Contents

Chapter One

Introduction

1.1 FROM LINES AND PLANES TO THE ZARISKI TOPOLOGY OF $\mathbb{P}^N$

Let K be any field. Choosing K^n as our set of points, and solution sets of systems of linear equations as our preferred subsets, we get what is called n-dimensional *affine geometry* over K, though n-dimensional *linear geometry* over K might be a better name. It is frequently denoted by Aff_K^n. Following Kepler (1571–1630) and Desargues (1591–1661) we add points at infinity to get n-dimensional *projective geometry* over K; the general case appears in the works of von Staudt [vS1857], Fano [Fan1892], and Veblen [Veb1906]. We denote it by Proj_K^n.

Thus the n-dimensional affine or projective geometries over K consist of

- point sets

$$\mathrm{Points}\big(\mathrm{Aff}_K^n\big) = \{(x_1, \dots, x_n) \in K^n\}, \text{ or}$$
$$\mathrm{Points}\big(\mathrm{Proj}_K^n\big) = \{(x_0{:}\cdots{:}x_n) \in (K^{n+1} \setminus \{\mathbf{0}\})/K^\times\}, \text{ and}$$

- the linear subspaces as distinguished subsets of the point set.

By definition, the algebra of the field K determines the affine and the projective geometries. The Fundamental Theorem of Projective Geometry—which should be called the Fundamental Theorem of *Linear* Geometry—says that, conversely, the geometry of Aff_K^n or of Proj_K^n determines the algebra of the field K.

The key ideas go back to Menelaus of Alexandria (c. 70–140 AD) and Giovanni Ceva (1647–1734). The first proof is due to von Staudt [vS1857]. A gap was noticed by Klein [Kle1874], and correct versions can be found in Reye's lectures [Rey1866] (starting with the second edition). General forms are given by Russell [Rus1903], Whitehead [Whi1906], and Veblen and Young [VY1908]; see also the books by Baer [Bae52] and Artin [Art57]. We state the two versions separately, although they are really the same.

Theorem 1.1.1 (Affine form). *Let K, L be fields and $n, m \geq 2$. Let*

$$\Phi\colon\ \mathrm{Points}\big(\mathrm{Aff}_K^n\big) \leftrightarrow \mathrm{Points}\big(\mathrm{Aff}_L^m\big)$$

be a bijection that maps linear subspaces to linear subspaces. Then $n = m$, and *there is a unique field isomorphism* $\varphi : K \cong L$, *vector* $(c_1, \dots, c_m) \in L^m$ *and matrix* $M \in \mathrm{GL}_m(L)$ *such that*

$$\Phi(x_1, \dots, x_n) = \big(\varphi(x_1) + c_1, \dots, \varphi(x_n) + c_n\big) \cdot M.$$

Theorem 1.1.2 (Projective form)**.** *Let* K, L *be fields and* $n, m \geq 2$*. Let*

$$\Phi\colon\ \mathrm{Points}\big(\mathrm{Proj}_K^n\big) \leftrightarrow \mathrm{Points}\big(\mathrm{Proj}_L^m\big)$$

be a bijection that maps linear subspaces to linear subspaces. Then $n = m$*, and there is a unique field isomorphism* $\varphi : K \cong L$ *and matrix* $M \in \mathrm{PGL}_{m+1}(L)$ *such that*

$$\Phi(x_0{:}\cdots{:}x_n) = \big(\varphi(x_0){:}\cdots{:}\varphi(x_n)\big) \cdot M.$$

Remark 1.1.3. The identity is the only automorphism of $\mathbb{R}$, thus for $K = L = \mathbb{R}$ we get that $\Phi(x_0{:}\cdots{:}x_n) = (x_0{:}\cdots{:}x_n) \cdot M$ for some $M \in \mathrm{PGL}_{m+1}(\mathbb{R})$. That is, the coordinatization of $\mathbb{RP}^n$ is unique, up to linear changes of the coordinates.

By contrast, the automorphism group of $\mathbb{C}$ is huge, of cardinality $2^{|\mathbb{C}|}$.

Remark 1.1.4. With some care, one can see that (1.1.1) and (1.1.2) also apply to non-commutative fields, but from now on we consider only commutative fields.

