Contents

CHAPTER 1
Introduction

We define Kähler groups in Section 1.1 and give a brief introduction to their study. In Section 1.2, we give a detailed description of the content of each chapter of the book. Section 1.3 lists a few topics related to the study of Kähler groups that we will not discuss here. Finally, a very short list of open problems is presented in Section 1.4.

1.1 Kähler Groups

Let M be a complex manifold and let h be a hermitian metric on M. The real part of h is a Riemannian metric on M and its imaginary part is a 2-form. We always write $g = \mathrm{Re}(h)$ and $\omega = -\mathrm{Im}(h)$. The metric h is said to be *Kähler* if the imaginary part of h is closed, i.e., if

$$d\omega = 0.$$

There are many characterizations of this property [447, §3.2]. Let us mention only one of them: the metric h is Kähler if and only if the complex structure $J\colon TM \to TM$ is parallel with respect to the Levi-Civita connection of g. A *Kähler manifold* is a complex manifold that admits a Kähler metric. We refer you to [264] for the original article introducing these metrics, to [265] for a historical account, and to [25, 250, 447, 449] for various more modern introductions. The *Fubini–Study metric* on the complex projective space $\mathbb{P}^n$ is Kähler. Indeed, if h_0 is the standard hermitian scalar product of $\mathbb{C}^{n+1}$, one can define the $(1, 1)$ form ω associated to the Fubini–Study metric as follows. We pick a point $p \in \mathbb{P}^n$ and a local holomorphic section s of the projection $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$, defined near p. One then has

$$\omega = \frac{i}{2}\partial\overline{\partial} \log h_0(s, s),$$

near p. This shows that ω is closed. In a standard affine chart

$$\begin{aligned} \mathbb{C}^n &\to \mathbb{P}^n, \\ z &\mapsto [1 : z] \end{aligned}$$

the corresponding hermitian metric reads

$$h_{[1:z]}(u,v) = \frac{(1+|z|^2)h_0(u,v) - h_0(u,z)h_0(z,v)}{(1+|z|^2)^2},$$

where $z \in \mathbb{C}^n$ and $u, v \in \mathbb{C}^n$ are tangent vectors at z. Since the restriction of a Kähler metric to a complex submanifold is again a Kähler metric, it follows that any smooth projective variety (i.e., any complex submanifold of a complex projective space) is Kähler. This provides plenty of examples of Kähler manifolds.

We are now ready to introduce the main characters of this book:

Definition 1.1 A *Kähler group* is a group that can be realized as the fundamental group of a closed Kähler manifold.

Let us describe a small assemblage of Kähler groups.

Example 1.2 Free abelian groups of even rank are Kähler; indeed, any complex torus $\mathbb{C}^n/\Lambda$ (where $\Lambda < \mathbb{C}^n$ is a lattice) admits a Kähler metric and its fundamental group Λ is isomorphic to $\mathbb{Z}^{2n}$.

Example 1.3 Fundamental groups of closed Riemann surfaces are Kähler groups.

Example 1.4 All finite groups are fundamental groups of smooth projective varieties, as was proved by Serre [393]. In particular, they are Kähler groups. See also [5] for an exposition of the proof.

Before introducing the next example, and since this notion will appear several times in the text, we recall that a *lattice* Γ in a Lie group G is a discrete subgroup such that the homogeneous space G/Γ has finite Haar measure.[1] We then say that Γ is cocompact or uniform if G/Γ is compact, and nonuniform otherwise.

Example 1.5 Let Z be a hermitian symmetric space of nonpositive curvature. As examples, one can think of the unit ball of $\mathbb{C}^n$—this is one model for the complex hyperbolic space—or the Siegel space

$$S_g := \{M \in \mathrm{M}_g(\mathbb{C}) : {}^tM = M, \operatorname{Im} M \gg 0\}$$

[1] A right-invariant Haar measure on G induces naturally a measure on G/Γ.

of symmetric $g \times g$ complex matrices with positive definite imaginary part. Then any discrete cocompact subgroup Γ of the group of holomorphic isometries of Z is Kähler. This is clear if Γ is torsion-free. In general this follows from a trick due to Kollár; see [5, p. 7]. Many nonuniform lattices are also known to be Kähler, thanks to a theorem of Toledo [435].

Example 1.6 There are examples of 2-steps nilpotent (non-virtually abelian) Kähler groups, discovered by Campana [80] and Carlson and Toledo [96]. Let us describe one family of examples. Consider the real Heisenberg group H_k of dimension $2k+1$. A model for it is the following. If ω_0 is the standard symplectic form of $\mathbb{R}^{2k}$, H_k identifies with $\mathbb{R}^{2k} \times \mathbb{R}$ with the product given by

$$(u,t)\cdot(v,s) = (u+v, s+t+\omega_0(u,v)).$$

Thus it fits into a central extension,

$$1 \longrightarrow \mathbb{R} \longrightarrow H_k \longrightarrow \mathbb{R}^{2k} \longrightarrow 1.$$

One can prove that a lattice $\Gamma < H_k$ is Kähler if and only if $k \geq 4$; see, for instance, [5, ch. 8].

