© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

CONTENTS

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

CHAPTER 1

Mr. Polytope Goes to Budapest

Geometry will draw the soul towards truth.

—Plato, *The Republic*

Bursts of white light lit up the splendidly restored auditorium of Hungary's Academy of Sciences as Donald Coxeter inched toward the lectern, leaning only slightly on his cane. Photographers had descended upon the academy, located on the east bank of the Danube in Budapest, to capture a few shots of the president of the Republic of Hungary, Ferenc Madl, who was there making a rare public appearance at the opening ceremony of the János Bolyai Conference on hyperbolic geometry, in July 2002. But afterward, the photographers stayed to snap a few shots of Coxeter as well.[1]

Flash from the cameras reflected off his pale pate and the bejeweled turtle brooch pinned to his lapel. Well into his nineties, Coxeter still traveled the international conference circuit. He had been invited to give the opening lecture at this event, commemorating the two hundredth birthday of Hungary's sainted Bolyai, who, with his discovery of non-Euclidean geometry in 1823, changed forever our perception of space.[2]

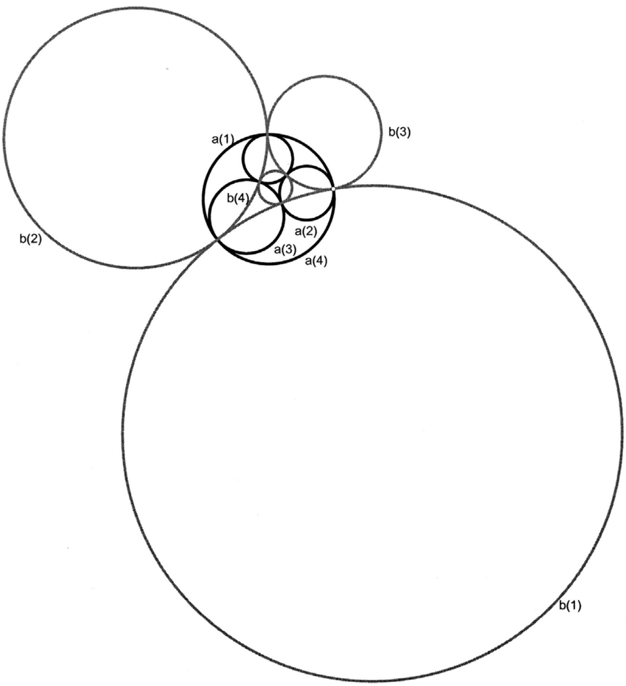
When a long-retired mathematician is asked to give an address at a conference, his audience would be forgiven in assuming that he'll provide an autobiographical synopsis of his career. Coxeter, however, wrote a scholarly paper, months and months in the preparation. Titled "An Absolute Property of Four Mutually Tangent Circles,"[3] it addressed a topic tangentially related to Descartes' Circle theorem, one of Coxeter's favorites.[4] As he was announced to the audience, Coxeter shuffled the pages of his talk, and readied his visual aids—numerous transparencies and a geometrical model, a cubic nexus of multicolored straws. Three hundred or so mathematicians awaited his presentation, a discrete group of individuals more than willing to forfeit July's summer sun for the somnolent glow cast by the lecture hall's overhead

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

projector. Most were a fraction of Coxeter's age. Many, the organizers included, had been skeptical that he would be able to make the journey. A similar number no doubt wondered whether he could possibly have anything left to profess.[5]

Coxeter began slowly, enunciating meticulously with his lingering British accent: "The absolute property of four mutually tangent circles that I am describing seems to have been discovered by Mr. Philip Beecroft, of Hyde Academy, Cheshire, England, and published in *The Lady and Gentleman's Diary* . . . In Beecroft's own words, [the theorem states,] 'If any four circles be described to touch each other mutually, another set of four circles of mutual contact may be described whose points of contact shall coincide with those of the first four.' "[6] He waded into an examination of what he believed to be his new proof—a simple, elegant proof—of Beecroft's theorem, delineating the four mutually tangent circles, a_1, a_2, a_3, a_4, and another set, b_1, b_2, b_3, b_4. "This figure makes the theorem almost obvious," he said, fixing his transparency into position, "but for the sake of completeness it seems desirable to



The diagram of four mutually tangent circles used by Coxeter in his Budapest talk.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

consider further details." He proceeded, pausing here and there, whistling lightly under his breath, as he often did to focus his concentration.[7]

When Donald Coxeter barely squeaked by his ninety-fifth birthday, his doctor diagnosed that he was in that final and waning stage of life warranting palliative care—there was no cure for what ailed him. He was creaky and tired, acute pointy parentheses wrinkling around his subtle smile. He had weathered cancer of the prostate and the right eye, and a heart attack, and now suffered chronic digestive troubles. Nonetheless, against doctor's orders, he was determined—ever the obstinate optimist—to make the trip to the Bolyai conference.[8]

In an attempt to be as prudent as possible in planning his journey, Coxeter tended with grandfatherly gumption to life-and-death details such as buying travel health insurance (he was refused), and determining what to do with his brain should he die while away. Coxeter's brain functioned so impressively over the years that he had received a request for its mass of synapses to be seized swiftly, no more than twelve hours' postmortem, in order to undergo scientific research at McMaster University, in Hamilton, Ontario,[9] where a specimen of Einstein's well-traveled and well-dissected brain resides.[10] As Coxeter recalled, McMaster's Dr. Sandra Witelson rang and asked with pardonable clinical insensitivity: "Dr. Coxeter, when you die can we have your brain?" He took it as a compliment and agreed.[11]

