Contents

1 The Fast-Track Introduction to Calculus 2

Chapter Preview. Calculus is a new way of thinking about mathematics. And Calculus 2 builds on that new perspective in new ways. This chapter introduces you to the calculus mindset, the core concepts of Calculus 2, and the sorts of problems these innovations help solve. The focus throughout is on the *ideas* behind Calculus 2 (the big picture of Calculus 2); the subsequent chapters discuss the *math* of Calculus 2. After reading this chapter, you'll have an intuitive understanding of Calculus 2 that'll ground your subsequent studies of the subject. Alright, let's start the adventure!

1.1 First Things First: What Is Calculus?

In *Calculus Simplified* [2] I gave this two-part answer to that question:

Calculus is a mindset—a dynamics mindset. Contentwise, calculus is the mathematics of infinitesimal change.

This frame on calculus applies as much to Calculus 2 as to Calculus 1 (and any mathematics that also uses calculus). So, let's unpack that answer, now in the specific context of Calculus 2.

Calculus as a Way of Thinking

The mathematics that precedes calculus—often called "precalculus," which includes algebra and geometry—largely focuses on *static* problems: problems lacking change. By contrast, change is central to calculus—calculus is about *dynamics*. Example:

- *Precalculus question:* Find the pattern in the sequence of numbers

$$1, 1, 2, 3, 5, 8, 13, 21, \dots$$

- *Calculus question:* Does the ratio of consecutive numbers in the sequence above *approach* a specific number?

The sequence of numbers above is the famous **Fibonacci sequence**. In this sequence, the nth number (let's denote that F_n) is the sum of the two preceding numbers (F_{n-1} and F_{n-2}), starting with $F_1 = 1$ and $F_2 = 1$. That's the precalculus answer to the precalculus question—a (static) formula. But the *calculus* answer to the *calculus* question reveals something magical and enlightening:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi = \frac{1+\sqrt{5}}{2} \approx 1.618\dots,$$

the **golden ratio** (figure 1.1(a)). Translation: The ratio of consecutive Fibonacci numbers tends to φ ("phi") as we get further into the sequence. "Tends" and "further into" here convey the *dynamics* of this calculus answer. And what about the

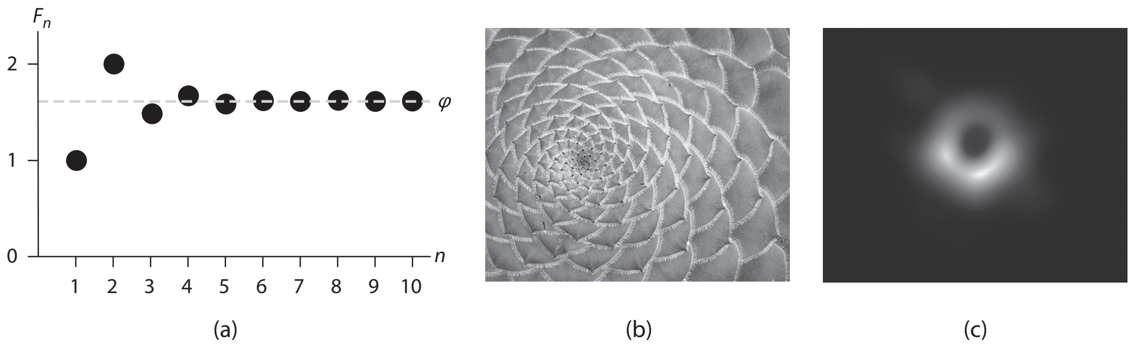

Figure 1.1: (a) The ratios of consecutive Fibonacci numbers F_{n+1}/F_n (black) approach the golden ratio φ as n increases. (b) Close-up of an *Aeonium* succulent by Max Ronnersjö showing spiral phyllotaxis, where successive leaves grow at approximately the "golden angle" $2\pi/\varphi^2$. Retrieved from Wikipedia Commons. (c) The black hole at the center of Messier 87, a galaxy in the constellation of Virgo. By the European Southern Observatory; retrieved from Wikipedia Commons.

magical and enlightening aspects I alluded to? Well, it turns out that *the golden ratio is hidden in many of Nature's patterns*. It's encoded in the spiral arrangements of leaves on plant stems (figure 1.1(b)), in the proportions of components in human hearts and brains, and in theoretical models of black hole physics (figure 1.1(c)) [3]. Takeaway: Calculus is hidden in Nature, *in you*.

This statics versus dynamics distinction between precalculus and calculus runs even deeper—change is the *mindset* of calculus. The subject trains you to approach a problem from a dynamics (versus statics) perspective. We saw this in Calculus 1 when we studied differentiation and interpreted derivatives as instantaneous *rates of change*. And we saw it again when we studied integration and *accumulation* functions. (Appendixes C–D review Calculus 1, in case you'd like a refresher.) This dynamics mindset carries over into Calculus 2. Let me illustrate this—and the continuing role of "infinitesimal change" in calculus—via Zeno's paradox.