Example 1.7 Stover and Toledo [421] recently constructed (in all complex dimensions ≥ 2) a wealth of examples of compact Kähler manifolds admitting Kähler metrics of negative sectional curvature that do not have the homotopy type of locally symmetric spaces. They are obtained by taking branched coverings of certain ball quotients. The fundamental groups of the corresponding manifolds are Kähler. Moreover, they are Gromov hyperbolic groups[2] and are not isomorphic to lattices in $\mathrm{PU}(n,1)$. Yet many of their group theoretical properties remain unclear. For instance, it is not known whether these groups are residually finite. Prior to [421], such examples of negatively curved Kähler manifolds were only known in complex dimensions 2 and 3, thanks to the work of Mostow and Siu [345], Zheng [457, 458], and Deraux [148, 149].

Example 1.8 Toledo [436] has discovered the first examples of nonresidually finite Kähler groups. See [5, ch. 8] for further examples.

Example 1.9 Let $\Gamma < \mathrm{PU}(n,1)$ be a torsion-free nonuniform lattice acting on the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$. If the parabolic subgroups of Γ are purely unipotent,

[2]The reader will find in [60] a definition and an introduction to this notion as well as to the notion of a CAT(0) group, which will appear in Section 1.3.3.

the ends of the space $\mathbb{B}^n/\Gamma$ are biholomorphic to the unit disk bundle minus the zero section in the total space of a line bundle over an abelian variety. More precisely, there exists a compact set K such that for each connected component O of

$$\mathbb{B}^n/\Gamma - K$$

there is an abelian variety A of dimension $n-1$ and a line bundle $L \to A$ endowed with a metric h such that O is biholomorphic to the open set

$$\{v \in L, 0 < h(v, v) < 1\}. \tag{1.1}$$

Hence such a quotient $\mathbb{B}^n/\Gamma$ can be naturally compactified into a manifold X_Γ by adding finitely many abelian varieties: the open set O is "closed" by adding the zero section to it, in the identification of O with (1.1); see [249, 333]. Hummel and Schroeder [249] have proved that after possibly replacing Γ by a deep enough finite index subgroup, the manifold X_Γ is Kähler and carries a Riemannian metric of nonpositive curvature. This provides interesting examples of nonpositively curved Kähler groups. We observe that the manifolds X_Γ, called toroidal compactifications of the quotients $\mathbb{B}^n/\Gamma$, have been studied by many authors, from many different points of view. See, for instance, [23, 48, 76, 156, 158, 186, 333, 376, 386] for various works related to these compactifications.

We will mention a few additional constructions of examples in Section 1.3.3.

As the fundamental group of a closed manifold, any Kähler group is finitely presented. A proof of this classical fact can be found in [60, I.8.10]. A natural problem is then to obtain restrictions on this class of groups among finitely presented groups and to build interesting examples. There are many results establishing restrictions on Kähler groups i.e., stating that certain finitely presented groups are not Kähler. On the other hand, new constructions of Kähler groups with interesting properties remain scarce and difficult. Building new examples is an arduous task; see, e.g., [412] for a discussion of this problem. Looking at the list of examples above, the reader can convince themselves that besides the cases of Examples 1.2 and 1.3, all the constructions that we have mentioned rely on nontrivial tools, either to construct the groups under consideration or to prove that they are Kähler. In Example 1.4 one makes a subtle use of Lefschetz's hyperplane theorem [5, p. 6]; the statement in Example 1.5 is easy but relies on the construction of lattices in Lie groups; Example 1.6 relies on some nontrivial results of complex Morse theory; etc.

What makes this topic interesting is the wide mix of techniques that it involves: classical Hodge theory, analytic techniques via L^2 methods or harmonic maps, topology, geometric group theory. Among the many authors who have worked on this topic, let us mention for now Campana, Carlson and Toledo, Delzant, Gromov, and Simpson. Many other names will appear in this book, notably in Section 1.3. Before going further in this introduction, we mention one elementary restriction on Kähler groups that follows from classical Hodge theory.

Proposition 1.10 *If Γ is a Kähler group, the space $H^1(\Gamma,\mathbb{R})$ is even-dimensional. In particular, a free group is never Kähler.*

Proof. Assume that Γ is realized as the fundamental group of a closed Kähler manifold X. Then the space $H^1(\Gamma,\mathbb{R})$ identifies with $H^1(X,\mathbb{R})$. The latter space can be thought of as a de Rham cohomology group. By Hodge theory, any class $a \in H^1(X,\mathbb{R})$ can be represented in a unique way as a sum,

$$a = [\alpha + \overline{\alpha}],$$

where α is a holomorphic 1-form on X. This identifies $H^1(X,\mathbb{R})$ with the *complex* vector space $\Omega^1(X)$ of holomorphic 1-forms on X, which obviously has even real dimension.

