CONTENTS

Chapter 1

KINEMATICS

Nature is written in mathematical language.
—GALILEO GALILEI

Computations must be based on a thorough theoretical framework, and in this chapter, we build the necessary foundation. We regard the Earth as a continuous distribution of matter, which can interact both through short-range and long-range forces governed by the laws of continuum mechanics. The mathematical description of such a continuum involves basic differential geometry and tensor calculus, which is prerequisite knowledge for this chapter of the book and reviewed in the appendices.

We develop a theory of continuum mechanics in which all physical quantities—for example, mass density, material velocity, and stress—are defined as unique tensors with respect to an *inertial* or *Galilean reference frame*, independent of any coordinate system. Guided by the theory of general relativity, there can be only one set of coordinate-free tensor equations that captures the laws of continuum mechanics, which include conservation of mass, linear momentum, angular momentum, and energy. The natural variables of tensor fields in continuum mechanics are Newtonian time and space positions in the Galilean frame.

To explore the governing tensor equations, we investigate two primary classes of coordinate systems within a *spatial manifold*.[1] The first class comprises *spatial* or *Eulerian* coordinates, which remain unaffected by the continuum's motion. The second class encompasses *comoving* or *Lagrangian* coordinates, which can be accelerated by the continuum's motion. The transformations between these two coordinate representations are governed by the well-established principles of standard tensor calculus, offering a rigorous mathematical foundation for our theoretical framework.

To describe *deformation* of the continuum, we introduce a quiescent *referential state* of matter characterized by some *referential time*, for example, the equilibrium configuration

[1] For our exploration, we define a manifold as a collection of interconnected "patches" that locally resemble Euclidean space, specifically $\mathbb{R}^3$, and are seamlessly "stitched" together. It is important to note that the manifolds we work with in this context possess a differentiable structure, enabling us to conduct calculus operations on these manifold spaces, thereby enriching our understanding and analytical capabilities.

of an elastic material at rest. To identify individual elements of the continuum in this referential state, we introduce *referential* coordinates in a *referential manifold*. The Lagrangian coordinates in the spatial manifold are chosen such that at the referential time, when the spatial and referential manifold describe the same state of the continuum, they coincide with the referential coordinates. In other words, the referential coordinates are identical to the comoving Lagrangian coordinates at the referential time.

In this chapter, we investigate the *kinematics* of a continuum, and in chapter 2 we explore its *dynamics*.

Notation

Throughout this book, we use bold Latin and Greek letters to denote vectors and tensors, for example, we use $\mathbf{v}$ to denote the material velocity and $\boldsymbol{\sigma}$ to denote the Cauchy stress tensor. We use a dot $\cdot$ to denote contraction between the last index of the first tensor and the first index of the second tensor (e.g., for the stress tensor $\boldsymbol{\sigma}$ and the material velocity $\mathbf{v}$, $\boldsymbol{\sigma}\cdot\mathbf{v}$), and a colon : to denote the contraction of two second-order tensors (e.g., for the stress tensor $\boldsymbol{\sigma}$ and the deformation-rate tensor $\mathbf{D}$, $\boldsymbol{\sigma}:\mathbf{D}$).

We introduce Cartesian spatial, or *Eulerian*, components of vectors, one-forms, and general tensors, which are denoted by Latin letters with lowercase Latin super- and subscripts (e.g., v^i, ω_i, or $\sigma^i{}_j$). Cartesian Eulerian or spatial coordinates are identified by a lowercase Latin r with lowercase Latin superscripts, r^i, Eulerian basis vectors by a bold lowercase e with lowercase Latin subscripts, $\mathbf{e}_i$, and Eulerian basis one-forms by a bold lowercase e with lowercase Latin superscripts, $\mathbf{e}^i$. These coordinates and associated basis vectors and one-forms are independent of the motion and rigidly attached to an inertial laboratory. The functional dependence of the spatial components of a tensor field on space and time is denoted by (r, t), for example, $v^i(r, t)$. Partial derivatives with respect to these coordinates are denoted by ∂_i and ∂_t, and such partial derivatives *can only act on the Eulerian components of tensor fields*.

Comoving, or *Lagrangian*, components of tensors are denoted by Latin letters with uppercase Latin super- and subscript (e.g., v^I or $\sigma^I{}_J$). Lagrangian coordinates are identified by the symbol X and uppercase Latin superscripts, X^I, Lagrangian basis vectors are denoted by a bold lowercase e with uppercase Latin subscripts, $\mathbf{e}_I$, and Lagrangian basis one-forms are denoted by a bold lowercase e with uppercase Latin superscripts, $\mathbf{e}^I$. The functional dependence of the comoving components of a tensor field on space and time is denoted by (X, T), for example, $\sigma^I{}_J(X, T)$. Partial derivatives with respect to these coordinates are denoted by ∂_I and ∂_T, and these partial derivatives *can only act on the Lagrangian components of a tensor*. One can think of t as *Newtonian time* and of T as a "comoving" or "convected" time. The distinction between the partial derivatives ∂_t and ∂_T is important for two reasons: (1) to make clear which kind of component of a tensor is being differentiated with respect to time, and (2) to indicate which remaining coordinates are held fixed.[2]

Scalar quantities—that is, tensors of rank zero—are denoted by Greek or Latin letters, for example, ρ for the mass density and q for a physical quantity "q-stuff". We use lowercase italicized letters to express the functional dependence of such a quantity in Cartesian Eulerian coordinates, for example, $\rho(r, t)$ and $q(r, t)$. To express the functional dependence of

[2]In general relativity, two four-dimensional coordinate systems may be expressed as $\{x^0, x^i\}$ and $\{x^{0'}, x^{i'}\}$, in which case x^0 and $x^{0'}$ indicate the two different time coordinates; in other words, one would naturally distinguish between the two time coordinates.

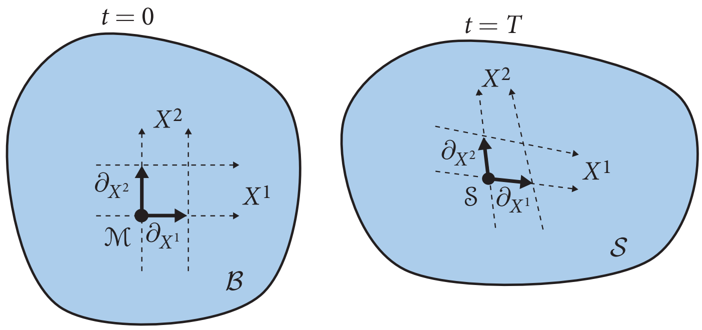

Figure 1.1: *Left*: *Referential state* of the continuum at time $t = 0$ captured by the *referential manifold* $\mathcal{B}$. A *material point* $\mathcal{M}$ may be identified with *referential coordinates* $\{X^I\}$, which define a *chart* in the referential manifold. A local vector basis in the tangent space of the referential manifold at $\mathcal{M}$ is denoted by the partial derivatives $\{\partial_{X^I}\}$. For convenience, the referential coordinates are chosen to be Cartesian, but this is not required. *Right*: Deformed state of the continuum at time $t = T$ captured by the *spatial manifold* $\mathcal{S}$. A *spatial point* $\mathcal{S}$, *not* tied to a specific element of the continuum, may be identified with the comoving Lagrangian coordinates $\{X^I\}$ of the material particle that happens to occupy its location at time $t = T$. Thus, Lagrangian coordinates define an evolving local chart in the spatial manifold at time $t = T$. A local non-orthonormal vector basis in the tangent space of the spatial manifold at $\mathcal{S}$ at time $t = T$ is denoted by the partial derivatives $\{\partial_{X^I}\}$. Importantly, Lagrangian coordinates in the spatial manifold are chosen such that they are identical to the referential coordinates in the referential manifold at the referential time $t = 0$, when the referential and spatial manifolds capture the same state of the continuum. Thus, the Lagrangian coordinates move along with the flow of matter.

such quantities in Lagrangian coordinates, we use uppercase italicized letters, for example, $\varrho(X, T)$ and $Q(X, T)$. A glossary of the notation is provided at the end of chapter 2.

Occasionally, we will need to explore an issue that has come up in the main text in further detail. When this occurs, we introduce a "box" in which we delve further into the topic.

1.1 Motion

Before we discuss the notion of *deformation* of a continuum, we need to introduce the concept of a *referential state* of the material. In seismology, this is typically the quiescent state of the Earth before an earthquake, which we identify with time $t = 0$ or sometimes $t = T_0$. As illustrated in figure 1.1 (*left*), an element of the continuum in the referential state is labeled by a *material point* $\mathcal{M}$, which may be identified with a set of Cartesian *referential coordinates* $\{X^I\}$. These referential coordinates define a *chart* in the *referential manifold* and remain associated with whichever element of the continuum they identify. The local vector basis in the tangent space of the referential manifold at material point $\mathcal{M}$ is identified with the partial derivatives ∂_{X^I}, analogous to the identification of vectors with tangents to curves, as discussed in appendix C.1.1.

The state of the continuum at a later time $t = T$ is captured by the *spatial manifold* $\mathcal{S}$, shown in figure 1.1 (*right*). A *spatial point* $\mathcal{S}$ in the spatial manifold is not tied to a specific element of the continuum: it simply denotes a location in inertial space. Thus, whereas a material point $\mathcal{M}$ labels a specific particle in the referential manifold at time $t = 0$, a spatial point $\mathcal{S}$ labels a location in the inertial spatial manifold not tied to any particular particle or

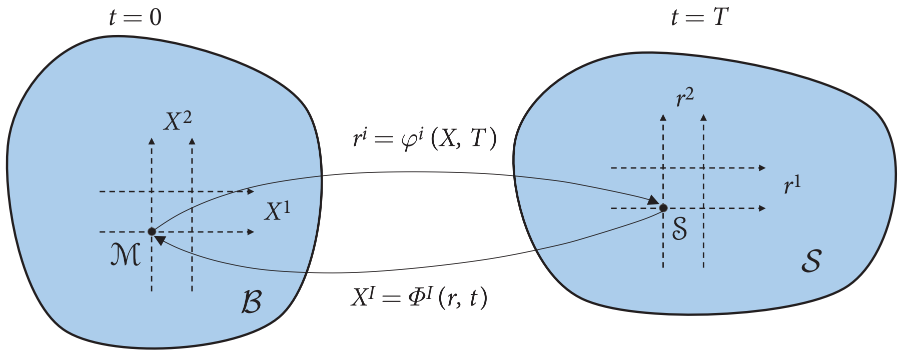

Figure 1.2: *Left*: *Referential state* of the continuum at time $t = 0$. A material point $\mathcal{M}$ may be identified with Lagrangian coordinates $\{X^I\}$ in the referential manifold $\mathcal{B}$. For convenience, in this figure, the Lagrangian coordinates are chosen to be Cartesian, but this is not required. *Right*: Deformed state of the continuum at time $t = T$. A spatial point $\mathcal{S}$ may be identified with Cartesian *spatial coordinates* or *Eulerian coordinates* $\{r^i\}$ in the inertial spatial manifold $\mathcal{S}$. The *motion* $\varphi^i(X, T)$ defines a map $\varphi : \mathcal{B} \to \mathcal{S}$, that is, from the referential manifold $\mathcal{B}$ to the spatial manifold $\mathcal{S}$, identifying the spatial location r^i of a particular element of the continuum at time $t = T$. Its inverse $\Phi^I(r, t)$ defines a map $\Phi : \mathcal{S} \to \mathcal{B}$.

time. Spatial points are identified by a set of Cartesian Eulerian coordinates $\{r^i\}$, with an associated orthonormal Cartesian vector basis (see appendix C.1)

$$\mathbf{e}_i \equiv \partial_i, \tag{1.1}$$

where we introduced the compact notation $\partial_i = \partial_{r^i}$.

Alternatively, a spatial point $\mathcal{S}$ may also be identified with the Lagrangian coordinates $\{X^I\}$ of whatever material particle happens to occupy location $\mathcal{S}$ at time $t = T$ (see, e.g., Sedov, 1966; Weile et al., 2013). Thus, Lagrangian coordinates define an evolving local chart in the spatial manifold $\mathcal{S}$. Lagrangian coordinates in the spatial manifold are chosen such that they are identical to the Cartesian referential coordinates in the referential manifold $\mathcal{B}$ at the referential time $t = 0$, when the referential and spatial manifolds capture the same state of the continuum. In other words, at times $t \geq 0$ the referential coordinates *comove* or *convect* with the material to evolve into a set of *Lagrangian* or *convected* or *comoving* coordinates. A local nonorthogonal Lagrangian vector basis in the tangent space of the spatial manifold $\mathcal{S}$ at $\mathcal{S}$ at time $t = T$ may be defined in terms of the evolving partial derivatives ∂_{X^I}, namely,

$$\mathbf{e}_I \equiv \partial_I, \tag{1.2}$$

where we introduced the compact notation $\partial_I = \partial_{X^I}$.