The next natural geometry to consider is *circle geometry,* where we work with lines and circles in the plane. It was discovered by Hipparchus of Nicaea (c. 190-120 BC) that, using stereographic projection, it is better to view this as the geometry whose points are given by the sphere

$$\mathbb{S}^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\},$$

and whose subsets are the circles contained in $\mathbb{S}^2$. These are also the intersections of $\mathbb{S}^2$ with planes.

More generally, let K be any field of characteristic $\neq 2$. Let Sph_K^n denote *spherical geometry* of dimension n over K. That is, its points are

$$\mathbb{S}_K^n := \{(x_0{:}x_1{:}\cdots{:}x_{n+1}) \in \mathrm{Proj}_K^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = x_0^2\},$$

and its distinguished subsets are the intersections of $\mathbb{S}_K^n$ with linear subspaces. (These are spheres if K is a subfield of $\mathbb{R}$. However, if $K = \mathbb{C}$ then the intersection with $(x_3 = \cdots = x_{n-1} = x_n - x_0 = 0)$ is a pair of lines, so the name 'spherical' may be misleading.) The Fundamental Theorem of Spherical Geometry now says the following.

Theorem 1.1.5. *Let* K, L *be fields and* $n, m \geq 2$*. Let*

$$\Phi\colon\ \mathrm{Points}\big(\mathrm{Sph}_K^n\big) \leftrightarrow \mathrm{Points}\big(\mathrm{Sph}_L^m\big)$$

be a bijection that maps linear intersections to linear intersections. Then $n = m$ *and there is a unique field isomorphism* $\varphi : K \cong L$ *and matrix* $M \in \mathrm{PO}_{n+1,1}(L)$

such that

$$\Phi(x_0{:}\cdots{:}x_{n+1}) = \bigl(\varphi(x_0){:}\cdots{:}\varphi(x_{n+1})\bigr)\cdot M.$$

Here $\mathrm{PO}_{n+1,1}(L) \subset \mathrm{PGL}_{n+2}(L)$ is the projective othogonal group, that is, the subgroup of those matrices that leave the sphere $\mathbb{S}^n_K$ invariant.

Although this seems like a new result, it easily reduces to the linear geometry case as follows. Fix a point $p \in \mathbb{S}^n_K$. If K is a subfield of $\mathbb{R}$ then stereographic projection shows that $\mathbb{S}^n_K \setminus \{p\}$ (with the spherical subsets containing p as our subsets) is isomorphic to Aff^n_K. For arbitrary K, we get the same conclusion for $\mathbb{S}^n_K \setminus \{\text{all lines through } p \text{ in } \mathbb{S}^n_K\}$.

The next natural topic could be *conic geometry.* Here we start with sets of points $\mathrm{Points}\bigl(\mathrm{Aff}^2_K\bigr)$ or $\mathrm{Points}\bigl(\mathrm{Proj}^2_K\bigr)$, but we work with lines and conics as distinguished subsets.

However, nothing new happens, since we can tell which curves are conics and which are lines. Indeed, in conic geometry, C is a line if and only if $C \cap C'$ consists of at most two points for every other curve C' (which is a conic or a line, not containing C). Thus we recover affine geometry.

What if we fix a degree d and consider *degree-d geometry* in the plane? It has the same point set as before, but we use all algebraic curves of degree $\le d$ as distinguished subsets. That is, solution sets of the form

- $\{(x, y) \in \mathrm{Aff}^2_K\colon f(x, y) = 0\}$ where $\deg f \le d$ (affine case), or
- $\{(x{:}y{:}z) \in \mathrm{Proj}^2_K\colon F(x{:}y{:}z) = 0\}$ where $\deg F \le d$ (projective case).

As before, it is not hard to show that if $|K| \ge d + 1$, then C is a line if and only if it has at least $d + 1$ points and $C \cap C'$ consists of at most d points for every other curve C' (of degree $\le d$ that does not contain C). Thus we get the same fundamental theorems as in the linear case.

While restricting to small values of d may be natural, it is very unlikely that specific large values of d are of much interest. So we should instead let d become infinite and work with all algebraic plane curves and their K-points. This is planar algebraic geometry. We focus now on the projective case; see (2.2.15) for some comments on the affine setting. As the natural continuation of (1.1.1)–(1.1.5), the next question to consider is the following.