For the second assertion of the proposition, we consider a free group F_k of rank k. If k is odd, the previous discussion implies that F_k is not Kähler. If it is even, we pick an index 2 subgroup $H < F_k$. The Euler characteristic[3] of H is then equal to

$$\chi(H) = 2\chi(F_k) = 2(1-k). \tag{1.2}$$

Since H is free, its rank is equal to $1-\chi(H)$, which is odd according to (1.2). Hence H cannot be Kähler. Since finite index subgroups of Kähler groups are Kähler, too, this proves that Γ cannot be Kähler. □

We now summarize a number of the results on Kähler groups that are contained in the book [5], before turning in Section 1.2 to a description of the material treated in this book.

- There exist many criteria that ensure the existence of a fibration onto a hyperbolic Riemann surface for a given closed Kähler manifold [5, ch. 2]. Some of these criteria are purely topological and sometimes depend only on the fundamental group.

[3] The Euler characteristic of a group G is defined as the Euler characteristic of any $K(G,1)$, i.e., of any CW-complex whose fundamental group is isomorphic to G and whose higher homotopy groups vanish. When there is a $K(G,1)$ which is a finite complex, this is well defined, i.e., independent from the choice of the finite $K(G,1)$.

- There are many restrictions on nilpotent quotients of Kähler groups, and in particular on nilpotent Kähler groups [5, ch. 3].
- Gromov [223] introduced the use of L^2 methods in the study of Kähler groups and proved that an infinite Kähler group has only one end; these methods are discussed in [5, ch. 4].
- Several chapters of [5] deal with the study of harmonic maps from Kähler manifolds to locally symmetric spaces and their applications to the study of linear representations of Kähler groups; see [5, ch. 5–7]. This includes a presentation of Corlette and Simpson's *nonabelian Hodge theory*.
- Finally, the last chapter of [5] contains descriptions of many examples of Kähler groups.

1.2 What Is to Be Found in This Book?

As was already mentioned in the preface, this text should be considered as a place to find proofs of a few key theorems for people entering the field of Kähler groups. Of course, our exposition is biased by our personal tastes and expertise. We are mostly guided by four principles:

(1) Study the actions of Kähler groups on spaces of nonpositive curvature. Among these, trees and symmetric spaces (possibly infinite dimensional) play the most important role.

(2) Study the connectivity properties at infinity of infinite covering spaces of closed Kähler manifolds (in other words, study their ends) and the connectivity properties of level sets of pluriharmonic functions defined on these covering spaces.

We follow the philosophy developed by Delzant and Gromov [141], which consists in applying ideas from large scale geometry or geometric group theory to the field of Kähler groups. We quote them, as in [73]. According to [141], a central problem in the study of Kähler groups is to

> identify the constraints imposed by the Kähler nature of the space on the asymptotic invariants of its fundamental group and then express these invariants in terms of algebraic properties.

The works [138, 139, 141, 223, 350] are representative of this philosophy.

(3) Kähler manifolds are complex manifolds whose geometry reduces to linear algebra. This sentence is taken from [5, p. 2]. It refers to the following characterization of Kähler metrics. A hermitian metric h on a complex manifold M is Kähler if and only if for each point p of

M one can find holomorphic coordinates centered at p such that the coefficients $(h_{ij}(z))$ of h in these coordinates are of the form

$$h_{ij}(z) = h_{ij}(0) + O(|z|^2).$$

It is easy to see that a hermitian metric with this property is Kähler, and a proof of this characterization can be found in [447, Prop. 3.14]. It implies that any identity involving only the metric and its first order derivatives is true on a Kähler manifold if and only if it is true in $\mathbb{C}^n$ with its flat metric. Illustrations of this principle can be found in the study of the formality of closed Kähler manifolds [5, ch. 3] or in the classical Castelnuovo–de Franchis theorem that we present in Chapter 4. We will not discuss this principle at length here but refer instead to [5] for more on this philosophy.

(4) Establish factorization theorems! This refers to the following idea. Let G be a "target" group. Suppose that we want to study representations of Kähler groups into G. One tries to associate to G a model space B (in general a projective or a Kähler manifold), or a family of model spaces, with the following property.

If X is a closed Kähler manifold and if $\phi\colon \pi_1(X) \to G$ is a homomorphism, then there exists a holomorphic map $f\colon X \to B$ to (one of) the model space(s) such that ϕ can be decomposed as $\phi = \theta \circ f_*$, where θ is a homomorphism from the fundamental group of B to G.

Definition 1.11 In the situation above, one says that the representation ϕ *factors* through the holomorphic map f.

One should thus try to identify some model Kähler manifolds whose fundamental group admits natural representations into G and such that any representation of a Kähler group into G factors through a holomorphic map to one of the model spaces. Of course, there are many variations on this principle: one can restrict to certain classes of homomorphisms (e.g., reductive, Zariski dense if G is an algebraic group); one can consider the orbifold fundamental group if B is an orbifold; etc. An elementary illustration of this principle is given by the case where G is torsion-free abelian. In this case one can take the family of model spaces to be the family of all complex tori. The factorization is then realized by the Albanese map (see Section 11.4 or [447, ch. 12] for the definition of this map). Further illustrations can be found in Chapters 6, 8, and 9. Earlier descriptions of this general principle can be found in [5, ch. 2] and in [228].