On the day of departure, Coxeter sat at his cluttered kitchen table and set about the task of testing his hearing aids. He snapped his fingers from one ear to the other, then tried a ticking pocket watch. "Dead as a doornail! I get the best results when I don't have any hearing aid in at all!" he concluded, at once confused and bemused. He glanced out the window to find the airport limousine waiting at the sidewalk. "Oh, bother!" he cursed (that was the extent of his cursing, for a minor mishap or a major flood in the basement). "How very, very awkward. I'm not ready!" Coxeter checked and rechecked that his passport and airline ticket and envelope stuffed with Hungarian money were safely stowed in his briefcase. He packed his hearing aids and the Tupperware container full of dead and fresh batteries ("A terrible nuisance that they were somehow mixed together"). He gathered all the parts of his electric shaver and stowed his high-altitude antiembolismic socks in the waistband of his daughter's skirt, like Kleenex at the ready under the cuff of a sleeve.[12]

Coxeter's daughter, Susan Thomas, a retired nurse, was his escort. With the chauffeur patiently standing by outside, Coxeter checked for his passport yet again, then snapped his briefcase shut. He inched down the stairs, taking

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

each step two feet at a time, and finally passed the cuckoo clock in the foyer embossed with the motto "Delay Not the Hour Flies." He shuffled along the front walk, and slid his stiff, angular body, not an ounce of body fat for cushioning, onto the limo's leather seat. He was off, venturing forth on one more journey into what he termed "the wild wicked world"—a world, according to the classical definition of "geo-metry," which he had spent more than three-quarters of a century measuring.[13]

One day after Coxeter arrived in Budapest, he attended a welcoming luncheon at his posh Hyatt hotel. There to greet him was the conference organizer, András Prékopa. A member of the Hungarian Academy of Sciences, and a professor of mathematics and operations research at Rutgers University, Prékopa had never before met Coxeter. When he did he shook his hand and announced with a beaming smile: "Dr. Coxeter is *the* world's greatest living classical geometer. No question!"[14]

Later that evening, relaxing in the hotel lobby, Coxeter met with another fan, Texan Glenn Smith, a self-described "geometry groupie," who makes a successful living in the sesame business. Smith brought to Budapest the geometric model for Coxeter's presentation—constructed by special order—as well as an antique set of wooden geometric solids, circa 1850, which he had purchased during a stopover in London. Smith always travels with models in his suitcase that can be assembled and disassembled like LEGO; it's how he kills time in airports and keeps himself company in hotel rooms.[15]

Even with his hobbyist's perspective, Smith had a cogent argument for Coxeter's designation as the savior of classical geometry. "Coxeter so understood the importance of geometry that he stuck with it. He went out on a hilltop—when all the rest of us were down in the valley—and he saw what was out in front of us and how important geometry was going to become, and he led us out of the darkness. We've been in a dark age," said Smith. "And I think we're still trying to come out of that age. The more we investigate geometry, the better off we all will be." Smith also provided an interesting way of explaining the importance of geometry in the world. "Geometry is at the root of everything, whether we recognize it or not. If you take everything and strip it down—start out with the universe and galaxies and stars and planets and solar system and the Earth, then the Earth is organized into countries and countries become communities and communities are made of families, families are made up of people, people have organs, organs have cells, molecules, atoms, subatomic—strip all that away, and at every stage there are certain geometries or configurations of patterns. If you study those patterns, you will see them almost wherever you go, they will always exist. That's the nice thing about geometry, about polytopes or polyhedra—we

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

could be anywhere in the universe and have the same thoughts. In other words, geometry is not particular to this planet we live on."[16]

"What I told my children when they were young," he continued, "is that you need to learn geometry because if you are ever picked up by a flying saucer, you'll need to show the aliens that you know geometry. They will know geometry for sure. You'll need to be able to make a tetrahedron like this"—he placed his right hand on his forehead and his left hand on his right elbow, forming the frame of a tetrahedron. "If you see somebody from another planet, do that and they'll know you have some intelligence, and they won't treat you like an insect and pull off your arms and legs."[17]

Coxeter, not long before, had articulated much the same sentiment when speaking of the Platonic solids: "I don't think they were invented. I think they were discovered. Somebody on a different planet, with the right kind of mind, would find the same thing."[18] That evening in Budapest, Coxeter added as a footnote: "It was Plato's idea that everything that is true has always been true and people simply reconstructed true things by thinking about them."[19]

Researching the family tree of geometry, tracing the ancestry from Thales who begat Pythagoras who begat Plato, is comparable to retelling tales from the Bible, since most of what is known about these single-name ancients comes from unattributed or biased sources, anecdotes passed down and spun together to form a grand mythology.[20]

The five regular polyhedra, for example, the mainstays of geometry, are also called Platonic solids even though they were known before Plato (427–347 BC). But Plato took a special interest in these solids and left us the earliest surviving description in his book *Timaeus* (the sequel to his *Republic*).[21] In Scotland, a complete set of five carved out of stone have been attributed to Neolithic people dating back some 4,000–6,000 years. In fact, according to George Hart's Web-based *Encyclopedia of Polyhedra*,[22] hundreds of stone spheres have been found with carved edges roughly corresponding to the regular polyhedra, and ranging in material from sandstone to granite and quartzite. Ornate bronze dodecahedra by the dozens, dating to Roman times from the second to fourth century, have been unearthed across Europe, in the United Kingdom, Belgium, Germany, France, Luxembourg, the Netherlands, Austria, Switzerland, and Hungary. Their function has not been confirmed—perhaps candle stands, flower stands, staff or scepter decorations, surveying instruments, leveling instruments, finger ring–size gauges, or just plain geometric sculpture.[23]