The *motion* of the continuum is captured by the map

$$\begin{aligned} r^i &= \varphi^i(X, T), \\ t &= T. \end{aligned} \tag{1.3}$$

This map, $\varphi : \mathcal{B} \to \mathcal{S}$, takes us from Cartesian referential coordinates $\{X^I\}$ assigned in the referential manifold $\mathcal{B}$ at time $t = 0$ to Cartesian *spatial coordinates* or *Eulerian coordinates* $\{r^i\}$ in the inertial spatial manifold $\mathcal{S}$ at time $t = T$, as illustrated in figure 1.2.[3]

[3]Because, unlike Euclidean space, a manifold has no origin, one cannot define "position vectors" $\mathbf{r}$ or $\mathbf{X}$. Motion in Euclidean space is discussed in box 1.1.

We assume that the motion is invertible (no tearing of the continuum, a topic we explore in chapter 3), such that

$$\begin{aligned} X^I &= \Phi^I(r, t), \\ T &= t, \end{aligned} \tag{1.4}$$

as illustrated in figure 1.2. The inverse map, $\Phi : \mathcal{S} \to \mathcal{B}$, takes us from inertial Cartesian spatial coordinates $\{r^i\}$ assigned in the spatial manifold $\mathcal{S}$ at time $t = T$ to Cartesian referential coordinates $\{X^I\}$ in the referential manifold $\mathcal{B}$ at time $t = 0$.

Alternatively, we may regard the motion (1.3) and its inverse (1.4) as a coordinate transformation between Cartesian Eulerian coordinates and evolving curvilinear Lagrangian coordinates in the spatial manifold, as illustrated in figure 1.4 and discussed in appendix B.3. It is important to recognize the dual role of the motion (1.3), describing both the location of a specific material particle in the spatial manifold and a coordinate transformation between Eulerian and Lagrangian coordinates in the spatial manifold.

In box 1.1, we consider the motion of a particle in *Euclidean space* with origin O. Particles in the material are labeled by their position "vector" $\mathbf{X}$ at time $T = 0$, and the position "vector" $\mathbf{r}$ of particle $\mathbf{X}$ at time $T \geq 0$ is denoted by $\mathbf{r} = \phi(\mathbf{X}, T)$, as illustrated in figure 1.3. More generally, in a manifold, the motion (1.3) *does not* define the components of a vector, rather it is a map between Eulerian and Lagrangian *coordinates*.

Box 1.1 Motion in Euclidean Space

In this box, we consider motion in *Euclidean space*. In such a space, the motion $\phi(\mathbf{X}, T)$ may be regarded as a "position vector" relative to an origin O, giving the spatial position $\mathbf{r}$ of the particle originally located at $\mathbf{X}$ at a later time T:

$$\mathbf{r} = \phi(\mathbf{X}, T), \tag{1.5}$$

as illustrated in figure 1.3. In this expression, bold quantities are interpreted as "position vectors." Such an approach is only permissible in Euclidean space when an origin O may be defined and vectors associated with distinct spatial locations may be combined. Dahlen and Tromp (1998) abbreviate the description further by expressing the position of particle $\mathbf{X}$ at time T as $\mathbf{r}(\mathbf{X}, T)$.

In future boxes, we will further explore other aspects of continuum mechanics in Euclidean space, thereby drawing parallels with the approach used in Dahlen and Tromp (1998).

1.1.1 Compatibility

A smooth motion satisfies the *compatibility conditions*

$$\begin{aligned} \alpha_{IJ}{}^i &\equiv (\partial_I \partial_J - \partial_J \partial_I)\, \varphi^i \\ &= 0. \end{aligned} \tag{1.6}$$

Equation (1.6) states that partial derivatives of the motion with respect to comoving coordinates *commute*. For an *incompatible motion*, the tensor $\alpha_{IJ}{}^i$ is nonzero and is referred to as the *incompatibility tensor*. Such a situation involves *material defects* in the form of *dislocations* and *disclinations*, as discussed extensively in chapter 3.

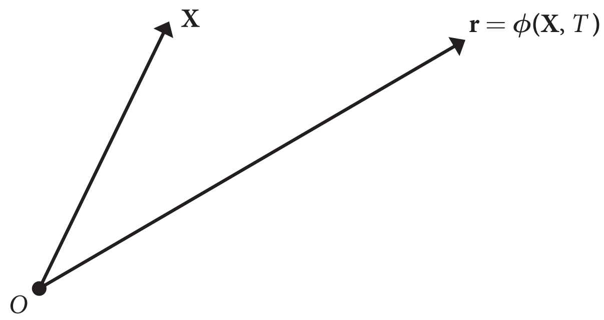

Figure 1.3: Motion in Euclidean space with origin O. Particles in the material are labeled by their referential position "vector" $\mathbf{X}$ at time $T = 0$, and the position "vector" $\mathbf{r}$ of particle $\mathbf{X}$ at time $T \geq 0$ is denoted by $\mathbf{r} = \phi(\mathbf{X}, T)$. The generalization of Euclidean space is a manifold, which has no origin, and one cannot define "position vectors."

Compatibility is related to the *Lie bracket* (see appendix C.4). The Lie bracket of two Eulerian or Lagrangian basis vectors is zero

$$[\mathbf{e}_i, \mathbf{e}_j] = \mathbf{0}, \qquad [\mathbf{e}_I, \mathbf{e}_J] = \mathbf{0}, \tag{1.7}$$

due to the commutativity of partial derivatives

$$\partial_i \partial_j - \partial_j \partial_i = 0, \qquad \partial_I \partial_J - \partial_J \partial_I = 0. \tag{1.8}$$

Such a basis is called *holonomic*. The Lie bracket is related to the *autonomous Lie derivative*, discussed in appendix F.3.3, in the sense $\pounds_{\mathbf{e}_i} \mathbf{e}_j = [\mathbf{e}_i, \mathbf{e}_j]$.

When basis vectors fail to commute, the basis is called *nonholonomic* or *anholonomic*. In that case,

$$[\mathbf{e}_i, \mathbf{e}_j] = \tau_{ij}^k \, \mathbf{e}_k, \tag{1.9}$$

or

$$\partial_i \partial_j - \partial_j \partial_i = \tau_{ij}^k \, \partial_k. \tag{1.10}$$

The parameters τ_{ij}^k are known as *structure coefficients*. We discuss an example of an anholonomic basis in spherical coordinates in box 1.2.

It is important to note that anholonomicity does not imply incompatibility. The former is a property of a vector basis, whereas the latter is a property of a motion.

1.2 Vectors: Material Velocity

The temporal derivative of the motion (1.3), that is, the partial derivative of $\varphi^i(X, T)$ with respect to time T, holding the Lagrangian coordinates X^I fixed, defines the Eulerian components of the *material velocity*:

$$v^i \equiv \partial_T \varphi^i. \tag{1.11}$$

The Eulerian components of the material velocity, v^i, are a function of the Eulerian variables $\{r^i, t\}$, and the motion is a function of the Lagrangian variables $\{X^I, T\}$, so the equality (1.11) should be understood explicitly as

Box 1.2 Anholonomicity

Consider the transformation from Cartesian coordinates $\{x, y, z\}$ to spherical coordinates $\{r, \theta, \phi\}$. The associated holonomic basis vectors are $\{\partial_x, \partial_y, \partial_z\}$ and $\{\partial_r, \partial_\theta, \partial_\phi\}$, respectively. We have the relationships

$$x = r\,\sin\theta\cos\phi, \qquad y = r\,\sin\theta\sin\phi, \qquad z = r\,\cos\theta,$$

with inverse

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \arctan(\sqrt{x^2 + y^2}/z), \qquad \phi = \arctan(y/x).$$

The spherical basis vectors are related to the Cartesian basis vectors via

$$\mathbf{e}_r \equiv \partial_r = \sin\theta\cos\phi\,\partial_x + \sin\theta\sin\phi\,\partial_y + \cos\theta\,\partial_z,$$

$$\mathbf{e}_\theta \equiv \partial_\theta = r\,\cos\theta\cos\phi\,\partial_x + r\,\cos\theta\sin\phi\,\partial_y - r\,\sin\theta\,\partial_z,$$

$$\mathbf{e}_\phi \equiv \partial_\phi = -r\,\sin\theta\sin\phi\,\partial_x + r\,\sin\theta\cos\phi\,\partial_y.$$

It is important to note that these basis vectors are not all "unit"[a] vectors. Specifically,

$$\mathbf{e}_r = \hat{\mathbf{r}}, \qquad \mathbf{e}_\theta = r\,\hat{\boldsymbol{\theta}}, \qquad \mathbf{e}_\phi = r\,\sin\theta\,\hat{\boldsymbol{\phi}},$$

where $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ denote traditional unit vectors in the directions of increasing r, θ, and ϕ, respectively. The basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ are holonomic, thanks to the commutativity of the partial derivatives ∂_r, ∂_θ, and ∂_ϕ. However, the unit basis vectors

$$\hat{\mathbf{r}} = \partial_r, \qquad \hat{\boldsymbol{\theta}} = r^{-1}\,\partial_\theta, \qquad \hat{\boldsymbol{\phi}} = (r\,\sin\theta)^{-1}\,\partial_\phi,$$

are anholonomic:

$$[\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}] = -r^{-1}\,\hat{\boldsymbol{\theta}}, \qquad [\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}] = -r^{-1}\,\hat{\boldsymbol{\phi}}, \qquad [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}] = -r^{-1}\,\cot\theta\,\hat{\boldsymbol{\phi}}.$$

The main conceptual difference is that a holonomic basis is integrable, whereas an anholonomic basis is non-integrable.

[a] See section 1.6 for a discussion on *length*.

$$v^i(\,\varphi^i(X, T)\,, T) = \partial_T\,\varphi^i(X, T), \tag{1.12}$$

so that both sides are evaluated at a particle labeled by $\{X^I, T\}$. Even though the motion φ^i itself *does not* define the components of a vector, its temporal derivatives $\partial_T\,\varphi^i$ *do* define the components of the material velocity *vector*.

At this point, we have introduced two sets of coordinate systems in the spatial manifold, namely, Cartesian Eulerian or spatial coordinates $\{r^i\}$ and Lagrangian or comoving coordinates $\{X^I\}$, related via the motion (1.3) and its inverse (1.4), as illustrated in figure 1.4. As discussed extensively in appendix C, vectors, and their generalization in the form of tensors, should be viewed as geometrical objects *independent of any coordinate system*. For practical applications, we choose to express tensors in a basis, and in continuum mechanics, the two most commonly used bases are the ones we just introduced, namely, those associated with Eulerian and Lagrangian coordinates. Consequently, as illustrated in figure 1.4, we may express the material velocity $\mathbf{v}$ in the following two equivalent component forms (see appendix C.1.4):

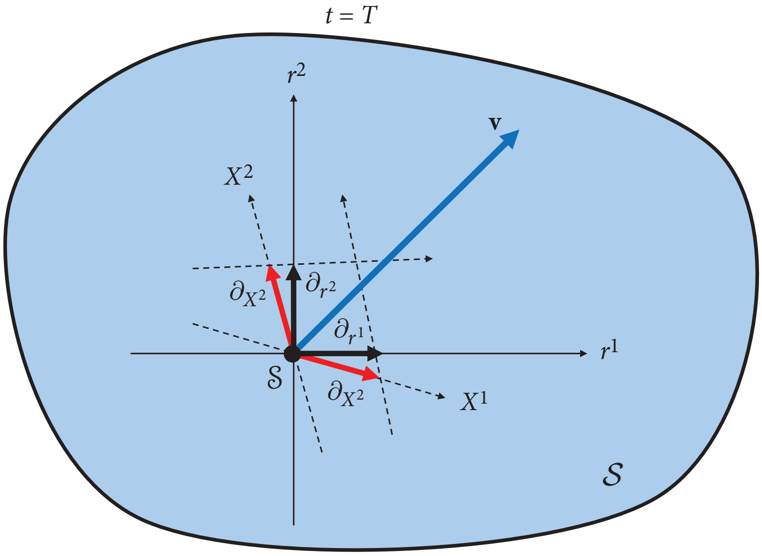

Figure 1.4: In continuum mechanics, a spatial point $\mathcal{S}$ in the spatial manifold $\mathcal{S}$ at time $t = T$ may be identified with either a set of Eulerian coordinates $\{r^i\}$ or a set of Lagrangian coordinates $\{X^I\}$. These coordinates induce a set of Eulerian basis vectors $\{\partial_{r^i}\}$ (shown in solid arrows) or Lagrangian basis vectors $\{\partial_{X^I}\}$ (shown in dashed arrows) in the tangent space at $\mathcal{S}$. The material velocity $\mathbf{v}$ (shown by the thin arrow) is a geometrical object that lives in the tangent space at $\mathcal{S}$ and may be expressed in either set of basis vectors, as stated mathematically in equation (1.13). The Eulerian coordinates are chosen to be Cartesian in this example, but this is not required.

$$\begin{aligned} \mathbf{v} &= v^i\,\mathbf{e}_i \\ &= v^I\,\mathbf{e}_I. \end{aligned} \tag{1.13}$$

In these expressions, we introduced the usual *Einstein summation convention*, in which a sum must be performed over a repeated upper and lower index, in this case the index i in the first equality and the index I in the second equality. The Eulerian components of the material velocity $\mathbf{v}$ are defined by (1.11), whereas its Lagrangian components are identified by the set $\{v^I\}$.