The Continuing Role of Infinitesimals in Calculus 2

Zeno of Elea (ca. 490–430 BC) was a Greek philosopher who devised a set of paradoxes arguing that motion is not possible. (Clearly, Zeno did not have a calculus mindset.) One such paradox—the Dichotomy Paradox—can be stated as follows:

To travel a certain distance you must first traverse half of it.

Figure 1.2 illustrates this. Therein Zeno is trying to walk a distance of 2 feet. But because of Zeno's mindset, with his first step he only walks half the distance: 1 foot (figure (b)). He then walks half of the remaining distance in his second step (0.5 foot) and reaches the 1.5-foot mark (figure (c)). After Zeno has taken n steps, the distance d_n he's traveled is given by

$$d_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}. \tag{1.1}$$

As Zeno continues his walk, the total distance walked d_n always gets closer to 2 yet never reaches 2 (because Zeno's steps always traverse *half* the remaining distance).

If we checked back in with Zeno after he's taken an *infinite* number of steps, however, his total distance traveled d would be . . . drum roll please . . . *infinitesimally close* to 2—as close to 2 as you can imagine but not equal to 2.

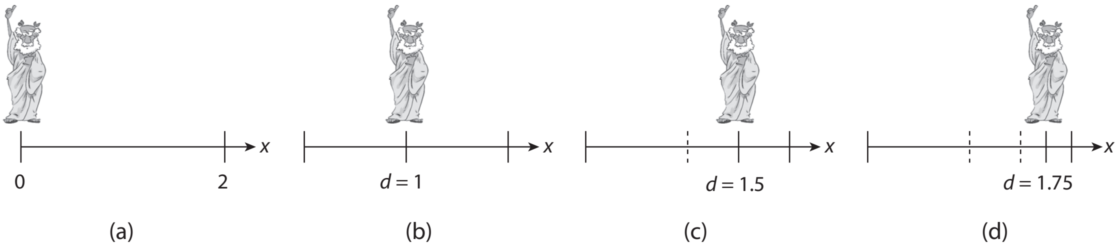

Figure 1.2: Zeno trying to walk a distance of 2 feet by traversing half the remaining distance with each step.

This example illustrates the dynamics mindset of calculus. We discussed Zeno *walking*; we thought about the *change* in the distance he traveled; we visualized the situation with a figure that conveyed *movement*. (Calculus is full of action verbs!) But the example also challenges us. What seems intuitively clear to us is that after an infinite number of steps Zeno would've covered a distance equal to

$$1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}+\cdots=2. \tag{1.2}$$

That is, *the infinite sum of the distances Zeno covers with each step should equal 2*. But how can we add up an infinite number of numbers? And how/why does the particular infinite sum above "yield" 2? (Those are the challenges.) To tackle these new calculus questions requires new calculus concepts and methods that leverage their inherently *dynamic* nature. Luckily, one of the pillars of Calculus 1 provides a stable foundation on which to build these new concepts and methods: limits.

1.2 Limits: (Still) The Foundation of Calculus

Let's return to equation (1.1). It turns out that we can express the sum therein much more compactly as

$$d_n=2-\frac{1}{2^{n-1}}=2\left(1-\frac{1}{2^n}\right). \tag{1.3}$$

(We'll learn how to do this in chapter 4.) With the help of the Limit Laws from Calculus 1,* it follows that

* These are reviewed in appendix C.

$$\lim_{n\to\infty} d_n=\lim_{n\to\infty}\left(2-\frac{2}{2^n}\right)=2-\lim_{n\to\infty}\frac{2}{2^n}=2. \tag{1.4}$$

This is Calculus 2's answer to the mystery of (1.2). It expresses the intuitive idea that the 2-foot mark is the *limiting* value of the total distance Zeno's traversing. Equation (1.4), therefore, is a statement about the *dynamics* of Zeno's walk, in contrast to (1.1), which is a statement about the *static* snapshots of how far Zeno has traveled after n steps. Moreover, (1.4) reminds us that d_n is always *approaching* 2 yet never *arrives* at 2. Indeed, as you may recall from Calculus 1:

Limits approach indefinitely (and thus never arrive).

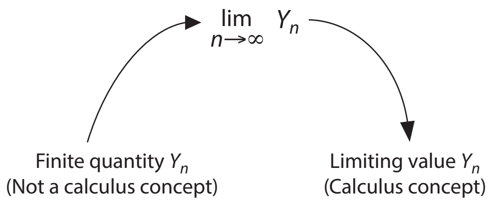

Figure 1.3: The Calculus 2 workflow.

