Contents

Chapter One

Introduction

We consider the homogeneous incompressible Euler equations

$$\partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0 \tag{1.1a}$$

$$\operatorname{div} v = 0 \tag{1.1b}$$

for the unknown velocity vector field v and scalar pressure field p, posed on the three-dimensional box $\mathbb{T}^3 = [-\pi, \pi]^3$ with periodic boundary conditions. We consider weak solutions of (1.1), which may be defined in the usual way for $v \in L^2_t L^2_x$.

We show that within the class of weak solutions of regularity $C^0_t H^{1/2-}_x$, the 3D Euler system (1.1) is *flexible*.[1] An example of this flexibility is provided by:

Theorem 1.1 (Main result). *Fix $\beta \in (0, 1/2)$. For any divergence-free vector fields $v_{\text{start}}, v_{\text{end}} \in L^2(\mathbb{T}^3)$ which have the same mean, any $T > 0$, and any $\epsilon > 0$, there exists a weak solution $v \in C([0,T]; H^\beta(\mathbb{T}^3))$ to the 3D Euler equations (1.1) such that $\|v(\cdot,0) - v_{\text{start}}\|_{L^2(\mathbb{T}^3)} \leq \epsilon$ and $\|v(\cdot,T) - v_{\text{end}}\|_{L^2(\mathbb{T}^3)} \leq \epsilon$.*

Since the vector field v_{end} may be chosen to have a much higher (or much lower) kinetic energy than the vector field v_{start}, the above result shows the existence of infinitely many *non-conservative* weak solutions of 3D Euler in the regularity class $C^0_t H^{1/2-}_x$. Theorem 1.1 further shows that the set of so-called *wild initial data* is dense in the space of L^2 periodic functions of given mean. The novelty of this result is that these weak solutions have *more than 1/3 regularity*, when measured on a L^2_x-based Banach scale.

Remark 1.2. We have chosen to state the flexibility of the 3D Euler equations as in Theorem 1.1 because it is a simple way to exhibit weak solutions which are non-conservative, leaving the entire emphasis of the proof on the *regularity class* in which the weak solutions lie. Using by now standard approaches encountered in convex integration constructions for the Euler equations, we may alternatively establish the following variants of flexibility for (1.1) within the class of $C^0_t H^{1/2-}_x$ weak solutions:

[1] Loosely speaking, we consider a system of partial differential equations of physical origin to be *flexible* in a certain regularity class if at this regularity level the PDEs are not anymore predictive: there exist infinitely many solutions, which behave in a non-physical way, in stark contrast to the behavior of the PDE in the smooth category. We refer the interested reader to the discussion in the surveys of De Lellis and Székelyhidi Jr. [30, 32], which draw the analogy with the flexibility in Gromov's h-principle [40].

1. The proof of Theorem 1.1 also shows that *given any* $\beta < 1/2$, $T > 0$, *and* $E > 0$, *there exists a weak solution* $v \in C(\mathbb{R}, H^\beta(T^3))$ *of the 3D Euler equations such that* $\operatorname{supp}_t v \subset [-T, T]$, *and* $\|v(\cdot, 0)\|_{L^2} \geq E$. Such weak solutions are nontrivial and have compact support in time, thereby implying the *non-uniqueness* of weak solutions to (1.1) in the regularity class $C_t^0 H_x^{1/2-}$. The argument is sketched in Remark 3.7 below.
2. The proof of Theorem 1.1 may be modified to show that *given any* $\beta \in (0, 1/2)$, *and any* C^∞ *smooth function* $e\colon [0, T] \to (0, \infty)$, *there exists a weak solution* $v \in C^0([0, T]; H^\beta(\mathbb{T}^3))$ *of the 3D Euler equations, such that* $v(\cdot, t)$ *has kinetic energy* $e(t)$, *for all* $t \in [0, T]$. In particular, the flexibility of 3D Euler in $C_t^0 H_x^{1/2-}$ may be shown to also hold within the class of *dissipative* weak solutions, by choosing e to be a non-increasing function of time. This is further discussed in Remark 3.8 below.