Problem 1.1 *By differentiating the inverse motion* (1.4) *with respect to time T, show that the Lagrangian components of the material velocity are given in terms of the inverse motion Φ^I by*

$$v^I = -\,\partial_t\,\Phi^I. \tag{1.14}$$

As discussed in detail in appendix C.1.4, Eulerian and Lagrangian basis vectors are related via the transformations

$$\mathbf{e}_I = F^i{}_I\,\mathbf{e}_i \qquad \text{and} \qquad \mathbf{e}_i = (F^{-1})^I{}_i\,\mathbf{e}_I, \tag{1.15}$$

where we have defined the *deformation gradient*[4]

[4]The nomenclature deformation “gradient” is not ideal; in view of definitions (C.14), deformation “matrix” would be preferable.

$$F^i{}_I \equiv \partial_I \varphi^i, \tag{1.16}$$

with inverse

$$(F^{-1})^I{}_i \equiv \partial_i \Phi^I. \tag{1.17}$$

Our nomenclature and notation for the deformation gradient, $F^i{}_I$, are chosen to coincide with those of Malvern (1969, section 4.5) and Marsden and Hughes (1983, section 1.3). Unlike these authors, we do not regard the deformation gradient as a tensor, eschewing the introduction of *two-point tensors* (see box 1.4 for further discussion). Matrices $F^i{}_I$ and $(F^{-1})^I{}_i$ are inverses of one another, in the sense that

$$F^i{}_I\,(F^{-1})^I{}_j = \delta^i{}_j \qquad \text{and} \qquad (F^{-1})^I{}_i\,F^i{}_J = \delta^I{}_J, \tag{1.18}$$

where $\delta^i{}_j$ and $\delta^I{}_J$ denote the Kronecker-delta symbol (for further details, see appendix D.4):

$$\delta^i{}_j = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases} \qquad \text{and} \qquad \delta^I{}_J = \begin{cases} 1 & \text{if} \quad I = J, \\ 0 & \text{if} \quad I \neq J. \end{cases} \tag{1.19}$$

Since the basis vectors transform according to rules (1.15), it follows from equation (1.13) that the components of the material velocity $\mathbf{v}$ transform according to

$$v^I = (F^{-1})^I{}_i\,v^i, \qquad v^i = F^i{}_I\,v^I. \tag{1.20}$$

Examining equations (1.15) and (1.20), we see that bases transform with the inverses of matrices used in the transformation of components and vice versa.

1.3 One-Forms

To describe the physics of a continuum, we are going to need more than vectors. To see why this is the case, consider the differential[5] of a scalar field q. In Eulerian coordinates, this field has the functional dependence $q(r, t)$, so we have[6]

$$\mathrm{dq} = \partial_i q\,\mathrm{d}r^i. \tag{1.21}$$

Alternatively, in Lagrangian coordinates, this field has the functional dependence $Q(X, T)$, and we have

$$\mathrm{dq} = \partial_I Q\,\mathrm{d}X^I. \tag{1.22}$$

Using the chain rule, being mindful of the nature of coordinate "slots," as discussed in box 1.3, we have the relationship

$$\partial_I Q = F^i{}_I\,\partial_i q, \tag{1.23}$$

where $F^i{}_I$ denotes elements of the deformation gradient (1.16).

Because equations (1.21) and (1.22) both represent the *same* differential scalar field dq, the Eulerian and Lagrangian differentials, $\mathrm{d}r^i$ and $\mathrm{d}X^I$, must be related via

$$\mathrm{d}X^I = (F^{-1})^I{}_i\,\mathrm{d}r^i, \qquad \mathrm{d}r^i = F^i{}_I\,\mathrm{d}X^I. \tag{1.24}$$

[5] Strictly speaking, the *exterior derivative*, discussed in appendix G.7.

[6] We generally use the notation d to denote the exterior derivative in three dimensions. However, when working in four dimensions, as in general relativity or to describe the dynamics and kinematics of defects, discussed in chapter 3, we use d to denote the four-dimensional exterior derivative and $\bar{\mathrm{d}}$ to denote its restriction to three dimensions.

Box 1.3 Scalars as Tensors of Rank Zero

In this book, scalar fields, such as the mass density, are viewed as geometrical objects independent of any coordinate system, just like all other tensors. As noted in the introduction to this chapter, to indicate the status of a scalar field as a tensor, we use Greek or Latin letters. Thus, the value of a scalar field q at location $\mathcal{S}$ and time t in the spatial manifold is written in the coordinate-free form $\mathrm{q}(\mathcal{S}, t)$. If we introduce a set of Cartesian Eulerian coordinates $\{r^i\}$ in the spatial manifold, then we may express the scalar field at time t in these coordinates as $q(r, t)$, with partial derivatives $\partial_i q$ and $\partial_t q$. Next, if we introduce a complementary set of curvilinear Lagrangian coordinates $\{X^I\}$ in the spatial manifold, then we may also express the scalar field at time T in these coordinates as $Q(X, T)$, with partial derivatives $\partial_I Q$ and $\partial_T Q$. The motion (1.3) enables us to relate the two descriptions of this scalar field because

$$q(\varphi^i(X, T), T) = Q(X, T). \tag{1.25}$$

This result is self-evident, inasmuch as both sides give the value of q recorded by particle X^I, which is at point $r^i = \varphi^i(X, T)$ at time $t = T$.

Whenever one takes partial derivatives of a tensor field in a specific coordinate system, one needs to be mindful of the nature of its coordinate and time "slots." For example, a scalar field q expressed in Eulerian coordinates, $\{r^i, t\}$, has slots that accept *only* such coordinates, $q(r, t)$, and one can take only partial derivatives of the field with respect to these coordinates. For this reason, the partial time derivative of a scalar field q in Lagrangian coordinates, $\partial_T Q$, is related to partial derivatives $\partial_t q$ and $\partial_i q$ in Cartesian Eulerian coordinates via

$$\partial_T Q = \partial_t q + (\partial_T \varphi^i)\, \partial_i q.$$

This relationship may be readily obtained by differentiating equation (1.25) with respect to time T and using the chain rule.

Similarly, the spatial partial derivatives $\partial_I Q$ and $\partial_i q$ are related via

$$\partial_I Q = (\partial_I \varphi^i)\, \partial_i q,$$

where, again, the left- and right-hand sides are evaluated at the location of material particle X^I at time T.

Upon comparing these expressions to the transformation rule for basis vectors (1.15), we note that the rules appear to be "reversed." This motivates us to introduce two new sets of basis elements defined in terms of differentials $\mathrm{d}r^i$ and $\mathrm{d}X^I$, namely,

$$\mathbf{e}^i \equiv \mathrm{d}r^i, \tag{1.26}$$

and

$$\mathbf{e}^I \equiv \mathrm{d}X^I. \tag{1.27}$$

These basis elements are referred to as *one-forms* and are discussed in detail in appendix C.2. The transformations (1.24) may now be rewritten in the forms

$$\mathbf{e}^I = (F^{-1})^I{}_i\, \mathbf{e}^i, \qquad \mathbf{e}^i = F^i{}_I\, \mathbf{e}^I. \tag{1.28}$$

These one-form basis transformation rules should be contrasted with the vector basis transformation rules (1.15).

We conclude that the differential $\mathrm{d}q$ should be regarded as a new form of tensor called a *one-form*,[7] discussed in appendix C.2. Such a tensor, say $\boldsymbol{\omega}$, may be expressed in either Eulerian or Lagrangian coordinates as

$$\begin{aligned}\boldsymbol{\omega} &= \omega_i\,\mathbf{e}^i \\ &= \omega_I\,\mathbf{e}^I,\end{aligned} \tag{1.29}$$

and its components transform according to the rules

$$\omega_I = \omega_i\,F^i{}_I, \qquad \omega_i = \omega_I\,(F^{-1})^I{}_i. \tag{1.30}$$

We note that one-forms have components labeled with subscripts, to clearly distinguish them from vectors, which have components labeled with superscripts.

1.3.1 Duality Product

As discussed in appendix C.2.1, spatial basis vectors and spatial basis one-forms are *duals* of each other; namely, for Eulerian vector basis elements $\mathbf{e}_i$ and one-form basis elements $\mathbf{e}^j$, we have

$$\langle\, \mathbf{e}^i\,,\, \mathbf{e}_j\,\rangle = \delta^i{}_j, \tag{1.31}$$

and for the Lagrangian bases

$$\langle\, \mathbf{e}^I\,,\, \mathbf{e}_J\,\rangle = \delta^I{}_J. \tag{1.32}$$

Here $\langle\cdot,\cdot\rangle$ denotes the bilinear[8] *duality product* between a one-form, to be placed in the first slot, and a vector, inserted in the second (see, e.g., Schutz, 1980; Dubrovin et al., 1985).

In the language of differential geometry, we may express the products (1.31) and (1.32) fancifully as

$$\mathrm{d}r^i(\,\partial_j\,) = \delta^i{}_j \qquad \text{or} \qquad \partial_i(\,\mathrm{d}r^j\,) = \delta^j{}_i. \tag{1.33}$$

and

$$\mathrm{d}X^I(\,\partial_J\,) = \delta^I{}_J \qquad \text{or} \qquad \partial_I(\,\mathrm{d}X^J\,) = \delta^J{}_I. \tag{1.34}$$

These expressions reflect the view discussed in appendix C.2.1 and in the next section that vectors are linear "machines" with a "slot" that accepts one-forms, whereas one-forms are linear machines with a slot that accepts vectors (Misner et al., 1973). For a discussion of linear spaces and linear transformations, the reader is referred to appendix A.

The Eulerian and Lagrangian components of a vector field $\mathbf{u}$ may now be defined in terms of the duality product as, respectively,

$$u^i \equiv \langle\, \mathbf{e}^i\,,\, \mathbf{u}\,\rangle, \qquad u^I \equiv \langle\, \mathbf{e}^I\,,\, \mathbf{u}\,\rangle. \tag{1.35}$$

Problem 1.2 *Express the vector* $\mathbf{u}$ *in Eulerian or Lagrangian components and use the duality product* (1.31) *or* (1.32) *to verify* (1.35).

---

[7]In old terminology, one-forms were called *covariant vectors*, *covectors*, and *dual vectors*. However, we do not use these terms because they contain the word "vector," which contradicts our understanding of a one-form as a collection of sheets.
[8]See appendix A for a description of linear spaces.

Similarly, the Eulerian and Lagrangian components of a one-form field $\boldsymbol{\omega}$ may be defined in terms of the duality product as, respectively,

$$\omega_i \equiv \langle\, \boldsymbol{\omega}\,,\, \mathbf{e}_i \,\rangle, \qquad \omega_I \equiv \langle\, \boldsymbol{\omega}\,,\, \mathbf{e}_I \,\rangle. \tag{1.36}$$

1.4 Tensors

Based on the discussion in the previous section, specifically the dualities (1.33) and (1.34), any vector, $\mathbf{u}$, may be viewed as a linear "machine" with one slot that accepts a one-form, $\boldsymbol{\omega}$, and returns a number:

$$\mathbf{u}(\boldsymbol{\omega}) = u^i\, \omega_i = u^I\, \omega_I. \tag{1.37}$$

Such a machine is an example of a $(1, 0)$-tensor: a machine with one one-form slot. Similarly, a one-form, $\boldsymbol{\omega}$, may be viewed as a linear machine with one slot that accepts a vector, $\mathbf{u}$, and returns a number:

$$\boldsymbol{\omega}(\mathbf{u}) = \omega_i\, u^i = \omega_I\, u^I. \tag{1.38}$$

A one-form is an example of a $(0, 1)$-tensor: a machine with one vector slot.