Finally, the reader will certainly notice that fibrations onto Riemann surfaces play a prominent role in this book. They appear in every single chapter, with the exception of Chapter 10. Indeed, surface groups[4] are very flexible; for instance, they surject onto free groups and so can be represented in any group. Many structure theorems on Kähler groups take the form of a factorization theorem in the sense of the last principle above, with the family of model spaces being the family of hyperbolic Riemann surfaces or hyperbolic two-dimensional orbifolds and the holomorphic map f being a fibration. This can be seen as a manifestation of the rigidity of Kähler groups. There are many spaces on which they do not act, unless it is through a fibration onto an (orbi-)Riemann surface.

We now turn to a detailed description of the content of each chapter of this book.

- Chapter 2 introduces the notion of orbifold strucure on a real surface or on a Riemann surface. It describes the orbifold structure induced by a fibration and proves a finiteness result for fibrations onto two-dimensional hyperbolic orbifolds.
- Chapter 3 proves that if an infinite covering space of a closed Kähler manifold admits a proper fibration onto an open Riemann surface, then the fibration necessarily descends to a finite cover of the original closed manifold. This is a classical result, which appears in several places in the literature, and which can be proved by various methods.
- Chapter 4 studies the classical Castelnuovo–de Franchis theorem. In a nutshell, this theorem states that the existence of two independent holomorphic 1-forms with vanishing exterior product on a closed Kähler manifold implies the existence of a fibration of the given manifold onto a Riemann surface. We also present several classical variants of this statement: when one of the differential forms takes values in a flat holomorphic line bundle, or when the closed Kähler manifold is replaced by an open complete Kähler manifold with bounded geometry and one considers L^2 differential forms.
- Chapter 5 presents a few other criteria which give sufficient conditions for the existence of a fibration of a Kähler manifold onto a Riemann surface.
- Chapter 6 gives a complete proof of Gromov and Schoen's theorem [230] describing nonelementary actions of Kähler groups on simplicial trees: any such action factors through a fibration onto a Riemann surface. This proof, inspired by [143], does not rely on the theory of harmonic maps to trees.

[4]See Definition 2.25.

- Chapter 7 gives a complete proof of a theorem due to Napier and Ramachandran [350] stating that if an infinite covering space Y of a closed Kähler manifold has at least three ends, then Y admits a proper fibration onto an open Riemann surface. The proof in general relies on some nontrivial potential theoretic notions (notably hyperbolicity/parabolicity of Riemannian manifolds). We have tried to be as self-contained as possible and have included a brief exposition of some of the necessary prerequisites in the appendices.
- Chapter 8 is devoted to Corlette and Simpson's description of irreducible representations of Kähler groups in $\mathrm{PSL}_2(\mathbb{C})$. We do not treat the case of quasi-projective manifolds as in [118]. Our approach in the closed case differs slightly from the one in [118]. Equipped with our simpler proof of Gromov and Schoen's theorem on Kähler group actions on trees, also valid in the nonlocally finite case, we can consider the Bruhat–Tits tree of SL_2 for arbitrary fields endowed with a discrete valuation. This makes the approach more direct. We also replace the original use of variations of Hodge structure by an argument taken from [142].
- Chapter 9 constitutes an introduction to the theory of harmonic maps from Kähler manifolds to locally symmetric spaces. We expose in details two theorems: firstly, a factorization result for harmonic maps of low rank, which was established in [142], and secondly, a classical theorem of Carlson and Toledo [94] saying that a harmonic map from a Kähler manifold to complex hyperbolic space that is of real rank 3 at a point must be holomorphic or anti-holomorphics.[5] Note that the latter result can also be deduced from the work of Siu [413]. We give a down-to-earth proof that uses less Lie theory than in the original article and mainly relies on the structure of the curvature tensor of complex hyperbolic space. This can serve as an introduction to the article [94].
- In Chapter 10, we build on the theory of harmonic maps studied in the previous chapter and prove a theorem due to Simpson [406], stating that if a lattice in a semisimple Lie group G is a Kähler group, then G must be of *Hodge type*. This notion, involving the structure theory of semisimple Lie groups, is defined and studied along the way.
- Chapter 11 contains an exposition of Delzant's result about the *Bieri–Neumann–Strebel invariant* of a Kähler group [139]. We have followed closely the original article as well as [73], but have added a detailed introduction to the theory of the BNS invariant. This takes roughly one half of

[5] We abuse notation and what we really mean is either a harmonic map from a closed Kähler manifold to a locally symmetric space or an equivariant harmonic map from the universal cover of a closed Kähler manifold to a symmetric space.

the chapter and is completely independent of the world of Kähler groups. We follow mostly [47] for this exposition. We believe that this can be useful for the reader who has no previous knowledge of the BNS invariant but who is interested in understanding Delzant's result.

