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Introduction

Trial on air quashed as unsound (10)

1 Down, *Daily Telegraph* crossword 26,488, 1 March 2011

Irrational numbers have been acknowledged for about 2,500 years, yet properly understood for only the past 150 of them. This book is a guided tour of some of the important ideas, people and places associated with this long-term struggle.

The chronology must start around 450 B.C.E. and the geography in Greece, for it was then and there that the foundation stones of pure mathematics were laid, with one of them destined for highly premature collapse. And the first character to be identified must be Pythagoras of Samos, the mystic about whom very little is known with certainty, but in whom pure mathematics may have found its earliest promulgator. It is the constant that sometimes bears his name, $\sqrt{2}$, that is generally (although not universally) accepted as the elemental irrational number and, as such, there is concord that it was this number that dislodged his crucial mathematical–philosophical keystone: positive integers do not rule the universe. Yet those ancient Greeks had not discovered irrational numbers as we would recognize them, much less the symbol $\sqrt{2}$ (which would not appear until 1525); they had demonstrated that the side and diagonal of a square cannot simultaneously be measured by the same unit or, put another way, that the diagonal is *incommensurable* with any unit that measures the side. An early responsibility for us is to reconcile the incommensurable with the irrational.

This story must begin, then, in a predictable way and sometimes it progresses predictably too, but as often it meanders along roads less travelled, roads long since abandoned or concealed in the dense undergrowth of the mathematical monograph. As the pages turn so we unfold detail of some of the myriad results which have shaped the history of irrational numbers, both great and small, famous and obscure, modern and classical – and these last we give in their near original form, costly though that can be. Mathematics can have known no greater aesthete than G. H. Hardy, with one of his most widely used quotations[1]:

> There is no permanent place in the world for ugly mathematics.

Perhaps not, but it is in the nature of things that first proofs are often mirror-shy.[2] They should not be lost, however, and this great opportunity has been taken to garner some of them, massage them a little, and set them beside the approaches of others, whose advantage it has been to use later mathematical ideas.

At journey's end we hope that the reader will have gained an insight into the importance of irrational numbers in the development of pure mathematics,[3] and also the very great challenges sometimes offered up by them; some of these challenges have been met, others intone the siren's call.

What, then, is meant by the term *irrational number*? Surely the answer is obvious:

> It is a number which cannot be expressed as the ratio of two integers.

Or, alternatively:

> It is a number the decimal expansion of which is neither finite nor recurring.

Yet, in both cases, irrationality is defined in terms of what it is not, rather like defining an odd number to be one that is not even. Graver still, these answers are fraught with limitations: for example, how do we use them to define equality between, or arithmetic operations on, two irrational numbers? Although these are familiar, convenient and harmless definitions, they are quite useless in practice. By them, irrational numbers are being defined in terms of one of their characteristic qualities, not as entities in their own right. Who is to say that they exist at all? For novelty, let us adopt a third, less well-known approach:

> Since every rational number r can be written
>
> $$r = \frac{(r-1) + (r+1)}{2},$$

[1] *A Mathematician's Apology* (Cambridge University Press, 1993).

[2] As indeed was Hardy.

[3] Even if they have no accepted symbol to represent them.

> every rational number is equidistant from two other rational numbers (in this case $r - 1$ and $r + 1$); therefore, no rational number is such that it is a different distance from all other rational numbers.

With this observation we define the irrational numbers as:

> The set of all real numbers having different distances from all rational numbers.

With its novelty acknowledged, the list of limitations of the definition is as least as long as before. It is an uncomfortable fact that, if we allow ourselves the integers (and we may not), a rigorous and workable definition of the rational numbers is quite straightforward, but the move from them to the irrational numbers is a problem of quite another magnitude, literally as well as figuratively: the set of rational numbers is the same size as the set of integers but the irrational numbers are vastly more numerous. This problem alone simmered for centuries and analysis waited ever more impatiently for its resolution, with the nineteenth-century rigorists posing ever more challenging questions and ever more perplexing contradictions, following Zeno of Elea more than 2,000 years earlier. In the end the resolution was decidedly Germanic, with various German mathematicians providing three near-simultaneous answers, rather like the arrival of belated buses. We discuss them in the penultimate chapter, not in the detail needed to convince the most skeptical, for that would occupy too many pages with tedious checking, but we hope with sufficient conviction for hand-waving to be a positive signal.

