**
 - of coherent sheaf on curve, **392**
 - additive in exact sequences, 392
 - of discriminant of degree d
 - polynomials, 480
 - of divisor on regular curve over k , **391**
 - of finite flat morphism is locally constant, 215, 360, 446, 480, 507, 542, 543, 552, 592
 - of finite map from curve to regular curve is constant, 360, 361, 392, 421, 431, 558
 - of finite morphism at point, **315**, 531
 - of generically finite morphism, 552
 - of generically finite rational map, **256**
 - multiplicative under composition, 256
 - of homogeneous element, 104, **104**
 - of hypersurface, **192**
 - of line bundle on curve, 361, **391**, 443, 531
 - additive for $\otimes$, 393
 - is homomorphism, 393
 - of line bundle on projective space, **342**
 - of projective morphism of curves, 359, **360**, 543
 - of projective variety, 285
 - of torsion sheaf on curve, 392, 393
- del Pezzo surfaces, 427, **483**, 597, *see also* cubic, surface; Fano variety (classification theory); quadric surface; surfaces, Enriques–Kodaira classification
- delta-functor, *see* δ -functor
- depth, 134, 581, **581**, 582, 585, 587
 - as cohomological property, 582
 - bounded by dimension of associated primes, 583
 - bounded by dimension of support, 581
 - of regular local ring = dimension, 583
- derivation, 466, **466**, 468, 469, 474
 - in differential geometry, 271
 - universal $d : A \rightarrow \Omega_{A/B}$, 466
- derived category, 384, 574, 627, 629
- derived functors, xix, 32, 379, 387, 511
 - = Čech cohomology, 511, 523, 525
 - and spectral sequences, 517
 - computed with acyclic resolutions, 517
 - left, 404, **514**, **515**
 - right, 404, **515**
- derived pushforward, *see* higher pushforward
- descending chain condition, **84**, 85, 262
- descent, 122, 212, 436, 532, *see* gluing
- desingularization, **492**, *see* resolution
- det $\mathcal{F}$ (determinant bundle), **310**
- determinant (line) bundles, **310**, 396, 619
 - behave well in exact sequences, 311
- determinant map $GL_n \rightarrow \mathbb{G}_m$, **166**
- dévissage, xix, 444
- diagonal Δ (locally closed subscheme of $X \times_Y X$), **236**
- diagonal morphism δ_π , **13**, 14, 15, 233, **233**, 235, 238, 240, 411, 468, 489, 490
 - is a locally closed embedding, 236
- Diagonal-Base-Change diagram, **14**, 15, 17, 234, 235, 240
- diagonalizing quadratic forms, 118, 193, 194, 227, 273, 451
- diagram indexed by, *see* category, index
- Dieudonné (completely functorised), 23
- different ideal, **490**, *see also* relative different and discriminant ideals
- differentiable manifolds, *see* manifolds, differentiable
- differentials, 233, **271**, **460**, *see also* cotangent exact sequence; universal property = cotangent vectors, 271
 - behave well with respect to base change, 467, 470
 - explicit description, **460**
 - fiber at a rational point, 462
 - module of (for ring maps), **460**
 - on hyperelliptic curves, 473
 - on projective space, *see* Euler exact sequence
 - pulling back, 467, 470
- dimension, 70, *see also* pure dimension = transcendence degree, 255
 - additive for products of varieties, 257
 - at a point, **260**
 - constant in flat families, 557
 - Krull, **251**
 - of a ring, **251**
 - of a scheme, xvii, 115
 - topological in nature, 110
 - of a topological space, **251**
 - of a variety at a point (upper semicontinuous), 260
 - of a variety preserved by field extension, 260
 - of fiber at a point (upper semicontinuous), 269
 - of fiber of closed map (upper semicontinuous), 269
 - of fibers, 266
 - never lower than expected, 592
- dimensional cohomology vanishing, **380**, 387, 546, 630
 - for quasiprojective schemes, 507
 - more generally, 387
 - relative, 406
- dimensional filtration on $K(X)$, **453**
- Diophantine equations, 154, 160–162, 418, 424, 589
- direct image sheaf, *see* sheaf, pushforward
- direct limit, *see* colimits
- discrete topology, **45**
- discrete valuation, **287**, 295, 297, *see also* discrete valuation rings; valuations
- discrete valuation rings, 73, 229, 271, 277, **288**, 295, 297, *see also* finitely generated modules, over discrete valuation rings (classification)
 - differentials over a field, 474, 478
 - flatness criterion, 541
- discriminant ideal, **490**, *see also* relative different and discriminant ideals
- disjoint union of schemes, 92, **96**, 563, *see also* recurring (counter)examples, infinite disjoint union of schemes
 - is coproduct, 205
- distinguished affine base, **123**
- distinguished open set, *see* open set, distinguished
- $\text{div}(s)$ (Weil divisor of zeros and poles), **338**
- divisible abelian group, **515**
- divisor, *see also* branch divisor; effective Cartier divisors; exceptional divisor; $\mathbb{Q}$ -Cartier divisor; Weil divisor
 - Cartier, xix, **348**
- Doesn't-vanish set, *see* open set, distinguished
- domain of definition, **142**, **242**, *see also* indeterminacy locus
- dominant morphism, **158**
- dominant rational map, **158**
- double complex, *see* complex
- double cover, 74, 177, 421, **421**, 422–425, 429, 430, 433, 439, 440, 473, 476, 531, *see also* branched cover; hyperelliptic map
- dual
 - of coherent sheaf, 312
 - of locally free sheaf, **306**
 - of $\mathcal{O}$ -module, **48**
 - projective space, **283**
 - variety, **286**
- dual numbers, **69**, 75, 93, 96, 111, 155, 187, 196, 241, 251, 274, 315, 349, 532, 541, 561
 - flatness criterion, 535, 541
- dualizing sheaf, 380, 395, **619**, **621**, *see also* canonical, sheaf
- DVR, *see* discrete valuation rings
- Dynkin diagrams, 503, 597
- E_6 , 589, 594, 603
- E_8 , 454, 603
- E_n , 603
- E_n -singularities, 503, **603**
- $E^{p,q}$ (entry in double complex), **33**
- $E_r^{p,q}$ (entry (p, q) in r th page of spectral sequence), **34**
- $E_X Y$ (exceptional divisor), **492**, **493**
- effaceable functor, *see* functor, effaceable
- effective Cartier divisors, xix, 134, 199, **199**, 200–203, 216, 243, 276, 335, 344–347, 387, 399, 444, 445, 447, 448, 492–500, 503–507, 537, 553,

- 555, 581, 584, 585, 595, 608, 613, 632, *see also* regular sequence; slicing criterion
- adding, 347
- and Cohen–Macaulayness, 585
- and invertible sheaves, 325, 346, 347
- contain no associated point, 200
- e.g., exceptional divisor $E_X Y$ on blow-up, 493
- how to restrict, 344
- is pure codimension 1, 262
- normal (line) bundle to, 347, 465
- not easily affine-local condition, 200
- principal, 498
- relative, 555, 555
- slicing by, 199, 199, 401, 403, 563, 581, 629
- effective cone, 450, 450
- Eisenstein’s Criterion, 281
- elementary transformation (ruled surface), 614, 614
- elimination of quantifiers, 184, 185
- elimination theory, 181, 185, *see also* Fundamental Theorem of Elimination Theory
- ellipse, 420, 438
- elliptic curves, 117, 155, 161, 162, 246, 307, 422, 424, 428, 428, 429–431, 433, 434, 436–442, 449, 568
- are group varieties, 434
- counterexamples and pathologies, 439
- degenerate, 437
- degree 1 points = degree 0 line bundles, 429
- group law, 429, 430, 433, 434, 440, 595
- geometrically, 433
- level n structure, 430
- non-torsion point, 440
- not same as genus 1 curves, 429
- elliptic fibration, 504
- embedded points/primes, 135, 135, 140, 140, 291, 360, 362, 400, 401, 447, 448, 472, 583–585, 587, 632
- of locally Noetherian scheme, 141
- embedding (unwise terminology), 190
- enough injectives, 515–517, 519
- e.g., Ab , 515
- e.g., Ab_X , 523
- e.g., Mod_A , xx , 515, 516
- e.g., $Mod_{\mathcal{O}_X}$, 523, 623, 624
- e.g., $QCoh_X$, 530
- enough projectives, 514, 515, 517, 525
- not $Mod_{\mathcal{O}_X}$, 525, 624
- epimorphisms, 15, 15
- may not be surjective, 15
- equalizer exact sequence, 44, 89, 209
- equidimensional, *see* pure dimension
- equivalence of categories, 8, 9, 55, 124, 151, 160, 322, 359
- espace étalé, *see* sheaf, space of sections
- essentially surjective functor, 9
- étale cohomology, 127, 511
- étale cover, 293, 479, 563
- étale morphisms, 271, 278, 292, 292, 293, 490, 560, 561, 563, 565, 592, 594, 600, *see also* formally étale
- = locally finitely presented + flat + unramified, 563
- = smooth + unramified, 563
- differential geometric motivation, 292
- local on source and target, 293
- not same as “local isomorphism”, 294
- open condition, 293
- preserved by base change, 293
- preserved by composition, 293
- reasonable class, 293
- étale topology, 127, 479, 560, 601
- Euclidean algorithm/Euclidean domain, 69, 70, 117
- Euler characteristic $\chi(X, \mathcal{F})$, 327, 390, 401, 402, 447, 531, 532, 546
- additive in exact sequences, 390, 391, 399, 453
- constant in projective flat families, 532, 546, 557, 573, 574
- constant in proper flat families, 403, 573
- without Noetherian conditions, 576
- is finite for proper X , 453
- polynomiality, 398, 428
- Euler exact sequence, 474, 474, 476, 483, 485
- generalizations, 476
- exact functor, 29, *see also* π^{-1} , π_1 , FHHF Theorem, flat morphisms, localization, stalkification
- exact modules/exact morphisms, 531
- exact sequence, *see also* Euler exact sequence; Grassmannian, universal exact sequence; excision exact sequence; equalizer exact sequence; exponential exact sequence; long exact sequence; normal exact sequence
- factored into short exact sequences, 27
- grafting, 37
- split, 522
- exceptional divisor $E_X Y$, 216, 448, 492, 493, 497–504, 506, 508, 595–597, 603, 608, 612, 614
- in minimal model program, 493
- normal bundle to, 498
- excision exact sequence for class groups, 341
- exponential exact sequence, 51, 53, 55, 57, 58, 392
- Ext functors, 511
- first version, 515
- for $\mathcal{O}$ -modules, 624
- computable by acyclics, 624
- long exact sequence, 624
- second version, 515
- two definitions are same, 517
- $\mathcal{E}xt$ functors, 624
- $\mathcal{E}xt_X^i$ (coherent, coherent) = coherent, 625
- $\mathcal{E}xt_X^i$ (coherent, quasicohherent) = quasicohherent, 625
- behaves like derived functor in first argument, 625
- computable with locally free resolutions, 625
- long exact sequence in first argument, 625
- may not preserve quasicohherence, 624
- extension by zero π_1 , 61, 369, 524, 524, 525, 526, 530
- is exact, 524
- left adjoint to inverse image π^{-1} , 61, 524
- extension of an ideal, 210
- extension of fields, *see* field extension
- extension of scalars, 23
- exterior algebra, 310
- $\mathbb{F}_n$ (Hirzebruch surface), 373, 450
- $\mathcal{F}|_p$ (stalk of $\mathcal{F}$ at p), 97
- factorial, 117
- $\neq$ normal, 118, 274
- but no affine open is UFD (example), 439
- implied by regular (Auslander–Buchsbaum Theorem), 298
- implies normal, 117
- open condition, 117
- ring, 117
- stalk-local, 117
- weaker than regular (example), 298
- faithful functor, 7, 7, *see also* fully faithful functor
- faithful module, 174
- faithfully flat algebra, 549
- faithfully flat morphisms, 122, 548, 549
- local on the target, 549
- preserved by base change, 549
- preserved by composition, 549
- reasonable class, 549
- faithfully flat sheaves, 549
- Faltings’s Theorem (Mordell’s Conjecture), 424
- Fano variety (classification theory), 266, 482, 482, 483, 594, *see also* del Pezzo surfaces; trichotomy
- Fano variety (of lines), 594
- $f.d. Vec_k$, 9
- Fermat cubic surface, 590, 592, 594
- Fermat curve, 162, 281
- Fermat’s Last Theorem, 162, 424
- FHHF Theorem, 29, 31, 387, 403–406, 538, 539, 624
- fibered coproduct, 14, *see also* universal property
- fibered diagram/square, 13, *see also* base change diagram/square; Cartesian diagram/square; pullback, diagram/square
- fibered product, 13, *see also* base change; product; pullback; universal property
- “commutes with localization”, 212
- in category of open sets, 13
- of functors, 208
- of schemes, 205
- of sets, 13
- with open embedding, 170