Coxeter liked to note that a pair of icosahedral dice of the Ptolemaic

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

dynasty reside in one of the Egyptian rooms of the British Museum in London, and that excavations on the Monte Loffa, near Verona, extracted an Etruscan dodecahedron, revealing that this figure was enjoyed as a toy at least 2,500 years ago.[24] Known for meticulously sourcing his ideas, Coxeter provided perhaps the best summation of origins: "The early history of these polyhedra is lost in the shadows of antiquity. To ask who first constructed them is almost as futile as to ask who first used fire."[25]

In the ancient Greek tradition, geometry was elevated beyond its practical Egyptian and Babylonian usage (5000–500 BC) to the rank of science.[26] The Greek word *mathemata* translated to "science of learning"—and mathematics in those days essentially comprised geometry.[27] Geometry was the purest measure of truth and the highest form of knowledge, with schools dedicated to its study. The Pythagorean School, which became part of the zeitgeist,[28] was attended by citizens of all social strata, especially the upper class. Women disregarded a law forbidding their presence at public meetings and flocked to hear Pythagoras speak.[29] The ingenuity of the Pythagorean theorem—stating that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides—provided early affirmation of the direct relationship between number and space. However, when Pythagoras squared the hypotenuse, he did not do "modern" mathematics, multiplying the hypotenuse by itself. Rather, he literally constructed a geometrical square on top of the hypotenuse. Likewise, the sum of two squares being equal to a third meant that the two squares could be physically cut up and reassembled to form the third square.[30]

Pythagoras (569–475 BC) believed that mathematics was religion, capable of purifying the spirit and uniting the soul with the Divine. He made the study of geometry part of a liberal education, probing theorems in an intellectual manner.[31] His heir was Plato, who proclaimed, "God ever geometrizes."[32] And when Plato started his own school, the Academy, the sign hanging over the entrance indicated he did not suffer geometrical fools gladly: "Let none ignorant of geometry enter my door."[33]

Plato, too, held that mathematics was the finest training for the mind, the secrets of the universe being embedded in number and form. He believed the ideal geometrical shapes—circles, spheres, squares, cubes—did not exist in reality but only in a higher realm of their own, independent from the physical world; a sphere in the physical world was only an approximation of the perfect form of a sphere.

"The ideal notion is the mathematical concept," said mathematical physicist Sir Roger Penrose. "A mathematical concept or mathematical structure, in a certain sense, conjures itself into existence. Mathematicians tend to think of mathematics as having its own existence . . . of mathematical notions and

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

mathematical truths as having a timeless existence. And mathematicians are somehow explorers in that world." The notion that mathematical structures contain an inviolable reality of their own is somehow reassuring. The human mind operates with significant margin of error, so often imprecise, inconsistent, and selective in its judgments. In mathematics, there exists logical rigor, an absolute purity. Plato's world of mathematical forms provided a methodology that modern science has followed ever since—scientists propose models of the world, and the models are tested against observations from previous or new experiments.[34]

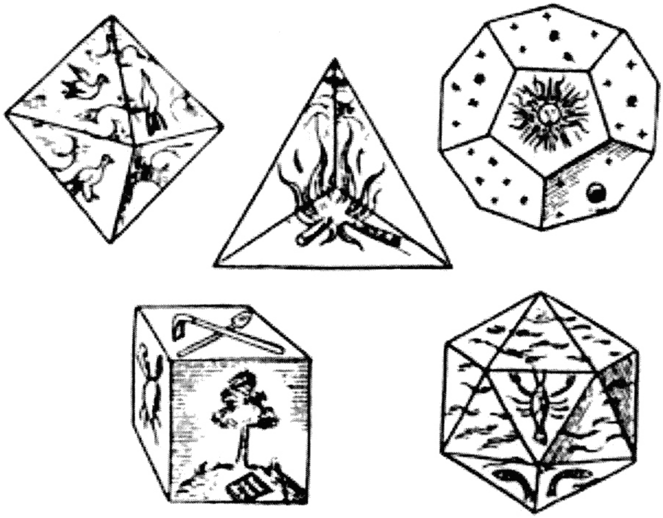
Plato himself had a model of the world, based on his namesake solids. In his book *Timaeus*, four interlocutors gather to discuss cosmology and natural science. The main character, Timaeus, constructs a story for the creation and composition of the universe. As one Plato biographer, A. E. Taylor, recounted, "What Timaeus is really trying to formulate is no fairy tale, but, as we shall see, a geometrical science of nature." In devising his theory of everything, Plato paired the classical elements with the five regular solids.[35] These shapes, Plato said, were "forms of bodies which excel in beauty,"[36] their beauty residing in the criteria they meet for being "regular," or uniform. First, each solid's surfaces are all the same regular polygon—a shape with all sides and all angles equal (the equilateral triangle, the square . . .). The classification of the Platonic solids as "regular" also depends on a second criterion: the same number of regular polygon faces must meet in the same way at each corner, or vertex.[37]

There are three Platonic solids constructed solely with the equilateral triangle. The simplest is the tetrahedron, composed of four equilateral triangles, three at each of its four vertices. In his scheme of the elements, Plato chose the tetrahedron, due to its simplicity and sharp corners, to represent fire, the fiercest and most basic of the elements—with its "penetrating acuteness . . . the pyramid is the solid which is the original element and seed of fire."[38] The octahedron is built from eight equilateral triangles, four at each vertex, and Plato considered it symbolic of air, because this solid spins nicely in the wind (or by blowing on it) when you hold it between finger and thumb.[39] The icosahedron has twenty equilateral triangles, five at each vertex, which combine to make it the roundest of the regular polyhedra. As a result, Plato associated the icosahedron with a drop of water, "the densest and least penetrating of the three fluid elements."[40]

The cube, Plato assigned to earth: "for earth is the most immovable of the four and the most plastic of all bodies, and that which has the most stable bases must of necessity be of such a nature."[41] Thus four of the five convex regular polyhedra symbolized the four elements: fire, air, water, and earth. "The discrepancy between four elements and five solids did not upset Plato's

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.



