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CHAPTER 1

Five

> The Pythagoreans associated the number five with marriage, because it is the sum of what were to them the first even, female number 2, and the first odd, male number 3.
>
> —DAVID WELLS, *THE PENGUIN DICTIONARY OF CURIOUS AND INTERESTING NUMBERS* (1986)

EVEN AMONG THE FIRST TEN INTEGERS, five stands out. The number one is, well, one, the generator of all integers. Two is one doubled; it is the natural cycle that governs our lives. We walk in steps of one-two, one-two, we breathe in an inhale-exhale cycle, our daily activities are regulated by the diurnal cycle of day and night, our body has a nearly perfect left-right symmetry, and our sense of direction is based on a left-right, forward-backward movement. The Chinese yin-yang is a symbol of all things that come in contrasting pairs—yes-no, on-off, good-evil, love-hate. Two is the numeration base on which all our computers are based, the binary system. We also note that two has some unique mathematical properties: $2+2=2\times 2=2^2$. And it has the distinction of being the first prime number and the only even prime. The exponent two is probably the most common power in all of mathematics, appearing in the Pythagorean theorem $a^2+b^2=c^2$, in the Mersenne numbers 2^n-1 and Fermat numbers $2^{2^n}+1$, and in numerous theorems in almost every branch of mathematics. It is just as prevalent in physics as the exponent in all inverse-square

laws, and it stars in the most famous equation in all of science, $E=mc^2$.

Three is next in line, being the sum of one and two (although we often perceive it as a single unit in counting: 1-2-3, 1-2-3,…). Dances based on a triple meter are very common, from Haydn and Mozart's minuets to Beethoven's scherzos to the waltzes of the Strauss family. It is the first odd prime, as well as the first Mersenne prime ($3=2^2-1$) and the first Fermat prime ($3=2^{2^0}+1$). It is also known as the "biblical value of π" due to a verse in I Kings 7:23: "And he made a molten sea, ten cubits from one brim to the other; it was round all about … and a line of thirty cubits did compass it round about." The "he" refers to King Solomon, and the "sea" alludes to a pond he ordered to be constructed at the outer entrance to Solomon's Temple in Jerusalem.

Next comes four, the smallest composite number and the only square integer of the form $p+1$, where p is a prime (this is because $n^2-1=(n+1)\cdot(n-1)$, a composite number except when $n=2$). In the decimal numeration system, a number is divisible by four if and only if its last two-digit number is divisible by four (for example, 1536 is divisible by four because 36 is, but 1541 is not because 41 is not). The first of the regular or Platonic solids, the tetrahedron, has four vertices and four faces, each an equilateral triangle. Four colors are sufficient to color any planar map such that two regions sharing a common border will have different colors (this famous theorem was first conjectured in 1852 but was not proved until 1977). We view the world as comprising four dimensions, three of space (length, width, and height) and one of time, all merged into a single entity, spacetime. There are four cardinal directions, designated (in clockwise direction) as N, E, S, and W. Four is the number of letters in the ineffable YHWH (the so-called tetragrammaton), one of the names of God in the Judeo-Christian tradition.