- ul style="list-style-type: none; padding-left: 0;">
- fibers of a morphism of schemes, 169, 213, **214**, *see also* generic fiber; scheme-theoretic
- connected, 222, 607, 608
 - from Zariski's Connectedness Lemma, 606
- dimension of, 266
 - well-behaved for flat morphisms, 536, 548, 550
- geometric, 220, **220**, 532, 562, 565, 571
- geometrically connected (etc.), **220**
 - properties preserved by base change, 220
- fibers of an $\mathcal{O}_X$ -module, 97
- field
 - imperfect, 246
 - perfect, 225, 226, 298, 485, 588
 - separably closed, 220, 223, 224, 239
- field extension, *see also* transcendence theory
 - algebraic, 171, 172, 230, 252, 254, 260, 462
 - and differentials, 462, 467
 - finite, 71, 83, 114, 118, 177, 183, 211, 215, 220, 225, 230, 231, 252, 255, 256, 259, 280, 357, 421, 439, 462, 489, 490, 560, 563
 - finitely generated, 160, **160**, 225, 255, 267, 402, *see also* transcendence degree
 - Galois, 211, 259, 479
 - normal, 259
 - purely inseparable, 219, 222, **222**, 231, 486
 - purely transcendental, 160, 260
 - separable, 225, 231, 462, 477, 489, 490, 560, 563
 - separably generated, 225, **225**, **467**
- filtered, **19**, *see also* colimits, filtered
- final object, *see* object, in a category, initial/final/zero
- finite B-algebra, **172**, **176**
- finite extension, *see also* field extension
 - of integrally closed domains, 258, 259, 549, *see also* Going-Down Theorem
 - of rings, 177, 258
- finite fibers not preserved by base change, 218
- finite flat morphisms, 542, 576, 589, 627, 628
 - have locally constant degree, 215, 360, 446, 480, 507, 542, 543, 552, 592
- finite global generation, *see* global generation
- finite length, **131**, *see also* Artinian; Jordan–Hölder package; length scheme, **133**
- finite morphisms, **176**
 - = integral + finite type, 179
 - = projective + affine, **382**
 - = projective + finite fibers, 360, 375, **382**, 383, 393, 414, 592, 614
 - = projective + quasifinite, **382**
 - = proper + affine, 246, 382, 409, 610
 - = proper + affine (Noetherian setting), **610**
 - = proper + finite fibers, 382
 - = proper + finite fibers (Noetherian setting), 610
 - = proper + quasifinite, 246
 - = proper + quasifinite (Noetherian setting), **610**
 - = proper + quasifinite + locally finitely presented, 610
- affine-local on target, 176
- are affine, 176
- are closed, 178, 193
- are integral, 172, 178
- are projective, 177, 374
- are proper, 244
- are quasifinite, 178, 179
- are separated, 236
- e.g., closed embeddings, 187
- have finite fibers, 178, 214, 215
- not same as finite fibers, 178
- preserved by base change, 218
- preserved by composition, 177
- reasonable class, 218
- separable, **477**
- to Spec k , 177
- finite presentation, *see* finitely presented (algebras, sheaves, modules, morphisms)
- finite type
 - A-scheme, 114, **114**
 - implied by quasiprojective, 114
 - over a ring, **114**
 - quasicoherent sheaves, **130**, *see also* modules, finitely generated
 - affine-local, 130
 - constant rank implies locally free on reduced, 314, 361, 542, 558
 - support is closed, 312
- finite type morphisms, **179**
 - affine-local on the target, 179
 - e.g., proper morphisms, 244
 - preserved by base change, 218
 - preserved by composition, 179
 - reasonable class, 218
- finitely generated domain, **255**
- finitely generated ideal extension, *see* field extension
- finitely generated modules, **128**, *see also* finite type, quasicoherent sheaves
 - over discrete valuation rings (classification), 129, 308, 313, 327, 362, 429, 474
 - over principal ideal domains (classification), xx, 129, 131, 308, 313, 327, 395, 429, 474, 533
 - over regular local rings have finite free resolutions, 547
- finitely globally generated, *see* global generation
- finitely presented, *see also* local finite presentation
 - = finite type under Noetherian hypotheses, 180, 181, 217
- algebras (ring map), 180, **180**, 181, **181**, 460
 - affine-local property, 180
 - implies always finitely presented, 180
- sheaves, **130**, 576, *see also* modules, finitely presented
 - affine-local, 130
 - flat = locally free, 542
 - local freeness is stalk-local property, 313
- finitely presented modules, **29**, **128**, 143, 144, 460, 541, 542, *see also* finitely presented, sheaves
 - = coherent for Noetherian rings, 144
 - “finitely presented implies always finitely presented”, 129, 313, 542
- graded, 323
- localization and Hom commute, 11
- finitely presented morphisms, 174, 179, 180, **181**, 354, 532, 543, 552, 564, 565, 570, 576–579
 - = “finite in all ways”, 181
- affine-local on the target, 181
- Chevalley's Theorem, 183, 217
- locally pullbacks of nice morphisms, 216, 217
- preserved by base change, 218
- preserved by composition, 181
- reasonable class, 235
- finiteness of integral closure, *see* integral closure
- finiteness of normalization, *see* normalization
- Five Lemma, 26, **37**, 38
 - subtler, **38**, 622
- flabby sheaf, *see* sheaf, flasque/flabby
- flag variety, **167**, 476
 - bundles, 476
 - partial, **167**, 365
- flasque sheaf, *see* sheaf, flasque/flabby
- flatness, 182, 245, 360, 387, 394, 403, 405, 443, 454, 508, 511–514, **534**, 559, *see also* Euler characteristic, constant in projective flat families; faithful flatness; finite flat morphisms; generic flatness; Going-Down Theorem; Miracle Flatness Theorem; resolution (complexes); slicing criterion
- = free = projective for finitely presented modules over local rings, 532, 541
- = locally free for finitely presented sheaves, 542
- = projective for finitely presented modules, 542
- = torsion-free for PID, 541
- and blowing up, 537
- and regular sequences, 200
- at a point, **534**
- constancy of fiber
 - degree/dimension/arithmetic genus, 557
- criterion over regular curve, 543
- e.g., affine space, 533
- e.g., completion, 601, 603
- e.g., free modules, 533
- e.g., localization, 533
- e.g., open embeddings, 534
- e.g., projective modules, 533
- equational criterion, 540

- fiber dimension well-behaved, 536, 550, 557
- fibril, 555, **555**, **556**
- flat limit, 543, **543**, 545, 546
 - explicit example, 544
- ideal-theoretic criteria, 532, 537, 540, **540**, 541, 552, 602
 - finitely generated version, 540
 - stronger form, 602
- implies torsion-free, 533, 537, 541, 553, 586
- local criterion, 532, 552, **552**, 569, 579
- no higher Tor, 513, 514, 532, 537, 538, 540, 541
- of modules, **30**, **532**
- of quasicohherent sheaf over a base, **534**
- of relative dimension n , 550, **550**, 551, 561, 562, 586
 - includes locally finite type hypotheses, 550
 - preserved by base change, 550
 - reasonable class, 551
- of ring morphism, 533
- open condition, **552**
- open map in good situations, 532, 549
- over a field, 533
- over discrete valuation rings, 532, 541
- over integral domains, 532, 533
- over local rings, 532
- over principal ideal domains, 532, 533, 541
- over regular curves, 542
- over the dual numbers, 532, 535, 541
- preserved by base change, 533, 535, 553
- preserved by composition, 533, 535
- reasonable class, 535
- regular embeddings pull back
 - under flat morphisms, 537
- sends associated points to
 - associated points, 535
- stalk/prime-local property, 533, 534
- topological aspects, 532, 548
- Tor₁-criterion, **513**
- valuative criterion, 543
- variation of cohomology, 544
- flex, 432, **432**, 433
- forgetful functor, *see* functor, forgetful
- form of degree d , **104**
- formal neighborhood, 600, 601, 605, **605**, 606, 613, 615
- formal power series, **18**, *see* power series
- formal schemes, xvii, xix, 605
- formally étale, 565, **565**, 601
- formally smooth, 565, **565**
- formally unramified, **489**, 565, **565**
 - e.g., localization, 489
- Four Squares Theorem, 162
- fraction field, **10**, *see also* total fraction ring
- fractional ideal, 303, **307**, 345
- fractional linear transformations, 343, **343**, 425
- free resolution, *see* resolution (complexes)
- free sheaf, 128, **303**, *see also* trivial bundle
- Freyd–Mitchell Embedding Theorem, **26**
- Frobenius, 246, 255, 278, 413, 477, 479
 - absolute, **179**
- full functor (unimportant), 7
- full rank quadratic form, *see* maximal rank
- full subcategory, *see* subcategory, full
- fully faithful functor, 7, 9, 16, 22, 26, 241
- function field, **111**, *see also* rational function
 - and birational maps, 159
 - and dominant rational maps, 158
 - determines irreducible regular projective curve, 359
- functions, **46**, *see also* differentiable functions; rational function
 - = maps to $\mathbb{A}^1$, 163
 - determined by values on reduced scheme, 110
 - global, **46**
 - multiplicity at regular point, **503**, 506
 - not determined by their values, 75
 - on a ringed space, **46**
 - on a scheme, **67**, **95**
 - on an open subset, **46**
 - on projective space, 101
 - value at a point, **67**, **96**
 - vanishes at a point, **96**
- functor, *see also* additive; derived; essentially surjective; exact; faithful; full; fully faithful; half-exact; left-exact; natural transformation; representable; right-exact
 - composition of, 7
 - contravariant, **8**
 - covariant, 7
 - effaceable, **517**
 - forgetful, 7, 8, 22, 23, 52, 58, 165, 348, 350
 - left adjoints to, 22, *see also* groupification; saturation; sheafification
 - functor category, **16**, 208
 - identity, 7
 - natural isomorphism, **8**
 - of points, **8**, **154**
- fundamental groupoid, 6
- fundamental point (of rational map), **242**
- Fundamental Theorem of Elimination Theory, 182, 185, **185**, 186, 193, 245, 266, 335, 382, 383
- $G(k, n)$ (Grassmannian), **166**
- $\mathbb{G}(k-1, n-1)$ (Grassmanian), **167**
- GAGA Theorem, 483, 531
- Galois cohomology, 127
- Galois theory, 69, 70, 72, 76, 211, 231, 259, 280, 421, 431, 479, 532, 594
- Gauss’s Lemma, 117
- Gaussian integers $\mathbb{Z}[i]$, 117, 215, 229, 282, 289
- general fiber, **216**, *see also* generic fiber
- general linear group scheme, **166**
- general point, **83**, 256, 286, 504, 595
- general type (variety), 266, 482, **482**, 483, *see also* trichotomy
- generated by (a finite number of) global sections, *see* global generation
- generated in degree 1, **105**
- generic fiber, **214**, 216, **216**, 268, 543
- generic flatness, **83**, 552, **552**
- generic point, 68, 69, 73, 76, 77, 81, **83**, **109**
 - and function field, 111
- generic smoothness, 83, 425, 485, 486, 488
 - of varieties, **485**
 - on the source, **486**
 - on the target, **487**
- generically finite, 216, **216**, 503
 - degree, **256**, 552
 - implies generally finite, 269, 552
 - usually means generally finite, 216
- generically separable morphisms, **477**, *see also* separable finite morphisms
- generization, **83**, **109**
 - stable under, 183, 549
- genus, 398, *see also* curves, genus 0, 1, ...
 - arithmetic, **392**, 427, 482, 531, 557, 576, 604
 - constant in projective flat families, 557
 - arithmetic = topological (for curves), 392
 - geometric, **472**, **482**
 - is a birational invariant, 481, 482
 - geometric = topological (for curves), 479
 - topological, 392
- geometric fiber, *see* fibers, geometric
- geometric genus, *see* genus, geometric
- Geometric Nakayama’s Lemma, *see* Nakayama’s Lemma
- geometric Noether normalization, *see* Noether normalization
- geometric point, **219**
- geometrically connected/integral/irreducible/reduced, **220**
- germs, **42**, **44**, *see also* compatible germs
 - determine section of sheaf, 50
- GL_n (group scheme), **166**
- global functions, **46**
- global generation (of quasicohherent sheaf), 351, **351**, 407, *see also* base-point-freeness
- at a point, **352**
- e.g., any quasicohherent sheaf on any affine scheme, 352
- finite, **351**
 - e.g., every finite type quasicohherent sheaf on every affine scheme, 352