Question 1.1.6. Let K, L be fields and

$$\Phi\colon\ \mathrm{Points}\bigl(\mathrm{Proj}^2_K\bigr) \leftrightarrow \mathrm{Points}\bigl(\mathrm{Proj}^2_L\bigr)$$

a bijection that maps algebraic curves to algebraic curves. Is there a field isomorphism $\varphi : K \cong L$ and a matrix $M \in \mathrm{PGL}_3(L)$ such that

$$\Phi(x_0{:}x_1{:}x_2) = \bigl(\varphi(x_0){:}\varphi(x_1){:}\varphi(x_2)\bigr)\cdot M?$$

In a surprising departure from the previous results, the answer is very field-dependent. For illustration, let us see what happens with finite fields, $\mathbb{R}$ and $\mathbb{C}$. For finite fields the answer is negative for trivial reasons (though of course the cardinality of $\mathrm{Points}(\mathrm{Proj}^2_K)$ determines K).

Proposition 1.1.7. *Let K be a finite field. For every subset $S \subset \mathrm{Points}(\mathrm{Proj}^2_K)$ there is a homogeneous polynomial F_S such that*

$$S = \{(x{:}y{:}z) \in \mathrm{Proj}^2_K \colon F_S(x{:}y{:}z) = 0\}.$$

Thus every subset of $\mathrm{Points}(\mathrm{Proj}^2_K)$ is an algebraic curve, hence every bijection $\mathrm{Points}(\mathrm{Proj}^2_K) \leftrightarrow \mathrm{Points}(\mathrm{Proj}^2_K)$ maps algebraic curves to algebraic curves.

Clearly, here the problem is that for finite fields K, the K-points of a high degree curve C tell us very little about C. While identifying a line with its K-points is harmless over any field, and identifying a (nonempty) conic with its K-points works whenever $|K| > 3$, thinking of C as its K-points tends to be helpful only if there are infinitely many K-points on C.

In the real case the answer is again negative, but this is more unexpected. We use $\mathbb{RP}^2$ to denote the real projective plane with its Euclidean topology.

Theorem 1.1.8 ([KM09]). *Every diffeomorphism $\Psi : \mathbb{RP}^2 \leftrightarrow \mathbb{RP}^2$ can be approximated by diffeomorphisms $\Phi : \mathbb{RP}^2 \leftrightarrow \mathbb{RP}^2$ that map algebraic curves to algebraic curves.*

As an example, the simplest non-linear algebraic diffeomorphisms of $\mathbb{RP}^2$ are given by

$$\begin{array}{rcl} x & \mapsto & x\big((c^6-1)y^2z^2 - c^2(c^2x^2+c^4y^2+z^2)^2\big), \\ y & \mapsto & y\big((c^6-1)z^2x^2 - c^2(c^2y^2+c^4z^2+x^2)^2\big), \\ z & \mapsto & z\big((c^6-1)x^2y^2 - c^2(c^2z^2+c^4x^2+y^2)^2\big), \end{array}$$

for any $c \in \mathbb{R} \setminus \{\pm 1\}$.

For $\mathbb{C}$, and more generally for algebraically closed fields of characteristic 0, we have a positive answer.

Theorem 1.1.9. *Let K, L be algebraically closed fields of characteristic 0, and*

$$\Phi\colon \mathrm{Points}(\mathrm{Proj}^2_K) \leftrightarrow \mathrm{Points}(\mathrm{Proj}^2_L)$$

a bijection that maps algebraic curves to algebraic curves. Then there is a unique field isomorphism $\varphi : K \cong L$ and a matrix $M \in \mathrm{PGL}_3(L)$ such that

$$\Phi(x_0{:}x_1{:}x_2) = \big(\varphi(x_0){:}\varphi(x_1){:}\varphi(x_2)\big) \cdot M.$$

It is not unexpected that there could be a difference between the real and complex cases, since we can get only limited information about a real polynomial if we ignore its complex roots. Thus we should not forget about the complex points when dealing with a projective space over $\mathbb{R}$.