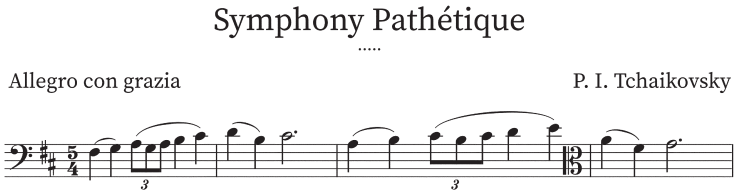

FIGURE 1.1. The "limping waltz" in Tchaikovsky's *Pathétique*

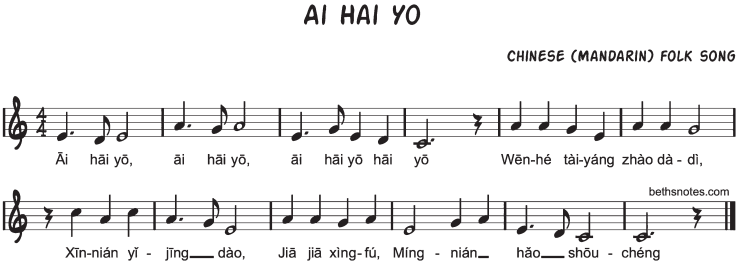

FIGURE 1.2. Chinese folk song in pentatonic scale

We now arrive at five. It feels somewhat awkward to walk in steps of five, let alone to keep a five-beat rhythm in music. A quintuple meter of five quarter-notes per bar (denoted by 5/4) in classical music is quite rare; a notable exception is the "limping waltz" from Tchaikovsky's Symphony no. 6, *Pathétique* (figure 1.1). Similarly, a person accustomed to Western classical music may find it unnatural to listen to a piece played in a *pentatonic scale* of five notes to the octave. There are several versions of this scale; in one version, the notes are C, D, E, G, A, C′ (where C′ is one octave above C), comprising the intervals 1, 1, 1½, 1, 1½ (where 1 and ½ denote a full tone and a half tone, respectively); another version starts with C-sharp and follows the black keys of the piano, with the sequence of intervals 1, 1½, 1, 1, 1½. Pentatonic melodies can be found in much of African and Asian music. Figure 1.2 shows an example of a Chinese (Mandarin) folk song in a pentatonic scale.

But while it may feel awkward to count by fives in music, it actually comes quite naturally in daily life. This is due to the fact that we are born with five fingers on each hand. We are therefore endowed with a natural calculating device—literally, a "pocket calculator," considering that many of us like to hold our hands in our pockets on a brisk, cold day. And it doesn't need to be recharged, it never runs out of power, and it is always available and ready to be used. If this sounds a bit trite, consider that many cultures have developed a kind of "finger arithmetic," and all of us, at one time or another, have used our ten fingers to count or do some mental calculation. Indeed, the word *digit* literally means "finger"; so every time you use the adjective *digital*, remember that it comes from our built-in natural calculator.

The Romans had a special symbol for five: V, perhaps resembling a fully opened hand, while one, two, and three were written as I, II, III, obviously a visual image of the raised fingers representing these numbers. For quick tallying, the symbol 𝍸 is often used even today, as can be seen on many prison walls where inmates counted the number of days already served. For multiples of five, the Romans used the letters X = 10, L = 50, C = 100, D = 500, and M = 1,000. Other numbers were written in combinations of these symbols, such as IV (= 4) and VI (= 6). The fact that smaller values sometimes precede larger values but follow them in other cases made the Roman numeration system awfully difficult to compute with, but it has nevertheless survived well into the Middle Ages and beyond. Even today you can often see the groundbreaking date of a public building chiseled in the cornerstone in Roman numerals. It was only in the Middle Ages that the Hindu-Arabic numeration system, with the numeral zero at its core, was gradually adopted in Europe and eventually became accepted internationally.

The Greek word for five is *πεντε* (spelled "pénte" in the Latin alphabet), from which numerous ancient and mod-

ern words derive; we encounter some of them later in this book. The Roman word for five was *quinque*, again the source of many ancient and current words. For example, *quincunx* describes a collection of five objects arranged in a square pattern, with one object located at the center and each of the others at a corner, as in the five-dot face on a die. And on the opposite side of the ancient world, the Chinese symbol for five was and still is 五 (pronounced like "me" in Mandarin), representing everything between heaven and earth and referring to the five elements that make up the universe: water, fire, earth, wood, and metal.

In the Hebrew alphabet, each letter is assigned a numerical value: א (aleph) = 1, ב (beith) = 2, ג (gimmel) = 3, ד (dalet) = 4, ה (heih) = 5, ו (vav) = 6, ז (zayin) = 7, ח (chet or het) = 8, ט (teth) = 9, and י (yod) = 10. Beyond ten, the system becomes additive (and read from right to left, as in all Semitic languages):

$$\text{י״א} = 10 + 1 = 11, \text{י״ב} = 10 + 2 = 12, \text{י״ג} = 10 + 3 = 13,$$
$$\text{י״ד} = 10 + 4 = 14.$$

But the next two numbers, 15 and 16, are written differently:

$$\text{ט״ו} = 9 + 6 = 15, \text{ט״ז} = 9 + 7 = 16,$$

this in order to avoid adjoining the letters י and ה, the first two Hebrew letters of the ineffable name YHWH, in accordance with the Third Commandment: "Thou shalt not take the Name of Hashem, your G'd, in vain." The remaining twelve letters after yod have the values 20, 30, 40,…, 100, 200, 300, 400.