- global generation (of quasicoherent sheaf) (cont.)
 - open condition, 352
 - preserved by $\oplus$, 352
 - preserved by $\otimes$, 332, 352
 - with respect to π (relatively globally generated), 357, **375**
 - affine-local on target, 376
 - for locally ringed spaces, 375
- global section, **43**
- gluability axiom, **44**, *see also* base gluability axiom
- gluing morphisms
 - of locally ringed spaces, 150
 - of ringed spaces, 148
 - of schemes, 151, 152, 206, 208
 - of topological spaces, 45
- gluing ringed spaces along open sets, 99
- gluing schemes
 - along closed subschemes, 332
 - along open subschemes, 98, **99**, 188, 206, 332
- $\mathbb{G}_m$ (multiplicative group scheme), **164**
- Going-Down Theorem, 173
 - for finite extensions of integrally closed domains, 258, **258**, 259, 549
 - for flat morphisms, 259, 548, 549, **549**, 550
 - for integral extensions of integrally closed domains, 259
- Going-Up Theorem, 172, **173**, 252, 259, 549
- Gorenstein, 633
- graded ideal, *see* homogeneous ideal
- graded modules, 23, 322, 342, 348, 349, *see also* saturated graded module; saturation functor
- quasicoherent sheaf corresponding to, 322, 348
- graded rings, 23, 102, **105**
 - $= \mathbb{Z}^{\geq 0}$ -graded ring, **104**
 - finitely generated over A , **105**
 - generated in degree 1, **105**
 - maps of, 156
 - over A , **105**
 - standing assumptions on, 105
 - $\mathbb{Z}$ -, **104**
- graph (of a) morphism, **237**, 411, 412, 449
 - is a locally closed embedding, 237, 412
 - to a separated scheme is a closed embedding, 237, 412
- graph of a rational map, **237**, 242, **242**, 243, 505
 - and blow-ups, 505
- Grassmannian, 108, 166, 167, **167**, 343, 362, 363, 365, 476, 576, 578–580, 592, *see also* Plücker
- as moduli space, 362
- bundle, **365**, 476
- functor (contravariant), 362, 363
- $G(2, 4) = G(1, 3)$, 261, 365, 591, 592
- $G(2, 5)$, 427, 434
- tautological bundles, **364**, 476, 577
- universal exact sequence, 363
- Grauert's Theorem, 568, **568**, 571–576
- Grothendieck
 - composition-of-functors spectral sequence, 38, 519, **519**, 523, 626
 - proof, 522
- Grothendieck functor, 163
- Grothendieck group of coherent sheaves on X ($K(X)$), 453
- Grothendieck topology, 126, **126**, 127, 532, 560
- Grothendieck's "six operations", 627
- Grothendieck's Coherence Theorem, 245, 316, 351, 376, 380, 382, 390, 409, **409**, 410, 411, 453, 573, 604, 607, 615, 617
 - for projective morphisms, 382, 404
- Grothendieck's Theorem (vector bundles on $\mathbb{P}^1$), **397**
- Grothendieck–Riemann–Roch, 446
- group (perverse definition), **6**
- group law on elliptic curve, *see* elliptic curves, group law
- group object (in a category), **164**
- group variety / scheme, 163–165, **165**, 166, 246, **246**, 248, 343, 365, 434, 437, 438, 487, 488, 571, *see also* action of group variety / scheme
- abelian, **165**
- morphism of, **166**
- structure on elliptic curves, 436
- groupification (of abelian semigroup), **22**, 23
 - left adjoint to forgetful functor, 22
- groupoid, **5**
- Groups (the category of groups), 165
- H ((co)homology), **26**
- $H^0(U, \mathcal{F})$ (section of sheaf $\mathcal{F}$ over U), **43**
- $h_{\mathcal{F}}(m)$ (Hilbert function), **398**
- h^A (functor of maps from A), **7**
- h_A , *see* functor, of points
- Hairy Ball Theorem, 473
- half-exact functor, 552
- Hartogs's Lemma, 98, 290, *see also* Algebraic Hartogs's Lemma
- Hausdorff, **233**, *see also* separated morphisms
 - in terms of separatedness, **236**
- height, *see* codimension
- Hensel's Lemma, 600, 601
- Hermitian metric, 354
- higher direct image sheaf, *see* sheaf, higher pushforward
- Hilbert Basis Theorem, **85**, 86
- Hilbert function, 85, 398, 402, 428, 557
 - of coherent sheaf, **398**
 - of projective scheme, **398**
- Hilbert functor, **576**, 577
- Hilbert polynomial, 381, 398, **398**, 399, 400, 402, 427, 444, 446, 531, 577
 - locally constant in flat families, 557, 558
- Hilbert scheme, 532, 576, 577, 580, 594
- universal family, 577
- Hilbert Syzygy Theorem, 85, 351, **351**, 399, 546, 548
 - proof, 547
- Hilbert's (Weak) Nullstellensatz, *see* Nullstellensatz
- Hilbert's fourteenth problem, 441
- Hironaka's example, 245, **559**
- Hironaka's Theorem, 503, 506, 615, *see also* resolution (singularities)
- Hirzebruch surfaces $\mathbb{F}_n$, 373, **373**, 450, 501, 604, 614, *see also* elementary transformation; ruled surface
- Hirzebruch–Riemann–Roch, 446
- Hodge diamond, 484, **484**, 615
- Hodge Index Theorem, 395, 451, **451**, 452
- Hodge numbers, 311, 483, **483**, 484, 568
 - birational invariance of some, 483
 - not all are birational invariants, 484
- Hodge theory, 481, 483, 484, 568
- Hom
 - is left-exact, 29
 - may not commute with localization, 29
 - right-adjoint to tensor product, 21
- Hom , **48**, **311**
 - is left-exact, 58
 - may not commute with stalkification, 48
 - of quasicoherent sheaves not necessarily quasicoherent, 311
- homogeneous element, **104**
- homogeneous ideal, **104**
- homogeneous space, **487**
- homology, **26**, *see also* cohomology commutes with exact functors, 30, *see also* FHHF Theorem commutes with filtered colimits in Mod_A , 31
- homomorphism (in additive category), **24**
- homotopic maps of complexes give same map on homology, **28**, **28**, 512
- Hopf algebra, **166**
- horseshoe construction, 513, 521
- Hurwitz's Automorphisms Theorem, **480**
- hypercohomology, *see* cohomology, of a double complex
- hyperelliptic
 - curves, 413, 421, **421**, 422–426, 433, 473, 474, *see also* curves, nonhyperelliptic; Riemann–Hurwitz formula, hyperelliptic differentials on, 473
 - moduli spaces, 423
 - most not if $g > 2$, 423
- involution, 434
- map, **421**, 423
 - unique for $g \geq 2$, 423
- hyperplane, 104, **192**
 - universal, **283**, 489, 577
- hyperplane class, **340**, **401**
- hypersurface, 71, 102, 104, **191**, **253**, *see also* quadric associated points of, 136 degree, **192**

- discriminant, 591
- in $\mathbb{A}_k^n$ have no embedded points, 140
- id (identity morphism), 5
- $I(S)$ (ideal of functions vanishing on S), 87
- I -adic completion, 599
- I -adic filtration, 299
- I -adically separated, 600
- I -depth, *see* depth
- I -filtration of a module, 299
- I -stable filtration, 299
- ideal, *see also* different ideal;
 - discriminant ideal; fractional ideal; homogeneous ideal; maximal ideals; minimal prime ideals; prime ideals; principal ideal
 - extension of, 210
 - of denominators, 116, 116, 117, 291
 - product of two, 78
- ideal sheaves, 125, 187
 - product of, 347, 465, 497, 537
- idempotent, 81, 153, 223, 606
- idempotent-connectedness package, 81, 223, 223
- identity axiom, 44, *see also* base identity axiom
- identity functor, 7
- identity morphism id_A , 5
- image (of morphism in a category), 25
- image (of morphism of schemes)
 - set-theoretic, *see* Chevalley's Theorem; closed map; Fundamental Theorem of Elimination Theory; open maps; surjective; universally closed; universally open morphisms
- image (of morphism of schemes)
 - scheme-theoretic, *see* scheme-theoretic, image
- image presheaf, 49
- immersion
 - of manifolds, 66, 560
 - of schemes (unwise terminology), 190
- imperfect field, *see* field, imperfect
- incidence correspondence/variety, 261, 283, 489, 591
- inclusion-exclusion via sheaf cohomology, 528
- indeterminacy locus, 142, 242
- index category, 16
- index of symmetric bilinear form, 451
- induced reduced subscheme, *see* reduced subscheme structure
- inductive limit, *see* colimits
- infinitesimal neighborhood, *see* formal neighborhood
- initial object, *see* object, in a category, initial/final/zero
- injection of vector bundles, 309, 563, *see also* maps of vector bundles
- injective limit, *see* colimits
- injective maps of rings induce dominant maps of schemes, 80
- injective morphisms of schemes not preserved by base change, 218, *see also* universally injective morphisms
- injective objects, 515, *see also* enough injectives; projective objects; resolution (complexes)
 - closed under product, 523
 - have no Čech cohomology, 528
 - preserved by right adjoints, 526
- integral closure, 174, *see also* normalization
 - finiteness of, 231, 231
 - of Noetherian need not be Noetherian, 231
- integral element of B -algebra, 171
- integral extensions of rings, 171, *see also* Going-Down Theorem
 - preserved by localization of source, 171
- integral morphisms of rings, 171
 - does not imply finite, 172
 - e.g., finite, 172
 - preserved by composition, 172
 - preserved by localization of source, quotient of source and target, 171
- integral morphisms of schemes, 178
 - affine-local on target, 178
 - are affine, 178
 - are closed, 178
 - e.g., finite morphisms, 178
 - fibers of, 178, 252
 - may not be finite, 178
 - preserved by base change, 178, 218
 - preserved by composition, 178
 - reasonable class, 218
- integral scheme, 111
 - = irreducible and reduced, 111
 - almost a stalk-local property, 112, 114
- integrally closed, 115, *see also* integral closure
- intersection, *see* scheme-theoretic, intersection
- intersection number/product of $\mathcal{L}_1, \dots, \mathcal{L}_n$ with $\mathcal{F}$, 443, 443, 444–447, 451–453
 - deformation invariance, 443
 - is symmetric multilinear, 444
 - locally constant in flat families, 558
 - on a surface, 447
- intersection theory, 331, 443, 508, 531, 589
 - on a surface, 447
- inverse image (scheme), *see* scheme-theoretic preimage
- inverse image sheaf, 58, 59, 316
 - construction, 59
 - exact (for sheaves of abelian groups), 60
 - left adjoint to pushforward, 59
 - preserves stalks, 59
 - right adjoint to extension by zero, 61, 524
- inverse limit, *see* limit
- invertible sheaves, *see* line bundles
- irreducible, 81, 81, *see also* component criterion to be, 269
- schemes, 109
 - Weil divisor, 337
- irreducible object in abelian category, 131, *see also* Jordan–Hölder package
- irregularity (surface), 484
- irrelevant ideal, 105, 105, 107, 193, 195
- isomorphism
 - (category), 5
 - of ringed spaces, 95
 - of schemes, 95, 151
- j -invariant, 422, 428, 431, 431, 433, 437
 - and the cross-ratio, 430
- Jacobian
 - computes cotangent space at rational points, 274
 - computes cotangent space at separable closed points, 462, 472
 - criterion for regularity, 276, 276, 277–279, 282, 421, 461, 471
 - for projective hypersurfaces, 281
 - description of Ω , 462, 470
 - matrix, 274
- Jacobian corank function J_C , 275
 - bounds dimension of variety, 275
 - for projective k -schemes, 275
 - independent of presentation, 275
 - preserved by field extension, 275
- Jacobson radical, 174
- J_C (Jacobian corank function), 275
- Jordan–Hölder package, 130
- Jordan–Hölder Theorem, 130, 131, 131
- $K_\bullet$ (the Koszul complex), 547
- $k[\epsilon]/(\epsilon^2)$, *see* dual numbers
- K_X (canonical divisor), 448
- $\mathcal{K}_X$ (canonical sheaf/bundle), 396, 481
- $k[[x_1, \dots, x_n]]$, 599
- $K(A)$ (fraction field), 10, 111
- $\kappa(p)$ (residue field), 96
- $\kappa(X)$ (Kodaira dimension), 482
- $K(X)$ (function field of X), 111
- $K(X)$ (Grothendieck group of coherent sheaves on X), 453
- $K(X)^{\leq d}$ (dimensional filtration), 453
- k -point/ k -rational point, *see* k -valued point
- k -scheme, *see* A -scheme
- k -smooth, *see* smoothness, over a field
- k -valued point, 154
- k -variety, *see* variety
- $K3$ surfaces, 482, 483, 483
- Kähler differentials, *see* differentials
- ker (kernel), 25
- $\ker_{\text{pre}}$ (presheaf kernel), 49
 - is a sheaf, 50
- kernels, 25, *see also* universal property
 - are limits, 30
 - commute with limits, 30
 - presheaf, 49
 - sheaf, 50
- Kleiman's criterion for ampleness, 395, 451, 454, 456, 456, *see also* Nakai–Moishezon criterion
- Kleiman's Theorem, 456, 456