- Chapter 12 describes results of Beauville and Delzant around the *Green–Lazarsfeld set* of Kähler groups [33, 138]. We give a complete description of this set, which was first obtained by combining the work of Beauville, Campana, and Simpson. Delzant's contribution allows us to simplify the proofs and to state the results elegantly in terms of fibrations onto orbifolds. We conclude the chapter with a proof of Delzant's theorem stating that solvable Kähler groups are virtually nilpotent [139].
- Chapter 13 proves a factorization result for Kähler group actions on real trees. Here again, the statement of the theorem is not new. The theorem was already known to hold, thanks to the theory of harmonic maps to real trees. However, we give again a proof that is "free of harmonic maps to singular spaces". Note that the theory of harmonic maps to CAT(0) spaces is only briefly alluded to in Section 9.1.3.

Next come the appendices!

- Appendices A and B introduce basic notions concerning ends of groups and spaces and groups acting on trees that are needed in the text. Appendix C introduces classical facts about unitary representations and amenability.
- Appendices D, E, and F all deal with the construction of harmonic functions on Riemannian manifolds. The first two appendices explain how to build harmonic functions with prescribed behavior along the ends of a Riemannian manifold, first in the case where the manifold satisfies a linear isoperimetric inequality, then in a more general setting. In the Kähler case, we also explain how to prove that these functions are pluriharmonic if they are of finite energy, or at least of energy not growing too fast. Appendix F explains how to build proper harmonic functions on parabolic ends of Riemannian manifolds, following Nakai's work [348].
- Appendix G explains the proof of a theorem due to Diederich and Mazzilli dealing with complex analytic sets contained in a real analytic set of $\mathbb{C}^n$. This result is needed in the proof of one of the factorization results established in Chapter 9.
- Finally, the short Appendix H gives a detailed proof of a curvature identity related to the so-called *Bochner–Siu–Sampson* formula, which shows that harmonic maps from Kähler manifolds to Riemannian manifolds satisfying a certain curvature condition are automatically pluriharmonic.

This is needed in Chapter 9, but we have presented the proof separately to avoid making that chapter too lengthy. The proof only involves linear algebra computations and is independent of the other techniques presented in Chapter 9.

Although we discuss many developments that have appeared after the book [5], there is unavoidably some overlap between this book and [5]. This concerns mainly the chapters on the Castelnuovo–de Franchis theorem and on harmonic maps as well as the various sections dealing with L^2 methods. We hope that this will help make this text slightly more self-contained. However, we strongly advise the reader to read [5] in parallel to this book, for a more thorough introduction to the world of Kähler groups.

We close this section with one important note for the reader: we do believe that the construction of *examples* is a fundamental task in this field. However, we have not devoted any chapter to this topic. This is mainly because the techniques involved are essentially disjoint from the ones used to establish constraints on Kähler groups. Besides the examples mentioned so far, we discuss in Section 1.3.3 some of the new examples that have appeared since [5]. Progress has been slow, but some new examples have appeared!

1.3 What Is Not to Be Found in This Book?

We give in this section a short survey of some developments in the field of Kähler groups that occurred after the publication of the book [5] and that we will not discuss in detail.

1.3.1 3-Manifold Groups and Kähler Groups

Carlson and Toledo [94] proved in the 1980s that the fundamental group of a closed hyperbolic n-manifold is not isomorphic to a Kähler group if $n \geq 3$. Their result is actually more general and will be discussed in Chapter 9. Since many closed 3-manifolds carry hyperbolic structures, this result motivated Reznikov [380] and Donaldson and Goldman (unpublished) to ask, in the years following [94], the following question: which groups are at the same time Kähler and the fundamental group of a closed 3-manifold? See [300, 380] for an extensive historical discussion. This question was completely answered in [165] by Dimca and Suciu: a group sharing these two properties must be finite. Different proofs of their result were later given by Kotschick [300] and by Biswas, Mj, and Seshadri [51]. Later on, generalizations of this result (obtained by dealing with fundamental groups of quasi-projective manifolds or quasi-Kähler manifolds or open 3-manifolds) were obtained [192, 301].

1.3.2 The Kodaira Problem

The following question is a well-known open problem in the field:

> Is is true that any Kähler group is also the fundamental group of a smooth projective variety?

It already appears implicitly in [5, p. 5] and a positive answer seems to be expected by many people and is conjectured, e.g., in [114]. It is sometime called the *Kodaira problem for the fundamental group*. This refers to the (general) Kodaira problem, which asks whether a compact Kähler manifold can always be deformed to a smooth projective manifold; see, e.g., [145, 251] for a discussion. Kodaira proved that the answer is positive for compact Kähler surfaces [291]. Voisin proved in 2004 [445] that the answer is negative in complex dimension at least 4. She achieved this by building, in any complex dimension ≥ 4, examples of compact Kähler manifolds that do not have the homotopy type of smooth projective varieties. In dimension 3, the answer to the Kodaira problem is positive by the work of Lin [319]. For further work related to the Kodaira problem, see [318, 446]. Further references can be found in the bibliography of [318]. As for the Kodaira problem for the fundamental group, it is open in full generality but a positive answer is known in some particular cases [87, 88, 113, 114], for instance, for groups that are linear over $\mathbb{C}$, thanks to the work of Campana, Claudon, and Eyssidieux.