The Hebrew word for five is חמש, pronounced "Ha'mesh."[1] Several words derive from it: חומש (Hu'mash), standing for the Torah—the Five Books of Moses, known in the Western world as the Pentateuch; חמישית (Hami'sheet, one-fifth);

מחומש (Mehu'mash, a five-sided polygon), and חמסה (Hamsah), an amulet resembling the open palm of a hand, symbolizing divine protection, fortune, and good luck; it usually comes in vibrant colors dominated by blue (plate 4), and is commonly found among Middle Eastern and North African cultures.

In the Talmud, the compilation of Jewish law written simultaneously in Jerusalem and in Babylon around the third century CE, it says "One should not donate more than a fifth of one's assets" (Babylonian Talmud, Tractate *Ketuvot*, p. 50a). The intention, no doubt, was to forewarn overgenerous donors against the possibility that they themselves might one day become dependent on charity.

★ ⬟ ★

The ten fingers on our hands are the very reason why the decimal system has become the universal numeration system of the human race. Perhaps it isn't the best choice: had we been endowed with six fingers on each hand, a duodecimal (base 12) system would have been the natural choice, and a much better one indeed. For one, twelve has five proper divisors, 1, 2, 3, 4, and 6, whereas ten has only three, 1, 2, and 5. As a result, division in base 12 would be much simpler, avoiding, for example, a repeating decimal like 0.333 … when dividing by 3. Second, many things in our lives already come in multiples of six—an egg carton contains twelve eggs, a pack of beer holds six cans, and our days and clocks are divided into twelve hours,[2] an hour has sixty minutes, and a minute has sixty seconds.

Around the middle of the twentieth century, the Duodecimal Society of America and a similarly named British society (both later renamed the Dozenal Societies) launched a vigorous campaign to change our numeration system from decimal to duodecimal. They issued decimal-to-duodecimal conversion tables, not just for integers but also for common

and decimal fractions, special numbers like $\sqrt{2}$, π, and e, and even base 12 logarithmic tables. These were all well-intended goals, and logic stood on their side. In the end, however, five hundred years of familiarity with the decimal system have prevailed, and we are still holding on to the good old base 10 numerals.

Here is one small benefit of using base 10 as our numeration base. Because $2 \times 5 = 10$, we have $10/5 = 2$ and $10/2 = 5$. These last relations can be put to use for a quick, mental multiplication and division of a number by five: for multiplication, divide the number by two and move the decimal point one place to the right; for division, multiply the number by two and move the point one place to the left. For example, $38 \times 5 = (38/2) \times 10 = 19 \times 10 = 190$, and $47/5 = (47 \times 2)/10 = 94/10 = 9.4$. Yes, I know, everyone nowadays has a calculator on their smartphones, but still it is fun—and sometimes quicker—to do it mentally.

The ancient Babylonians used a hybrid of the base 10 numeration system for numbers from one to fifty-nine and a base 60 system—called the *sexagesimal* system—for numbers greater than or equal to sixty (presumably because sixty has ten proper divisors, 1, 2, 3, 4, 5, 6, 10, 12, 15, and 30, making division easier by reducing the need to use fractions). The Mayans preferred a smaller base: a hybrid system based on five for integers up to nineteen—the *quinary* system—and powers of twenty (the combined number of fingers and toes) for numbers greater than or equal to twenty (figure 1.3). The few written documents

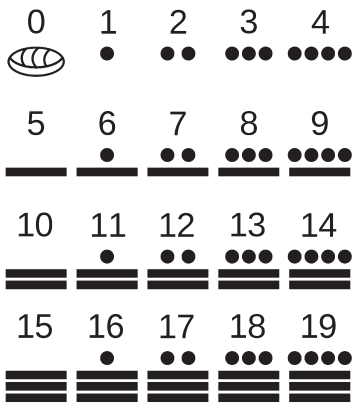

FIGURE 1.3. The integers one through nineteen in Mayan representation

that survived the Spanish conquest of their land include calendars and astronomical records using this *vigesimal* system.[3]