As discussed in appendix D, the generalization of vectors and one-forms is a *tensor*, $\mathbf{T}$, which is a multilinear machine with multiple slots that accept either vectors or one-forms. Specifically, a (p, q) tensor is a machine with p one-form slots and q vector slots in any order. For example, the rank-4 tensor $\mathbf{r}$ with three vector slots followed by a one-form slot returns

$$\mathbf{r}(\mathbf{u}, \mathbf{v}, \mathbf{w}, \boldsymbol{\omega}) = r_{ijk}{}^{\ell}\, u^i\, v^j\, w^k\, \omega_\ell = r_{IJK}{}^{L}\, u^I\, v^J\, w^K\, \omega_L \tag{1.39}$$

when fed three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and one one-form $\boldsymbol{\omega}$.

The components of a tensor may be obtained by inserting the appropriate basis vectors and one-forms, in this instance

$$\mathbf{r}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}^\ell) = r_{ijk}{}^{\ell}, \qquad \mathbf{r}(\mathbf{e}_I, \mathbf{e}_J, \mathbf{e}_K, \mathbf{e}^L) = r_{IJK}{}^{L}, \tag{1.40}$$

such that we have

$$\begin{aligned} \mathbf{r} &= r_{ijk}{}^{\ell}\, \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \mathbf{e}_\ell \\ &= r_{IJK}{}^{L}\, \mathbf{e}^I \otimes \mathbf{e}^J \otimes \mathbf{e}^K \otimes \mathbf{e}_L, \end{aligned} \tag{1.41}$$

where the symbol $\otimes$ designates a *tensor product* (see appendix D.2.2). Equation (1.41) illustrates how higher-order tensor fields may be expressed in terms of combinations of Eulerian or Lagrangian basis vectors and basis one-forms.

The transformation of a tensor from spatial to Lagrangian coordinates and vice versa may be accomplished based on a generalization of the vector and one-form transformation rules (1.20) and (1.30), for example,

$$r_{IJK}{}^{L} = F^i{}_I\, F^j{}_J\, F^k{}_K\, (F^{-1})^L{}_\ell\, r_{ijk}{}^{\ell}, \tag{1.42}$$

and

$$r_{ijk}{}^{\ell} = (F^{-1})^I{}_i\, (F^{-1})^J{}_j\, (F^{-1})^K{}_k\, F^\ell{}_L\, r_{IJK}{}^{L}. \tag{1.43}$$

In some textbooks, tensors are *defined* as objects that transform according to these rules.

1.5 Covariant Derivative

In this section, we introduce a tensorial description of the "gradient of a vector field," such as the material velocity $\mathbf{v}$. This is accomplished by means of the *covariant derivative*, which is denoted by the symbol $\boldsymbol{\nabla}$ and discussed in detail in appendix F.1. In spatial coordinates, the covariant derivative is expressed as ∇_i, and in these Cartesian coordinates it is identical to the partial derivative ∂_i:

$$\nabla_i v^j \equiv \partial_i v^j. \tag{1.44}$$

If the elements $\nabla_i v^j$ are to define the spatial components of a tensor, we have

$$\boldsymbol{\nabla}\mathbf{v} = \nabla_i v^j \, \mathbf{e}^i \otimes \mathbf{e}_j. \tag{1.45}$$

In Lagrangian coordinates the components of this tensor must be given by the tensor transformation rule discussed in section 1.4, namely,

$$\nabla_I v^J = F^i{}_I \, (F^{-1})^J{}_j \, \nabla_i v^j. \tag{1.46}$$

Problem 1.3 *Show that the Lagrangian components of the material velocity gradient are given by*

$$\nabla_I v^J = \partial_I v^J + \Gamma^J_{IK} \, v^K, \tag{1.47}$$

where the Lagrangian connection coefficients are defined by

$$\begin{aligned} \Gamma^J_{IK} &\equiv (F^{-1})^J{}_j \, \partial_I F^j{}_K \\ &= (\partial_j \Phi^J) \, \partial_I \partial_K \varphi^j. \end{aligned} \tag{1.48}$$

Problem 1.4 *For a non-Cartesian Eulerian basis, as discussed in appendix F.1.2, show that the relation between Eulerian and Lagrangian connection coefficients, Γ^k_{ij} and Γ^K_{IJ}, is*

$$\Gamma^K_{IJ} = (F^{-1})^K{}_k \, F^i{}_I \, F^j{}_J \, \Gamma^k_{ij} + (F^{-1})^K{}_k \, \partial_I F^k{}_J. \tag{1.49}$$

We conclude from equation (1.47) that the covariant derivative of the material velocity in Lagrangian coordinates is $\partial_I v^J$ augmented by a *connection* to ensure the covariant derivative of the vector field is tensorial, that is, independent of the chosen coordinate system. We note that, thanks to the compatibility condition (1.6), the Lagrangian connection coefficients (1.48) exhibit the symmetry,

$$\Gamma^K_{IJ} = \Gamma^K_{JI}, \tag{1.50}$$

which means that the connection is *torsion-free*, as discussed in appendix F.1.7.

Problem 1.5 *For a non-Cartesian Eulerian basis, use relation* (1.49) *between Eulerian and Lagrangian connection coefficients to show that vanishing torsion requires the symmetry*

$$\Gamma^k_{ij} = \Gamma^k_{ji}. \tag{1.51}$$

We conclude that if the Eulerian connection coefficients are torsion-free, then so are the Lagrangian connection coefficients. In problem 1.7, we demonstrate that this property corresponds to a vanishing *torsion tensor*.

Based on this discussion, we may now regard the gradient of the material velocity, $\nabla\mathbf{v}$, as a $(1,1)$ tensor,

$$\nabla\mathbf{v} = \nabla_i v^j\, \mathbf{e}^i \otimes \mathbf{e}_j = \nabla_I v^J\, \mathbf{e}^I \otimes \mathbf{e}_J, \tag{1.52}$$

subject to the transformation rules

$$\nabla_i v^j = (F^{-1})^I{}_i\, F^j{}_J\, \nabla_I v^J, \qquad \nabla_I v^J = F^i{}_I\, (F^{-1})^J{}_j\, \nabla_i v^j. \tag{1.53}$$

The covariant derivative of higher-rank tensors involves additional terms with connection coefficients, as we will see in the next section when we apply the covariant derivative operator twice to the material velocity.

1.5.1 Evolution of Connection Coefficients

The Lagrangian connection coefficients Γ^K_{IJ} evolve over time with the flow of matter.

Problem 1.6 *To see what form this evolution takes, differentiate the relationship* $\Gamma^K_{IJ}\,\partial_K\varphi^i = \partial_I\partial_J\varphi^i$, *easily obtained from* (1.48), *with respect to time* T, *and show, using the general definition of the covariant derivative of a tensor* (F.20), *that*

$$F^i{}_K\,\partial_T\,\Gamma^K_{IJ} = F^i{}_K\,\nabla_I\nabla_J v^K. \tag{1.54}$$

We conclude that

$$\partial_T\,\Gamma^K_{IJ} = \nabla_I\nabla_J v^K. \tag{1.55}$$

Although the Lagrangian connection coefficients Γ^K_{IJ} do not define a tensor, their time rate of change $\partial_T\,\Gamma^K_{IJ}$ does, because the right-hand side of equation (1.55) is the tensor $\nabla\nabla\mathbf{v}$.[9]

Equation (1.55) has important consequences. For example, it implies that

$$\partial_T\,(\Gamma^K_{IJ} - \Gamma^K_{JI}) = (\nabla_I\nabla_J - \nabla_J\nabla_I)v^K = 0, \tag{1.56}$$

because the connection is torsion-free, as expressed by equation (1.50). Thus, we find that, in continuum mechanics, covariant derivatives commute,

$$[\nabla_I, \nabla_J] = 0. \tag{1.57}$$

Problem 1.7 *Show, based on the expression for the general covariant derivative* (F.20), *that, more generally for a vector* $\mathbf{u} = u^I\,\mathbf{e}_I$, *we have the Ricci identity*

$$(\nabla_I\nabla_J - \nabla_J\nabla_I)u^K = r_{IJL}{}^K\,u^L - t_{IJ}{}^L\,\nabla_L\,u^K, \tag{1.58}$$

where

$$t_{IJ}{}^K = \Gamma^K_{IJ} - \Gamma^K_{JI} \tag{1.59}$$

denotes the components of the torsion tensor, and where

$$r_{IJL}{}^K \equiv \partial_I\Gamma^K_{JL} - \partial_J\Gamma^K_{IL} + \Gamma^K_{IM}\,\Gamma^M_{JL} - \Gamma^K_{JM}\,\Gamma^M_{IL} \tag{1.60}$$

[9] The rate of change with respect to the convected time T corresponds to the *Lie derivative* relative to the material velocity, as discussed in section 1.8.

denotes the components of the curvature tensor or Riemann tensor. In continuum mechanics, space has zero torsion, $t_{IJ}{}^{K} = 0$, and zero curvature, $r_{IJL}{}^{K} = 0$, which leads to expression (1.56) *and the commutability of covariant derivatives* (1.57).

Note that although the Lagrangian connection coefficients Γ^{K}_{IJ} do not define a tensor, their difference $\Gamma^{K}_{IJ} - \Gamma^{K}_{JI}$ does. This difference in status is reflected in the placement of the indices on the connection coefficients Γ^{K}_{IJ} (not tensorial) and the elements of the torsion tensor $t_{IJ}{}^{K}$ (tensorial).

1.6 Metric

To introduce a notion of *length*, let us consider the squared norm of a vector **u**. In Cartesian Eulerian coordinates r^i, this squared norm is given by

$$\begin{aligned} \|\mathbf{u}\|^2 &= u^i\, u^j\, g_{ij} \\ &= (u^1)^2 + (u^2)^2 + (u^3)^2, \end{aligned} \tag{1.61}$$

where we have used the fact that the Cartesian components of the *metric tensor* (see appendix D.10) are defined in terms of the Kronecker-delta symbol:

$$g_{ij} = \delta_{ij}. \tag{1.62}$$

In convected Lagrangian coordinates we have

$$\|\mathbf{u}\|^2 = u^I\, u^J\, g_{IJ}, \tag{1.63}$$

where g_{IJ} denotes the Lagrangian components of the metric tensor. Equations (1.61) and (1.63) measure the length of the same vector, which implies that the Eulerian and Lagrangian components of the metric tensor,

$$\mathbf{g} = g_{ij}\, \mathbf{e}^i \otimes \mathbf{e}^j = g_{IJ}\, \mathbf{e}^I \otimes \mathbf{e}^J, \tag{1.64}$$

are related via

$$g_{ij} = (F^{-1})^I{}_i\, (F^{-1})^J{}_j\, g_{IJ}, \tag{1.65}$$

$$g_{IJ} = F^i{}_I\, F^j{}_J\, g_{ij}. \tag{1.66}$$

Writing expressions (1.65) and (1.66) out explicitly, being mindful of the functional dependencies, we obtain

$$g_{ij}(r) = (F^{-1})^I{}_i(r,t)\, (F^{-1})^J{}_j(r,t)\, g_{IJ}(\,\Phi(r,t)\,,t\,), \tag{1.67}$$

$$g_{IJ}(X,T) = F^i{}_I(X,T)\, F^j{}_J(X,T)\, g_{ij}(\,\varphi(X,T)\,). \tag{1.68}$$

Here we have used the fact that the Eulerian components of the metric tensor may depend on the Eulerian spatial coordinates $\{r^i\}$, but they do not depend on time: $g_{ij} = g_{ij}(r)$. The Lagrangian components of the metric tensor, on the other hand, depend both on the Lagrangian coordinates $\{X^I\}$ and time T: $g_{IJ} = g_{IJ}(X,T)$. In classical continuum mechanics (see, e.g., Malvern, 1969), the combination of terms on the right-hand side of equation (1.66) is called the *right Cauchy–Green deformation tensor* or *Green deformation* *tensor*, with components denoted as $C_{IJ} = F^i{}_I F^j{}_J \delta_{ij}$. This tensor is discussed extensively in section 1.20.2.