Kepler's Platonic solids etched with the classical elements, from *Harmonice Mundi*, 1619.

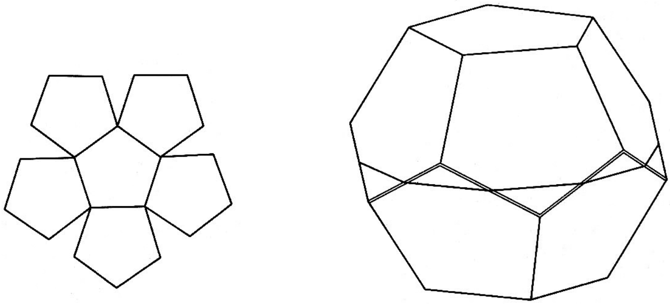
scheme," Coxeter noted. "He described the fifth as a shape that envelops the whole universe."[42] The dodecahedron, with twelve pentagonal or five-sided faces, was the model of the universe as a whole. "There remained a fifth construction," said Plato, "which God used for embroidering the constellations on the whole heaven."[43] Plato's scheme demonstrated considerable prescience, because the Platonic solids, even though they did not turn out to be the exact elements of all existence, are in many ways elemental, or fundamental, components of the universe, emerging on both microscopic and macroscopic dimensions in the most unexpected places—a recent cosmological hypothesis revisited Plato's notion that the universe might be dodecahedral; and in astrochemistry, the shape of the Nobel-winning C_{60} molecule is a truncated icosahedron. (See chapter 10 for C_{60} and chapter 12 for the dodecahedral universe.)[44]

The Fabergé-egg feature of Platonic solids—what makes them such exquisite treasures—is the fact that only five regular solids can physically exist.* This cunning act of geometric sorcery is explained by the solids' regularity

*Another awe-inspiring feature of the Platonic solids is their interconnectedness. The dodecahedron, with twelve faces and twenty vertices, is the mate, or dual, of the icosahedron, which has twenty faces and twelve vertices. Similarly, the cube, with six faces and eight vertices, is the dual of the octahedron, which has eight faces and six vertices. The fact that these solids are dual to one another has the result that they also share their symmetries.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.



The elasticized popping dodecahedron, from *Introduction to Geometry*.

(faces all the same regular polygon, with the same grouping of polygons around each vertex). It is best appreciated by constructing the Platonic solids for oneself, piece by piece—simply taping the component polygons together. "Any intelligent child who plays with regular polygons (cut out of paper or thick cardboard, with adhesive flaps to stick them together) can hardly fail to rediscover the Platonic solids," said Coxeter. "They were built up that 'childish' way by Plato himself."[45]

Models of these highly symmetric solids can also be constructed from "nets," made by tracing a flat pattern of adjoined component polygons. Coxeter provided instructions in his book *Introduction to Geometry*[46] for a springy dodecahedron model made by fitting together two nets, folded into "bowls" of pentagons that are then strung together by an elastic band. When assembled, the dodecahedron model becomes alive—animated by its crude spring-release system, it can be pushed flat and stored in a book, but when not compressed by sufficient weight it spontaneously pounces back into shape. During class, Coxeter made a stunt of pretending to have lost his dodecahedron model. "Oh, bother!" he'd mutter mid-lecture. "Now, where is my dodecahedron?" He'd look around, opening a book or lifting a stack of papers and then—POP!—there it was, springing into being.[47] (Endnote 47 contains illustrated instructions for constructing a popping dodecahedron.)

Euclid (365–25 BC) proved there are only five Platonic solids.[48] And given the above-mentioned restrictions, only three regular polygons (the equilateral triangle, square, and pentagon) can be used in the construction of the Platonic solids. This is because the sum of polygon angles that meet at a vertex must be less than 360° in order to form a convex solid. This can be

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

proved algebraically, or by physically putting the component polygons together and discovering what works. For example, if you try to fit three, four, or five triangles around a vertex, there is still a gap, and the triangles then can be folded down to meet one another, forming a corner of the respective solid (the tetrahedron, octahedron, or icosahedron). All other options with the equilateral triangle would not work: two triangles around a vertex cannot possibly meet at all edges to form a solid, while six triangles add up to 360° exactly, thus leaving no gaps and forming a flat tiling, and seven, eight, or more triangles overlap or meet in accordion-like folds.[49]

Euclid's seminal contribution to geometry was his book *The Elements*. But Euclid was not the author of *The Elements* so much as its editor. He compiled and organized the fundamentals of geometry, work done by Thales, Pythagoras, and other predecessors. Euclidean geometry in general, to loosely define it, encompassed the study of familiar shapes, their areas and angles, and filled thirteen books. The first book covered triangles; the next, rectangles; followed by circles, polygons, proportion, similarity; with four books on number theory, and one each on solid geometry and pyramids, culminating with the properties of the majestic five regular polyhedra—here Euclid placed the Platonic solids on a pedestal and gave his proof that there are only five.[50]