1.3.3 Examples of Kähler Groups

As already mentioned above, constructions of new examples of Kähler groups are rare and difficult. We mention below some recent works giving new examples of Kähler groups. Note that all the examples mentioned at the beginning of the introduction are quite old, except for Example 1.7, which highlights the recent work [421].

The articles [8, 187] by Aramayona–Funar and Eyssidieux–Funar prove that certain quotients of the mapping class group of a closed surface are (possibly virtually) fundamental groups of smooth projective varieties and study some of their properties. It would be interesting to study further these groups. It is an open question whether the mapping class group of a closed oriented surface of genus at least 3 has a vanishing virtual first Betti number, i.e., whether the first Betti number of all its finite index subgroups vanishes [253]. If this vanishing were confirmed, the constructions in [8, 187] would provide examples of Kähler groups with vanishing virtual first Betti numbers. In particular the corresponding projective varieties would not fiber onto hyperbolic 2-orbifolds and would have certain rigidity properties.

Besides the construction of negatively curved Kähler manifolds mentioned in Example 1.7, there are other constructions in the literature that yield aspherical smooth projective surfaces: Panov [366] introduced the notion of a *polyhedral Kähler metric* on a complex surface and built examples of such metrics on $\mathbb{P}^2$. He used this construction to show the asphericity of certain (desingularized) ramified covers of $\mathbb{P}^2$ [367]; see also [430]. It would be interesting to study whether one can build new examples of Kähler groups that are hyperbolic using these constructions (and that are not complex hyperbolic lattices). This is a subtle problem. Gromov suggested a long time ago [222, §4.4] a way to build spaces with hyperbolic fundamental groups by considering ramified covers of tori (instead of $\mathbb{P}^2$). The idea is to take a ramification locus that is totally geodesic, so that the new space will carry a locally CAT(0) metric, and that is complicated enough so as to kill all flats. A delicate problem is that new flats can appear in this construction; see [57, §2.6.3] for a discussion of this issue.

In the same vein, Stadler [418] studied ramified covers of the product $\Sigma \times \Sigma$, where Σ is a Riemann surface of genus at least 2, and the ramification locus is the diagonal. Such complex surfaces carry a CAT(0) metric. Stadler proved that they do not carry any smooth Riemannian metric of nonpositive curvature (answering an exercise raised by Gromov [26, p. 2]). These ramified covers are examples of Kodaira fibrations. The arguments certainly apply to other constructions of Kodaira fibrations, such as the classical ones in [20, 292]. This provides interesting examples of Kähler groups that are CAT(0).

Finally let us mention another line of research, concerning *Kähler groups with exotic finiteness properties*. We first recall a definition due to C.T.C. Wall [448].

Definition 1.12 A group G is of type $\mathscr{F}_n$ if it admits a classifying space (i.e., a $K(G, 1)$) that is a CW-complex with finite n-skeleton.

Conditions $\mathscr{F}_1$ and $\mathscr{F}_2$ are equivalent respectively to being finitely generated and being finitely presented. The study of the finiteness conditions $\mathscr{F}_n$ is a classical topic in geometric group theory; see, for instance, [43] and the references therein for an introduction. Note that if G is of type $\mathscr{F}_n$, all its homology groups $H_i(G, \mathbb{Q})$ are finite-dimensional for $i \leq n$. In general, one says that a group has exotic finiteness properties if it violates the condition $\mathscr{F}_n$ for some n.

In [164], Dimca, Papadima, and Suciu constructed the first examples of smooth projective varieties whose fundamental groups have exotic finiteness properties. Actually, they prove that the homology of the groups under consideration is infinite-dimensional in a certain degree. These groups are subgroups of direct products of surface groups. Later on, further examples of Kähler groups with exotic finiteness properties were constructed: first by Llosa Isenrich [320, 321]—these examples are again subgroups of direct products of surfaces groups—and then by Bridson–Llosa Isenrich [62] and Nicolás–Py [362]. The examples from [62, 362] are not subgroups of direct products of surface groups. We also refer the reader to [268] for earlier related work. For the study of the finiteness properties of certain subgroups of complex hyperbolic lattices, we refer the reader to [322].

1.3.4 Lattices in PU(n, 1)

This is a very active topic. We mention the classification of fake projective planes [281, 374, 379], the study of deformations of linear representations of complex hyperbolic lattices [282, 283], and the recent discovery of a new arithmeticity criterion for these lattices [22, 24]. There is also some activity around the construction of new examples of non-arithmetic lattices in low dimension [121, 151, 152, 153], after the discovery of the first such examples in the 1980s by Mostow and Deligne–Mostow [136, 137, 342, 343, 344].