Note that if i is a root of a real polynomial, then so is $-i$. In general, working with real polynomials only, we can detect conjugate pairs of complex numbers, but not individual complex numbers. In order to understand how this works for other fields, we need to think about what the basic objects of algebraic geometry are.

Definition 1.1.10 (Affine n-space in algebraic geometry)**.** Let K be a field and $\overline{K} \supset K$ an algebraic closure. We denote the Zariski topological space of affine n-space by $|\mathbb{A}^n_K|$. It consists of the following.

(1) A point set, which can be given in two equivalent ways.

(a) (Geometric form) Points in $\overline{K}^n$ modulo conjugation. That is,

$$|\mathbb{A}^n_K|^{\text{set}} := \{(x_1, \dots, x_n) \in \overline{K}^n\}/(x_1 \dots, x_n) \sim (\sigma(x_1), \dots, \sigma(x_n)),$$

where $\sigma \in \operatorname{Gal}(\overline{K}/K)$ is any automorphism of $\overline{K}$ that fixes K.

(b) (Algebraic form) The set of maximal ideals of $K[x_1, \dots, x_n]$.

(2) A topology whose closed sets are the solution sets of systems of equations

$$\{(x_1 \dots, x_n) \in \overline{K}^n : f_1(x_1, \dots, x_n) = \cdots = f_r(x_1, \dots, x_n) = 0\},$$

where $f_i \in K[x_1, \dots, x_n]$ are polynomials.

The advantage of this definition is that the connection between geometry and algebra is now very tight. For example, Hilbert's Nullstelensatz implies that two polynomials $f, g \in K[x_1, \dots, x_n]$ have the same zero sets in $|\mathbb{A}^n_K|$ if and only if they have the same irreducible factors.

A disadvantage is that it is no longer clear how to distinguish K-points from $\overline{K}$-points. In fact, the arguments of [WK81] show that if K is a finite field, then the group of homeomorphisms is transitive on $|\mathbb{A}^2_K|$; see (10.3.1) for details. While this may be the only such example, we have a good solution of this problem only in the projective case.

Remark 1.1.11 (Non-closed points)**.** The above is the traditional definition of $\mathbb{A}^n_K$; see [Sha74]. In the modern scheme-theoretic version, the points of $\mathbb{A}^n_K$ correspond to all prime ideals of $K[x_1, \dots, x_n]$. Thus our $|\mathbb{A}^n_K|$ is the set of closed points of the scheme-theoretic $\mathbb{A}^n_K$; see (2.3.1) for details. For our current purposes, the distinction is not important.

Definition 1.1.12 (Projective n-space in algebraic geometry)**.** Let K be a field and $\overline{K} \supset K$ an algebraic closure. We denote the underlying Zariski topological space of projective n-space by $|\mathbb{P}^n_K|$. It consists of

(1) a point set

$$|\mathbb{P}^n_K|^{\text{set}} := \{(x_0{:}\cdots{:}x_n) \in \overline{K}^{n+1} \setminus \{\mathbf{0}\}\}/(x_0{:}\cdots{:}x_n) \sim (c\sigma(x_0){:}\cdots{:}c\sigma(x_n)),$$

where $c \in \overline{K}^\times$ and $\sigma \in \operatorname{Gal}(\overline{K}/K)$, and

(2) a topology, whose closed sets are

$$\{(x_0{:}\cdots{:}x_n) \in \overline{K}^{n+1} : F_1(x_0{:}\cdots{:}x_n) = \cdots = F_r(x_0{:}\cdots{:}x_n) = 0\},$$

where $F_i \in K[x_0, \dots, x_n]$ are homogeneous polynomials.

One of our theorems is the following answer to the higher dimensional version of (1.1.6). We prove it in (8.1.4).