By the middle of the nineteenth century, Euclid's *Elements* had been the bible of mathematics for two millennia. Arabian mathematicians and authors, providing one of few sources of information on Euclid's life, translated his name as "Uclides," *ucli* meaning "key" and *des* meaning "measurement"—Euclid was the "key of geometry."[51] And the Euclidean framework was assumed to be the geometry of the real world. Immanuel Kant's philosophy still dominated metaphysical beliefs, and in his *Critique of Pure Reason* he asserted that the Euclidean system was "a priori"—meaning "prior to experience," based on synthetic, theoretical deduction rather than empirical observation, or, as Kant translated it, "an inevitable necessity of thought."[52]

In 1847, Oliver Byrne, a mathematics schoolteacher and Queen Victoria's surveyor of the Falkland Islands, published a beautiful new edition of Euclid's *Elements*, with color diagrams replacing equations (this in addition to the simple line drawings of previous editions).[53] Byrne's book, *The First Six Books of the Elements of Euclid*, stated on its title page, "Colored Diagrams and Symbols Are Used Instead of Letters for the Greater Ease of Learners." In the preface, Byrne elaborated: "The arts and sciences have become so extensive, that to facilitate their acquirement is of as much importance as to extend their boundaries. Illustration, if it does not shorten the time of study, will at least make it more agreeable. This work has a greater aim than mere illustration; we do not introduce colors for the purpose of entertainment . . .

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

but to assist the mind in its researches after truth, [and] to increase the facilities of instruction."[54]

Euclid, then, was enjoying continued popularity, but there were undercurrents of dissent. In *The Elements*, Euclid had outlined his exalted five postulates, and the first four were simple enough:

1. A straight line may be drawn between any two points.
2. A piece of straight line may be extended indefinitely.
3. A circle may be drawn with any given radius and an arbitrary center.
4. All right angles are equal.[55]

But the fifth postulate—the parallel postulate—was unlike the others, and allegedly Euclid himself had been hesitant to include it in his *Elements*. "His reluctance to introduce it," Coxeter observed, "provides a case for calling [Euclid] the first non-Euclidean geometer!"[56] It stated:

5. If a straight line crossing two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which are the angles less than the two right angles.[57]

Coxeter deemed it "unnecessarily complicated."[58] Indeed, since Euclid's time, the parallel postulate had dogged mathematicians, and annoyed them. It was not intuitively obvious and required mathematicians to suspend disbelief; it stumped them because it could in no way be verified by experience.

Another way of expressing the parallel postulate is to say that, given a line and a point not on the line, every line through the point will meet the line, except in one "freaky case": when the two lines are parallel to each other. But, as Jeremy Gray, historian of mathematics at the Open University, pointed out, who is to say what happens to two parallel lines when extended to infinity, or off 10^{10} light-years away, where strange things might alter the laws of space? Maybe parallel lines could meet somewhere in the "vaguer cluster," said Gray. Regardless, it is impossible to check. "So it's a very strange statement," he said. "It's a blot. Because it's a leap of faith unlike all the other postulates."[59]

Over the years, most mathematicians ignored this blot, for if they didn't the reign of Euclidean geometry threatened to collapse like scaffolding with one faulty strut. Some mathematicians, the more daring, courageous, and foolhardy—Greek, Arab, Islamic, and eventually Western mathematicians—tried and failed to prove the parallel postulate using the other four postulates. As the failures accumulated, these attempts of geometrical derring-do only

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

continued, forming a procession of doomed parallel postulators throughout history.[60] The predicament was decried in the mid–eighteenth century as "the scandal of elementary geometry."[61]

Hungary's János Bolyai (1802–60) was one of the adventurers who went in search of geometry's Holy Grail. He first tried to prove the fifth postulate, with no success. He then wondered whether the postulate was perhaps false. Bolyai became infatuated, convinced he was closing in on the chase for geometry's mercurial axiom. His efforts dismayed his father, Farkas Bolyai, who himself had exercised self-destroying due diligence with the parallel postulate.[62] "I have traveled past all reefs of this infernal Dead Sea," he told his son, "and have always come back with broken mast and torn sail." He tried desperately to disabuse János of his interest.[63]

> You must not attempt this approach to parallels. I know this way to the very end. I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallels alone . . . I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. I accomplished monstrous, enormous labours . . . I turned back when I saw that no man can reach the bottom of this night. I turned back unconsoled, pitying myself and all mankind. Learn from my example: I wanted to know about parallels, I remain ignorant, this has taken all the flowers of my life and all my time from me.[64]

His son, however, ignored the warnings:

> I am determined to publish a work on parallels as soon as I can put it in order, complete it, and the opportunity arises. I have not yet made the discovery but the path that I am following is almost certain to lead to my goal, provided this goal is possible. I do not yet have it but I have found things so magnificent that I was astounded . . .[65]

Eventually Farkas relented and encouraged his son to publish whatever he had as soon as possible, lest the ideas pass to someone else. "There is some truth in this," János agreed, "that certain things ripen at the same time and then appear in different places in the manner of violets coming to light in early spring."[66] They published János's findings in 1832, as an appendix to a book on geometry his father had long been preparing.[67]

János's findings proved that the fifth postulate was not a theorem—not a consequence of Euclid's first four postulates—by showing that there are geometries in which Euclid's first four postulates hold true but the fifth does not. He had discovered a consistent and self-contained system of geometry

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

that differed from Euclid's in its properties of parallelism; in Bolyai's non-Euclidean geometry, there are infinitely many lines through a given point that do not meet a given line. With this, Bolyai had performed a seemingly impossible feat.[68] He had discovered a new geometry—"one of the most momentous discoveries ever made," said Gray—but the world simply ignored it. By the time János Bolyai died, in 1860, he had received no recognition for his discovery of non-Euclidean geometry.[69]