The metric tensor is an example of a symmetric $(0, 2)$ tensor:

$$g_{ij} = g_{ji}, \qquad g_{IJ} = g_{JI}, \tag{1.69}$$

or expressed as a tensor equation

$$\mathbf{g}^t = \mathbf{g}. \tag{1.70}$$

A superscript t denotes the transpose of a rank-2 tensor, which is discussed in appendix D.2.5.

A manifold endowed with a metric is called a *Riemannian manifold*. As discussed in appendix D.10, it may be used to define the *dot product* between two vectors:

$$(\mathbf{u}, \mathbf{w}) \equiv \mathbf{g}(\mathbf{u}, \mathbf{w}) = \mathbf{u} \cdot \mathbf{w}, \tag{1.71}$$

such that the norm of a vector is obtained by taking the dot product of a vector $\mathbf{u}$ with itself:

$$\|\mathbf{u}\|^2 \equiv (\mathbf{u}, \mathbf{u}) = \mathbf{g}(\mathbf{u}, \mathbf{u}) = \mathbf{u} \cdot \mathbf{u}. \tag{1.72}$$

In spatial and comoving components, we have

$$\mathbf{u} \cdot \mathbf{w} = u^i w^j g_{ij} = u^I w^J g_{IJ}, \tag{1.73}$$

and the component expressions for the squared norm of a vector are given by equations (1.61) and (1.63). The two vectors are *orthogonal* to one another if $\mathbf{g}(\mathbf{u}, \mathbf{w}) = \mathbf{0}$.

The metric tensor has an inverse

$$\mathbf{g}^{-1} = g^{ij} \mathbf{e}_i \otimes \mathbf{e}_j = g^{IJ} \mathbf{e}_I \otimes \mathbf{e}_J, \tag{1.74}$$

such that $\mathbf{g} \cdot \mathbf{g}^{-1} = \mathbf{g}^{-1} \cdot \mathbf{g} = \mathbf{I}$, or, in components,

$$g^{ik} g_{kj} = \delta^i{}_j, \qquad g^{IK} g_{KJ} = \delta^I{}_J. \tag{1.75}$$

Here $\mathbf{I}$ denotes the *identity tensor* or *Kronecker tensor* (see appendix D.4)

$$\mathbf{I} = \delta^i{}_j \mathbf{e}_i \otimes \mathbf{e}^j = \delta^I{}_J \mathbf{e}_I \otimes \mathbf{e}^J, \tag{1.76}$$

which is defined in terms of the Kronecker-delta symbols (1.19).

Problem 1.8 *Show that the identity tensor* $\mathbf{I}$ *transforms as a tensor from Eulerian to Lagrangian coordinates and vice versa.*

Like the metric tensor, the inverse metric tensor is symmetric:

$$(\mathbf{g}^{-1})^t = \mathbf{g}^{-1}. \tag{1.77}$$

It is conventional to denote elements of the inverse metric by g^{ij}, rather than—more accurately but more cumbersomely—by $(g^{-1})^{ij}$. The metric tensor and its inverse may be used to “raise” or “lower” indices, for example, $\omega^i = g^{ij} \omega_j$ or $D_{IJ} = g_{IK} D^K{}_J$, or in the dot product: $u^I v^J g_{IJ} = u^I v_I = u_I v^I$. An implication is that, in a Riemannian manifold, the nature of a tensor becomes less important because one can change the character of its “slots” with the metric tensor.

1.6.1 Covariant Derivative of the Metric

An important property of the metric tensor is that its covariant derivative vanishes:

$$\boldsymbol{\nabla}\mathbf{g} = \mathbf{0}. \tag{1.78}$$

This is obvious in Cartesian Eulerian coordinates: $\nabla_i g_{jk} = \nabla_i \delta_{jk} = \partial_i \delta_{jk} = 0$. In comoving Lagrangian coordinates, we must therefore have

$$\nabla_I g_{JK} = \partial_I g_{JK} - \Gamma^M_{IJ}\, g_{MK} - \Gamma^M_{IK}\, g_{JM} = 0, \tag{1.79}$$

where we have used expression (F.20) for the covariant derivative of a general tensor. We demonstrate in appendix F.1.8 that the implication is that the Lagrangian connection coefficients may be expressed in terms of the Lagrangian components of the metric tensor and its inverse as

$$\Gamma^I_{JK} = \tfrac{1}{2}\, g^{IL}\, (\partial_K g_{LJ} + \partial_J g_{KL} - \partial_L g_{JK}). \tag{1.80}$$

In classical tensor calculus, the torsion-free connection coefficients Γ^I_{JK} are also referred to as the *Christoffel symbols of the second kind* and are denoted by $\left\{ {I \atop JK} \right\}$. If the covariant derivative of the metric vanishes, then so does the covariant derivative of the inverse metric, which may be confirmed by taking the covariant derivative of the expression $g^{IK}\, g_{KJ} = \delta^I{}_J$. An important consequence of the vanishing of the covariant derivative of the metric and its inverse is that *raising and lowering indices commutes with covariant differentiation*.

A non-vanishing covariant derivative of the metric tensor is captured by the *nonmetricity tensor*, as discussed in appendices F.1.13 and G.9.10. As an example, nonmetricity is used to capture *point defects* in crystals, as discussed in chapter 3.

1.7 Deformation Rate and Vorticity

The gradient of the material velocity $\mathbf{v}$ is captured by the expression

$$G^i{}_j \equiv \nabla_j v^i, \tag{1.81}$$

which defines the Cartesian Eulerian components of a $(1, 1)$ tensor:

$$\mathbf{G} = G^i{}_j\, \mathbf{e}_i \otimes \mathbf{e}^j = \nabla_j v^i\, \mathbf{e}_i \otimes \mathbf{e}^j = (\boldsymbol{\nabla}\mathbf{v})^t. \tag{1.82}$$

In the first equality, we used the spatial Eulerian vector and one-form basis elements. In the second equality, we used the transpose of the covariant derivative of the material velocity, $\boldsymbol{\nabla}\mathbf{v}$, which we introduced in section 1.5. The transpose of a $(1, 1)$ tensor is discussed in appendices D.2.5 and D.11. In a nutshell, we have

$$(G^t\,)^i{}_j \equiv g^{i\ell}\, G^k{}_\ell\, g_{kj} = G_j{}^i, \tag{1.83}$$

where in the last expression we used the metric to raise and lower the indices. If we think of a $(1, 1)$ tensor as a matrix, then the last equality in expression (1.83) implies exchanging its rows and columns to obtain its transpose. Taking the transpose a second time returns the original tensor: $(\mathbf{G}^t\,)^t = \mathbf{G}$.

We can write the material velocity gradient (1.82) as the sum of a symmetric and an antisymmetric tensor, as discussed in appendices D.2.4 and D.2.5, namely

$$\mathbf{G} = \mathbf{D} + \mathbf{W}, \tag{1.84}$$

where

$$\mathbf{D} \equiv \tfrac{1}{2}\left(\mathbf{G} + \mathbf{G}^t\right) = \widehat{\mathbf{G}} = \tfrac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{v})^t + \boldsymbol{\nabla}\mathbf{v}\right] \tag{1.85}$$

is the *symmetric deformation-rate tensor*,

$$\mathbf{D} = \mathbf{D}^t, \tag{1.86}$$

and

$$\mathbf{W} \equiv \tfrac{1}{2}\left(\mathbf{G} - \mathbf{G}^t\right) = \widetilde{\mathbf{G}} = \tfrac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{v})^t - \boldsymbol{\nabla}\mathbf{v}\right] \tag{1.87}$$

the *antisymmetric vorticity tensor*,

$$\mathbf{W} = -\mathbf{W}^t. \tag{1.88}$$

Problem 1.9 *Express equations* (1.85) *and* (1.87) *in general spatial components, using the definition of the transpose of a* $(1,1)$ *tensor* (D.22).

Problem 1.10 *Show that*

$$\mathrm{tr}(\mathbf{D}) = \boldsymbol{\nabla}\cdot\mathbf{v} \qquad \text{and} \qquad \mathrm{tr}(\mathbf{W}) = 0, \tag{1.89}$$

where the trace operation, an example of the contraction of a tensor, is defined in appendix D.2.3.

The action of the vorticity tensor on a vector $\mathbf{u}$ may be expressed as

$$\mathbf{W}\cdot\mathbf{u} = \tfrac{1}{2}\left(\boldsymbol{\nabla}\times\mathbf{v}\right)\times\mathbf{u}, \tag{1.90}$$

involving the dot product between the vorticity tensor $\mathbf{W}$ and the vector $\mathbf{u}$, that is $W^i{}_j\, u^j$. We also encounter the *cross product* " $\times$ " denoting the "curl" or the cross product of two vectors; this is an element of classical tensor calculus we will eschew in favor of the more general Levi-Civita pseudotensor, as discussed in appendices D.7 and D.13. Equation (1.90) justifies the use of the name "vorticity tensor" for $\mathbf{W}$ and shows that $\boldsymbol{\nabla}\times\mathbf{v}$ is twice the instantaneous angular velocity of the material in the vicinity of point $\mathcal{S}$ at time t.

Problem 1.11 *Prove expression* (1.90) *by writing it out in index notation in Cartesian Eulerian coordinates, using the properties of the alternating tensor (see appendix D.14).*

Problem 1.12 *By differentiating the deformation gradient* (1.16) *with respect to time T, show that*

$$\partial_T F^i{}_I = G^i{}_k\, F^k{}_I. \tag{1.91}$$

We conclude from equation (1.91) that the velocity gradient $\mathbf{G}$ and the deformation gradient are related to each other via

$$G^i{}_j = (F^{-1})^I{}_j\, \partial_T F^i{}_I. \tag{1.92}$$

Problem 1.13 *Show that, equivalently,*

$$G^i{}_j = -F^i{}_I\,\partial_T\,(F^{-1})^I{}_j. \tag{1.93}$$

The Lagrangian components of the material velocity gradient may be obtained based on the transformation

$$G^I{}_J = (F^{-1})^I{}_i\,G^i{}_j\,F^j{}_J = (F^{-1})^I{}_i\,\partial_T\,F^i{}_J. \tag{1.94}$$

In box 1.4, we investigate deformation in Euclidean space.

Box 1.4 Deformation in Euclidean Space

In box 1.1, we expressed the motion in Euclidean space in terms of the "position vector" form (1.5), namely, $\mathbf{r} = \phi(\mathbf{X}, T)$. In this box, we consider deformation in Euclidean space. The *deformation tensor* is now defined as the two-point tensor

$$\mathbf{F} = (\boldsymbol{\nabla}_X\phi)^t, \tag{1.95}$$

where $\boldsymbol{\nabla}_X$ denotes the gradient with respect to the particle labeled $\mathbf{X}$. If two material particles currently located at Euclidean positions $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ were initially located at $\mathbf{X}$ and $\mathbf{X} + d\,\mathbf{X}$, then the relative current and initial position vectors $d\mathbf{r}$ and $d\,\mathbf{X}$ are related via the deformation tensor:

$$d\mathbf{r} = \mathbf{F}\cdot d\,\mathbf{X}. \tag{1.96}$$

The velocity gradient $\mathbf{G}$ and the deformation tensor are related to each other via

$$\partial_T\,\mathbf{F} = \mathbf{G}\cdot\mathbf{F}, \tag{1.97}$$

which is a tensorial form of equation (1.91).

The suitability of the name "deformation-rate tensor" for $\mathbf{D}$ may be appreciated by considering the rate of change of the Lagrangian components of the metric tensor (1.66).

Problem 1.14 *Show that*

$$\partial_T\,g_{IJ} = 2\,g_{IK}\,D^K{}_J = 2\,D_{IJ}. \tag{1.98}$$

We conclude that the rate of change of the Lagrangian components of the metric tensor equals two times the Lagrangian components of the deformation-rate tensor, thereby justifying the latter's name. Expression (1.98) is an example of what is called a *Lie derivative*, which we discuss in the next section.

1.8 Lie Derivative

As discussed in appendix F.3, the *Lie derivative* of a tensor field $\mathbf{T}$ relative to the flow of matter $\mathbf{v}$ takes a very simple form in Lagrangian coordinates, namely,

$$\mathcal{L}_{\mathbf{v}}\mathbf{T} = \partial_T T^{I_1\cdots I_p}{}_{J_1\cdots J_q}\, \mathbf{e}_{I_1} \otimes \cdots \otimes \mathbf{e}_{I_p} \otimes \mathbf{e}^{J_1} \otimes \cdots \otimes \mathbf{e}^{J_q}. \tag{1.99}$$

This definition implies that Lie derivatives with respect to the flow of Lagrangian basis vectors and one-forms vanish:

$$\mathcal{L}_{\mathbf{v}}\mathbf{e}_I = \mathbf{0}, \qquad \mathcal{L}_{\mathbf{v}}\mathbf{e}^I = \mathbf{0}. \tag{1.100}$$

The Eulerian components are less intuitive, as discussed in appendix F.3.