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Before we turn to the number-theoretic properties of the number five relevant to the pentagon, here is a brief aside. The famous painting *I Saw the Figure 5 in Gold* by American artist Charles Demuth was first exhibited in New York in 1929 and is now in the permanent collection of the Metropolitan Museum of Art (see plate 5). Demuth (1883–1935) painted it as a tribute to a poem, *The Great Figure*, written by his friend William Carlos Williams describing a fire truck racing down the streets of New York on a rainy night. Demuth's painting became an American icon and appears on a US postage stamp issued in 2013. It also features on the cover of a mathematical novel, *Uncle Petros and Goldbach's Conjecture* by Greek author Apostolos Doxiadis, published in 1992. The title refers to German mathematician Christian Goldbach (1690–1764), who in 1742 wrote a letter to Leonhard Euler, then Europe's most famous mathematician, in which he claimed that every even integer greater than two can be written as a sum of two primes (sometimes in more than one way). For example, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$, $10 = 3 + 7 = 5 + 5$, and so on. Euler, being occupied by more pressing problems, ignored Goldbach's letter; it was only found after his death in 1783. Despite its seeming simplicity and the fact that it has been confirmed for all even integers up to 4×10^{18}, the conjecture remains unproved. And while we are still on the artistic side of our story, Eugen Jost has depicted many of the daily occurrences of five in his painting *All Is Five*, which shows several whimsical allusions to various aspects of this number (plate 6).

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Five has some interesting mathematical features. It is the hypotenuse of the right triangle (3, 4, 5)—the smallest Pythagorean triangle and the only primitive one whose sides form an arithmetic progression (a primitive triple is one whose members have no common divisors other than one). Also, the sequence 5, 11, 17, 23, and 29 is the smallest sequence of five primes forming an arithmetic progression.

Five is the second Fermat prime ($5 = 2^{2^1} + 1$), and, consequently, a regular pentagon can be constructed with the Euclidean tools—a straightedge (an unmarked ruler) and compass. This is due to a discovery made by Carl Friedrich Gauss (1777–1855) when he was just nineteen years old: a regular polygon of n sides—a regular n-gon, for short—can be constructed with Euclidean tools if n is a product of nonnegative powers of 2 and/or *distinct* primes of the form $2^{2^k} + 1$, where k is a nonnegative integer. Primes of this form are called *Fermat primes*, after the great French number theorist Pierre Fermat (1601–1665).

The only regular polygons the Greeks knew how to construct with Euclidean tools were an equilateral triangle, a square, a pentagon, and a fifteen-sided gon, plus any polygons obtained from these by repeatedly doubling the number of sides (for example, the hexagon, octagon, and twelve-sided gon). Imagine the surprise when young Gauss added a new member to that list—a regular seventeen-sided polygon; that's because seventeen is the third Fermat prime ($17 = 2^{2^2} + 1$). As the story goes, Gauss was deeply impressed by this discovery and asked that a seventeen-sided gon be engraved on his tombstone after his death. But the stone cutter, fearing that a polygon with so many sides would be mistaken for a circle, chiseled a seventeen-pointed star instead. The original star is no longer visible, but Gauss's

hometown of Brunswick, Germany, erected a statue in his honor, with a seventeen-sided star polygon engraved on its base. Plate 7, *Homage to Gauss*, is an artistic rendition of it by Eugen Jost.

Fermat conjectured that the expression $2^{2^k}+1$ yields a prime for every nonnegative value of k. Indeed, for $k=0$, 1, 2, 3, 4 we get the primes 3, 5, 17, 257, and 65,537, and therefore regular polygons with these numbers of sides are constructable with the Euclidean tools. Well, at least in principle. Even the seventeen-sided gon is fairly complicated to construct, and I wouldn't recommend anyone try the 257-sided gon.

Fermat's conjecture stood unchallenged until 1732, when Leonhard Euler showed that for $k=5$ we get the Fermat number $2^{2^5}+1=4,294,967,297=641\times 6,700,417$—a composite number. As of this writing, it is not known if any other Fermat primes exist, leaving the possibility that there are other, as yet undiscovered regular polygons constructable with Euclidean tools. Needless to say, such polygons would have a huge number of sides, making any actual construction totally out of the question.[4]

Gauss's discovery provided a *sufficient* condition for constructing a regular n-gon with Euclidean tools. In 1837, Pierre Laurent Wantzel (1814–1848) proved that it is also a *necessary* condition, so the Fermat-prime polygons, and those obtained from them by repeatedly doubling the number of sides, are the *only* constructable n-gons. Thus, a fifteen-sided gon is constructable because $15=3\times 5$, and both 3 and 5 are Fermat primes. But a seven-sided gon (a heptagon) is not, because 7 is not a Fermat prime. Nor is a fifty-sided gon, because $50=2\times 5\times 5$, and the double presence of 5 makes it ineligible. But a fifty-one-sided gon, practicably indistinguishable from its fifty-sided neighbor, *is* constructable, because $51=3\times 17$, each of the factors being a Fermat prime.