- Kleiman–Bertini Theorem, 284, 487, 488, **488**, *see also* Bertini’s Theorem
- poor man’s, 488
- Klein quartic curve, 426
- Kodaira dimension $\kappa(X)$, 482, **482**
 - is a birational invariant, 482
- Kodaira Embedding Theorem, 354
- Kodaira Vanishing Theorem, 380, 483, **483**
- Koszul complex, 201, 475, 546, **546**, 547, 555, 581
- Koszul homology, **547**
- Koszul resolution, **547**, **547**
- Krull dimension, *see* dimension
- Krull Intersection Theorem, **300**, 599, 600
- Krull’s Height Theorem, 202, 254, **264**, 265, 283, 294, 403
- Krull’s Principal Ideal Theorem, 202, 256, 261, 262, **262**, 263–266, 272, 282, 286, 399, 585
 - for tangent spaces, **272**, 276
 - proof, 262
- Künneth formula for quasicohherent sheaves, 389, 538, **540**
- $L_i F$ (left derived functors), **514**, **515**
- $\ell(\cdot)$ (length of module or coherent sheaf), **131**, **133**
- Lefschetz principle, 483, 568
- left derived functors, *see* derived functors, left
- left-adjoints, *see* adjoint, left
- left-exact functors, **28**, **29**
 - and derived functors, 379
 - and right derived functors, 515
 - and the FHHF Theorem, 30
 - and universal δ -functors, 517
 - e.g., global section functor, 58
 - e.g., Hom , 29
 - e.g., $\mathcal{H}om$, 58
 - e.g., limits, 30
 - e.g., pushforward, 58
 - e.g., right adjoints, 30, 31
 - e.g., sections over $\mathcal{U}$, 58
 - not pullback of quasicohherent sheaves, 320
- Leibniz rule, **271**
- length, **131**, *see also* Jordan–Hölder package
 - additive in exact sequences, 132
 - at a point, **133**
 - of a coherent sheaf, **133**
 - of a scheme, **133**
 - of dimension 0 scheme, 448
 - of regular sequence, **200**
- Leray spectral sequence, 380, 405, 409, 410, 511, 519, **523**, 525
- Lie groups, 246, 248, 589
- $\varinjlim$ (colimit), **18**
- $\varprojlim$ (limit), **17**
- limits, **17**, *see also* universal property
 - are left-exact, 30, 617
 - commute with kernels, 30
 - commute with limits, 23, 30, 31
 - commute with right adjoints, 23, 30, 31
 - e.g., kernels, 30
- lim is exact if transition maps of source are surjective, **32**
- line, *see also* affine line; projective line; ruling of quadric; tangent in $\mathbb{P}_k^n$, **192**
 - on cone, 273
 - with doubled origin, 92, 99, 114, 233, 238, 239, 242, 296, *see also* recurring (counter)examples
- line bundles, xix, 107, 155, 290, 303, **304**, 305, *see also* algebraic equivalence; ampleness; base-point-freeness; degree of line bundle on curve; global generation; numerically trivial; Picard group; $\mathbb{Q}$ -line bundle; total space; very ampleness
 - $\cong$ invertible sheaves, 305
- additive notation, 444
- and maps to projective schemes, 328
- nef, 394, **395**, 456, *see also* nef cone
 - = limit of amples, 456
- numerical property, **394**
- numerically effective, *see* line bundles, nef
- numerically trivial, 394, **394**, 451, 452, 559
- on $\mathbb{P}_k^n$, 327
 - cohomology, 387
- on projective schemes, 327
- semiample, **356**
- twisting by, **328**
- twisting by divisors, 342
- linear independence of characters, 211, 231
- linear series, 329, **329**, 331, 332, 356, 395, 400, 416, 419, 420, 433, 436, 441, 447, 449, 451, 488, 489, 504, 590
 - (non)degenerate, **329**
 - anticanonical, 597
 - base points, base-point-free, base locus, . . . , 329
 - canonical, 423
 - complete, **329**, 330, 331, 415, 423, 434, 436, 612
 - is always nondegenerate, 329
- linear space, **192**
- linear system, **329**, *see* linear series
- local, *see also* affine-local; stalk-local
 - on the source, **171**
 - on the target, **169**
- local finite presentation, *see also* finite presentation
 - morphisms, 174, 179, 180, **181**, 293, 489, 490, 550, 552, 561–563, 565, 610
 - “limit-preserving”, 181
 - affine-local on target, 181
 - Chevalley’s Theorem, **217**
 - preserved by base change, 218
 - preserved by composition, 181
 - reasonable class, 218
 - over B , **180**
- local freeness of finitely presented sheaf is stalk-local, 313
- local isomorphism, 560
- local rings, **xx**, *see also* regular local rings
 - morphisms of, **150**
 - are faithfully flat, 549
- localization, **10**, *see also* universal property
 - as colimit, 20
 - commutes with direct sums, 11
 - give monomorphisms of schemes, 212
 - is exact, 29–31, 122, 314, 477, 533
 - is flat, 533
 - may not commute with Hom , 11, 29
 - may not commute with infinite products, 11
 - of regular local rings are regular local rings, 298
 - “preserved by base change”, 212
- locally (of) finite presentation, *see* local finite presentation
- locally closed embeddings, 170, 187, **189**, **190**
 - are locally of finite type, 190
 - are monomorphisms, 190, 212
 - are separated, 236
 - finite intersections of, 190, 211, **211**
 - preserved by base change, 211
 - reasonable class, 217
- locally closed immersions, *see* locally closed embeddings
- locally closed subschemes, **190**, *see also* locally closed embeddings
- locally closed subset, **182**
- locally constructible subsets, *see* constructible subsets
- locally finite type
 - A-scheme, **114**
 - over a ring, **114**
- locally finite type morphisms, **178**
 - affine-local on the target, 179
 - preserved by base change, 218
 - preserved by composition, 179
 - reasonable class, 218
- locally finitely presented, *see* local finite presentation
- locally free resolution, *see* resolution
- locally free sheaf, **303–305**, 306, *see also* vector bundle
 - = finitely presented sheaf with free stalks, 313
 - in short exact sequences, 308
 - is quasicohherent, 307
 - rational/regular section, **308**
 - $\sim$ vector bundle, **369**
- locally of finite type, *see* locally finite type
- locally principal closed subscheme, *see* closed subscheme, locally principal
- locally principal divisor, *see* Weil divisor, locally principal
- locally ringed spaces, 67, **96**, *see also* gluing morphisms
 - functions have values; values pull back; zeros of functions pull back, 150
 - morphisms of, **150**, 375
- LocPrin (group of locally principal divisors), **340**
- locus where two morphisms agree, **241**, 242, 247, 248
- long exact sequence, **28**
 - for Ext , 515, 582, 584, 624

- for Ext-functors of $\mathcal{O}$ -modules, 624
- for Tor, 511, 513, 538
- for higher pushforward sheaves, 404, 579
- lower semicontinuity, *see* semicontinuity
- Lüroth's Theorem, 431, 479, **479**
- Lutz–Nagell Theorem, 440
- Lying Over Theorem, 173, 173, 178, 228, 256, 259, 269, 296
 - geometric translation, 173
- $\mathfrak{m}_{\mathbb{P}}$ (maximal ideal of stalk of functions), **42**
- $\mathcal{M}_g$ (moduli space of curves), 428
- manifolds, xvii, xvii, xx, 4, 41, 45, 47, 65–67, 96, 97, 147, 150, 155, 163, 166, 167, 169, 233, 251, 292, 303, 305, 307, 441, 459, 465, 468, 560, *see also* immersion; submersion
 - analytic, 42, 65
 - complex, xx, 67, 74, **97**, 98, 130, 143, 155, 156, 162, 244, 290, 341, 354, 428, 431, 439, 471, 478, 492, 604, *see also* varieties, complex
 - differentiable, 42, 65–67, **97**, 147, 271, 274, 292
 - morphisms of, 66
 - smooth, **97**
 - vector bundle on, **303**
- map, *see also* morphism; rational map (category), **5**, *see* morphism (category)
 - of graded rings, **156**
 - of vector bundles, 309, **309**, 310, 315
- mapping cone, 34, 38, **38**, 386, 573, 574
- Max Noether's $AF + BG$ Theorem, **585**
- maximal ideals of A
 - = closed points of $\text{Spec } A$, 82
 - exist if $A \neq 0$, xx
- maximal rank (quadratic form), **118**, 194, 227
 - = smooth, 281
- maximum principle, 245
- minimal prime ideals, **88**
 - = irreducible components of $\text{Spec } A$, 88
 - are associated primes, 139
 - Noetherian ring has finitely many, 88
- Miracle Flatness Theorem, 581, 585, 586, **586**, 627
 - algebraic version, **586**
- Möbius strip, 303
- Mod_A (category of A -modules), **6**
 - has enough injectives, xx
 - is an abelian category, 24
- $\text{Mod}_{\mathcal{O}_X}$ (category of $\mathcal{O}_X$ -modules), **48**, 127
 - has enough injectives, 523
 - is an abelian category, 58
 - not enough projectives, 525
- modular curve, 430
- modular form, 430
- modules, *see also* coherent; depth; finitely generated; finitely presented; flatness; graded; length; Noetherian; Kähler differentials; projective
 - Mod_A has enough injectives, **515**, 516
 - Cohen–Macaulay, **584**, *see also* Cohen–Macaulayness
 - free implies projective implies flat, 533
 - topological, **599**
- moduli space, 167, 296, 297, 329, 423–425, 427, 428, 430, 434, 532, 571, 575, 576, 589, 592, 594, *see also* parameter space
 - of curves $\mathcal{M}_g$, 428, 532, 569
 - universal families, 577
- monoid, **6**
- monomorphisms, **14**
 - are separated, 236
 - are universally injective, 221
 - diagonal characterization, 15
 - e.g., closed embeddings, 189
 - e.g., locally closed embeddings, 190
 - e.g., open embeddings, 148
 - e.g., open embeddings of schemes, 171
 - local on the target, 235
 - preserved by base change, 235
 - preserved by composition, 15, 212
 - reasonable class, 235
- Mor, **5**
- Mordell's conjecture (Faltings's Theorem), 424
- Mordell–Weil Theorem, group, and rank, 429, 439
- morphisms, **5**, *see also* diagonal morphism; gluing morphisms; morphisms of schemes
 - in a category, **5**
 - more fundamental than objects, 169
 - of (pre)sheaves, **47**
 - induce morphisms of stalks, 48
 - of graded rings, 156
 - of local rings, **150**
 - of locally ringed spaces, **150**
 - of manifolds, 66
 - of ringed spaces, **148**
 - of sheaves determined by stalks, **51**
- morphisms of schemes, xvii, 149, **151**, *see also* affine morphisms; closed embeddings; gluing morphisms; locally closed embeddings; open embeddings; projective morphisms; proper morphisms; quasilinear morphisms; quasiseparated morphisms; reasonable class; separated morphisms; smooth morphisms
 - and tangent spaces, 274
 - criterion to be a closed embedding, 413, **413**
 - dominant, **158**
 - from Spec of local ring, 153
 - glue, 150
 - of group schemes, **166**
 - to affine schemes, 152
- multiplicative group $\mathbb{G}_m$, **164**
- multiplicative subset, **10**
- multiplicity of function at regular point, **503**
- multiplicity of zero or pole, *see* order of pole/zero
- $\text{mult}_{\mathbb{P}} D$, **506**
- Mumford complex, 572
- $\mathfrak{N}$ (nilradical), **75**
- $N^1(X) = \text{Pic } X / \text{Pic}^{\tau} X$, **394**, 395
- $N_{\mathbb{Q}}^1(x) = N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, **395**, 395, 449–451, 454, 455, 506
- $\mathcal{N}_{D/X}$ (normal bundle to effective Cartier divisor), **347**
- $NS(X)$ (Néron–Severi group), 394
- $N_X Y$ (normal cone), **499**
- $\mathcal{N}_{X/Y}^{\vee}$ (conormal sheaf), **465**
- $\mathcal{N}_{X/Y}$ (normal sheaf), **465**
- n -fold, **252**
- n -plane, **192**
- Nagata's Compactification Theorem, 611, **611**
- Nagata's Lemma, 117, 345, **345**, 346
- Nagell–Lutz Theorem, 440
- Nakai–Moishezon criterion for ampleness, 417, 443, 447, 454, **454**, 455–457, *see also* Kleiman's criterion
- Nakayama's Lemma, 130, 171, 173, 174, **174**, 202, 262, 265, 279, 286–288, 300, 314, 415, 474, 490, 542, 553, 564, 570, 584
 - for half-exact functors, 552, 553
- geometric, 312–314, 352, 506
- graded version, 548
- natural isomorphism of functors, **8**
- natural transformation of functors, **8**, 9, 16, 21, 30, 48, 208, 209, 364, 517, 620
- nef, *see* line bundles, nef
- nef cone, 394, 395, **395**, 449, 451, 456
 - = closure of ample cone, 456
- Néron–Severi group $NS(X)$, 394, **559**
- Néron–Severi Theorem (Theorem of the Base), **394**, 395, *see also* Picard number $\rho(X)$
- nilpotent, xvi, **75**, **75**
 - = vanishes at every associated point, 135
- nilradical, **75**
 - is intersection of all primes, 75
- nodal cubic curve, 177, **229**, 499, 502, 531, 542, 544, 546
- normalization of, 229, 544
- node, 229, 281, **281**, 503, 599, **603**, 604
- normalization, 414
- Noether normalization, 231, 254, 256, **256**, 257, 258, 267, 268, 357
 - geometric, 256
 - projective, **627**
- Noether–Lefschetz Theorem, 261
- Noetherian, *see also* scheme, locally Noetherian; scheme, Noetherian
 - hypotheses, xvii, xix, 86, 87, 575
 - induction, **85**
 - local ring has finite dimension, 254, 265
 - module, **86**
 - rings, 82, 84, **85**