1.1 CONTEXT AND MOTIVATION

Classical solutions of the Cauchy problem for the 3D Euler equations (1.1) are known to exist, locally in time, for initial velocities which lie in $C^{1,\alpha}$ for some $\alpha > 0$ (see, e.g., Lichtenstein [48]). These solutions are unique, and they conserve (in time) the kinetic energy $\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{T}^3} |v(x,t)|^2 dx$, giving two manifestations of *rigidity* of the Euler equations within the class of smooth solutions.

Motivated by hydrodynamic turbulence, it is natural to consider a much broader class of solutions to the 3D Euler system; these are the *distributional* or *weak* solutions of (1.1), which may be defined in the natural way as soon as $v \in L_t^2 L_x^2$, since (1.1) is in divergence form. Indeed, one of the fundamental assumptions of Kolmogorov's '41 theory of turbulence [46] is that in the infinite Reynolds number limit, turbulent solutions of the 3D Navier-Stokes equations exhibit anomalous dissipation of kinetic energy; by now, this is considered to be an experimental fact; see, e.g., the book of Frisch [39] for a detailed account. In particular, this anomalous dissipation of energy necessitates that the family of Navier-Stokes solutions does not remain uniformly bounded in the topology of $L_t^3 B_{3,\infty,x}^\alpha$ for any $\alpha > 1/3$, as the Reynolds number diverges, as was alluded to in the work of Onsager [58].[2] Thus, in the infinite Reynolds number limit for turbulent solutions of 3D Navier-Stokes, one expects the convergence to *weak* solutions of 3D Euler, not classical ones.

It turns out that even in the context of weak solutions, the 3D Euler equa-

[2] Onsager did not use the Besov norm

$$\|v\|_{B^\alpha_{p,\infty}} = \|v\|_{L^p} + \sup_{|z|>0} |z|^{-\alpha} \|v(\cdot + z) - v(\cdot)\|_{L^p};$$

here we use this modern notation and the sharp version of this conclusion, cf. Constantin, E, and Titi [22], Duchon and Robert [35], and Drivas and Eyink [34].

tions enjoy some conditional variants of rigidity. An example is the classical *weak-strong* uniqueness property.[3] Another example is the question of whether weak solutions of the 3D Euler equation conserve kinetic energy. This is the subject of the Onsager conjecture [58], one of the most celebrated connections between phenomenological theories in turbulence and the rigorous mathematical analysis of the PDEs of fluid dynamics. For a detailed account we refer the reader to the reviews [37, 21, 61, 30, 64, 32, 33, 12, 14] and mention here only a few of the results in the Onsager program for 3D Euler.

Constantin, E, and Titi [22] established the rigid side of the Onsager conjecture, which states that if a weak solution v of (1.1) lies in $L^3_t B^{\beta}_{3,\infty,x}$ for some $\beta > 1/3$, then v conserves its kinetic energy. The endpoint case $\beta = 1/3$ was addressed by Cheskidov, Constantin, Friedlander, and Shvydkoy [16], who established a criterion which is known to be sharp in the context of 1D Burgers. By using the Bernstein inequality to transfer information from L^2_x into L^3_x , the authors of [16] also prove energy-rigidity for weak solutions based on a regularity condition for an L^2_x-based scale: if $v \in L^3_t H^{\beta}_x$ with $\beta > 5/6$, then v conserves kinetic energy (see also the work of Sulem and Frisch [63]). We emphasize the discrepancy between the energy-rigidity threshold exponents $5/6$ for the L^2-based Sobolev scale, and $1/3$ for L^p-based regularity scales with $p \geq 3$.