Problem 1.15 *The Lagrangian components of the Lie derivative of a scalar* q *are* $\partial_T Q$*. Show that its Eulerian components are given in terms of the material velocity* $\mathbf{v}$ *by* $\partial_t q + v^j\,\partial_j q$.

Problem 1.16 *Show that the Eulerian components of the Lie derivative of a vector* $\mathbf{u}$ *relative to the material velocity* $\mathbf{v}$ *are determined by*

$$(\mathcal{L}_{\mathbf{v}}\mathbf{u})^i = \partial_t u^i + v^j\,\partial_j u^i - u^j\,\partial_j v^i.$$

Problem 1.17 *Show that the autonomous Lie derivative, discussed in appendix F.3.3, of a vector* $\mathbf{u}$ *relative to the material velocity* $\mathbf{v}$ *equals their Lie bracket:*

$$\pounds_{\mathbf{v}}\mathbf{u} = [\mathbf{v}, \mathbf{u}].$$

Given the definition of the Lie derivative with respect to the material velocity (1.99), we observe that equation (1.98) may be expressed in terms of this derivative in the tensor form

$$\mathcal{L}_{\mathbf{v}}\mathbf{g} = 2\,\mathbf{g}\cdot\mathbf{D}. \tag{1.101}$$

It is important to note that *raising and lowering indices does not commute with the Lie derivative*, that is, $g_{IK}\,\partial_T T^{KJ} \neq \partial_T T_I{}^J$, as a consequence of (1.101). Thus, we should know the precise nature of the tensor of which we are taking the Lie derivative. For this reason, it is sometimes useful to distinguish between the three versions of a rank-2 tensor, as explored in box 1.5.

In section 1.14, we will give an alternative definition of the Lie derivative relative to the flow of matter in terms of the pullback and pushforward.

Box 1.5 Distinguishing Rank-2 Tensors

A rank-2 tensor, such as the deformation-rate tensor $\mathbf{D}$, may be expressed as a $(2,0)$, $(1,1)$, or $(0,2)$ tensor, with components D^{IJ}, $D^I{}_J = D_J{}^I$, or D_{IJ}, respectively. Most of the time, these distinctions are immaterial because, in a Riemannian manifold, we can raise and lower indices with the metric tensor and its inverse. However, we encountered the problem of raising and lowering indices not commuting with the Lie derivative, which led to a desire to be clear about the nature of the tensor of which we are taking the Lie derivative. This may be accomplished by borrowing notation from music in the form of the *accidentals* "sharp" ($\sharp$), "natural" ($\natural$), and "flat" ($\flat$) to label $(2,0)$, $(1,1)$, and $(0,2)$ versions of a rank-2 tensor. Thus, the deformation-rate tensor $\mathbf{D}$ has three versions, a sharp version $\mathbf{D}^\sharp$ with elements D^{IJ}, a natural version $\mathbf{D}^\natural$ with elements $D^I{}_J = D_J{}^I$, and a flat version $\mathbf{D}^\flat$ with elements D_{IJ}. In this notation, equation (1.101) for the Lie derivative of the metric relative to the flow of matter becomes

$$\mathcal{L}_\mathbf{v}\mathbf{g} = 2\,\mathbf{g}\cdot\mathbf{D}^\natural = 2\,\mathbf{D}^\flat. \tag{1.102}$$

To reduce clutter, we continue to use $\mathbf{D}$, $\mathbf{W}$, and $\mathbf{G}$ to denote the natural forms $\mathbf{D}^\natural$, $\mathbf{W}^\natural$, and $\mathbf{G}^\natural$.

Problem 1.18 *Using the notation in box 1.5, show that*

$$\mathcal{L}_\mathbf{v}\mathbf{g}^{-1} = -2\,\mathbf{D}\cdot\mathbf{g}^{-1} = -2\,\mathbf{D}^\sharp. \tag{1.103}$$

Problem 1.19 *Let $\mathbf{T}$ denote a $(1,1)$ tensor with Lie derivative $\mathcal{L}_\mathbf{v}\mathbf{T}$ relative to the flow of matter. Show that*

$$(\mathcal{L}_\mathbf{v}\,\mathbf{T})^t = \mathcal{L}_\mathbf{v}\,\mathbf{T}^t + 2\,\mathbf{D}\cdot\mathbf{T}^t - 2\,\mathbf{T}^t\cdot\mathbf{D}, \tag{1.104}$$

which illustrates that raising and lowering indices does not commute with taking the Lie derivative:

$$(\mathcal{L}_\mathbf{v}\,\mathbf{T})^t \neq \mathcal{L}_\mathbf{v}\,\mathbf{T}^t.$$

If we take the transpose of equation (1.104), remembering that for two $(1,1)$ tensors $\mathbf{A}$ and $\mathbf{B}$ we have $(\mathbf{A}\cdot\mathbf{B})^t = \mathbf{B}^t\cdot\mathbf{A}^t$, we find that

$$\mathcal{L}_\mathbf{v}\,\mathbf{T} - \left(\mathcal{L}_\mathbf{v}\,\mathbf{T}^t\right)^t = 2\,[\mathbf{T},\mathbf{D}], \tag{1.105}$$

where we have introduced the *commutator* of two $(1,1)$ tensors:

$$[\mathbf{T},\mathbf{D}] \equiv \mathbf{T}\cdot\mathbf{D} - \mathbf{D}\cdot\mathbf{T}. \tag{1.106}$$

We conclude that if a $(1,1)$ tensor commutes with the deformation rate tensor, $[\mathbf{T},\mathbf{D}] = \mathbf{0}$, then $(\mathcal{L}_\mathbf{v}\,\mathbf{T})^t = \mathcal{L}_\mathbf{v}\,\mathbf{T}^t$. We encountered an example of the Lie derivative previously when we investigated the time evolution of the connection coefficients. Specifically, equation (1.55) may be expressed in terms of the Lie derivative in the tensor form

$$\mathcal{L}_{\mathbf{v}}\Gamma = \nabla\nabla\mathbf{v}. \tag{1.107}$$

1.9 Euler Derivative

In complementary fashion to the Lie derivative discussed in the previous section, the *Euler derivative* of a general tensor field $\mathbf{T}$ takes a very simple form in Cartesian Eulerian coordinates:

$$\mathrm{d}_t\mathbf{T} \equiv \partial_t T^{i_1\cdots i_p}{}_{j_1\cdots j_q}\,\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_p}\otimes\mathbf{e}^{j_1}\otimes\cdots\otimes\mathbf{e}^{j_q}. \tag{1.108}$$

In this book, we will not use the commonly used notation $\partial_t\mathbf{T}$ to denote the Euler derivative of a tensor field $\mathbf{T}$, because we reserve ∂_t strictly for partial differentiation of tensor field components expressed in Eulerian coordinates.

In this case, the Lagrangian components of the Euler derivative are less intuitive.

Problem 1.20 *The Eulerian components of the Euler derivative of a scalar* q *are* $\partial_t q$*. Show that its Lagrangian components are given by* $\partial_T Q - v^J\,\partial_J Q$.

Problem 1.21 *Show that the Lagrangian components of the Euler derivative of a vector* $\mathbf{u}$ *are given by*

$$(\mathrm{d}_t\mathbf{u})^I = \partial_T\, u^I - v^J\,\partial_J u^I + u^J\,\partial_J v^I\,. \tag{1.109}$$

Problem 1.22 *Use the fact that the Lagrangian connection coefficients are torsion-free to demonstrate that the Lagrangian components of the Euler derivative of a vector may also be expressed as*

$$(\mathrm{d}_t\mathbf{u})^I = \partial_T\, u^I - v^J\,\nabla_J u^I + u^J\,\nabla_J v^I. \tag{1.110}$$

Thus, in tensor notation, for a torsion-free connection, we have the following relationship between the Euler derivative and the Lie derivative of a vector:

$$\mathrm{d}_t\mathbf{u} = \mathcal{L}_{\mathbf{v}}\mathbf{u} - \mathbf{v}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{v}, \tag{1.111}$$

a result we could have anticipated based on problem 1.16. In particular, for the material velocity

$$\mathrm{d}_t\mathbf{v} = \mathcal{L}_{\mathbf{v}}\mathbf{v}. \tag{1.112}$$

Problem 1.23 *Show that*

$$\mathrm{d}_t\mathbf{v} = \partial_t v^i\,\mathbf{e}_i = \partial_T\, v^I\,\mathbf{e}_I. \tag{1.113}$$

1.10 Material Derivative

The *material derivative* or *substantial derivative* of a general tensor field $\mathbf{T}$ is defined in terms of the Euler derivative, the material velocity $\mathbf{v}$, and the covariant derivative as

$$\mathrm{D}_t \mathbf{T} \equiv \mathrm{d}_t \mathbf{T} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{T}. \tag{1.114}$$

This derivative combines the local change of a tensor field, $\mathrm{d}_t \mathbf{T}$, with a term due to advection, $\mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{T}$. Thus, it captures the rate of change of a tensor field experienced by an observer who "rides along" with an element of the continuum.

Problem 1.24 *Show that for a $(0,0)$ tensor, that is, a scalar, q, its material derivative and Lie derivative relative to the motion are equivalent:*

$$\mathrm{D}_t \mathrm{q} = \mathcal{L}_{\mathbf{v}} \mathrm{q}. \tag{1.115}$$

Problem 1.25 *Show that*

$$\begin{aligned} \mathrm{D}_t \mathbf{v} &= \mathrm{d}_t \mathbf{v} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} \\ &= (\partial_t v^i + v^j \nabla_j v^i)\, \mathbf{e}_i \\ &= (\partial_T v^I + v^J \nabla_J v^I)\, \mathbf{e}_I. \end{aligned} \tag{1.116}$$

It is important to note that the material derivative involves the covariant derivative, and thus a connection, whereas the Lie Euler derivatives require no connection.

1.11 Corotational Material Derivative

Corotational material derivatives capture the rate of change experienced by an observer who comoves and *corotates* with an element of the continuum; we use the notation $\mathring{\mathrm{D}}_t$ to denote a generic corotational material derivative. Depending on the rate of rotation, these derivatives carry different names. The most obvious option is to consider a corotation rate determined by the vorticity tensor (1.87). These derivatives are most commonly used to capture the rate of change of $(1,1)$ tensors.

Problem 1.26 *Let $\mathbf{T}$ be a $(1,1)$ tensor. Show that for a torsion-free connection*

$$\mathrm{D}_t \mathbf{T} = \mathcal{L}_{\mathbf{v}} \mathbf{T} + \mathbf{G} \cdot \mathbf{T} - \mathbf{T} \cdot \mathbf{G}. \tag{1.117}$$

Using the decomposition (1.84), we see that equation (1.117) implies the identity

$$\mathrm{D}_t \mathbf{T} + \mathbf{T} \cdot \mathbf{W} - \mathbf{W} \cdot \mathbf{T} = \mathcal{L}_{\mathbf{v}} \mathbf{T} + \mathbf{D} \cdot \mathbf{T} - \mathbf{T} \cdot \mathbf{D}. \tag{1.118}$$

The terms on the left-hand side of this equation define the *Zaremba–Jaumann rate* (Zaremba, 1903; Jaumann, 1911) of the $(1,1)$ tensor $\mathbf{T}$, which is an example of a corotational material derivative:

$$\mathring{\mathrm{D}}_t^{\mathrm{J}} \mathbf{T} \equiv \mathrm{D}_t \mathbf{T} + \mathbf{T} \cdot \mathbf{W} - \mathbf{W} \cdot \mathbf{T}. \tag{1.119}$$

The Zaremba–Jaumann rate is an example of a so-called *objective rate*, which means that it is a measure of a time rate of change that is *independent of the frame of reference*. In other words, an objective rate is unaffected by rigid translations and rotations of the reference frame. The material time derivative is not objective, but the Lie derivative is.