Theorem 1.1.13. *Let K, L be fields. Assume that* $\operatorname{char} K = 0$ *and fix $n, m \geq 2$. Let*

$$\Phi\colon |\mathbb{P}^n_K| \leftrightarrow |\mathbb{P}^m_L|$$

be a homemorphism (that is, a bijection that maps closed, algebraic subsets to closed, algebraic subsets). Then $n = m$ and there is a unique field isomorphism $\varphi\colon K \cong L$, and a unique matrix $M \in \mathrm{PGL}_{m+1}(L)$ such that

$$\Phi(x_0{:}\cdots{:}x_n) = \big(\varphi(x_0){:}\cdots{:}\varphi(x_n)\big) \cdot M.$$

Clarification 1.1.14. We need to explain what $\varphi(x_i)$ means if x_i is not in K. Fix algebraic closures $\overline{K} \supset K$ and $\overline{L} \supset L$. Then φ extends (non-uniquely) to a field isomorphism $\overline{\varphi} : \overline{K} \cong \overline{L}$. However, if $(x_0{:}\cdots{:}x_n) \in |\mathbb{P}^n_K|$ then

$$\big(\overline{\varphi}(x_0){:}\cdots{:}\overline{\varphi}(x_n)\big) \in |\mathbb{P}^m_L|$$

is independent of the choice of $\overline{\varphi}$ and of the representative of $(x_0{:}\cdots{:}x_n)$.

1.2 THE MAIN THEOREM

The book is devoted to extending (1.1.13) from $\mathbb{P}^n$ to other algebraic varieties by proving that in most cases the topological space $|X|$ determines X. It is the culmination of several reconstruction results proved in [KLOS20, Kol20]. (The precise definition of a normal, projective, geometrically irreducible variety is given in Section 2.1.)

Main Theorem 1.2.1. Let K be a field of characteristic 0, and X_K, Y_L normal, projective, geometrically irreducible varieties over K (resp. over an arbitrary field L). Let $\Phi\colon |X_K| \to |Y_L|$ be a homeomorphism. Assume that

(1) $\dim X \geq 4$, or

(2) $\dim X \geq 3$ and K is a finitely generated field extension of $\mathbb{Q}$, or

(3) $\dim X \geq 2$ and K is uncountable.

Then there is a field isomorphism $\varphi\colon K \xrightarrow{\sim} L$ and embeddings $j_K\colon X_K \hookrightarrow \mathbb{P}^N_K$

and $j_L\colon Y_L \hookrightarrow \mathbb{P}^N_L$ for $N = 2\dim X + 1$, such that we get a commutative diagram

$$\begin{tikzcd} {|X_K|} \arrow[r, "\Phi"] \arrow[d, "j_K"'] & {|Y_L|} \arrow[d, "j_L"] \\ {|\mathbb{P}^N_K|} \arrow[r, "\Phi'"] & {|\mathbb{P}^N_L|}, \end{tikzcd}$$

where $\Phi'(x_0{:}\cdots{:}x_N) = \bigl(\varphi(x_0){:}\cdots{:}\varphi(x_N)\bigr)$.

Examples (2.2.1) to (2.2.6) show that normality and geometric irreducibility are necessary assumptions, but it is possible that $\dim X \geq 4$ can always be weakened to $\dim X \geq 2$.

Projectivity is crucial for our proof, but may not be necessary. The characteristic 0 assumption is also necessary, but there are natural conjectural versions in positive characteristic; we elaborate on these in Section 2.2.

As an illustratrion of some of the methods, we prove a special case of (1.1.13). The key is the following characterization of smooth rational curves in $\mathbb{P}^n_{\mathbb{C}}$.

Lemma 1.2.2. *Let $C \subset \mathbb{P}^m_K$ be an irreducible curve. Consider the properties:*

(1) *C is smooth and rational.*

(2) *For every point $p \in C$ there is a hypersurface H such that $C \cap H = \{p\}$.*

Then (1) $\Rightarrow$ (2) and if $K = \mathbb{C}$ then (2) $\Rightarrow$ (1).

Proof. Set $c := \deg C$. Fix d such that $H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d)) \to H^0\bigl(C, \mathcal{O}_{\mathbb{P}^m}(d)|_C\bigr)$ is surjective. If C is smooth and rational then $\mathcal{O}_C(c[p]) \cong \mathcal{O}_{\mathbb{P}^m}(1)|_C$. Thus $\mathcal{O}_{\mathbb{P}^m}(d)|_C$ has a section that vanishes only at p (with multiplicity cd), and it lifts to a section of $\mathcal{O}_{\mathbb{P}^m}(d)$ as needed.