The first flexibility results were obtained by Scheffer [59], who constructed nontrivial weak solutions of the 2D Euler system, which lie in $L^2_t L^2_x$ and have compact support in space and time. The existence of infinitely many dissipative weak solutions to the Euler equations was first proven by Shnirelman in [60], in the regularity class $L^\infty_t L^2_x$. Inspired by the work [53] of Müller and Šverak for Lipschitz differential inclusions, in [29] De Lellis and Székelyhidi Jr. have constructed infinitely many dissipative weak solutions of (1.1) in the regularity class $L^\infty_t L^\infty_x$ and have developed a systematic program towards the resolution of the flexible part of the Onsager conjecture, using the technique of *convex integration*. Inspired by Nash's paradoxical constructions for the isometric embedding problem [54], the first proof of flexibility of the 3D Euler system in a Hölder space was given by De Lellis and Székelyhidi Jr. in the work [31]. This breakthrough or crossing of the L^∞_x to C^0_x barrier in convex integration for 3D Euler [31] has in turn spurred a number of results [8, 6, 9, 27] which have used finer properties of the Euler equations to increase the regularity of the wild weak solutions being constructed. The flexible part of the Onsager conjecture was finally resolved by Isett [43, 42] in the context of weak solutions with compact support in time (see also the subsequent work by the first and last authors with De Lellis and Székelyhidi Jr. [11] for dissipative weak solutions), by showing that for any regularity parameter $\beta < 1/3$, the 3D Euler system (1.1) is flexible in the class of $C^{\beta}_{t,x}$ weak solutions. We refer the reader to the review

[3] If v is a strong solution of the Cauchy problem for (1.1) with initial datum $v_0 \in L^2$, and $w \in L^\infty_t L^2_x$ is merely a weak solution of the Cauchy problem for (1.1), which has the additional property that its kinetic energy $\mathcal{E}(t)$ is less than the kinetic energy of v_0, for a.e. $t > 0$, then in fact $v \equiv w$. See, e.g., the review [66] for a detailed account.

papers [30, 64, 32, 33, 12, 14] for more details concerning convex integration constructions in fluid dynamics, and for open problems in this area. We note that the situation in two dimensions appears considerably more difficult, as the full flexible side of the Onsager conjecture remains open in this setting [56]. Successfully extending either the homogeneous $C^{1/3-}$ constructions, or the present construction, to the 2D Euler equations appears to require new ideas.

Since the aforementioned convex integration constructions are spatially homogenous, they yield weak solutions whose Hölder regularity index cannot be taken to be larger than $1/3$ (recall that weak solutions in $L^3_t C^\beta_x$ with $\beta > 1/3$ must conserve kinetic energy). However, *the exponent* $1/3$ *is not expected to be a sharp threshold for energy rigidity/flexibility if the weak solutions' regularity is measured on an* L^p_x*-based Banach scale with* $p < 3$. This expectation stems from the measured intermittent nature of turbulent flows; see, e.g., Frisch [39, Figure 8.8, page 132]. In broad terms, intermittency is characterized as a deviation from the Kolmogorov '41 scaling laws, which were derived under the assumptions of homogeneity and isotropy (for a rigorous way to measure this deviation, see Cheskidov and Shvydkoy [20]). A common signature of intermittency is that for $p \neq 3$, the p^{th} order structure function[4] exponents ζ_p deviate from the Kolmogorov-predicted values of $p/3$. We note that the regularity statement $v \in C^0_t B^s_{p,\infty}$ corresponds to a structure function exponent $\zeta_p = sp$; that is, Kolmogorov '41 predicts that $s = 1/3$ for all p. The exponent $p = 2$ plays a special role, as it allows one to measure the intermittent nature of turbulent flows on the Fourier side as a power-law decay of the energy spectrum. Throughout the last five decades, the experimentally measured values of ζ_2 (in the inertial range, for viscous flows at very high Reynolds numbers) have been consistently observed to *exceed* the Kolmogorov-predicted value of $2/3$ [1, 50, 62, 45, 15, 44, 55], thus showing a steeper decay rate in the inertial range power spectrum than the one predicted by the Kolmogorov-Obhukov 5/3 law. Moreover, in the mathematical literature, Constantin and Fefferman [23] and Constantin, Nie, and Tanveer [24] have used the 3D Navier-Stokes equations to show that the Kolmogorov '41 prediction $\zeta_2 = 2/3$ is only consistent with a lower bound for ζ_2, instead of an exact equality.