We'll learn much more about infinite sums in chapter 4. For now, the Zeno example is sufficient to illustrate one key idea: *Limits are also the foundation of Calculus 2.* The Calculus 1 mansion was built on limits, and so will be the Calculus 2 mansion. And the workflow we'll use for building new Calculus 2 concepts from limits will be similar to the one introduced in *Calculus Simplified* [2]: *Start with a finite quantity Y_n that depends on an integer n, and then let n tend to infinity (i.e., take the limit as $n \to \infty$) to arrive at a calculus result* (see figure 1.3). Working through this process—like we just did with the Zeno example—for various quantities Y_n of real-world and mathematical interest is part of what *doing* Calculus 2 is all about.

1.3 The Three Difficult Questions That Drove the Development of Calculus 2

Calculus 2 developed out of a need to answer the following three Big Questions.

1. ***The Geometry Question:*** *Can we calculate the length of any curve, area of any surface, and volume of any solid?* The ancient Greeks (and other ancient civilizations) could calculate lengths, areas, and volumes for polyhedra and some curvy shapes (e.g., spheres), but that was about it. For example, for thousands of years mathematicians struggled to calculate the surface areas and volumes of objects as simple as the flower vase shown in figure 1.4(a). They could paint them and fill them with water but not know beforehand how much paint or water they'd need.

2. ***The Infinite Sum Question:*** *Does an infinite sum have a sum, and if so, what's the sum?* For centuries mathematicians have used clever geometric arguments to tackle infinite sums. For example, adding up all the areas inside the square in figure 1.4(b) shows that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1. \tag{1.5}$$

   Notice that if we multiply this equation by 2 we get (1.2), yielding a second (this time geometric) verification of that sum. But does a general infinite sum, like the one shown in the cloud in figure 1.4(b), have a sum? And if so, how can we calculate that sum? Tough questions.

3. ***The Approximation Question:*** *Without knowing the exact value of a function, can we accurately approximate it?* Before calculators and computers, accurately approximating quantities like the ones shown in figure 1.4(c) was a difficult

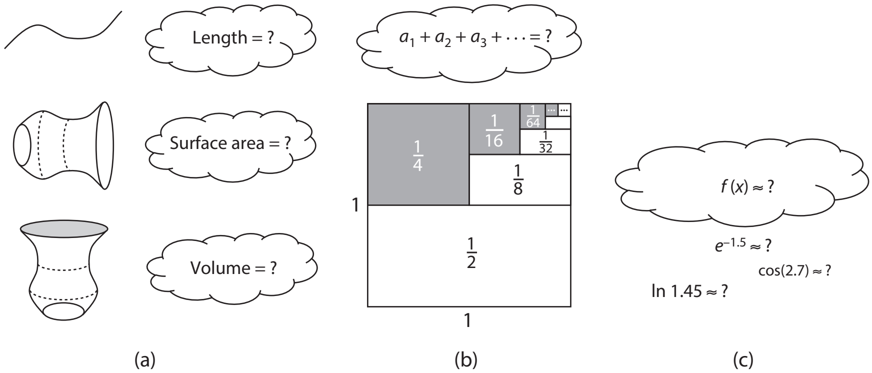

Figure 1.4: The three sets of Big Questions that drove Calculus 2's development.

problem. And though today's technology makes that easy, the algorithms that technology uses to produce those accurate approximations trace back to the early work on this third Big Question. For example, how can we accurately approximate quantities like $\sqrt{1.05}$, or the other quantities under the cloud in figure 1.4(c), without using a calculator?

I hope these descriptions impart the mystery and difficulty involved in trying to answer these Big Questions. Indeed, *the resolution of these questions took millennia*. But it won't take that long for you. By the time you're done reading the next three pages you'll have developed an intuitive understanding of how to resolve each of those questions. That understanding is grounded in my earlier answer to the question, What is calculus? *A dynamics mindset*. Let's see how.

First, note that nothing about figure 1.4 says "dynamics." Every image is a static depiction of something (e.g., a volume). Yet in the real world we *fill* beakers with liquids to measure volume and *add up* numbers to obtain a sum. (There are those action verbs again.) We've turned on our calculus (dynamics) mindset. The next step is to look for the Y_n we'll need so we can apply the Calculus 2 workflow (figure 1.3). To illustrate this process, let's *calculus* three of the questions in figure 1.4—yep, I'm encouraging you to think of calculus as a verb—and search for the finite quantities Y_n that, in the limit as $n \to \infty$, yield the desired quantities (e.g., volume). Figure 1.5 illustrates the results. Let me give you a tour of that figure now.

- *Row 1:* The length of the curve in the third column—called the **arc length** of that curve—is realized as the infinite limit (second column) of the total length s_n of n line segments (the gray ones in the figure) created by choosing suitable points on the curve (first column). Thus, here $Y_n = s_n$.