Five is the fifth member of the Fibonacci series, a simple-looking sequence of numbers with many remarkable properties. The sequence starts with 1 and 1, then continues by adding the two previous numbers to get the next number:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots,$$

and in general

$$F_1 = F_2 = 1, F_{n+2} = F_n + F_{n+1}, n = 1, 2, 3, \ldots. \qquad (1)$$

The sequence grows very fast: the twentieth member is 6,765, and the thirtieth member is 832,040. It is named after the Italian mathematician Leonardo of Pisa, born ca. 1170 to a Pisan merchant; he later became known by the name Fibonacci, meaning the son of Bonacci. In 1202 he published a book by the title *Liber Abaci* ("The Book of Calculation"), in which he advocated use of the Hindu-Arabic numeration system, known already for some time in the East but not yet widely accepted in Europe. The book became an instant hit and helped greatly in adopting the new system by merchants, then by scholars, and eventually by most of the learned world. The Fibonacci numbers appear in his book as a recreational problem: a pair of rabbits produce an offspring at the end of their first month and every month thereafter. The offsprings then repeat the same schedule. How many rabbits will there be at the end of the first year? It is easy to see that the number of rabbits follows the Fibonacci sequence, whose twelfth member is 144.[5]

It is somewhat ironic that Fibonacci's name is remembered today mainly for this little aside, rather than for his promotion of the Hindu-Arabic numeration system. His sequence enjoys numerous interesting properties, and a

FIGURE 1.4. The five Platonic solids, Bagno Steinfurt, Germany

scholarly publication, the *Fibonacci Journal*, is dedicated to their study. We will have much more to say about this sequence in chapter 3.

Five is the number of *Platonic* or *regular polyhedra*, symmetrical solids whose faces are all identical regular polygons that meet each other at the same angle (figure 1.4): the tetrahedron (four faces, each an equilateral triangle), the hexahedron, more commonly known as the cube (six faces, each a square), the octahedron (eight equilateral triangles), the dodecahedron (twelve regular pentagons), and the icosahedron (twenty equilateral triangles). That there exist exactly five regular solids—unlike the infinitely many regular polygons in the plane—is surprising and has made these solids an object of endless fascination (for a proof, see appendix C). The Pythagoreans were familiar with all five solids and knew how to construct them, using only the Euclidean tools. Four of these solids involve either equilateral triangles or squares, which are easy to construct; but the dodecahedron has pentagonal faces, whose construction is not at all obvious. It was this problem that most likely led them to discover the golden ratio or divine proportion—the key to constructing the regular pentagon.

NOTES AND SOURCES

1. It is somewhat difficult to transliterate the guttural Hebrew consonant ח (het) into English; it is variously written as "ch" or just "h."
2. Or twenty-four hours, known in the United States as "military time" but in common usage throughout the rest of the world, where no one has any trouble reading 17:00 as 5:00 p.m.
3. For more on the Mayan numeration system, see Georges Ifrah, *The Universal History of Numbers* (New York: John Wiley, 2000), pp. 44–46, 94–95, 308–12, 339; and Frank Swetz, *From Five Fingers to Infinity* (Chicago: Open Court, 1994), pp. 71–79.
4. Around 1980, when the first programmable calculators appeared on the market, I bought Texas Instruments' latest version, the SR 56 (the designation SR stood for "slide rule," until then the trade tool of every scientist and engineer for the past 350 years). It had a ten-digit display, so I programmed it to factor a number into its prime factors, punched in 4,294,967,297 and hit the "start" key. For the next 28 minutes the machine did its calculations, and then the smaller of the two factors, 641, appeared in the display, to my great delight. Needless to say, a modern computer can do it in a tiny fraction of a second. (There are several factorization sites available online, such as Prime Factors Decomposition at https://www.dcode.fr/prime-factors-decomposition.)
5. The sequence can be extended to negative indices as well, by rewriting equation (1) as $F_n = F_{n+2} - F_{n+1}$: …, –8, 5, –3, 2, –1, 1, 0, 1, 1, 2, 3, 5, 8, …, and in general $F_0 = 0$ and $F_{-n} = (-1)^{n+1} F_n$, where n is a positive integer. For more on the Fibonacci numbers, see chapter 2 and appendix B.

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