Conversely, if (2) holds and $d := \deg H$, then $\mathcal{O}_C\bigl(cd[p]\bigr) \cong \mathcal{O}_{\mathbb{P}^m}(d)|_C$. Since the smooth points of C generate $\mathrm{Pic}(C)$, this implies that $\mathrm{Pic}(C)/\langle \mathcal{O}_{\mathbb{P}^m}(1)|_C\rangle$ is a torsion group. Equivalently, the connected component $\mathrm{Pic}^\circ(C)$ is a torsion group. Over $\mathbb{C}$ this holds only if $\mathrm{Pic}^\circ(C)$ is trivial, hence C is smooth and rational. □

This already shows why varieties over different fields behave differently. If $K = \overline{\mathbb{F}}_p$ then $\mathrm{Pic}^\circ(C)$ is a torsion group for every curve C over K. If K is any field of positive characteristic, then $\mathrm{Pic}^\circ(C)$ is a torsion group whenever C is rational with only cusps. If K is a number field then sometimes $\mathrm{Pic}^\circ(C)$ is a torsion group even if C has large genus.

Corollary 1.2.3. *Let K be a field and fix $n, m \geq 2$. Let*

$$\Phi\colon |\mathbb{P}^n_K| \leftrightarrow |\mathbb{P}^m_{\mathbb{C}}|$$

be a homemorphism. Then

(1) *Φ maps smooth rational curves to smooth rational curves, and*

(2) *if $n = 2$ then it maps lines to lines.*

Proof. Let $C \subset \mathbb{P}^n_K$ be a smooth rational curve. By (1.2.2) it satisfies (1.2.2(2)). The latter is a purely topological property, so $\Phi(C) \subset \mathbb{P}^n_{\mathbb{C}}$ also satisfies (1.2.2(2)). Thus $\Phi(C)$ is a smooth rational curve by (1.2.2).

If $n = 2$ then lines and conics are the only smooth rational curves; so we are in the case of conic geometry, discussed after (1.1.5). □

1.3 ORGANIZATION OF THE BOOK

Our approach naturally breaks into two, mostly independent, parts.

- Reconstruction of X from $|X|$ together with the additional information of the linear equivalence relation on divisors.
- Reconstruction of linear equivalence of divisors from $|X|$.

We briefly describe each of these two parts.

1.3.1 Reconstruction of X from $|X|$ and its divisorial structure

Recall that a (Weil) divisor on a variety is a $\mathbb{Z}$-linear combination of irreducible closed subsets of codimension 1. Since 'irreducible closed subset of codimension 1' is a purely topological notion (the codimension 1 irreducible closed subsets being the maximal proper ones), the group of Weil divisors on X is determined by $|X|$. However, the linear equivalence relation on the group of divisors depends, a priori, on more than just $|X|$.

The *divisorial structure* of X is the topological space $|X|$ together with the linear equivalence relation $\sim$ on the group of Weil divisors of X. Our main reconstruction result for varieties together with the divisorial structure is as follows (this is a slightly simplified version—see (4.1.14)).

Theorem 1.3.1. *Let K, L be fields and let X_K, Y_L be normal, proper, geometrically integral varieties over K (resp. L). Let $\Phi\colon |X_K| \to |Y_L|$ be a homeomorphism such that for D_1, D_2 effective divisors on X, $\Phi(D_1) \sim \Phi(D_2)$ if and only if $D_1 \sim D_2$. Assume that*

(1) *either K is infinite and* $\dim X \geq 2$,

(2) *or K is a finite field of cardinality > 2 and* $\dim X \geq 3$,

(3) *or $K \cong \mathbb{F}_2$,* $\dim X \geq 3$, *and X is Cohen-Macaulay.*

Then Φ is the composite of a field isomorphism $\varphi\colon K \to L$ and an algebraic *isomorphism of* L*-varieties* $X_L^\varphi \to Y_L$.

Here, X_L^φ refers to the base change of X_K via the isomorphism φ. See (2.1.6) for a concrete description of this construction.

1.3.2 Reconstruction of divisorial structure from $|X|$

Over fields of characteristic 0 one can often recover the linear equivalence relation on divisors from the topological space $|X|$. Our main result in this regard is the following, which is a slightly simplified version of (9.8.18).