- Noetherian (cont.)
 - e.g., principal ideal domains, 85
 - finitely generated module over, 136, 138
 - finitely many minimal primes, 88
 - important facts about, 85, 86
 - infinite-dimensional, 251, 254, 265
 - preserved by quotients and localization, 85
 - topological space, 84, **84**, 85, 86, 109, 113, 182, 387
 - finitely many irreducible components, 84
 - open subsets are quasicompact, 86
- non-affine scheme (example), 96, 98–101, 178
- non-Archimedean, 127, 130, 143, 287
- non-zero-divisor, 10, **10**, 199–202, 347, 466, 495, 498, 500, 501, 504, 533, 535, 553, 554, 582–586, *see also* zero-divisor
- nondegenerate linear series, **329**
- nonprojective proper variety, *see* proper nonprojective variety (examples)
- nonsingular, *see* regular
- normal (ring/scheme), 98, 115, 116, **116**, *see also* normalization; Serre’s criterion for normality
 - = $R_1 + S_2$, *see* Serre’s criterion for normality
 - implied by factorial, 117
 - may not be integral, 116
 - may not be UFD, 118
 - stalk-local property, 116
- normal bundle, **465**, *see also* normal exact sequence; normal sheaf
 - projectivized, 498, 500
 - to effective Cartier divisor, **347**, 447–449, 465
 - to exceptional divisors, 498–500, 502, 612
 - to regular embedding, 465
- normal cone, **499**, *see also* deformation to the normal cone
 - projective completion, 508
- normal exact sequence for smooth varieties, 468, **471**, *see also* conormal, exact sequence
- normal form for quadrics, *see* diagonalizing quadratic forms
- normal sheaf, **465**, 632, *see also* normal bundle
 - to regular embedding is locally free, 465
- normalization, 177, 228, **228**, 229, 231, 252, 290, 296, 358–360, 367, 368, 370, 371, 407, 417, 421, 425, 482, 489, 542, 544, 604, 608, 613, *see also* integral closure; universal property
 - finiteness of, 231, 357, 360
 - in field extension, 230, **230**, 231, 282, 357
 - is a birational morphism, 228, 231
 - is integral and surjective, 228
 - not flat, 542
 - of cusp, 413, 489, 502, 550
 - of node, 360, 414
- Nullstellensatz, 71, **71**, 72, 76, 77, 82, 87, 109, 185, 205, 280, 429
 - proofs, 183, 255
 - weak, **71**, 76, 83, 155, 219
- number field, **230**, *see also* ring, of integers in number field
- numerical criterion
 - for ampleness, *see* Kleiman’s criterion
 - for line bundle on curve to be base-point-free, 416, 417
 - for line bundle on curve to be very ample, 417
- numerical equivalence, **394**, 445, 454
 - implied by algebraic equivalence, 559
- numerically effective line bundles, *see* line bundles, nef
- numerically trivial line bundles, *see* line bundle, numerically trivial
- $\mathcal{O}$ (structure sheaf), **46**, **89**
- $\mathcal{O}$ -modules, **47**, 121
 - Ext-functors for, *see* Ext functors
 - may not have enough projectives, 525
 - not always quasicohherent, 123
 - on X form an abelian category, 58, 305
 - on a base, 55
 - tensor product, 58
- $\mathcal{O}(1)$ on relative Proj, **372**
- $\mathcal{O}_{\mathbb{P}^1}(\mathfrak{m})$, $\mathcal{O}_{\mathbb{P}^n}(\mathfrak{m})$, **325**, **326**
- $\mathcal{O}^{\oplus 1} = \mathcal{O}^1$, free sheaf of “rank 1”, 303
- $\mathcal{O}_K$ (ring of integers in number field K), **230**
- $\mathcal{O}_{\text{Spec } \Lambda}$, 89
- $\mathcal{O}_X(-D)$ for D an effective Cartier divisor, **347**
- $\mathcal{O}_X(D)$, **339**
 - for D an effective Cartier divisor, **347**
 - is line bundle if X Noetherian factorial, 341
 - is quasicohherent, 339
- $\mathcal{O}_{X,p}$, **46**
- $\mathcal{O}$ -connected morphisms, 570, **570**, 571, 607–610
 - + proper implies connected fibers, 606
 - + proper implies surjective, 570, 610
 - + proper not reasonable class, 570
 - local on the target, 570
 - not preserved by base change, 570
 - preserved by composition, 570
 - preserved by flat base change, 570
- object
 - in a category, **5**, *see also* automorphism; group object
 - initial/final/zero, **9**
 - in an abelian category, *see* acyclic; injective; irreducible; projective; quotient; simple; subobject
- Oka’s Theorem, 130, 143
- open embeddings, **170**, 611
 - are flat, 534
 - are locally finitely presented, 181
 - are locally of finite type, 179
 - are monomorphisms, 148, 212
 - are separated, 240
 - base change by, 210
 - local on the target, 170
 - not local on the source, 171
 - of ringed spaces, 46, **148**
 - of schemes are monomorphisms, 171
 - preserved by base change, 170, 217
 - preserved by composition, 170
 - reasonable class, 170
- open immersions, *see* open embeddings
- open maps of topological spaces, **222**, **532**, *see also* universally open morphisms
 - e.g., flat morphisms (in reasonable situations), 549
- open subfunctors, **209**
- open subschemes, **96**, **170**, *see also* affine open subscheme/subset
- open (sub)sets, *see also* affine open subscheme/subset
 - distinguished $D(f)$, 73, 77, **79**
 - form base for Zariski topology, 80
 - projective distinguished $D_+(f)$, 80, 106, **106**
 - form base for Zariski topology, **106**
- opp (opposite category), **8**
- opposite category, **8**, 151, 160, 359
 - Rings* and affine schemes, 151
- orbifolds, xvii, 428
- order of pole/zero, **288**, **289**
- ordinary double point, 603
- ordinary multiple (m -fold) point, **603**
- orientation (of spectral sequence), **33**, **36**
- $\mathbb{P}_k^1$ (projective line), **100**
- $\mathbb{P}(d_1, \dots, d_n)$ (weighted projective space), **196**
- $\mathbb{P}\mathcal{F}$ (projectivization of finite type quasicohherent sheaf, with caution over notation), **372**
- $\mathbb{P}^{n\vee}$ (dual projective space), **283**
- $\mathbb{P}^n$ (projective space), **107**
- $\mathbb{P}^n$ -bundle (projective bundle), **373**
- $\mathbb{P}_\Lambda^n$ (projective space over Λ), **101**
 - is quasicompact, 109
- $\mathbb{P}_k^n$ (projective space over a field k), **100**
- $\mathbb{P}V$ (projectivization of vector space), **108**
- p -adics, 16, **18**, 30, 288, 298, 541, 599
- page of spectral sequence, **33**
- Pappus’s Theorem, 419, **419**, 420
- parabola, 70, 73, 74, 83, 84, 94, 104, 214, 215, 220, 361, 401, 560
- parameter space, 261, 266, 393, 571, 578, 580, 591, *see also* moduli space
- partial flag variety, *see* flag variety
- partially ordered set (poset), **6**, *see also* filtered
- partition of unity, 90, 115, 171, 385, 539
- Pascal’s “Mystical Hexagon” Theorem, 419, **419**, 420, 589
- perfect field, *see* field, perfect

- perfect pairing of vector bundles, **311**
- Pfaffian, **434**
- $\text{Pic}(\mathbb{A}_k^n) = 0$, **341**
- $\text{Pic}(\mathbb{P}_k^n) \cong \mathbb{Z}$, **342**
- $\text{Pic}^0(X)$, **558**
- $\text{Pic}^0(E)$, **429, 434, 441**
 $\cong E$, **433**
- $\text{Pic}^d(C)$ (the degree d line bundles on C), **391**
- $\text{Pic}^\tau(X)$ (group of numerically trivial line bundles), **394**
- $\text{Pic}(X)$ (Picard group), **306**
- Picard group, **306, 307, 326, 338, 340–342, 344, 345, 347, 392, 394, 395, 429, 437, 439, 441, 447–450, 453, 506, 558, 559, 571**, *see also* line bundles
 - of Spec of UFD is 0, **341**
 - of cone, **344**
 - of projective space, **340**
 - torsion example, **344**
- Picard number $\rho(X)$, **395, 484, 614**
 finiteness of, *see also* Néron–Severi Theorem (Theorem of the Base) complex case, **395**
- PID, *see* principal ideal domains
- pinched plane, **290, 328, 587**
 normalization of, **229, 290**
 not Cohen–Macaulay, **588**
- plane, **192**, *see also* affine space;
 curves, plane
 with doubled origin, *see* recurring (counter)examples
- Plücker
 embedding, **167, 364, 365, 427, 434**
 equations, **364**
- plurigenus (birational invariant), **482**
- point, *see also* closed point; general point; generic point; geometric point
 field-valued / ring-valued / scheme-valued, **154**
- pole, **115, 288, 289**
 of order n for discrete valuation rings, **288**
 of order n for Noetherian scheme, **289**
- Poncellet’s Porism, **438, 438, 439**
- poset, *see* partially ordered set
- power series (formal, convergent, etc.), **18, 18, 502, 599, 612**
- preimage, **214**, *see also* scheme-theoretic preimage
- preserved by base change
 e.g., any reasonable class of morphisms, **169**
 e.g., projective morphisms, **374, 375**
 not “has finite fibers”, **218**
 not closed maps, **244**
 not injective morphisms of schemes, **218**
 not $\mathcal{O}$ -connected morphisms, **570**
- preserved by composition
 e.g., any reasonable class of morphisms, **169**
 e.g., $\mathcal{O}$ -connected morphisms, **570**
- projective morphisms, usually, **377**
 commutes with base change, **372**
 finite generation in degree 1, **371**
 no great universal property, **370**
 $\mathcal{O}(1)$, **372**
 projection formula, **400, 405, 407, 446, 446, 454, 629**
 for quasicoherent sheaves, **321**
 more general, **446**
- projective
 A-scheme, **104**
 S-scheme, **335**
 Y-scheme, **374**
 over Y , **374**
- projective morphisms, **375**
- projective bundle, **155, 261, 285, 373, 571**
- projective change of coordinates, **343**, *see also* coordinates, projective; projective transformation
- projective completion, **195, 546**
 of normal cone, **508**
- projective cone, **103, 108, 195, 195, 196, 257**
 relative version, **373**
- projective coordinates, *see* coordinates
- projective distinguished open set, *see* open set
- projective limit, *see* limit
- projective line, **100, 472**
- projective module
 = direct summand of free module, **514**
 = free if finitely generated over local ring, **541**
 e.g., free module, **514**
 is flat, **514**
- projective morphisms, **335, 359, 367, 373, 374, 374, 375, 377, 382, 383, 404, 406, 410, 411, 413, 414, 543, 557–559, 592, 607, 613, 614, 627**
 are proper, **374**
 not a reasonable class, **375**
 notion not local on target, **375, 559**
 preserved by base change, **374, 375**
 preserved by product (despite unreasonableness), **375**
 usually preserved by composition, **375, 377, 411, 497, 559**
- projective objects, **514**, *see also* injective objects
- projective resolution, *see* resolution
- projective schemes, *see* scheme, projective
- projective space, **100, 107**, *see also* coordinates, projective x_i ;
 projective, change of coordinates; x_i / x_j (coordinates on patches for projective space)
 automorphisms, **343, 425**
 cohomology of line bundles on, **387**
 functions on, **101**
 functorial definition, **329**
 is finite type, **179**
 is separated, **238**
 line bundles on, **325**
 Serre duality for, **621, 625**
 tautological bundle, **369**
- projective transformation, **343**, *see also* projective change of coordinates
- preserved by products
 e.g., any reasonable class of morphisms, **169, 233**
 e.g., projective morphisms (despite unreasonableness), **375**
- presheaf, **43**
 $\neq$ sheaf (examples), **44**
 as contravariant functor, **43**
 cokernel, **49**
 espace étalé, *see* sheaf, space of sections
 germ, **44**, *see* germ
 image, **49**
 kernel, **49**
 is a sheaf, **50**
 on a base, **54**
 stalk, **54**
 quotient, **49**
 section of, **43**
 separated, **44**
 stalk, **44**
- prevarieties (complex analytic), **155**, *see also* varieties, complex analytic
- primary decomposition, **134, 142, 143**
- primary ideal, **134, 142, 142, 143, 262**
- prime avoidance, **259, 259, 263, 265, 535**
- prime ideals, **88**, *see also* maximal ideals; minimal prime ideals; primary ideals
 are the points of Spec, **67**
 correspond to irreducible closed subsets, **87**
- Prin (group of principal Weil divisors), **340**
- principal divisor, *see* effective Cartier divisors, principal; Weil divisor, principal
- principal fractional ideal, **307**, *see* fractional ideal
- principal ideal, **70, 98, 136, 199, 273, 286, 289, 307**, *see also* Krull’s Principal Ideal Theorem
- principal ideal domains, **70, 282, 286, 287, 313, 327, 429, 532, 533, 541**, *see also* finitely generated modules, over principal ideal domains (classification)
 are Noetherian, **85**
- Principal Ideal Theorem, *see* Krull’s Principal Ideal Theorem
- principal Weil divisor, *see* Weil divisor, principal
- principality not affine-local, **199, 200, 441**
- product, **3, 3, 17**, *see also* fibered product; ideal sheaves, product of; universal property
 of irreducible varieties / $\bar{k}$ is irreducible, **218**
 of schemes, **205**
 of schemes isn’t product of sets, **205**
 of varieties, **239**
- profinite topology, **211**
- $\text{Proj } S_\bullet$, **102, 105, 107**
 $\text{Proj}_\Lambda S_\bullet$, **370**
- Proj (relative Proj), **367, 370, 371**
 and affine base change, **370**