With Bolyai's discovery, there were then two types of geometry, Euclidean and non-Euclidean, each rooted in the classical tradition. As a classical geometer, Coxeter carved a unique, surprisingly productive, and far-reaching career from Euclidean geometry, elevating it to complex and hyper-dimensional levels, and he made forays in the non-Euclidean realm as well.[70] Thus, Coxeter was what you might call a modern classical geometer, according to Sir Michael Atiyah, one of the finest mathematicians of our day: "Coxeter's geometry was classical flat geometry, geometry of ordinary space. Then he moved into variations on that, with group theory. And this brings geometry into touch with modern algebra in lots of interesting ways. He was the master of that bridge," said Sir Michael. "But Coxeter stayed in the old world. He didn't become a modern geometer. He didn't embrace modern geometry as a whole. He stayed very close to the spirit of classical geometry . . . He was a virtuoso in that area. Quite unique. He's almost the last classical geometer more than the first modern geometer."[71]

Since Bolyai's time, many more types of non-Euclidean geometry have been discovered. Geometry, broadly speaking, is anything that shares the general ideas of Euclidean geometry. If a few rules are changed, however, then a slightly different "non-Euclidean" geometry results. There is a seemingly infinite diversity of geometries, either classical or contemporary in origin—each logical systems unto themselves and devised for a specific purpose.[72] Some of them, Coxeter waded into headlong (especially projective geometry); some, he approached in spurts (such as topology, also known as "rubber-sheet" geometry, with the four-color problem, regarding the theory of maps); and other areas he touched on scarcely at all (modern curved complex geometry, fractal geometry, and taxicab geometry*).[73] The different geometries evolved slowly, like a genealogy, responding to ideas of the times, and sometimes pushing the envelope. For example, the study of knots

*Taxicab geometry measures distances by vertical and horizontal steps—east–west and north–south increments—the way taxis traverse city blocks, rather than by the shortest distance between two points, as the crow flies. Fittingly, the distance units in taxicab geometry are known as the "Manhattan metric."

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

required the development of topology, which in turn required the development of metric spaces. Whereas differential geometry, the study of curved surfaces via calculus, originated in the mid-1800s and was found to be relevant (along with non-Euclidean geometry) at the turn of the next century in the space-time geometry of Einstein's relativity theory.[74] Many of these branches were "beyond my powers," Coxeter once admitted,

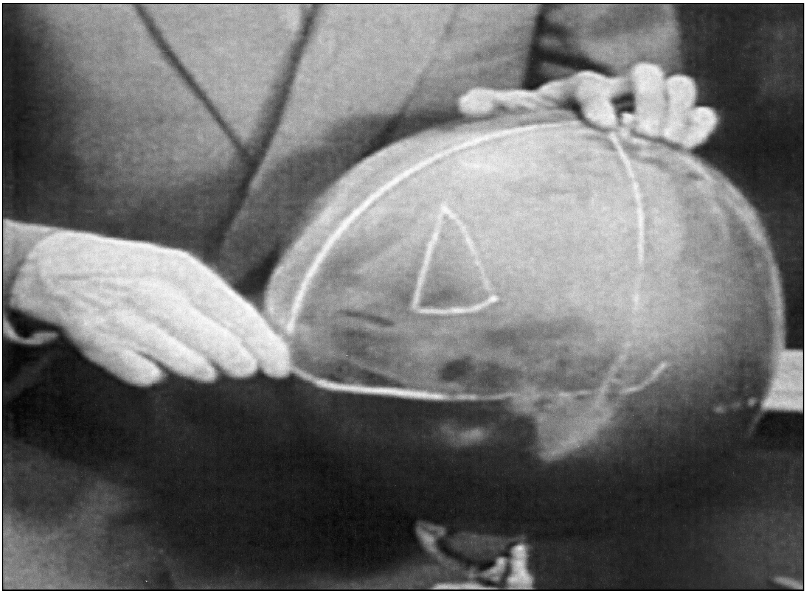
> There are so many branches of the subject in which I am almost as ignorant as the proverbial man in the street. I must ask you to forgive me if I concentrate on my own favorite branches, and I must take the risk of offending various geometers who will ask why I have not dealt with algebraic geometry, differential geometry, symplectic geometry, continuous geometry, metric spaces, Banach spaces, linear programming, and so on . . . Thus there are many geometries, each describing another world: wonderlands and Utopias, refreshingly different from the world we live in.[75]

A different non-Euclidean geometry from Bolyai's, for instance, occurs when you assume there are no parallel lines at all—every pair of lines intersects. One way to illustrate this mind-bending geometry is with a query that Coxeter entertained (one long a part of geometry folklore): If you had your pilot's license and flew ten hours due south, then ten hours directly west, and then ten hours due north, how could it transpire that you would find yourself right back at your starting place?[76]

Flummoxed disbelief is the usual reaction to this question, because the directions are envisioned in the flat Euclidean plane. Coxeter demonstrated this warped perspective in 1957 with a grainy black-and-white television appearance on a Canadian news magazine. In comparing the "nature of space" and alternative geometries to Euclidean, he made use of two blackboards—a standard flat blackboard on the wall, and a swiveling globe of the world painted black. First, Coxeter said, consider an ordinary triangle in the plane. He gestured to his triangle drawn on the regular chalkboard—a traditional Euclidean triangle with angles summing to 180°. Another kind of geometry, he continued, moving toward his globe, is geometry on the surface of a sphere. And then, beginning at the North Pole, he chalked lines on the globe running due south, then traveling due west, and finally due north, leading directly back to his starting point and forming a triangle with his path—a triangle constructed from three 90° angles.[77] So if we choose, Coxeter concluded, "we can find a triangle having right angles at each vertex, and the sum of the three is 270°." This is spherical geometry, one example of a non-Euclidean geometry. Non-Euclidean geometry exists in worlds where, tinkering with

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.