Prior to this work, it was not known whether the 3D Euler equation can sustain weak solutions which have kinetic energy that is uniformly bounded in time but not conserved, and which have spatial regularity equal to or exceeding $H^{1/3}_x$, corresponding to $\zeta_2 \geq 2/3$; see [12, Open Problem 5] and [14, Conjecture 2.6]. Theorem 1.1 gives the first such existence result. The solutions in Theo-

[4] In analogy with L^p-based Besov spaces, absolute p^{th} order structure functions are typically defined as $S_p(\ell) = \fint_0^T \fint_{\mathbb{T}^3} \fint_{\mathbb{S}^2} |v(x+\ell z,t) - v(x,t)|^p dz dx dt$. The structure function exponents in Kolmogorov's '41 theory are then given by $\zeta_p = \limsup_{\ell \to 0^+} \frac{\log S_p(\ell)}{\log(\epsilon \ell)}$, where $\epsilon > 0$ is the postulated anomalous dissipation rate in the infinite Reynolds number limit. Of course, for any non-conservative weak solution we may define a positive number $\epsilon = \fint_0^T |\frac{d}{dt}\mathcal{E}(t)| dt$ as a substitute for Kolmogorov's ϵ, which allows one to define ζ_p accordingly.

rem 1.1 may be constructed to have second-order structure function exponent ζ_2 an arbitrary number in $(0, 1)$, showing that (1.1) exhibits weak solutions which severely deviate from the Kolmogorov-Obhukov 5/3 power spectrum.

We note that in a recent work [18], Cheskidov and Luo established the sharpness of the $L^2_t L^\infty_x$ endpoint of the Prodi-Serrin criteria for the 3D Navier-Stokes equations, by constructing non-unique weak (mild) solutions of these equations in $L^p_t L^\infty_x$, for any $p < 2$.[5] As noted in [18, Theorem 1.10], their approach also applies to the 3D Euler equations, yielding weak solutions that lie in $L^1_t C^\beta_x$ for any $\beta < 1$, and thus these weak solutions also have more than $1/3$ regularity. The drawback is that the solutions constructed in [18] do not have bounded (in time) kinetic energy, in contrast to Theorem 1.1, which yields weak solutions with kinetic energy that is continuous in time.

Theorem 1.1 is proven by using an intermittent convex integration scheme, which is necessary in order to reach beyond the $1/3$ regularity exponent, uniformly in time. Intermittent convex integration schemes have been introduced by the first and last authors in [13] in order to prove the non-uniqueness of weak (mild) solutions of the 3D Navier-Stokes equations with bounded kinetic energy, and then refined in collaboration with Colombo [7] to construct solutions which have partial regularity in time. Recently, intermittent convex integration techniques have been used successfully to construct non-unique weak solutions for the transport equation (cf. Modena and Székelyhidi Jr. [52, 51], Brué, Colombo, and De Lellis [5], and Cheskidov and Luo [17]), the 2D Euler equations with vorticity in a Lorentz space (cf. [4]), the stationary 4D Navier-Stokes equations (cf. Luo [49]), the α-Euler equations (cf. [3]), and the MHD equations and related variants (cf. Dai [26], the first and last authors with Beekie [2]), and the effect of temporal intermittency has recently been studied by Cheskidov and Luo [18]. We refer the reader to the reviews [12, 14] for further references, and for a comparison between intermittent and homogenous convex integration.

When applied to three-dimensional nonlinear problems, intermittent convex integration has insofar only been successful at producing weak solutions with negligible spatial regularity indices, uniformly in time. As we explain in Section 1.2, there is a fundamental obstruction to achieving high regularity: in physical space, intermittency causes concentrations that result in the formation of intermittent peaks, and to handle these peaks the existing techniques have used an extremely large separation between the frequencies in consecutive steps of the convex integration scheme.[6] This book is the first to successfully implement a high-regularity (in L^2), spatially intermittent, temporally homogenous, convex integration scheme in three space dimensions, and shows that for the 3D Euler system any regularity exponent $\beta < 1/2$ may be achieved.[7] In fact, the

[5]See also [19] for a proof that the space $C^0_t L^p_x$ is critical for uniqueness at $p = 2$, in two space dimensions.

[6]This becomes less of an issue when one considers the equations of fluid dynamics in very high space dimensions; cf. Tao [65].