- projective variety, *see* variety, projective
- projectively normal, **588**
- projectivization
 - of finite type quasicoherent sheaf, **372**, *see also* normal bundle, projectivized; tangent cone, projectivized
 - of vector space, **108**
- projectivized tangent cone, *see* tangent cone, projectivized
- proper (=strict) transform (of blow-up), **448**, **492**, **493**, **495**, **496**, **502**, **506**, **559**, **597**, **614**
 - of nodal curve, **502**
- proper (=strict=birational) transform (of blow-up), **493**
- proper morphisms, **82**, **102**, **233**, **244**, *see also* Chow's Lemma; Grothendieck's Coherence Theorem; valuative criteria
 - e.g., projective morphisms, **374**
 - local on the target, **244**
 - of projective schemes, **245**
 - over A , **244**
 - preserved by base change, **244**, **247**
 - preserved by composition, **244**
 - preserved by product, **244**
 - reasonable class, **244**
- proper nonprojective variety (examples), **245**, **332**, **449**, **559**
- property R_k , *see* R_k
- property S_k , *see* S_k
- pseudo-morphisms, **158**
- Puiseux series, **288**
- pullback, *see also* π^* ; universal property
 - diagram/square, **13**, **213**, *see also* base change diagram/square; Cartesian diagram/square; fibered diagram/square
- in cohomology, **380**
- of $\mathcal{O}$ -modules, **148**, *see also* π^* of a section, **319**
- of quasicoherent sheaves, **315**, **327**, *see also* π^*
 - construction via affines, **317**
 - explicit construction, **319**
 - left-adjoint to pushforward for qcqs morphisms, **319**
 - not left-exact, **320**
 - of ideal sheaves (subtleties), **321**
 - preserves $\otimes$, **320**
 - preserves finite type and finite presentation, **320**
 - pullback of vector bundle is vector bundle, **319**
 - right-exact functor, **320**
 - stalks and fibers, **320**
- of schemes, **213**, *see also* base change; scheme-theoretic, pullback
 - of closed subschemes by flat morphisms, **536**
- pure dimension, **252**
- purely inseparable, *see* field extension
- push-pull map
 - for $\mathcal{O}$ -modules, **149**
 - for cohomology, **405**, **405**, **567**, *see also* Cohomology and Base Change Theorem
 - isomorphism for flat base change, **536**
 - for quasicoherent sheaves, **321**
 - for sheaves, **60**
- pushforward
 - occasionally left-adjoint to $\pi^!$?, **630**, **631**
 - of $\mathcal{O}$ -modules, **148**
 - of quasicoherent sheaves, **315**
 - is exact for affine morphisms, **316**
 - right adjoint to inverse image, **59**
 - sheaf, **46**, *see* sheaf, pushforward
- Pythagorean triples, **160**–**162**, **418**
- $\mathbb{Q}$ -Cartier Weil divisor, **345**
- $\mathbb{Q}$ -line bundles, **395**
 - ampleness, **395**
 - nef, **395**
- $QCoh_X$ (category of quasicoherent sheaves on X), **121**, **127**
 - is an abelian category, **127**
- qcqs (quasicoherent quasiseparated), **xix**, **110**, **110**, **126**, **175**, **175**, **176**, **181**, **297**, **316**, **319**–**321**, **342**, **355**, **375**, **378**, **408**, **611**
 - down-to-earth interpretation, **110**
- Qcqs Lemma, **126**, **176**
 - generalization, **342**
- quadratic forms, *see* diagonalizing
 - quadratic forms; maximal rank; rank
- quadric hypersurface, **104**, **192**, **194**, **365**, **427**
- quadric surface, **92**, **102**, **116**, **118**, **161**, **194**, **227**, **256**, **273**, **279**, **290**, **346**, **403**, **426**, **427**, **438**, **505**, *see also* del Pezzo surfaces; cone over quadric surface; recurring (counter)examples
- quartic, **482**
 - hypersurface, **192**
- plane curve, **221**, **425**–**427**, **433**, **589**, *see also* Klein quartic curve
- surface, **261**, **482**
 - most contain no lines, **261**
- quasiaffine morphisms, **354**, **378**, **378**
 - affine-local on target, **378**
 - e.g., quasicoherent locally closed embeddings, **378**
 - preserved by base change, **378**
 - preserved by composition, **378**
 - properties of, **378**
 - reasonable class, **378**
- quasiaffine schemes, **378**
 - $= \mathcal{O}_X$ ample, **378**
- quasicoherent sheaves, **xvii**, **91**, **121**, **121**, **303**, **305**, *see also* coherent; finite type; finitely presented; global generation; Künneth formula
 - characterization using distinguished affine base, **123**, **125**
 - corresponding to graded modules, **322**, **322**, **348**
 - corresponding to modules, **121**
- higher pushforward, *see* sheaf, higher pushforward
- local nature of, **322**
- morphisms of, **127**
- not all locally free, **307**
- not necessarily preserved by $\mathcal{H}om$, **311**
- on X form an abelian category $QCoh_X$, **127**, **188**, **303**
- on affines have no derived functor cohomology, **528**
- on projective schemes, **322**
- on ringed spaces, **128**
- pullback of, **319**
- pushforward, *see* sheaf, pushforward
- pushforward usually quasicoherent, **316**
- restriction of, **317**
- quasicoherent morphisms, **xix**, **175**
 - affine-local on the target, **175**
 - preserved by base change, **218**
 - preserved by composition, **175**
 - reasonable class, **218**
- quasicoherent schemes, **109**
 - e.g., affine schemes, **82**
 - have closed points, **109**
- quasicoherent topological space, **81**
- quasifinite morphisms, **177**, **179**, **610**, **611**
 - from proper to separated is finite, **246**
- how to picture, **180**, **611**
- may not be finite, **179**
- most are open subsets of finite morphisms, **611**
- preserved by base change, **218**
- reasonable class, **218**
- to fields are finite, **179**, **183**, **216**, **218**, **611**
- quasisomorphism, **384**, **572**, **573**
- quasiprojective, *see also* scheme; variety
 - implies finite type, **114**
 - implies separated, **240**
 - morphisms, **377**
- quasiseparated morphisms, **169**, **174**, **175**, **175**, **176**, **181**, **235**, **235**, **237**, **296**, **297**, **316**, **319**–**321**, **375**, **530**, **536**
 - affine-local on target, **175**
 - e.g., morphisms from locally Noetherian schemes, **175**
 - e.g., morphisms from quasiseparated schemes, **175**
 - preserved by base change, **218**
 - preserved by composition, **175**
 - reasonable class, **235**
- quasiseparated schemes, **xix**, **109**, **109**, **110**, **126**, **175**, **176**, **342**, **355**, **357**, **378**, **611**
 - e.g., affine schemes, **110**
 - e.g., locally Noetherian schemes, **113**, **175**
 - e.g., projective A -schemes, **110**
 - not A^∞ with doubled origin, **110**, *see also* recurring (counter)examples
- quasiseparated topological space, **109**

- Quillen–Suslin Theorem, 341
- quotient
 - object, **25**
 - of monomorphism in abelian category, **25**
 - presheaf, **49**
 - sheaf, **53**
- $R^1\pi_*$ (higher pushforward sheaf), **404**
- R^1F (right derived functor), **515**
- R_k (regular in codimension k)
 - R_0 = generically reduced, **587**
- R_1 (regular in codimension 1), **289**
- radical ideal, **78**
- radical of ideal, **78**
 - = intersection of primes containing ideal, **78**
 - commutes with finite intersections, **78**
- radicial morphisms, *see* universally injective (radicial) morphisms
- ramification
 - divisor, **477**
 - and number of preimages, **478**
 - locus, **477, 489**
 - order, **478**
 - point, **421, 478**
 - tame/wild, **478**
- rank
 - of (locally) free sheaf, **303**
 - of coherent sheaf is additive in exact sequences, **314**
 - of finite type quasicoherent sheaf at point, **314**
 - of $\mathcal{O}_X$ -module, **97**
 - of quadratic form, **118**, *see also* maximal rank (quadratic form)
 - of quasicoherent sheaf, **314**
 - of vector bundle on manifold, **304**
- rank_p (rank of $\mathcal{O}$ -module at p), **97**
- rational function, **46, 67–69, 98, 111, 115, 142**
- rational map, **158**, *see also* birational; degree; domain of definition; dominant; graph; Reduced-to-Separated Theorem
- rational normal curve, **81, 194, 196, 331, 403, 423, 451**, *see also* twisted cubic curve; Veronese embedding
 - not complete intersection for $d > 2$, **403**
- rational point of k -scheme, *see* k -valued point
- rational section
 - of line bundle, **337**
 - of locally free sheaf, **308**
- rational variety, **159, 161, 595**
- reasonable classes of morphisms, **169**
 - not projective morphisms, **375**
 - by definition include all isomorphisms, are preserved by composition, are preserved by base change, are local on the target, **169**
 - e.g., closed embeddings, **217**
 - e.g., étale morphisms, **293**
 - e.g., faithfully flat morphisms, **549**
 - e.g., finitely presented, **235**
 - e.g., flat morphisms, **535**
 - e.g., flat of relative dimension n , **551**
 - e.g., isomorphisms, **170**
 - e.g., locally closed embeddings, **217**
 - e.g., monomorphisms, **235**
 - e.g., open embeddings, **170**
 - e.g., proper morphisms, **244**
 - e.g., quasiffine morphisms, **378**
 - e.g., quasicompact; affine; finite; integral; locally finite type; finite type; locally finitely presented, **218**
 - e.g., quasifinite morphisms, **218**
 - e.g., quasiseparated morphisms, **235**
 - e.g., separated morphisms, **236**
 - e.g., smooth morphisms, **293**
 - e.g., surjective morphisms, **218**
 - e.g., universally closed morphisms, **243**
 - e.g., universally injective (radicial) morphisms, **222, 240**
 - e.g., unramified morphisms, **489**
 - P reasonable implies $P\delta$ reasonable, **234**
 - preserved by products, **169**
 - satisfy the Cancellation Theorem, **234**
- recurring (counter)examples, xviii, **92**
 - $\prod_{i=1}^{\infty} \mathbb{F}_2$, **82, 84, 92, 542**
 - $k(x) \otimes_k k(y)$, **92, 212, 261, 551**
 - $x^2 = 0$ in the projective plane, **92, 111, 246, 403**
 - (cone over) quadric surface, **92, 102, 116, 118, 161, 192, 194, 227, 256, 273, 279, 281, 290, 344–346, 426, 438, 505**
 - affine space minus the origin, **82, 86, 92, 98, 178, 179, 196, 387**
 - affine space with doubled origin, **92, 99, 100, 110, 178, 233, 238, 239, 242, 296**
 - embedded point on line, **92, 94, 111, 134, 141, 142, 242, 536**
 - infinite disjoint union of schemes, **92, 96, 98, 111, 126, 197, 199, 561, 601**
 - $\text{Spec } \mathbb{Q} \rightarrow \text{Spec } \mathbb{Q}$, **92, 172, 178, 179, 206, 211, 218, 220**
 - two planes meeting at a point, **91, 92, 396, 403, 542, 543, 554, 583, 584, 586–588, 623**
- reduced subscheme structure, **199, 296, 487, 507, 635**
- Reduced-to-Separated Theorem, **157, 241, 242, 243, 295, 335, 595**
- reducedness, **110**
 - is stalk-local, **110, 113, 116, 225**
 - not open condition in general, **111**
 - of ring, **75**
 - often open condition, **199**
 - reduced rings have no embedded primes, **140**
- reducible, *see* irreducible
- reduction
 - of a morphism, **232**
 - of a scheme, **199, 232, 371, 393, 407, 546**, *see also* universal property
- Rees algebra, **299, 497, 616**
- reflexive sheaf, **312, 465**
- regular
 - $\mathbb{A}_k^n$ is, **283**
 - does not imply smooth over field, **278**
 - implies Cohen–Macaulay, **585**
 - in codimension 1, **289**, *see* R_1
 - in codimension k , **587**, *see* R_k
 - ring, **276**, *see also* regular local ring scheme, **276**
 - stalk-local property, **276**
- regular embeddings, **199, 202, 202, 263, 274, 278, 279, 465, 466, 472, 498–500, 537, 581, 585, 587, 619, 632, 633**
- blowing up, **499**
- e.g., regular in regular, **278**
- in smooth k -schemes are Cohen–Macaulay, **585**
- normal bundle is locally free, **465**
- open condition, **202**
- pull back under flat morphisms, **537**
- regular function, **46, 142**, *see* function
- regular local rings, **276, 276, 278, 279, 283, 286–289, 474, 563, 586, 600, 608**
- are Cohen–Macaulay, **299, 581, 585, 588**
- are integral domains, **278**
- are integrally closed, **299, 581, 588**
- are Noetherian by definition, **276**
- are unique factorization domains, *see* Auslander–Buchsbaum Theorem
- finitely generated modules have finite free resolutions, **547**
- preserved by localization, **277, 298, 461, 485, 587, 588**
- regular in regular is regular embedding, **278**
- regular section of locally free sheaf, **308**
- regular sequence, **134, 199, 200, 200, 201–203, 465, 466, 499–501, 555, 581–585**
- and the Koszul complex, **547**
- cohomological criterion for existence, **582**
- length, **200**
- maximal, **582**
- order doesn’t matter (over Noetherian local rings), **201**
- order matters, **201**
- preserved by flat base change, **200**
- relative cotangent sheaf, **468**
- relative different and discriminant ideals, **490**
- relative dimension, *see* flat morphisms, relative dimension n ; smooth morphisms, of relative dimension n
- relative Proj and relative Spec, *see* Proj and Spec

- relative tangent sheaf, **468**
- relatively ample, *see* ampleness, with respect to π
- relatively base-point-free, *see* base-point-freeness, with respect to π
- relatively globally generated, *see* global generation, with respect to π
- relatively very ample, *see* very ample, with respect to π
- Remainder Theorem, 500
- representable functor, 15, **16**, **163**, **163**, 208–210, 343, 362, 363, 365, 367, 577–580, *see also* fibered product; Grassmannian; Hilbert scheme; Yoneda’s Lemma; Zariski sheaf
- residue field $\kappa(p)$ of a point p , **96**
- Residue Theorem, 361, 622
- resolution (complexes), 351, *see also* Cartan-Eilenberg resolution; Čech resolution; Ext functors
 - acyclic, 517, **519**
 - flat, 519
 - free, 512, 517, 537, 547, 625
 - graded, 548
 - injective, 517, 519–522, 526, 528, 582, 624, 625, 631
 - locally free, 624, 625, 632
 - projective, 514
 - resolution (singularities), 491, **492**, 502, 503, 604, 615, *see also* Alteration Theorem; Hironaka’s Theorem; Weak Factorization Theorem
 - for curves, 608
 - for surfaces, 608, 615
 - of indeterminacy of rational maps, 491
 - small, 477, 505, **505**
 - nonisomorphic, 505
- restriction, *see also* pullback; quasicoherent sheaves
 - map (sections of sheaves), **43**
 - of scalars, 23
 - of sheaf to open subset $\mathcal{U} \mid \mathcal{U}$, **45**
 - of Weil divisor, **337**
- resultant, 185
- Riemann surface, xx, 66, 359, 424, 479, 622
 - genus, 392
- Riemann–Hurwitz formula, 413, 432, 473, 474, 476, **478**, 479, 480, 482
 - applications, 479
 - hyperelliptic, **421**, 422, 424, 429, 474
- Riemann–Roch, 390, **390**, 395, 396, 400, 416, 426, 428, 432, 446, 448, *see also* Asymptotic Riemann–Roch; Grothendieck–Riemann–Roch; Hirzebruch–Riemann–Roch
 - common formulation, 392
 - for coherent sheaves or vector bundles, **392**
 - for nonreduced curves, **393**
 - for surfaces, 446, 448, **448**, 451
 - restatement using Serre duality, 396
- right adjoints, *see* adjoints, right
- right derived functors, *see* derived functors, right
- right-exact functors, **12**, **28**, **29**
 - and derived functors, 379
 - and left derived functors, 515
 - and the FHHF Theorem, 29
 - e.g., left adjoints, 30, 31
 - e.g., tensor product, 12, 29
 - may not commute with colimits, 32
- Rigidity Lemma, 246, 247, **247**, 248, 383
- ring, *see also* Noetherian; graded ring; local rings; regular local rings
 - catenary, **258**, 587
 - of integers in number field, 230, **230**, 289, 303, 307, 345, 490
 - topological, **599**
- ring scheme, **165**
- ringed spaces, xvii, **46**, *see also* locally ringed spaces; gluing morphisms; structure sheaf
 - (locally) free sheaf on, 303, 305
 - (quasi)coherent sheaves on, **128**, 130
 - functions on, **46**, 67, 96
 - gluing along open sets, 99
 - higher pushforwards for, 523
 - isomorphism of, 95, **95**, 96, 148, 610
 - Leray spectral sequence, 523
 - morphisms of, 147, 148, **148**, 149–151, 318, 319, 367, 523–525
 - open embedding of, 46, 148, 151, 170, 318
 - Picard functor for, 307
 - pullback for, 317
 - pushforward of $\mathcal{O}$ -modules, 148
- Rings (category of rings), **6**
- ruled surface, **373**, 449, 450, 614, *see also* elementary transformation; Hirzebruch surface
 - self-intersection of section of, 450
- ruling of quadric, 102, 194, **194**, 227, 261, 279, 346, 438, 439
 - of higher-dimensional quadrics, 194, **195**
 - of quadric hypersurfaces in $\mathbb{P}^5$, 365
- $S_\bullet$ (graded ring), **104**
- S_+ (irrelevant ideal), **105**
- S_κ , 587, **587**, 588
 - and Cohen–Macaulayness, 587
 - S_1 = no embedded points, 587
 - S_2 (elementary interpretation), 587
- saturated graded module, **350**
- saturation functor, 322, 348, 349, **349**, 350
 - left-adjoint to forgetful functor, 348, 350
 - may not be injective or surjective, 349
- Sch (category of schemes), **151**
- Sch_S (category of schemes over S), **152**
- scheme-theoretic, **141**, 189
 - base locus, **329**
 - closure, 141, 196, **198**, 242, 411, 494–496, 543
 - fiber, 141, 169, 214, **214**, 215, 222, 563, 593
- image, 141, 196, **196**, 197, 198, 243, 245, 329
 - and set-theoretic image, 198
 - of proper is proper, 245
 - usually computable affine-locally, 197
- intersection, 141
 - of closed subschemes, 94, **188**, 203, 273, 285, 329, 401, 447, 448, 493, 496
 - of locally closed subschemes, **218**
- inverse image, *see* scheme-theoretic, preimage
- preimage, 141, 213, **213**, 214, 315, 360, 448, 493, 496, 497, 593, 613
- pullback, *see* scheme-theoretic, preimage
- support (of a module or finite type quasicoherent sheaf), 141, **189**, 393, **410**
- union (finite) of closed subschemes, **188**, 273, 508
- scheme-theoretically dense, **141**
- schemes, **95**, *see also* disjoint union; finite type; functions; gluing; isomorphism; locally of finite type; morphisms; gluing morphisms; normal scheme; quasiaffine; quasiseparatedness; regular; ...
 - affine, **89**, **95**, *see also* Serre’s cohomological criterion for affineness
 - is quasicompact, 82
 - is quasiseparated, 110
 - morphisms to, 152
- locally Noetherian, 110, **113**
 - implies quasiseparated, 113
- Noetherian, xix, 109, **113**
 - fibered product need not be Noetherian, 206
 - finitely many connected and irreducible components, 113
 - finiteness of zeros and poles, 289
 - with non-Noetherian ring of functions, 442
 - non-affine, 96
 - over a ring, **114**
 - projective, 23, 102, 103, **108**
 - are qcqs, 110
 - line bundles on, 327
 - morphisms to, 330
 - quasiprojective, 102, **108**
 - visualizing, 70, 73, 76, 92, 93, 155, 180, 195, 315
- Schubert cell, 167
- Second Riemann Extension Theorem, 290
- section
 - of a morphism, **237**
 - of sheaf
 - means global section, 43
 - over open set, **43**
- Segre embedding, 226, 227, 238, 332, 353, 377, 415
 - as complete linear series, 331
 - coordinate-free description, 227
- Segre variety, **227**