Coxeter demonstrating non-Euclidean spherical geometry with a 270° triangle on a globe.

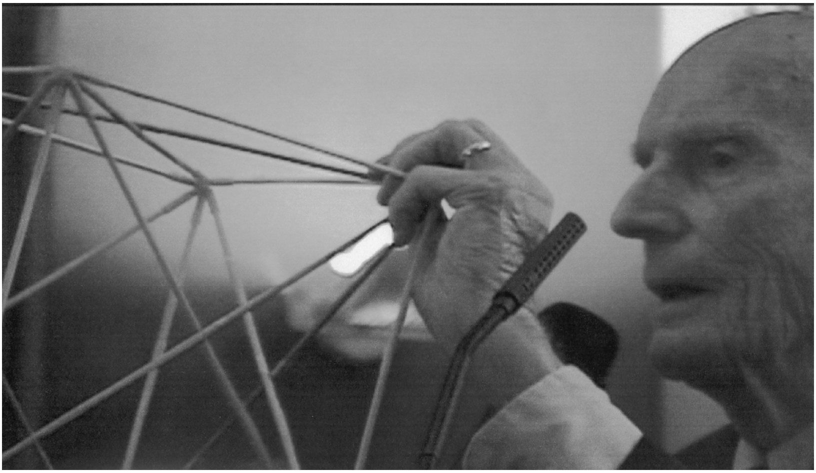
qualitative and quantitative factors, the angles of a triangle sum to more or less than the traditional Euclidean 180°. It is simply a matter of experiment; mathematicians invent new geometries and then it is left to the physicists to figure out which of these geometries, if any, apply in the real world.[78]

When Bolyai's non-Euclidean geometry eventually gained attention, people began asking, "Which geometry is valid in physical space—Euclidean or non-Euclidean?"[79] Bolyai's new geometry had exposed a firmly entrenched misunderstanding about the nature of space. For ages mathematicians had believed that Euclidean geometry was the one and only logical account of the way the world could be. But, as Bolyai announced: "All I can say now is that I have created a new and different world out of nothing."[80]

Wading into his talk at the commemorative Bolyai conference, Coxeter twirled in his fingers Smith's model, a nexus of multicolored straws—a skeleton of a cube surrounding the skeletons of two interlocked tetrahedra. As he proceeded with his lecture, a rumble of unease stirred in the audience, the skeptics straining to hear. "Louder! Louder, please! We cannot hear!" cried Coxeter's daughter. His microphone wasn't working. Neither was his hearing aid.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.



Coxeter at the microphone in Budapest.

Oblivious to the predicament as it was being resolved, Coxeter carried on, the audience scribbling bouquets of tangent circles into notepads on their laps.[81]

Coxeter was hardly a showman. He was a gentleman geometer who held his audience's attention with the beauty and elegance of his work. He was known for being birdlike, both in appearance and style—very delicate, very precise, very spare. "You have to know what's important," said Gray, who likened Coxeter to a violin rather than an entire orchestra. "He's not going to rhapsodize, he's not going to tell you that this is a huge big deal . . . he's not going to write you any advertising copy."[82] Coxeter meets the measure of an elegant and beautiful practitioner also because his mathematics flourishes in the minds of other mathematicians. When a piece of mathematics is called beautiful or elegant, it is presented in a way that conveys understanding, and one litmus test for understanding is whether other mathematicians can do something with it, fitting it nicely into the bigger picture. "It becomes elegant because it opens something up," said Gray. "The elegance is in the power it conveys to do something that couldn't, or hadn't, been done before."[83]

In the end, Coxeter's talk went over well, and it proved relevant—it was related to the hot applied topic of data mining. "His proof [of an absolute property of four tangent circles] is not an earth-shaking discovery," said Karoly Bezdek, the secretary of the Budapest conference committee. "But his proof is the simplest one, the ideal proof for Beecroft's theorem. Nowadays many mathematicians publish with very complicated proofs. It is important to have simple proofs that we can digest and really learn from. It's an art to

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

discover the right proof."[84] The conference organizer, Prékopa, was very pleased as well: "It is amazing that somebody who is 95 years old can invent new scientific results of such depth and present them at a meeting. I wish I could be such a fresh-minded person, and interested and active. Coxeter gets distracted and falls asleep during some of the other talks," he noted (many an audience member was caught nodding off), "but he always wakes up when he's interested."[85]

One widely accepted mathematical truth is that mathematics is a "young man's" game. "Young men should prove theorems, old men should write books," said the legendary G. H. Hardy, a professor of Coxeter's at Cambridge who penned *A Mathematician's Apology*, a lament for his waning mathematical prowess.[86] Hungarian mathematician Paul Erdös (1913–96) is the usual counterexample. Erdös was a prolific problem solver to the end of his life, publishing more than one thousand papers, more than any mathematician in history. He was "the man who loved only numbers," as the title of Paul Hoffman's biography proclaimed.[87] A close friend and collaborator, Ron Graham (introduced to Erdös by Coxeter in 1958), recalled that Erdös "was completely dedicated to, as he would say, 'taking a peek into *The Book*'—'*The Book*'* was this hypothetical book of the Almighty that contains all the best possible proofs, all the gems of mathematics that you can present in a page or two. Erdös really lived mathematics."[88]

Donald Coxeter is an equally good counterexample disproving the stereotype of a mathematician's "best-before date."[89] Coxeter's only professional regret, articulated at the end of his days, was that he had not collaborated with Erdös[90]—his Erdös Number was 2 (as was Einstein's).[91] A person who coauthored a paper with Erdös gained an Erdös Number 1; a person who coauthored a paper with such a person has an Erdös Number 2, and so on, forming an international nexus of Erdös's 485 coauthors. Erdös had no real home base and traveled the world with his battered Mexican leather briefcase of worldly possessions, landing on the doorstep of welcoming or unexpecting mathematicians. Upon arriving at his destination Erdös would announce: "My brain is open!" (the title of another Erdös biography, by Bruce Schechter).[92] His visits were so intensive that Graham often joked, "We had Erdös over for a month last weekend."[93] After Erdös squeezed all the mathematical juice from his host he moved on to his next stop.[94]

*Galileo Galilei (1564–1642) also referred to a "grand book" of the universe, and to the importance of geometry in gleaning knowledge of its contents. In *The Assayer* he wrote: "Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these one is wandering in a dark labyrinth."