1.3.5 Simpson's Integrality Conjecture

Let Γ be a finitely generated group and let $\varrho\colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$ be a linear representation. Recall that ϱ is said to be *rigid* if any close enough representation is conjugate to ϱ. Note that this property is sometimes called local rigidity. The representation ϱ is said to be integral if it is conjugate to a representation with values in

$$\mathrm{GL}_n(A),$$

where $A \subset \mathbb{C}$ is the ring of algebraic integers. Simpson [406] conjectured that rigid representations of Kähler groups should be integral (this is actually a particular case of a more general conjecture that we shall not state here). The case where $n = 2$ has been settled by Corlette and Simpson [118]; see Chapter 8. Very recently, Esnault and Groechenig proved this conjecture (under a slightly stronger assumption) [181]; see [182, 280, 309] for related work.

1.3.6 Open or Singular Varieties

Note that the class of fundamental groups of quasi-projective varieties is bigger than that of projective varieties. This is already apparent by looking at Riemann surfaces: free groups appear as fundamental groups of punctured Riemann surfaces, whereas they cannot appear as fundamental groups of any closed Kähler manifolds. Yet there are restrictions on fundamental groups of smooth quasi-projective varieties, although we will not deal with this topic at all here. We instead refer the reader to [10, 17, 18, 63, 77, 118, 163, 167, 273].

As for the study of fundamental groups of *singular varieties*, it is known that many restrictions that hold for smooth projective varieties also hold for normal projective varieties; see for instance, [15]. This topic is also briefly discussed in [5, p. 9].

1.3.7 The Shafarevich Conjecture

Recall that a complex manifold M is said to be *holomorphically convex* if for any sequence $(x_n)_{n\geq 0}$ of points of M that tends to infinity, there exists a holomorphic function $f\colon M \to \mathbb{C}$ such that

$$|f(x_n)| \underset{n\to+\infty}{\longrightarrow} +\infty.$$

The Shafarevich conjecture predicts that the universal cover of a smooth projective manifold (or more generally of a compact Kähler manifold) should be holomorphically convex.[6] The only general results concerning this conjecture use harmonic maps and linear representations. Indeed, the conjecture holds true for projective manifolds whose fundamental group is linear over $\mathbb{C}$; see [188] as well as [88] for the Kähler case and [184] for earlier results. We refer the reader to [185] for a survey on this question. See also [186, 189, 386] for proofs in some particular cases as well as [68, 147] for further developments. For arbitrary Kähler (or projective) manifolds, a "meromorphic" version of the conjecture was established by Campana [79] and Kollár [293].

1.3.8 Rational Homotopy Theory

In algebraic topology, Sullivan's theory of *minimal models* [426] has strong consequences concerning the homotopy type of compact Kähler manifolds, as shown by Deligne, Griffiths, Morgan, and Sullivan [135]. In particular, this theory yields restrictions on nilpotent quotients of Kähler groups. We shall not discuss this here but refer the reader instead to [9, §4] and to [5, ch.3]. See also [239, 340] in the same vein. For nonabelian aspects of this theory, see [207], as well as [190, 310, 311] for more recent developments.

1.3.9 Miscellaneous

We refer the reader to [11, 12, 13, 50, 97, 124, 220, 240, 260, 274, 275, 284, 285, 299, 303, 311, 330, 332, 353, 361, 408, 438, 460, 461] for a few more recent results around Kähler groups, harmonic maps, and linear representations. For older surveys adopting a different perspective, more centered around algebraic geometry and the Shafarevich conjecture, we refer

[6] Actually, this only appears as a question in [396].

the reader to [81, 294]. For a survey of results concerning discrete groups of isometries of complex hyperbolic spaces, see [272].

1.4 Problems

It is difficult to single out a coherent list of open problems about Kähler groups. Yet, here is a short and slightly arbitrary list of such problems.

Of course, the study of *examples* seems to be the key to better understand the world of Kähler groups. (Although we have decided not to deal with examples here!) As already mentioned in Section 1.3.3, it would be interesting to construct further examples of Gromov hyperbolic Kähler groups, or to study the properties of the fundamental groups of the negatively curved Kähler manifolds built by Stover and Toledo (see Example 1.7). Let us also mention some curious examples. Bogomolov and Tschinkel [53] as well as Schoen [387] have studied some examples of surfaces in some four-dimensional abelian varieties that are Lagrangian with respect to a holomorphic symplectic form on the abelian variety. Using Sullivan's theory of minimal models [5, ch. 3] or some more elementary algebraic topology [35], one can show that such surfaces admit non-virtually abelian 2-step nilpotent quotients. It would be interesting to study further these examples.

On the side of restrictions, it is apparently still unknown whether a cocompact lattice in the group $\mathrm{Sp}(n, 1)$ is Kähler. This is related to a conjecture of Carlson and Toledo [94, p. 178] that we will state in Chapter 10. This conjecture, together with partial results, is also discussed in [437, p. 524]. In another direction, it seems to be unknown whether there exists a single example of a Kähler group that does not admit any linear representation over $\mathbb{C}$ with infinite image. This is discussed, for instance, in [69].