[7]It was known within the community (see Section 2.4.1 for a detailed explanation) that

techniques developed in this book are the backbone for the recent work [57] of the last two authors, which gives an alternative, intermittent, proof of the Onsager conjecture. In general, we expect the framework developed in the present work to inspire future iterations requiring a combination of intermittency and sharp regularity estimates.

1.2 IDEAS AND DIFFICULTIES

As alluded to in the previous paragraph, the main difficulty in reaching a high regularity exponent for weak solutions of (1.1) is that the existing intermittent convex integration schemes do not allow for consecutive frequency parameters λ_q and λ_{q+1} to be close to each other. In essence, this is because intermittency smears out the set of active frequencies in the approximate solutions to the Euler system (instead of concentric spheres, they are more akin to thick concentric annuli), and several of the key estimates in the scheme require frequency separation to achieve L^p-decoupling (see Section 2.4.1). Indeed, high regularity exponents necessitate an almost geometric growth of frequencies ($\lambda_q = \lambda_0^q$), or at least a barely super-exponential growth rate $\lambda_{q+1} = \lambda_q^b$ with $0 < b - 1 \ll 1$ (in comparison, the schemes in [13, 7] require $b \approx 10^3$). Essentially every new idea in this manuscript is aimed either directly or indirectly at rectifying this issue: how does one take advantage of intermittency, and at the same time keep the frequency separation nearly geometric?

The building blocks used in the convex integration scheme are intermittent pipe flows,[8] which we describe in Section 2.3. Due to their spatial concentration and their periodization rate, quadratic interactions of these building blocks produce both the helpful low frequency term which is used to cancel the previous Reynolds stress $\mathring{R}_q$, and a number of other errors which live at intermediate frequencies. These errors are spread throughout the frequency annulus with inner radius λ_q and outer radius λ_{q+1}, and may have size only slightly less than that of $\mathring{R}_q$. If left untreated, these errors only allow for a very small regularity parameter β. In order to increase the regularity index of our weak solutions, we need to take full advantage of the frequency separation between the slow frequency λ_q and the fast frequency λ_{q+1}. As such, the intermediate-frequency errors need to be further corrected via velocity increments designed to push these residual stresses towards the frequency sphere of radius λ_{q+1}. The quadratic interactions among these higher order velocity corrections themselves, and in principle also

there is a key obstruction to reaching a regularity index in L^2 for a solution to the Euler equations larger than $1/2$ via convex integration.

[8]The moniker used in [27] and the rest of the literature for these stationary solutions has been "Mikado flows." However, we rely rather heavily on the geometric properties of these solutions, such as orientation and concentration around axes, and so to emphasize the tube-like nature of these objects, we will frequently use the term "pipe flows."

with the old velocity increments, in turn create *higher order Reynolds stresses*, which live again at intermediate frequencies (slightly higher than before), but whose amplitude is slightly smaller than before. This process of adding *higher order velocity perturbations* designed to cancel intermediate-frequency higher order stresses has to be repeated many times until all the resulting errors are either small or live at frequency $\approx \lambda_{q+1}$, and thus are also small upon inverting the divergence. See Sections 2.4 and 2.6 for a more thorough account of this iteration.

Throughout the process described in the above paragraph, we need to keep adding velocity increments, while at the same time keeping the high-high-high frequency interactions under control. The fundamental obstacle here is that when composing the intermittent pipe flows with the Lagrangian flow of the slow velocity field, the resulting deformations are not spatiotemporally homogenous. In essence, the intermittent nature of the approximate velocity fields implies that a sharp global control on their Lipschitz norm is unavailable, thus precluding us from implementing a gluing technique as in [42, 11]. Additionally, we are faced with the issue that pipe flows which were added at different stages of the higher order correction process have different periodization rates and different spatial concentration rates, and may a priori overlap. Our main idea here is to implement a *placement technique* which uses the *relative intermittency* of pipe flows from previous or same generations, in conjunction with a sharp bound on their local Lagrangian deformation rate, to determine suitable spatial shifts for the placement of new pipe flows so that they dodge all other bent pipes which live in a restricted space-time region. This geometric placement technique is discussed in Section 2.5.2.