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

Left to right: Unidentified woman, Coxeter, Branko Grünbaum, Paul Erdös, circa 1965.

Coxeter had plenty of opportunity to become an Erdös 1. He and Erdös often crossed paths. One day in 1935, when Erdös was at Cambridge, he rang Coxeter and asked him a question regarding a problem he was working on about parallelotopes.[95] Coxeter worked on Erdös's problem for a few days, but it didn't lead to collaboration. They met numerous times thereafter, in London, Toronto, and elsewhere. In 1965, Coxeter noted in his diary: "while shaving I solved Erdös's problem (of the dancing girls and boys)." But still no collaboration.[96]

For Erdös's sixty-eighth birthday, Coxeter dedicated a talk in his honor on "a symmetrical arrangement of eleven hemi-icosahedra,"[97] and the two bounced ideas back and forth in correspondence—a letter from Erdös, always written with a fountain pen, typically began with a brief pleasantry, promptly launching into pages of mathematical proposition, suppositions, equations, conclusions, and a diagram. Coxeter sent Erdös problems he might appreciate, and Erdös contributed ideas to a few of the problems Coxeter was working on,[98] but upon Erdös's death in 1996, Coxeter settled for an Erdös Number 2.[99]

Coxeter's daughter Susan, not being the least bit mathematically inclined (or even empathetic), wasn't so impressed with her father's intellectual longevity, and in general she ran hot and cold on his status as a mathematical

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

legend. Having solved the problem of the microphone malfunction during his lecture, Susan settled in and read her novel. And at the end of his talk, when Coxeter hightailed it to the loo, Susan gave her evaluation. "To think," she said. "We've come all this way to talk about circles touching circles when there are so many more important things going on in the world. Dad would hate to be equated with Elvis Presley, but Elvis gave people some moments of joy, happiness, inspiration. And if that's what Dad's work does for these people, that's wonderful. Personally, I get more from Elvis Presley."[100]

The day wound down with a reception in the Academy of Sciences ballroom. Conference-goers stood nibbling on a dinner buffet, and scrounged for a miscalculated supply of desserts. Coxeter found one of few seats in the house, a majestic dais elevated above the crowd. A steady stream of admirers stopped by, bowed at his side, and gave him praise. Ernest Vinberg, from Moscow State University, introduced himself and thanked Coxeter for long ago writing a letter to his Soviet-era PhD committee, reassuring them that Vinberg's field of study—Coxeter groups—was not politically suspect (after perestroika Vinberg's PhD was finally conferred, and he proceeded to do a second PhD, also on Coxeter groups).[101] Daina Taimina, a senior research associate at Cornell, approached Coxeter to show him her crocheted model of the hyperbolic plane, and to tell him that his *Introduction to Geometry* was a blessing—"it saved me," she said—when she started teaching high school geometry in Latvia in 1975.[102]

At the close of the festivities Coxeter plodded back to his hotel, the scorching July sun retreating over the Danube. Just then John Ratcliffe, from Vanderbilt University, in Nashville, Tennessee, caught up with him on the sidewalk. Ratcliffe told Coxeter he had two copies of his *Regular Polytopes*—one at work and another in his study at home for late-night consultations. "This is the modern-day Euclid's *Elements*," said Ratcliffe, pulling a copy of the book from his attaché case. "It's like the Bible for me. I refer to it all the time."[103]

All in all, it was a jubilant day for Coxeter. He had managed the trip, delivered an apropos presentation, and been showered with adulation. "It was very satisfactory!" he said, never one for hyperbole in language. Susan deposited her father in his hotel room and withdrew for some time on her own. Coxeter climbed out of his suit jacket, undid his shirt and tie, sat on the edge of the bed, and sipped on some champagne from the minibar.* After the high of the day he was stung by melancholy (as he was a few times during the

*His usual bedtime elixir, to fortify his constitution, was a stomach-curdling mixture of Kahlúa coffee liqueur, peach schnapps, sometimes a splash of vodka, and soy milk.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

Budapest conference, prompted by a documentary camera following him the entire trip). He thought of how he could have been a better husband, father, and grandfather, spending less time on his work. He thought of the recent invitation he passed up to return to his alma mater, Trinity College, Cambridge, as a newly minted emeritus fellow—a mark of honor that allowed him, if he chose, to live out the end of his days in a room in Great Court, kept company by all his old haunts. He thought of his childhood governess May Henderson, whom he had been known to confess he loved more than his mother. When he was in his late sixties, Coxeter planned to pay May a surprise visit on a trip home to England. He was devastated to find she had died of cancer only two weeks before he arrived. May had taught Coxeter French and Latin, multiplication, division, and quadratic equations. Little did she know, way back then, what a fine mathematical mind she was molding.[104]

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

Index

NOTE: DC refers to Donald Coxeter

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

For general queries, contact info@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

For general queries, contact info@press.princeton.edu