Theorem 1.3.1. *Let k be a field of characteristic* 0 *and X a normal, projective, geometrically irreducible k-variety. Assume that*

(1) $\dim X \geq 4$*, or*

(2) $\dim X \geq 3$ *and k is a finitely generated field extension of* $\mathbb{Q}$*, or*

(3) $\dim X \geq 2$ *and k is uncountable.*

Then $|X|$ determines linear equivalence of divisors.

Remark 1.3.2. Here is a very rough idea why small or very large fields help us. Assume that f is a rational function on a variety X, and we know its zero set $Z_0 := (f = 0)$ and its polar set $Z_\infty := (f = \infty)$. Note that if $g^n = c \cdot f^m$ for some $c \in k^\times$ and $m, n \in \mathbb{N}$, then g and f have the same zero and polar sets. If X is normal, projective, and geometrically irreducible, and Z_0, Z_∞ are both irreducible, then the converse also holds. Thus we are in a better situation if there are many rational functions with irreducible zero and polar sets.

If $\dim X \geq 2$ then Bertini's theorem guarantees that almost every rational function is such. If $\dim X = 1$ and k is algebraically closed, there may not be any such functions. However, if k is a finitely generated field extension of $\mathbb{Q}$, then Hilbert's irreducibility theorem guarantees that there are many such functions.

We need to apply such considerations not to the original variety X, but in the following setting: $C \subset X$ is a curve, $Y \subset X$ is an irreducible subvariety to which the above considerations apply, and $C \cap Y$ is a single point. Except in rare instances, this can be arranged only if $\dim X > 1 + \dim Y$.

Such considerations lead to the notion of *Bertini-Hilbert dimension* of a field (which is either 1 or 2; see (9.5.5)). Then (1.3.1) holds whenever the dimension of X is greater than $1 + \mathrm{BH}(k)$.

Finally, another problem occurs when the zero and polar sets have 'unexpected' irreducible components. In algebraic geometry it is usually easy to show that 'unexpected' things can happen in only countably many ways. So, over uncountable fields, most functions do not behave in 'unexpected' ways.

In combination with (1.3.1), this yields Main Theorem (1.2.1).

1.3.3 Structure of the chapters

The book is broadly organized into two parts, corresponding to Sections 1.3.1 and 1.3.2. In the first part, consisting of Chapters 3 through 6, we prove (1.3.1) by observing that the divisorial structure lets us define linear systems of effective divisors, reconstructing the projective structure on linear systems using variants of the Fundamental Theorem of Projective Geometry, and then reconstructing rings of functions using these linear systems.

In Chapters 8 and 9, we prove (1.3.1) by first reconstructing a weaker equivalence relation for divisors purely from the topology, then using that to reconstruct various types of geometric data, and finally reconstructing the usual linear equivalence relation for divisors. Beforehand, in Chapter 7, we give a simpler argument following a similar strategy for varieties over an uncountable algebraically closed field and also collect various results about pencils that are used in that chapter and subsequent ones.

Chapter 10 includes complements, counterexamples, and conjectures: a topological Gabriel theorem, various types of schemes for which results of the type we describe here fail, and several questions and conjectures about extensions of our results to larger classes of schemes and positive characteristic.

Ancillary results are collected in appendices. These are mostly known but are included as we found it hard to find references for the precise statements that we need. The reader may wish to consult the appendices only as needed while reading the main parts of the book.

The first appendix recalls the definitions and basic properties of locally finite, Mordell-Weil, anti-Mordell-Weil, and Hilbertian fields. This appendix is included at the end of Chapter 8, where these notions are first used. In the second appendix, which appears at the end of Chapter 9, we introduce the notion of weakly Hilbertian fields, (9.9.1). This notion is new and may be of independent interest.

The appendices included in Chapter 11 contain various background material that is used in the book, but follow more standard algebraic geometry terminology. In Sections 11.1 and 11.2 we summarize properties of complete intersections and various Bertini-type theorems. The theory of the Picard group, Picard variety, and Albanese variety is recalled in Section 11.3. The literature is much less complete about the class group and its scheme version, which does not even seem to have a name. Basic results on commutative algebraic groups and the multiplicative groups of Artin algebras are also studied in Section 11.3.

Index of Notation

Index of Terminology