A rigorous mathematical implementation of the heuristic ideas described in the previous two paragraphs, which crucially allows us to slow down the frequency growth to be almost geometric, requires extremely sharp information on all higher order errors and their associated velocity increments. For instance, in order to take advantage of the transport nature of the linearized Euler system while mitigating the loss of derivatives issue which is characteristic of convex integration schemes, we need to keep track of essentially *infinitely many sharp material derivative estimates* for all velocity increments and stresses. Such estimates are naturally only attainable on a *local inverse Lipschitz timescale*, which in turn necessitates keeping track of the precise location in space of the peaks in the densities of the pipe flows, and performing a frequency localization with respect to both the Eulerian and the Lagrangian coordinates. In order to achieve this, we introduce carefully designed *cutoff functions*, which are defined recursively for the velocity increments (in order to keep track of overlapping pipe flows from different stages of the iteration), and iteratively for the Reynolds stresses (in order to keep track of the correct amplitude of the perturbation which needs to be added to correct these stresses); see Section 2.5. The cutoff functions we construct effectively play the role of a joint Eulerian-and-Lagrangian Littlewood-Paley frequency decomposition, which in addition keeps track of both the position in space and the amplitude of various objects (akin to a wavelet decomposition). The analysis of these cutoff functions requires estimating very high order commutators between Lagrangian and Eulerian derivatives (see Chapter 6 and Appendix A). Lastly, we mention an additional technical complication: since the sharp control of the Lipschitz norm of the approximate velocities in our scheme is local in space and time, we need to work with an inverse divergence operator (e.g., for computing higher order stresses) which, up to much lower order error terms, maintains the spatial support of the vector fields that it is applied to. Additionally, we need to be able to estimate an essentially infinite number of material derivatives applied to the output of this inverse divergence operator. This issue is addressed in Section A.8.

1.3 ORGANIZATION OF THE BOOK

The goal of this book is to prove Theorem 1.1 through an explicit construction of satisfactory weak solutions of the 3D Euler equations. Many aspects of this construction are in fact predicated on several recent advancements in the field of convex integration, particularly for the Euler and Navier-Stokes equations. Readers wishing to familiarize themselves with the important concepts can consult the survey paper [12], which provides an excellent overview of the relevant literature, along with essentially complete proofs of some fundamental results. We also refer the reader to the foundational papers [31, 8, 43, 11, 13], in which much of the aforementioned theory for the Euler and Navier-Stokes equations was developed.

As the complete proof of Theorem 1.1 is quite intricate, we have provided in Chapter 2 a broad overview of the main ideas, and how they tie together in order to prove the end result. Any path through this book, whether a short sojourn or a deep dive, should begin here. Specifically, Chapter 2 contains an outline of the convex integration scheme, in which we replace some of the actual (and more complicated) estimates and definitions appearing in the proof with heuristic ones in order to highlight the new ideas at an intuitive level. Readers familiar with the aforementioned literature may read only this chapter and still encounter the inspiration behind every new idea in the proof.

For those readers wishing to move past heuristics, the proof of Theorem 1.1 is given in Chapter 3, assuming that a number of estimates hold true inductively for the solutions of the Euler-Reynolds system at every step of the convex integration iteration. The remainder of the book is dedicated to showing that the inductive bounds stated in Section 3.2 may indeed be propagated from step q to step $q+1$. Chapter 4 contains the construction of the intermittent pipe flows used in this book and describes the careful placement required to show that these pipe flows do not overlap on a suitable space-time set. The mollification step of the proof is performed in Chapter 5. Chapter 6 contains the definitions of the cutoff functions used in the proof and establishes their properties. Readers may skip the proofs in Chapters 5 and 6, simply take the results for granted, and read the rest of the book successfully. Chapter 7 breaks down the main inductive bounds from Section 3.2 into components which take into account the higher order stresses and perturbations. Chapter 8 then proves the constituent parts of the inductive bounds outlined in Chapter 7. Chapter 9 carefully defines the many parameters in the proof, states the precise order in which they are chosen, and lists a few consequences of their definitions. Finally, Appendix A contains the analytical toolshed to which we appeal throughout the book. Readers may also wish to read the proofs in the appendix sparingly, as the statements are generally sufficient for understanding most of the arguments.

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Index