For whom, then, is this story intended? At once to the reader who is comfortable with real variable calculus and its associated limits and series, for they might read it as one would read a history book: sequentially from start to finish. But also to those whose mathematical training is less but whose curiosity and enthusiasm are great; they might delve to the familiar and sometimes the new, filling gaps as one might attempt a jigsaw puzzle. In the end, the jigsaw might be incomplete but nonetheless its design should be clear enough for recognition. In as much as we have invested great effort in trying to explain sometimes difficult ideas, we must acknowledge that the reader must invest energy too. Borrowing the words of a former president of Princeton University, James McCosh:

> The book to read is not the one that thinks for you but the one that makes you think.[4]

The informed reader may be disappointed by the omission of some material, for example, the base φ number system, Phinary (which makes essential use of the defining identity of the Golden Ratio), and Farey sequences and Ford Circles, for example. These ideas and others have been omitted by design and undoubtedly there is much more that is missing by accident, with the high ideal of writing comprehensively diluted to one that has sought simply to be representative of a subject which is vast in its age, vast in its breadth and intrinsically difficult. Each chapter of this book could in itself be expanded into another book, with each of these books divided into several volumes.

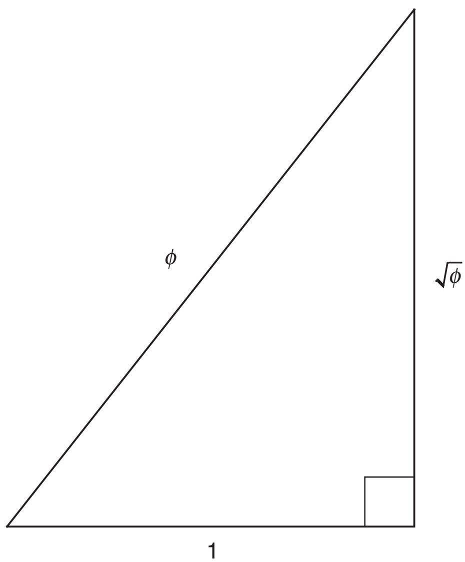
We apologize for any errors, typographic or otherwise, that have slipped through our mesh and we seek the reader's sympathy with a comment from Eric Baker:

> Proofreading is more effective after publication.

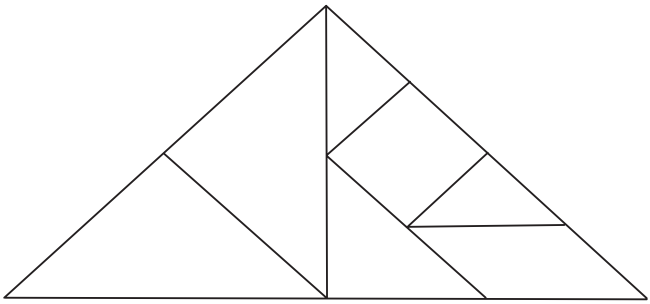
[4]He continued: "No book in the world equals the Bible for that." That acknowledged, we regard the sentiment as wider.

The moderation of men gaoled for fiddling pension at last (6,4)

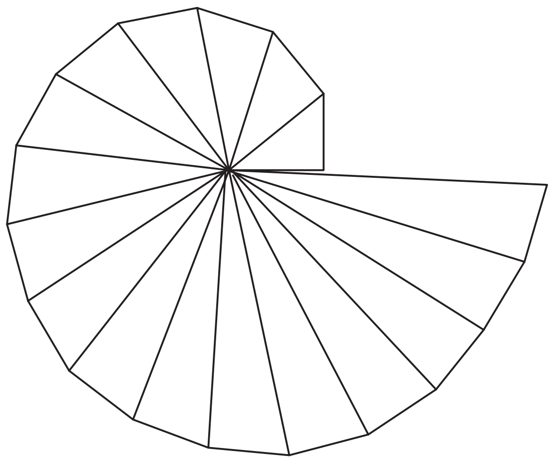
3 Down, *Daily Telegraph* crossword 26,501, 16 March 2011



Pythagϕras and the wϕrld's mϕst irratiϕnal number



Pythagoras, $\sqrt{2}$ and tangrams



The Spiral of Theodorus

$$\lim_{m\to\infty}\lim_{n\to\infty}\cos^{2n}(m!\pi x) = \begin{cases} 1: & x \text{ is rational} \\ 0: & x \text{ is irrational} \end{cases}$$

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