- self-intersection of curve on a surface, **447**
- semiample line bundle, **356**
- semicontinuity, **260**
 - (lower) rank of matrix, 260, 572–574
 - (upper) Jacobian corank function, 275
 - (upper) degree of finite morphism, 260, 315, 414
 - (upper) dimension of a variety at a point, 260
 - (upper) dimension of tangent space, 462
 - (upper) dimension of tangent space at closed points, 260
 - (upper) fiber dimension, 260, 267, 383, 414, 592, 609
 - (upper) fiber dimension (for closed maps), 269
 - (upper) fiber dimension at a point, 269
 - (upper) rank of cohomology groups, 260, *see* Semicontinuity Theorem
 - (upper) rank of sheaf, 260, 314, 315, 485, 568
 - list of examples, 260
- Semicontinuity Theorem, 260, 544, 546, 557, 568, **568**, 572, 573, 576
- semigroup (abelian), **21**, *see also* groupification
 - graded, 156
- separable field extension, *see* field extension
- separable finite morphisms, **477**
- separably closed, *see* field
- separably generated, *see* field extension
- separated morphisms, 99, 102, 109, 110, 114, 158, 175, 233, **236**, *see also* Associated-to-Separated Theorem; Reduced-to-Separated Theorem; valuative criteria
 - e.g., affine morphisms, 236
 - e.g., proper morphisms, 244
 - implies quasiseparated, 236
 - local on the target, 236
 - over A , **236**
 - preserved by base change, 236
 - preserved by composition, 240
 - reasonable class, 236
 - topological characterization, 236
- separated presheaf, **44**
- separating transcendence basis, **225**
 - proof of existence, 225
- Serre duality, 369, 381, 395–397, 413, 415, 426, 428, 448, 451, 452, 473, 474, 481, 484, 515, 516, 581, 585, 619, 623
 - for Cohen–Macaulay projective schemes, 396
 - for Ext, 619, **623**
 - functorial, 623, 626, 627, 629–633
 - functorial (for projective space), 625
 - trace version, 623, **623**
 - for Hom, 619, **620**, 621, **621**, 623, 628
 - for projective space, 621
 - functorial, **620**, 628, 630
 - functorial = trace version, 620
 - trace version, 628
 - for projective schemes, 626
 - for smooth projective varieties, **395**
 - for vector bundles, 619, 620, **620**, 623
 - functorial, 623, 626–629
 - functorial (for projective space), 626
 - trace version, 623, 629
 - functorial vs. trace version, 619
- Serre vanishing, **381**, 390, 397, 398, 407, 455, 456, 557, 612, 623, 629, 630
 - relative, **406**
- Serre’s cohomological criterion for affineness, **380**, 387, 407, 408, **408**
- Serre’s cohomological criterion for ampleness, 354, 407, **407**, 417
- Serre’s criterion for normality, 291, 581, 587, **587**, 588
- Serre’s property S_k , *see* S_k
- Serre’s Theorem A, **352**, 354, 397, 630
- Sets (category of sets), **5**
- Sets $\times$ (category of sheaves of sets on X), **48**
- sextic, 192, 426, 431
- sheaf, **44**, *see also* coherent; dualizing; extension by zero; finite presentation; finite type; germs; inverse image; kernels; line bundle; locally free; pullback; quasicohherent; quotient; section; skyscraper; structure sheaf; subsheaf; support; Zariski sheaf; ...
 - determined by sheaf on distinguished affine base, 54, 124
 - direct image, *see* sheaf, pushforward
 - espace étalé, *see* sheaf, space of sections
 - flasque (flabby), **523**
 - have no Čech cohomology, 528
 - implies acyclic, 524
 - preserved by pushforward, 524
 - gluability axiom, **44**
 - gluing, 55
 - higher pushforward, 351, 403–406, 410, 454, 525, 529, 536, 557, 567, 568, 572, 579
 - and base change, 405
 - as derived functor, **523**
 - commutes with affine flat base change, 405
 - commutes with flat base change, **536**
 - quasicohherent definition, **404**
 - identity axiom, **44**
 - injective, *see also* injective objects
 - have no Čech cohomology, 528
 - implies flasque, 524
 - morphisms determined by stalks, 51
 - of (relative) i -forms $\Omega^i_{X/Y}$, **481**
 - of differentiable functions, 65
 - of ideals, *see* ideal sheaves
 - of nilpotents, 125
 - of relative differentials, 459
 - of sections
 - of a map, **46**
 - of a vector bundle, 47, **304**
 - of smooth functions, 41
 - on a base, 53, **54**, 123
 - determines sheaf, 54
 - pushforward, **46**, **46**, 48, 58, 59, 66, 148, 315, 316, 318, 320, 321, 369, 382, 405, 454, 523, 568, 570, 615, 626
 - induces maps of stalks, 46
 - preserves coherence for projective morphisms, 382
 - preserves coherence for proper morphisms, *see* Grothendieck’s Coherence Theorem
 - right adjoint to π^{-1} (for sheaves of sets), 59
 - right-adjoint to π^* (for $\mathcal{O}$ -modules), 148
 - usually right-adjoint to π^* (for quasicohherent sheaves), 319
 - restriction to open subset $\mathcal{U}$, 316
 - sections determined by germs, 50
 - space of sections (espace étalé), 41, 43, **46**, 52, 60
- sheaf Hom, sheaf Proj, sheaf Spec, *see* Hom; Proj; Spec
- sheafification, 23, 49, **51**, *see also* universal property
 - construction, 52
 - induces isomorphism of stalks, 52
 - is a functor, 52
 - left-adjoint to forgetful functor, 52
- short exact sequence, **26**
- signature of symmetric bilinear form, **451**
- simple group (of order 168), 426
- simple object in abelian category, **131**, *see* Jordan–Hölder package
- simply connected (étale), **479**
- singular (point), *see* regular
- singularities, **276**, 599, 603, *see also* ADE singularities; A_n -singularities; curves; D_n -singularities; E_n -singularities; resolution (singularities)
- site, *see* Grothendieck topology
- skyscraper sheaf, **45**
- slicing criterion, 554, *see also* effective Cartier divisors, slicing by
 - for Cohen–Macaulayness, **584**, 585
 - geometric interpretation, 584
 - for flatness, 553, 584
 - on the source, **554**, 562
 - on the target, 554, **554**, 555, 586
 - for regularity, **276**, 435, 500, 563, 584
- SL_n (group scheme), **166**, 246
- small resolution, *see* resolution (singularities)
- smooth functions (sheaf), 41, 42, 45, 47, 65–67, 147
 - not coherent, 130
- smooth morphisms, 66, 271, 292, 293, **293**, 560, 563, 565, *see also* formally smooth; generic smoothness; smooth varieties
- differential geometric motivation, 292

- smooth morphism (cont.)
 - implies flat, 561
 - local on source and target, 293
 - of relative dimension n , 292, **292**, 294, 470, 488, 490, **561**, 563
 - open condition, 293
 - preserved by base change, 293
 - open condition, 425
 - preserved by base change, 293
 - preserved by composition, 293
 - preserved by product, 293
 - “reasonable” class, 293
- smooth varieties, 245, 246, 271, 276, 277, **277**, **471**, 487, 491, 563
 - are Cohen–Macaulay, 585
 - implies reduced, 279, 472
 - of dimension d , 277, 294
 - smoothness preserved by base field extension, 277
- smoothness over a ring, **294**
- Smoothness-Regularity Comparison Theorem, 278, 500
 - proof, 563
- $S_{n,\bullet} \subset S_\bullet$ (Veronese subring), **157**
- Snake Lemma, 26, 28, 35, **36**, 37, 143
- space of sections of sheaf, *see* sheaf, space of sections
- Spec* (relative *Spec*), **367**, 368, *see also* universal property
 - and affine morphisms, 368
 - and base change, 369
 - computable affine-locally, 368
 - universal property, 368
- $\text{Spec } 0 = \emptyset$, 69
- $\text{Spec } A$, **67**
- Spec adjoint to Γ , 152
- $\text{Spec } k$, **69**
- $\text{Spec } \mathbb{Z}$, **69**
- specialization, 83, **83**, 84, **109**, 182, 183, 265
- spectral functor, 39, 530
- spectral sequence, 28, 32, 33, **33**, 34–39, 201, 202, 410, 517, 519, 520, 522, 525, 526, 528–530, 540, 625, *see also* Grothendieck
 - composition-of-functors
 - spectral sequence; Leray
 - spectral sequence; orientation of spectral sequence
- abut, **35**
- and derived functors, 517
- collapse of, **35**
- convergence, **35**
- local-to-global for Ext, 626, **626**, 630
- page of, **33**
- split exact sequence, 522
- split node, 603
- stacks, xvii, 56, 127, 239, 428, 430, 569
- stalk-local properties, **50**
 - Cohen–Macaulayness, 584
 - exactness for sheaves, 57, 127
 - factoriality, 117
 - flatness, 534
 - implies affine-local, 113, 169
 - inverse image sheaf, 60
 - local freeness of finitely presented sheaves, 313
 - morphisms of sheaves, 51
 - being epimorphisms, 53
 - being injective, 52
 - being isomorphisms, 51
 - cokernels, 57
 - kernels, 56
 - normality, 116
 - reducedness, 110, 113, 116, 225
 - regularity of a locally Noetherian scheme, 276
 - sections of a sheaf, 50
 - support, 61
 - tensor product of sheaves, 58
 - torsion-freeness, 123
- stalkification, **48**
 - doesn’t commute with Hom , 48
 - is exact, 31, 57
- stalks
 - as colimit, **44**
 - determine isomorphisms, 51
 - determine morphisms, 51
 - of (pre)sheaf on base, **54**
 - of a presheaf, **44**
 - of a sheaf, **44**
 - preserved by inverse image, 59
- Stein factorization, 606, 607, **607**, 608, 609
 - commutes with flat base change, 608
- Stein Factorization Theorem, 599, 607, **607**
- Stein space, 409, 439
- stereographic projection, 160
- strict transform, *see* proper transform
- structure morphism, **152**
- structure sheaf, **89**
 - of ringed space, **46**
- subcategory, 7
 - full, 7, 8, 22, 26, 47, 51, 92, 127, 132, 208
- submersion, 66, 292, 459, 468, 560, 561, *see also* smooth morphisms
- subobject, **25**
- subpresheaf, **49**
- subschemes = locally closed
 - subschemes, **190**, *see also* closed/locally closed/open subschemes
- subsheaf, **53**
- subtler Five Lemma, *see* Five Lemma
- subvariety (e.g., open or closed), **239**
- $\text{Supp } D$ (support of divisor), **337**
- $\text{Supp } \mathcal{G}$ (support of a sheaf), **61**
- $\text{Supp } m$ (support of element of module), **92**
- $\text{Supp } M$ (support of module), **92**
- $\text{Supp } s$ (support of a section), **60**
- support
 - commutes with localization, 140
 - of $m \in M$, **92**
 - of a module, **92**
 - = closure of associated points, 134
 - of a section s , **60**, 312
 - is closed, 61
 - of a sheaf, **61**
 - of a Weil divisor, **337**
 - of finite type quasicohherent sheaf is closed, 312
 - scheme-theoretic, **189**
 - stalk-local notion, 61
- surfaces, **192**, **252**, 482, *see also* cubic; Fermat cubic surface; intersection theory; irregularity; quadric surface; quartic; resolution (singularities); Riemann–Roch; ruled surface
 - Enriques–Kodaira classification, 483, *see also* del Pezzo surfaces; Hirzebruch surfaces; K3 surfaces; minimal models, 615
- proper but no nontrivial line bundles, 441
- proper nonprojective, 441, 449
- surjective morphisms, **173**
 - checking on closed points, 183
 - preserved by base change, 218
 - reasonable class, 218
- Sylvester’s law of inertia, **451**
- $\text{Sym}^\bullet M$, $\text{Sym}^\bullet \mathcal{F}$ (symmetric algebra), **310**, **311**
- symbolic power of an ideal $\mathfrak{q}^{(n)}$, **262**
- symmetric algebra, 105, **310**
- system of parameters, 265, 267, 276
- $t: H^n(X, \omega_X) \rightarrow k$ (trace map), **619**
- $T^\bullet M$, $T^\bullet \mathcal{F}$ (tensor algebra), **310**, **311**
- $\mathcal{T}_{X/Y}$ (relative tangent sheaf), **468**
- tacnode, 229, 281, 503, **603**, 604
- tame ramification, **478**
- tangent
 - bundle, 303
 - cone, 499, **499**
 - = tangent space for regular point, 499, 500
 - projectivized, **499**
 - line, 272, 281, 282, **282**, 285, 432, 433, 435, 438, 439
 - map, 413
 - plane, **281**, 282
 - sheaf, **468**, *see also* cotangent, sheaf relative, 463, **468**
 - space, 66, 260, 271, **271**, 272–275, 282, 290, 294, 340, 414, 459, 462–464, 472, 499, 500, 560, 561
 - and morphisms, 274
 - vector, **271**, **459**
 - relative, **459**
- Taniyama–Shimura Conjecture, 424
- Tate curve, 431
- tautological bundle
 - on projective space, **369**
 - on the Grassmannian, **364**, **476**, 577
- tensor algebra, **310**
- tensor product, **11**, *see also* universal property
 - by locally free sheaf is exact, 306
 - commutes with direct sum, 13, 32, 512, 602
 - is not left-exact, 12
 - is right-exact, 12, 29, 534, 602
 - left-adjoint to Hom , 21
 - of $\mathcal{O}_X$ -modules, **58**
 - of quasicohherent sheaves, **125**
 - of rings, **12**
 - is fibered coproduct, 14, 206