For a general tensor $\mathbf{T}$, we define the corotational material derivative due to a generic spin tensor $\boldsymbol{\Omega}$ as

$$(\overset{\circ}{\mathrm{D}}{}^{\Omega}_t \mathbf{T})^{ij\cdots}{}_{k\ell\cdots} \equiv (\mathrm{D}_t \mathbf{T})^{ij\cdots}{}_{k\ell\cdots} - \Omega^i{}_m T^{mj\cdots}{}_{k\ell\cdots} - \Omega^j{}_m T^{im\cdots}{}_{k\ell\cdots} - \cdots \\ + T^{ij\cdots}{}_{m\ell\cdots}\Omega^m{}_k + T^{ij\cdots}{}_{km\cdots}\Omega^m{}_\ell + \cdots . \tag{1.120}$$

Note how each contravariant and covariant component is rotated by the spin tensor. The Zaremba–Jaumann rate of a general tensor is obtained by using the vorticity tensor as the spin tensor, $\boldsymbol{\Omega} = \mathbf{W}$, in equation (1.120).

1.12 Levi-Civita Density and Capacity

Most readers will be familiar with the *alternating symbol* or *Levi-Civita symbol* $\underline{\epsilon}_{ijk}$,[10] which is used to express cross products in index notation, for example, $\mathbf{u} \times \mathbf{w}$ has components $\underline{\epsilon}_{ijk}\, u^j\, w^k$. The indices on the alternating symbol suggest that it is tensorial, and in this section, we investigate to what extent this is the case.

We begin by defining two different alternating symbols in spatial coordinates with all lower and all upper indices:

$$\underline{\epsilon}_{ijk} = \begin{cases} +1 & \text{if} \quad i,j,k \quad \text{is an } \textit{even} \text{ permutation of} \quad 1,2,3, \\ \;\;0 & \text{if any indices are identical}\,, \\ -1 & \text{if} \quad i,j,k \quad \text{is an } \textit{odd} \text{ permutation of} \quad 1,2,3, \end{cases} \tag{1.121}$$

and[11]

$$\overline{\epsilon}^{ijk} = \begin{cases} +1 & \text{if} \quad i,j,k \quad \text{is an } \textit{even} \text{ permutation of} \quad 1,2,3, \\ \;\;0 & \text{if any indices are identical}\,, \\ -1 & \text{if} \quad i,j,k \quad \text{is an } \textit{odd} \text{ permutation of} \quad 1,2,3. \end{cases} \tag{1.122}$$

A complementary set of alternating symbols in Lagrangian coordinates is written as $\underline{\epsilon}_{IJK}$ and $\overline{\epsilon}^{IJK}$, and these symbols also take values of $+1$, 0, or -1.

The introduction of these alternating symbols enables us to define the determinant of the deformation gradient as

$$F = \begin{vmatrix} F^1{}_1 & F^1{}_2 & F^1{}_3 \\ F^2{}_1 & F^2{}_2 & F^2{}_3 \\ F^3{}_1 & F^3{}_2 & F^3{}_3 \end{vmatrix} \equiv \frac{1}{3!}\,\underline{\epsilon}_{ijk}\, F^i{}_I\, F^j{}_J\, F^k{}_K\, \overline{\epsilon}^{IJK}. \tag{1.123}$$

The expression for the determinant given by the second equality in equation (1.123) is more general than the first equality, which is borrowed from matrix algebra.

[10] The reason for using the underline will become clear in a moment.
[11] The reason for using the overline will become clear in a moment.

Let us transform the symbol $\underline{\epsilon}_{ijk}$ to Lagrangian coordinates as if it were a tensor. We have

$$\underline{\epsilon}'_{IJK} = F^i{}_I \, F^j{}_J \, F^k{}_K \, \underline{\epsilon}_{ijk}. \tag{1.124}$$

How is the transformation result $\underline{\epsilon}'_{IJK}$ related to the symbol $\underline{\epsilon}_{IJK}$? If we contract equation (1.124) with the alternating symbol $\overline{\epsilon}^{IJK}$ we obtain

$$\overline{\epsilon}^{IJK} \, \underline{\epsilon}'_{IJK} = \overline{\epsilon}^{IJK} \, F^i{}_I \, F^j{}_J \, F^k{}_K \, \underline{\epsilon}_{ijk} = 3! \, F, \tag{1.125}$$

where in the second equality we used equation (1.123). We conclude that $\underline{\epsilon}'_{IJK}$ must take the form

$$\underline{\epsilon}'_{IJK} = F \, \underline{\epsilon}_{IJK}, \tag{1.126}$$

such that equation (1.125) is satisfied. Thus, we conclude from equation (1.124) that the alternating symbols $\underline{\epsilon}_{ijk}$ and $\underline{\epsilon}_{IJK}$ are related via

$$\underline{\epsilon}_{IJK} = \frac{1}{F} \, F^i{}_I \, F^j{}_J \, F^k{}_K \, \underline{\epsilon}_{ijk}. \tag{1.127}$$

This transformation rule is modified from the usual tensor transformation rule by the factor $1/F$. Objects which transform according to this modified rule are called *tensor capacities* of *weight* one.

A similar approach shows that the alternating symbols $\overline{\epsilon}^{ijk}$ and $\overline{\epsilon}^{IJK}$ are related via

$$\overline{\epsilon}^{IJK} = F \, (F^{-1})^I{}_i \, (F^{-1})^J{}_j \, (F^{-1})^K{}_k \, \overline{\epsilon}^{ijk}. \tag{1.128}$$

This is the transformation rule for a *tensor density* of weight one. Tensor densities and capacities are discussed in detail in appendix D.6.

An important implication of the results in this section is that the cross product between two vectors, $\mathbf{u} \times \mathbf{w}$, expressed in terms of the alternating symbol, $\underline{\epsilon}_{ijk} \, u^j \, w^k$, defines a $(0, 1)$ tensor *capacity*.

1.13 Levi-Civita Pseudotensor and Volume Form

Upon calculating the determinant of the metric tensor in Lagrangian coordinates (1.66) based on definition (D.58), using the rule $\det(AB) = \det(A)\,\det(B)$ for the product of two matrices A and B (see box 1.6), and taking the square root of the result, we find

$$\overline{G} = F \, \overline{g}. \tag{1.129}$$

Here F is the determinant of the deformation gradient given by equation (1.123), and we defined the Lagrangian and Eulerian square roots of the determinant of the metric as

$$\overline{G} \equiv \left(\tfrac{1}{3!} \, \overline{\epsilon}^{IJK} \, g_{IL} \, g_{JM} \, g_{KN} \, \overline{\epsilon}^{LMN} \right)^{1/2}, \tag{1.130}$$

and

$$\overline{g} \equiv \left(\tfrac{1}{3!} \, \overline{\epsilon}^{ijk} \, g_{i\ell} \, g_{jm} \, g_{kn} \, \overline{\epsilon}^{\ell mn} \right)^{1/2}, \tag{1.131}$$

respectively. We deduce from (1.129) that the square root of the determinant of the metric $\overline{g}$ transforms like a scalar density of weight one.

Box 1.6 det(A · B)

In this box we demonstrate that $\det(\mathbf{A}\cdot\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B})$ for two $(1, 1)$ tensors $\mathbf{A}$ and $\mathbf{B}$. Using definition equation (D.58), the determinant of $\mathbf{A}\cdot\mathbf{B}$ is

$$\det(\mathbf{A}\cdot\mathbf{B}) = \tfrac{1}{3!}\,\overline{\epsilon}^{IJK}\,A^L{}_P\,B^P{}_I\,A^M{}_Q\,B^Q{}_J\,A^N{}_R\,B^R{}_K\,\underline{\epsilon}_{LMN}.$$

According to equation (1.127), we have

$$A^L{}_P\,A^M{}_Q\,A^N{}_R\,\underline{\epsilon}_{LMN} = \det(\mathbf{A})\,\underline{\epsilon}_{PQR},$$

and

$$\overline{\epsilon}^{IJK}\,B^P{}_I\,B^Q{}_J\,B^R{}_K = \det(\mathbf{B})\,\overline{\epsilon}^{PQR}.$$

Thus

$$\det(\mathbf{A}\cdot\mathbf{B}) = \tfrac{1}{3!}\det(\mathbf{A})\,\underline{\epsilon}_{PQR}\det(\mathbf{B})\,\overline{\epsilon}^{PQR} = \det(\mathbf{A})\det(\mathbf{B})\,,$$

as advertised.

Problem 1.27 *Show that the determinant of the inverse metric,*

$$\underline{g} \equiv \frac{1}{g}, \tag{1.132}$$

transforms as a scalar capacity of weight one:

$$\underline{G} = \frac{1}{F}\,\underline{g}. \tag{1.133}$$

Problem 1.28 *Show that*

$$\overline{\epsilon}^{IJK}\,g_{PL}\,g_{JM}\,g_{KN}\,\overline{\epsilon}^{LMN} = 2\,\overline{G}^2\,\delta^I{}_P. \tag{1.134}$$

Problem 1.29 *Demonstrate Cramer's rule for the inverse of the metric tensor:*

$$g^{NK} = \tfrac{1}{2}\,\underline{G}^2\,\overline{\epsilon}^{IJK}\,\overline{\epsilon}^{LMN}\,g_{IL}\,g_{JM}, \tag{1.135}$$

using equation (1.134)*. Cramer's rule may be used to solve a system of three simultaneous linear equations.*

Problem 1.30 *Show that*

$$\underline{g}\,\partial_i\overline{g} = \Gamma^{\,j}_{ij} \qquad \text{and} \qquad \underline{G}\,\partial_I\overline{G} = \Gamma^{\,J}_{IJ}. \tag{1.136}$$

The transformation rule (1.129) motivates the introduction of the tensor

$$\boldsymbol{\epsilon} = \epsilon_{ijk}\,\mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k = \epsilon_{IJK}\,\mathbf{e}^I \otimes \mathbf{e}^J \otimes \mathbf{e}^K, \tag{1.137}$$

where

$$\epsilon_{ijk} \equiv \overline{g}\,\underline{\epsilon}_{\,ijk} \qquad \text{and} \qquad \epsilon_{IJK} \equiv \overline{G}\,\underline{\epsilon}_{\,IJK}. \tag{1.138}$$

Equation (1.137) defines a tensor, since it transforms according to rules (D.26) and (D.27). Instead of $\{+1, 0, -1\}$, it takes values of $\{+\overline{g}, 0, -\overline{g}\}$ and $\{+\overline{G}, 0, -\overline{G}\}$, respectively. However, a change in the *orientation* or *handedness* of the coordinate system changes the sign of this tensor, which means it is a *pseudotensor* (Frankel, 2004), as discussed in appendix D.6.

The implication for the "cross product" in three dimensions is that one should use the pseudotensor $\boldsymbol{\epsilon}$ for this purpose, *not* the alternating symbols $\underline{\epsilon}_{\,ijk}$ and $\underline{\epsilon}_{\,IJK}$, especially in curvilinear Lagrangian coordinates.

We identify the Levi-Civita pseudotensor (1.137) with the *volume form*, a completely antisymmetric $(0, 3)$ *pseudoform*. To justify the nomenclature *volume form*, imagine feeding the $(0, 3)$-pseudotensor $\boldsymbol{\epsilon}$ three small vectors $\Delta\mathbf{u} = \Delta u^i\,\mathbf{e}_i$, $\Delta\mathbf{v} = \Delta v^j\,\mathbf{e}_j$, and $\Delta\mathbf{w} = \Delta w^k\,\mathbf{e}_k$. The result is the scalar

$$\boldsymbol{\epsilon}(\Delta\mathbf{u}, \Delta\mathbf{v}, \Delta\mathbf{w}) = \epsilon_{ijk}\,\Delta u^i\,\Delta v^j\,\Delta w^k = \Delta\mathbf{u} \cdot (\Delta\mathbf{v} \times \Delta\mathbf{w}). \tag{1.139}$$

This is, of course, precisely the volume of the parallelepiped spanned by the three vectors $\Delta\mathbf{u}$, $\Delta\mathbf{v}$, and $\Delta\mathbf{w}$.

In appendix G.5, we identify a three-dimensional volume with a *three-form* expressed using either Eulerian or Lagrangian basis one-forms:

$$\begin{aligned} \boldsymbol{\epsilon} &= \tfrac{1}{3!}\,\epsilon_{ijk}\,\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k \\ &= \overline{g}\,\underline{\epsilon}_{\,123}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &= \tfrac{1}{3!}\,\epsilon_{IJK}\,\mathbf{e}^I \wedge \mathbf{e}^J \wedge \mathbf{e}^K \\ &= \overline{G}\,\underline{\epsilon}_{\,123}\,\mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3. \end{aligned} \tag{1.140}$$

Problem 1.31 *Show that*

$$\partial_i\epsilon_{jk\ell} = \Gamma^m_{im}\,\epsilon_{jk\ell} \qquad \text{and} \qquad \partial_I\epsilon_{JKL} = \Gamma^M_{IM}\,\epsilon_{JKL}. \tag{1.141}$$

Problem 1.32 *Show that*

$$\nabla_i\epsilon_{jk\ell} = 0 \qquad \text{and} \qquad \nabla_I\epsilon_{JKL} = 0. \tag{1.142}$$

It is straightforward to show that the three-vector with Lagrangian and Eulerian elements $\epsilon^{ijk} = \underline{g}\,\overline{\epsilon}^{\,ijk}$ and $\epsilon^{IJK} = \underline{G}\,\overline{\epsilon}^{IJK}$, respectively, has similar properties as the volume form.