- *Row 2:* The infinite sum of a set of numbers (third column) is realized as the infinite limit (second column) of the sum of the first N of them, S_N (first column).[1] Thus, here $Y_N = S_N$.

- *Row 3:* The value $f(x)$ of a function for x-values near $x = 0$ (third column) is realized as the infinite limit (second column) of polynomials $T_n(x)$ of increasing degree n (first column). So, here $Y_n = T_n(x)$.

[1]This is exactly what we did in the Zeno example in equation (1.4).

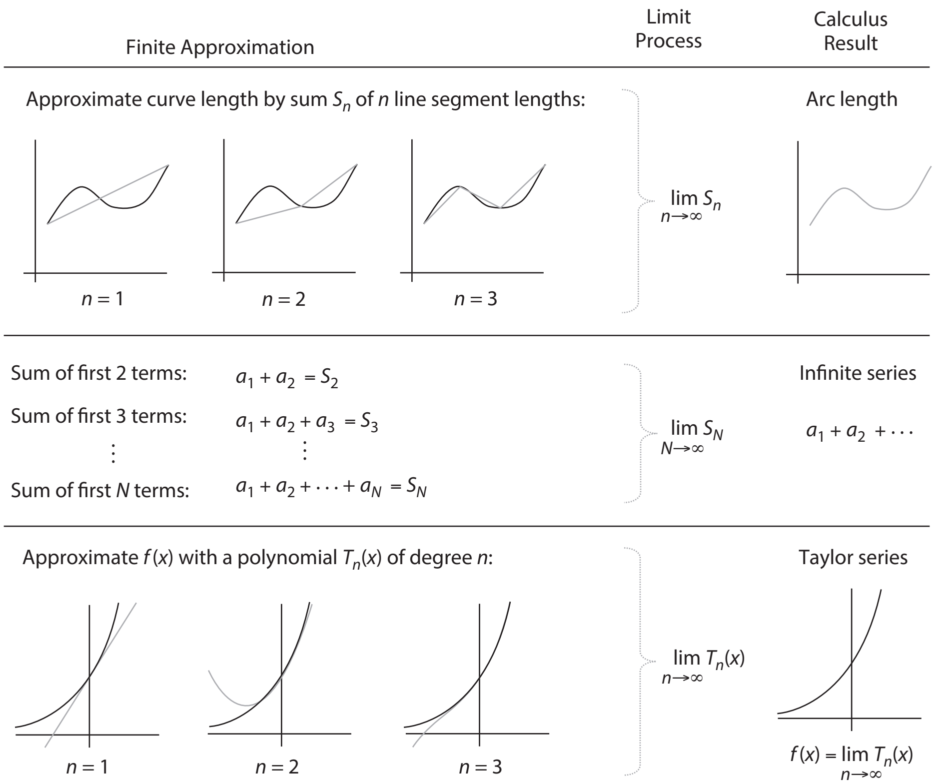

Figure 1.5: The Calculus 2 workflow (figure 1.3) applied to the three Big Questions.

We'll learn how to calculate arc length in chapter 3. The result will be a definite integral that in many cases will require advanced integration methods. We'll learn those at the end of chapter 2. At the beginning of that chapter we'll learn how to approximate definite integrals, which comes in handy when we can't evaluate them exactly. In chapter 4 we'll then revisit the last two rows of figure 1.5. The infinite sum of numbers in the second row of the figure is called an **infinite series**, and through our attempts to approximate the sums of certain infinite series we'll end up circling back to the third row of the figure. We'll call the polynomials $T_n(x)$ illustrated therein (the gray curves) **Taylor polynomials**. And by observing that as n increases they approximate $f(x)$ better, we'll build up to the climax of the chapter: the *remarkable* result that sometimes the limit as $n \to \infty$ of the Taylor polynomial $T_n(x)$ *is* $f(x)$. Translation: *Some functions can be represented as "infinite degree" polynomials!* (We'll call these **Taylor series**.)

I don't expect you to understand everything I've just described. My intent was instead to provide you with a roadmap of the main stops and a preview of the highlights of our upcoming calculus adventure. I also hope that the preceding discussion helps you appreciate this book's approach to Calculus 2. Figure 1.5 originated from switching to a dynamics mindset to tackle our three Big Questions. We then applied the Calculus 2 workflow to realize each of the calculus objects in the third column of the figure as infinite limits of appropriate Y_ns. Takeaway: *In* *scanning figure 1.5 from column to column (left to right) you're following the Calculus 2 workflow*.

This completes my big-picture overview of Calculus 2. We've by no means resolved the Big Questions illustrated in figure 1.4, but we've created a roadmap for tackling those Big Questions. What's left now is to apply the Calculus 2 workflow to our Big Questions to develop the mathematics of Calculus 2. That's what we'll do in the rest of the book. See you in the next chapter!

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