- tensor-finiteness trick, 223, **223**, 224–226
- Theorem on Formal Functions, 599, 604–606, **606**, 612–614
 - proof, 615
- thickening, *see* formal neighborhood
- Top* (category of topological spaces), 6
- topological spaces, *see* gluing morphisms; Noetherian
- topology, *see* base of a topology; discrete topology; étale topology; Grothendieck topology; profinite topology; Zariski topology
- topos, 3, **127**
- Tor functors, 511, **514**, 532, 537
 - computable with flat resolution, 519
 - criterion for flatness, **513**
 - long exact sequence, 513
 - symmetry of, 517, 538
- torsion
 - (sub)module, **123**
 - quasicoherent sheaf on reduced scheme, **123**, **313**
- torsion-freeness, **123**
 - = flat for PID, 541
 - implied by flatness, 533, 537, 541, 553, 586
 - is a stalk-local notion, 123
 - module, **123**
 - sheaf, **123**, **313**
- torsor, 211
- total complex (of a double complex), **33**
- total fraction ring (total quotient ring), 142, **142**, 500, *see also* fraction field
- total space
 - of line bundle, 439, 441, 450, 613
 - of vector bundle/locally free sheaf, **305**, 369, **369**, 442, 499
- total transform (of blow-up), **493**, 495, **495**, 502, 503
- trace map, 312, **619**, 620, 621, **621**, 622, 623, 628, 629
- transcendence basis, **255**, *see also* separating transcendence basis
- transcendence degree, **255**, 267
 - = dimension, 255
- transcendence theory, 172, 224, 254
- transition functions/matrices, 304, **304**, 305, 306, 308, 311, 325–327, 330, 337, 369, 371, 392, 398, 400, 450
- trdeg (transcendence degree), **255**
- trichotomy (classification of varieties), 424, 482, 483
- triple point, 603, **603**
- trivial bundle/sheaf of rank n , **303**, **304**
- trivialization of vector bundle/locally free sheaf, **303**
- tropical geometry, 87
- Tsen's Theorem, 571
- tubular neighborhood, 465, 499
- twisted cubic curve, 81, 192, **192**, 194, 255, 281, 282, 323, 400, 544, 546, *see also* rational normal curve; Veronese embedding
- two planes meeting at a point, *see* recurring (counter)examples
 - not S_2 or Cohen–Macaulay, 588
- ultrafilter, 82
- uniformizer, **286**, 288, 312, 336, 360, 362, 474, 478, 541
- union (scheme-theoretic), *see* scheme-theoretic, union
- unique factorization domain, 68, 69, 116–119, 131, 136, 162, 229, 263, 264, 273, 274, 286, 288–290, 298, 299, 341, 342, 344–346, 439, 485, 506, 588, *see also* factorial
 - = all codimension 1 primes
 - principal, 117, 253, 254, 264, 274, 341, 345
 - = integrally closed and class group 0, 345
 - is integrally closed, 117
 - list of useful criteria, 117
 - not affine-local property, 119, 344
- uniruled variety, **266**
- universal δ -functors, *see* δ -functors, universal
- universal family over moduli spaces, 577, **577**, 579
- universal hyperplane, *see* hyperplane, universal
- universal property, 3, 9
 - and Yoneda's Lemma, 15
 - of blow-up, 491, 492, **492**, 493, 494, 496, 497, 537, 613
 - of cokernel, **25**, 49, 56
 - of colimit, **18**, 56
 - of coproduct, **14**
 - of differentials, 460, 466, **466**, 467, 469
 - of fibered coproduct, 14, **14**
 - of fibered product, 13, **13**, 205, 207, 208, 217
 - of inverse image sheaf, 59, **59**
 - of kernel, **25**, 49, 50
 - of limit, **17**, 31
 - of localization, 10, **10**, 11, 125
 - of normalization, 228, **228**, 229, 358, 613
 - of product, 3, 4, 9
 - of *Proj* (not great), 370
 - of pullback of $\mathcal{O}$ -modules, **318**, 319
 - of pullback of quasicoherent sheaves, 317, 318, **318**, 319
 - of reduction of a scheme, **232**
 - of sheafification, 51, 51, 52, 57
 - of *Spec*, 367, **367**, 368
 - of stalks, **44**, 46
 - of Sym and $\wedge$, 310
 - of tensor product, **11**, **12**, 12, 21, 469
- universally, **219**
- universally closed morphisms, 219, **243**, 297, 333, 374, 411, *see also* valuative criteria
 - e.g., proper morphisms, 244
 - reasonable class, 243
- universally injective (radicial) morphisms, 219, 221, **221**, 222, 240
 - = surjective diagonal, 240
 - are separated, 240
 - e.g., monomorphisms, 221
- local on the target, 222
 - of varieties over k , 240
- preserved by composition, 222
- reasonable class, 222, 240
- universally open morphisms, **222**
 - e.g., maps to *Spec* k , 222–224, 551
- unramified morphisms, 66, 187, 190, 413, 481, **489**, 560, 561, 563, 565, 609
 - characterizations by fibers, 489
 - diagonal characterization, 489
 - e.g., étale morphisms, 563
 - examples, 489
 - formal, *see* formally unramified
 - in number theory, 490
 - open condition, 490
 - reasonable class, 489
- upper semicontinuity, *see* semicontinuity
- $V(S)$ (vanishing set), **77**
 - projective case, **106**
- $\text{val} = \text{val}_p$ (valuation), **287**
- valuation rings, 87, 143, 288, **288**, 290, 294–297, *see also* discrete valuation rings
 - rank 2 example, 334
- Serre makes the case, 297
- valuations, **287**, 337, 358, *see also* discrete valuations
- valuative criteria, xix, 271, 294, 295
 - for flatness, 543
 - for properness, 297, **297**, 335, 565
 - for separatedness, **295**, 296, **296**, 297, 565
 - for universal closedness, 297, **297**
- value of a function at a point, *see* functions, value at a point
- vanishing scheme, 104, 107, **107**, **189**, 191, 477
- vanishing set, 77, 77, 87, 96, 103, **106**
 - properties of, 78
- vanishing theorems, 390, *see also* affine cover cohomology vanishing; dimensional cohomology vanishing; Kodaira vanishing; Serre vanishing
- varieties, **114**, **239**
 - affine, **114**
 - classically, 83
 - complex algebraic, xvi, 65, 72, 74, 76, 102, 153–156, 237, 239, 246, 293, 307, 347, 360, 392, 395, 401, 439, 441, 447, 471, 479, 483, 484, *see also* analytification
 - complex analytic, xvi, xvii, 67, 128, 130, 155, 156, 239, 246, 293, 307, 347, 360, 392, 395, 401, 409, 429, 431, 439, 447, 471, 479, 483, 604
 - product of, **239**
 - projective, **114**
 - quasiprojective, 114
 - with non-finitely generated ring of global sections, 441
- Vec_k (category of vector spaces over k), 5
- vector bundles, xvii, 46, 47, 108, 130, 303–306, 308, 309, 312, 315, 351, 369, 373, 392, 398, 449, 465, 476,

- vector bundles (cont.)
 - 498, 532, 538, 568, 571, 572, 619, 620, 623, 626–629, *see also* Serre duality, for vector bundles; total space
 - ample, 377
 - classification on $\mathbb{P}^1$, 396
 - $\sim$ locally free sheaves, 369
 - not always direct sum of line bundles, 476
 - on manifold, 303
 - perfect pairing, 311
 - pulling back, 317
 - trivialization, 303
- Veronese embedding v_d , 81, 157, 192, 193, **194**, 227, 238, 284, 330, 331, 400, 401, 415, 423, 627, *see also* rational normal curve, twisted cubic curve
 - as complete linear series, 330, 415
- Veronese subring $S_{n\bullet}$, 157, **157**, 193
- Veronese surface, 194, **194**
- very ample, 353, **353**
 - $\boxtimes$ very ample is very ample, 353
 - $\otimes$ base-point-free is very ample, 353
 - $\otimes$ very ample is very ample, 353
 - implies base-point-free, 353
 - with respect to π (relatively very ample), **353**, **372**, 375
 - affine-local on the target, 377
 - worse-behaved than ample, 354
- very general, **261**
- Weak Factorization Theorem, 506, 615
- Weierstrass normal form, 433, **433**, 434, 436
 - and double cover of $\mathbb{P}^1$, 433
- weighted projective space, **196**
- Weil divisor, 325, 337, **337**
 - effective, 337
 - irreducible, **337**
 - locally principal, 340, **340**, 341, 342, 505, *see also* closed subscheme, locally principal
 - not necessarily locally principal, 340, 345
 - of zeros and poles, 338
 - principal, **340**, 341
 - $\mathbb{Q}$ -Cartier, **345**
 - restriction to open set, 337
 - support of, **337**
 - twisting line bundles by, 342
- Weil X (group of Weil divisors on X), **337**
- Weyl group, 589, 594, 597
- wild ramification, **478**
- X_{an} (analytification of X), **155**
- X_f ($f \in \Gamma(X, \mathcal{O}_X)$), 80, **126**
- $x_{i/j}$ (coordinates on patches for projective space), 100, **100**, 103, 105, 167, 191, 227, 238, 325, 326, 330, 364, 472, 473, 475, 476
- X_s ($s \in \Gamma(X, \mathcal{L})$), **342**
- Yoda embedding, 16
- Yoneda's Lemma, xvi, 15, **15**, 16, **16**, 154, 163, 205, **208**, 295, 329, 364, 493, 577
- $\mathbb{Z}$ -graded rings, *see* graded rings, $\mathbb{Z}$
- $\mathbb{Z}^{\geq 0}$ -graded rings =: graded rings, **105**, *see* graded rings
- $\mathbb{Z}_p$ (p -adics), **18**
- Zariski closure, **78**
- Zariski cotangent space, *see* cotangent, space
- Zariski sheaf (sheaf on the big Zariski site), 208, 209, **209**, 210, 363
- Zariski tangent space, *see* tangent, space
- Zariski topology, 76–81, 92, **95**
 - on $\bar{k}^n$, **184**, **260**
 - on $\text{Proj } S_{\bullet}$, **106**
 - on $\text{Spec } A$, **78**
 - on a scheme, **95**
- Zariski's Connectedness Lemma, 570, 599, 606, **606**, 607, 609
- Zariski's Lemma, **71**, *see* Nullstellensatz
- Zariski's Main Theorem, 180, 246, 595, 606, 610
 - for birational morphisms, **607**, 609
 - Grothendieck's form, 599, **609**, 610
 - misnamed, 608
 - other versions, 610–612
- zero object, *see* object, in a category, initial / final / zero
- zero / pole of order n
 - for discrete valuation rings, **288**
 - for Noetherian scheme, **289**
- zerodivisor, **10**, 134, *see also* non-zerodivisor
 - = element of associated prime, 137
 - = vanish at some associated point, 134, 137
- Zorn's Lemma, xx, 75, 84, 218, 515, *see also* Axiom of Choice