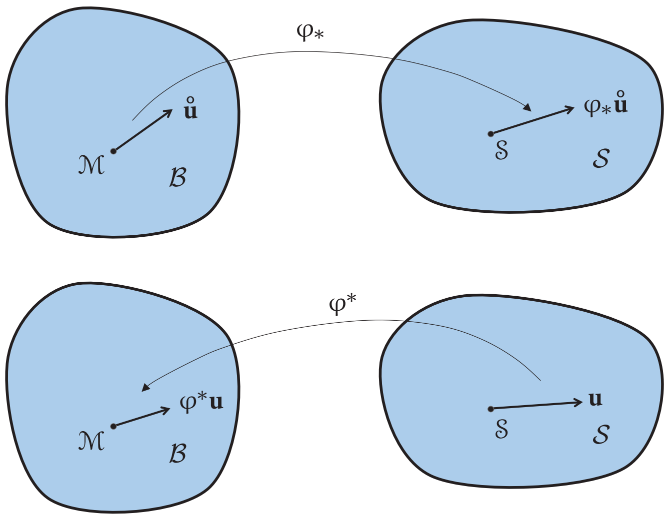

Figure 1.5: *Top*: Let $\mathring{\mathbf{u}} = \mathring{u}^I\, \mathring{\mathbf{e}}_I$ denote a vector in tangent space of the referential manifold $\mathscr{B}$ at material location $\mathcal{M}$. The *pushforward* of $\mathring{\mathbf{u}}$ from the referential manifold $\mathscr{B}$ to the spatial manifold $\mathscr{S}$ with the motion φ is defined as $\varphi_*\mathring{\mathbf{u}} \equiv \mathring{u}^I\, F^i{}_I\, \mathbf{e}_i$, which defines a vector in tangent space of the spatial manifold $\mathscr{S}$ at the current location of the particle $\mathcal{S}$. *Bottom*: Conversely, let $\mathbf{u} = u^i\, \mathbf{e}_i$ denote a vector in tangent space of the spatial manifold $\mathscr{S}$ at location $\mathcal{S}$. The *pullback* of $\mathbf{u}$ from the spatial manifold $\mathscr{S}$ to the material manifold $\mathscr{B}$ with the motion φ is defined as $\varphi^*\mathbf{u} \equiv u^i\, (F^{-1})^I{}_i\, \mathring{\mathbf{e}}_I = u^I\, \mathring{\mathbf{e}}_I$. If the referential vector $\mathring{\mathbf{u}}$ is the pullback of a spatial vector $\mathbf{u}$, that is, $\mathring{\mathbf{u}} = \varphi^*\mathbf{u}$, then $\mathring{u}^I = u^I$; in other words, the referential components of the referential vector equal the Lagrangian components of the spatial vector.

The volume form (1.137) and the metric (1.64) are the two most important entities capturing the structure of a Riemannian manifold.

1.14 Pullback and Pushforward

To introduce the notions of *strain* and *stress*, we need to define an equilibrium state at some referential time T_0, for example, the state of the Earth prior to an earthquake. This state is captured by the referential manifold $\mathscr{B}$ introduced in section 1.1 and illustrated in figure 1.1. Thus, we need to be able to compare tensors in the spatial manifold $\mathscr{S}$ at time $t = T$ with tensors in the referential manifold $\mathscr{B}$ at time $t = T_0$ and vice versa. This is accomplished based on "pushforwards" and "pullbacks" associated with the motion φ, which are concepts discussed in appendix E.3 and illustrated in figure 1.5.

To clearly distinguish tensors in the referential manifold $\mathscr{B}$ from tensors in the spatial manifold $\mathscr{S}$, we use a "$\circ$" for their identification, and we use $\mathring{\mathbf{e}}_I$ and $\mathring{\mathbf{e}}^I$ to denote referential basis vectors and one-forms, respectively. For example, the metric tensor in the referential manifold is denoted by

$$\mathring{\mathbf{g}} = \mathring{g}_{IJ}\, \mathring{\mathbf{e}}^I \otimes \mathring{\mathbf{e}}^J, \tag{1.143}$$

where

$$\mathring{g}_{IJ}(X) \equiv g_{IJ}(X, T_0), \tag{1.144}$$

and the square root of its determinant is

$$\overline{\mathring{G}}(X) \equiv \overline{G}(X, T_0). \tag{1.145}$$

We have thus far chosen this to be a Cartesian metric; here, we accommodate the more general case of a curvilinear referential coordinate system.

As an example, the pushforward of the metric in the referential manifold to the spatial manifold is defined by

$$\begin{aligned} \varphi_* \mathring{\mathbf{g}} &\equiv \mathring{g}_{IJ} \, (F^{-1})^I{}_i \, (F^{-1})^J{}_j \, \mathbf{e}^i \otimes \mathbf{e}^j \\ &= \mathring{g}_{IJ} \, \mathbf{e}^I \otimes \mathbf{e}^J, \end{aligned} \tag{1.146}$$

where in the second equality we expressed the pushforward in Lagrangian coordinates in the spatial manifold. Because referential coordinates in the referential manifold coincide with Lagrangian coordinates in the spatial manifold at time $t = T_0$, the Lagrangian components of the pushforward $\varphi_* \mathring{\mathbf{g}}$ are identical to the referential components of the referential metric tensor, $\mathring{g}_{IJ}$.

Similarly, the pullback of the metric tensor in the spatial manifold to the referential manifold is defined by

$$\begin{aligned} \varphi^* \mathbf{g} &\equiv g_{ij} \, F^i{}_I \, F^j{}_J \, \mathring{\mathbf{e}}^I \otimes \mathring{\mathbf{e}}^J \\ &= g_{IJ} \, \mathring{\mathbf{e}}^I \otimes \mathring{\mathbf{e}}^J. \end{aligned} \tag{1.147}$$

In this case, the referential components of the pullback $\varphi^* \mathbf{g}$ are identical to the Lagrangian components of the spatial metric tensor, g_{IJ}.

In general, the Lagrangian components of the pushforward of a tensor $\mathring{\mathbf{T}}$ from the referential to the spatial manifold, $\varphi_* \mathring{\mathbf{T}}$, are identical to its referential components, and the referential components of the pullback of a tensor $\mathbf{T}$ from the spatial to the referential manifold, $\varphi^* \mathbf{T}$, are identical to its Lagrangian components.

In terms of the inverse map $\Phi : \mathcal{S} \to \mathcal{B}$, we note that the pullback Φ^* is equivalent to the pushforward φ_*, and that the pushforward Φ_* is equivalent to the pullback φ^*. With this understanding, we will continue to use the notation φ_* and φ^* for pushforwards and pullbacks.

At the end of section 1.1, we noted the dual role of the motion (1.3), describing both a map between material points in the referential manifold and the location of that element of the material in the spatial manifold, and defining a coordinate transformation between Eulerian and Lagrangian coordinates in the spatial manifold. The pushforward and pullback involve a subtle third use of the motion, this time to provide a means of "pulling" or "pushing" tensors between the referential and spatial manifolds.

Finally, we note that the Lie derivative relative to the flow of matter (1.99) may be expressed in terms of a pullback and a pushforward as

$$\mathcal{L}_{\mathbf{v}} \mathbf{T} = \varphi_* \, \partial_T \, \varphi^* \mathbf{T}. \tag{1.148}$$

Problem 1.33 *Demonstrate equation* (1.148) *for a* (1, 1) *tensor.*

1.15 Volumes

If we feed the volume form (1.137) three infinitesimal vectors expressed in Eulerian basis vectors, $dr^1 \, \mathbf{e}_1$, $dr^2 \, \mathbf{e}_2$, and $dr^3 \, \mathbf{e}_3$, it returns

$$\epsilon(dr^1\,\mathbf{e}_1, dr^2\,\mathbf{e}_2, dr^3\,\mathbf{e}_3) = \epsilon(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\, dr^1\, dr^2\, dr^3 = \overline{g}\,\underline{\epsilon}_{123}\, dr^1\, dr^2\, dr^3. \tag{1.149}$$

The Eulerian and Lagrangian components of the infinitesimal vectors are related via $r^i = F^i{}_I\, dX^I$, such that

$$\begin{aligned}\overline{g}\,\underline{\epsilon}_{123}\, dr^1\, dr^2\, dr^3 &= \overline{g}\,\underline{\epsilon}_{123}\, F^1{}_I\, F^2{}_J\, F^3{}_K\, dX^I\, dX^J\, dX^K \\ &= \tfrac{1}{3!}\,\overline{g}\,\underline{\epsilon}_{ijk}\, F^i{}_I\, F^j{}_J\, F^k{}_K\, dX^I\, dX^J\, dX^K \\ &= \tfrac{1}{3!}\,\overline{g}\, F\,\underline{\epsilon}_{IJK}\, dX^I\, dX^J\, dX^K \\ &= \overline{g}\, F\,\underline{\epsilon}_{123}\, dX^1\, dX^2\, dX^3,\end{aligned} \tag{1.150}$$

where we used relationship (1.127). Thus, we find the relationship

$$\overline{g}\, dr^1\, dr^2\, dr^3 = F\,\overline{g}\, dX^1\, dX^2\, dX^3 = \overline{G}\, dX^1\, dX^2\, dX^3, \tag{1.151}$$

where, in the last equality, we used (1.129). This is, of course, precisely what happens in a change of variables during integration:

$$\overline{g}\, d^3 r = \overline{G}\, d^3 X. \tag{1.152}$$

We may wish to establish the *change in volume* when an element of the continuum transitions from the referential manifold to the spatial manifold. In the referential manifold $\mathcal{B}$, the volume element is

$$\begin{aligned}\mathring{\epsilon} &= \mathring{\epsilon}_{IJK}\, \mathring{\mathbf{e}}^I \otimes \mathring{\mathbf{e}}^J \otimes \mathring{\mathbf{e}}^K \\ &= \overline{\mathring{G}}\,\underline{\epsilon}_{IJK}\, \mathring{\mathbf{e}}^I \otimes \mathring{\mathbf{e}}^J \otimes \mathring{\mathbf{e}}^K,\end{aligned} \tag{1.153}$$

or, expressed as a three-form,

$$\begin{aligned}\mathring{\epsilon} &= \tfrac{1}{3!}\, \mathring{\epsilon}_{IJK}\, \mathring{\mathbf{e}}^I \wedge \mathring{\mathbf{e}}^J \wedge \mathring{\mathbf{e}}^K \\ &= \overline{\mathring{G}}\,\underline{\epsilon}_{123}\, \mathring{\mathbf{e}}^1 \wedge \mathring{\mathbf{e}}^2 \wedge \mathring{\mathbf{e}}^3,\end{aligned} \tag{1.154}$$

where $\overline{\mathring{G}}$ is the square root of the determinant of the referential metric given by (1.145). The pushforward of the referential volume element is

$$\begin{aligned}\varphi_* \mathring{\epsilon} &= \overline{\mathring{G}}\,\underline{\epsilon}_{IJK}\, (F^{-1})^I{}_i\, (F^{-1})^J{}_j\, (F^{-1})^K{}_k\, \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \\ &= \overline{\mathring{G}}\,\underline{G}\,\epsilon_{IJK}\, \mathbf{e}^I \otimes \mathbf{e}^j \otimes \mathbf{e}^K \\ &= F^{-1}\,\overline{\mathring{G}}\,\underline{g}\,\epsilon \\ &= J^{-1}\,\epsilon,\end{aligned} \tag{1.155}$$

where in the third equality we used equation (1.133), and where in the last equality we have introduced the *Jacobian of the motion* (see, e.g., Marsden and Hughes, 1983, Proposition 5.3)

$$J \equiv \overline{G}\,\underline{\mathring{G}} = F\,\overline{g}\,\underline{\mathring{G}}. \tag{1.156}$$

Note that at the referential time, we have

$$J(X, T_0) = 1. \tag{1.157}$$

(continued...)

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