The study of nilpotent Kähler groups is quite interesting. Although it was believed at first that nilpotent Kähler groups should be virtually abelian [226, p. 114], some truly 2-step nilpotent examples were built almost 30 years ago (see Example 1.6). It seems to be still unknown whether there can exist 3-step nilpotent examples (not virtually 2-step nilpotent). There are many constraints on these groups coming from (mixed) Hodge theory [5, ch. 3]. Classifying nilpotent Kähler groups is essentially the same thing as classifying solvable Kähler groups according to Delzant's theorem, stating that the latter are virtually nilpotent (see Chapter 12). It would also be interesting to establish restrictions on general *amenable* Kähler groups. We refer the reader to Definition C.18 for the notion of amenable group.

Another question concerns the study of subgroups of Kähler groups. Are there finitely generated groups that cannot be embedded in a Kähler group? What can be said about subgroups of Kähler groups? Kapovich conjectured that any finitely presented group can be embedded in a Kähler group;

see [302]. For restrictions on *normal* subgroups of Kähler groups, see [361]. For the study of some examples, see [322].

Finally, as already discussed, many theorems in this book give sufficient conditions for the existence of a fibration of a closed Kähler manifold onto a Riemann surface. This gives many restrictions on Kähler groups. But it is also interesting to try to establish *positive* results in this direction, i.e., to prove that some concrete known examples of closed Kähler manifolds admit fibrations onto Riemann surfaces. In this spirit, the following question is well known among people working on Kähler groups and is still widely open. Let $\Gamma < \mathrm{PU}(n,1)$ be a torsion-free cocompact lattice. Let $X = \mathbb{B}^n/\Gamma$ be the quotient of the unit ball of $\mathbb{C}^n$ by the action of Γ. Is it true that X has a finite cover that admits a fibration onto a Riemann surface? Note that examples of ball quotients admitting such fibrations are known to exist only when $n \leq 3$; see [150] and [272, §8]. It seems natural to investigate first this question for arithmetic lattices of the simplest type, which are known to have infinite virtual first Betti numbers [56, 276, 399]. See Section 11.4.2 in Chapter 11 for further discussion.

We have included a few more open problems in the text, and the reader will find many more in the literature.

1.5 Some Conventions

We close this introduction by introducing a few notions and definitions that will be used in several chapters of this book.

Definition 1.13 Let (X, h) be a hermitian manifold. We say that (X, h) has *bounded geometry* if there exist positive constants C_1 and C_2 such that for every point $x \in X$ there exists an open neighborhood U_x of x and a holomorphic diffeomorphism $f_x \colon U_x \to B(0,1) \subset \mathbb{C}^n$ such that

$$C_1 f_x^* h_0 \leq h \leq C_2 f_x^* h_0,$$

where h_0 is the standard hermitian metric on $\mathbb{C}^n$ and $B(0,1) \subset \mathbb{C}^n$ is the open unit ball.

Note that there exist other definitions of a hermitian (or Riemannian) manifold of bounded geometry in the literature, but we will use this one all along. This is also the definition used in [5]. Any covering space (with the induced metric) of a compact hermitian manifold has bounded geometry, and this is the type of example that we will consider here in almost all situations. Some rare exceptions will appear in Chapter 7.

Similarly, we shall make use of the following convention.

Convention 1.14 Unless otherwise stated, all the manifolds and the covering spaces that we shall deal with will be connected. If $p\colon Y \to X$ is a covering space of the manifold X, we identify the group $\pi_1(Y)$ to a subgroup of $\pi_1(X)$ via the map p_*. This means that we have implicitly chosen base points $y \in Y$ and $x \in X$ so that $p(y) = x$. In this situation, we write $\pi_1(Y) < \pi_1(X)$ without further notice.

We also say a word about *Stein factorization*. Let us consider a proper nonconstant holomorphic map $f\colon X \to S$ from a connected complex manifold to a Riemann surface. Then such a map f can always be written as a composition of holomorphic maps,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & S \\ & \Sigma \arrow[ur, "\pi"'] & \end{tikzcd}$$

where Σ is another Riemann surface, π has finite fibers, and g is surjective, is proper, and has *connected* fibers. This decomposition is called the Stein factorization of f. It is a special case of a much more general result dealing with holomorphic maps between arbitrary complex spaces; see [210, ch. 10]. We shall only use it in the special case above, which is much simpler to establish.

We are possibly being slightly demanding with the reader, expecting good knowledge of complex differential geometry, classical Hodge theory, and complex analytic geometry, as well as topology and (geometric) group theory. We have tried to be more or less coherent and to introduce notions in the order in which they are used. This principle suffers from some exceptions. For instance, harmonic maps are defined in Chapter 9, although they are alluded to plenty of times before. Similarly, Theorem 9.31 is used in Chapters 2 and 8, prior to its proof. Hopefully, the reader will survive these inconsistencies.

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