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Chapter One

A Comprehensive Introduction

One of the main purposes of this work is to prove *comparison principles* with respect to a constant-coefficient nonlinear potential theory, in a very straightforward manner, from *duality* and *monotonicity*. We will also show how to deduce comparison principles for nonlinear differential operators, a program which seems somewhat different from the first. However, we will marry these two points of view, for a wide variety of equations, under something we call the *correspondence principle*. This turns out to be interesting for several reasons. In potential theory, one is given a constraint set $\mathcal{F}$ on the 2-jets of a function, and the boundary of $\mathcal{F}$ gives a differential equation. There are many differential operators, suitably organized around $\mathcal{F}$, which give the same equation. So potential theory gives a great strengthening and simplification to the operator theory. Conversely, the set of operators associated to $\mathcal{F}$ can have much to say about the potential theory.

These comments are exemplified by the following two basic cases. Consider the constraint set $\mathcal{P} = \{D^2u \geq 0\}$. The potential theory associated to $\mathcal{P}$ is the full theory of convex functions. The equation given by $\partial\mathcal{P}$ is the homogeneous Monge–Ampère equation. However, it is also the equation $\lambda_1(D^2u) = 0$, where λ_1 is the first ordered eigenvalue, and there are many more examples. For the second case, consider the constraint set $\mathcal{P}_{\mathbb{C}} = \{D^2_{\mathbb{C}}u \geq 0\}$, where $D^2_{\mathbb{C}}$ is the complex Hessian. Here, the associated theory is the wide field of pluripotential theory. The operators include $\det(D^2_{\mathbb{C}}u)$, $\lambda_1(D^2_{\mathbb{C}}u)$, and many others.

One of the main objectives of the treatise is to bring together these two points of view for a large and important class of equations, where many new results have been established.

A motivation for this study comes from the following consideration (other motivations will be given below). For the Dirichlet problem (DP) on a bounded domain Ω in Euclidean space, it is proved in [49] that *existence always holds* in the constant-coefficient case[1] (assuming that Ω has a smooth C^2-boundary satisfying the appropriate strict boundary convexity conditions). This leaves uniqueness, which the *comparison principle*, that for u a subsolution and v a supersolution,

$$u \leq v \text{ on } \partial\Omega \;\;\Rightarrow\;\; u \leq v \text{ on } \Omega,$$

[1] The conclusion "existence always holds" is precisely defined in Theorem A.2.

obviously implies. Interestingly, in our constant-coefficient case, using the fact that existence always holds (see Theorem A.5) one can show that uniqueness and comparison are actually equivalent.

An object of central interest here is that of *monotonicity*. It is monotonicity that explains and unifies much of the theory. For each constraint set $\mathcal{F}$ there exists a maximal set $\mathcal{M}$ with the monotonicity property $\mathcal{F}+\mathcal{M}=\mathcal{F}$. In many interesting cases, $\mathcal{M}$ is itself a constraint set. In simpler cases, such as pure second-order equations, or gradient-free equations, monotonicity comes down to the standard degenerate ellipticity and negativity assumptions. To explain this in more detail we need some notation. (The reader should note the many examples below, starting with Example 1.7, which may illuminate the following sections.)

1.1 THE POTENTIAL THEORY SETTING

Set $\mathcal{J}^2 := \mathbb{R}\times\mathbb{R}^n\times\mathcal{S}(n)$, the space of 2-jets with standard jet coordinates (r,p,A), where $\mathcal{S}(n)$ is the space of symmetric $(n\times n)$-matrices with real entries, and consider a set $\mathcal{F}\subset\mathcal{J}^2$. Then $\mathcal{F}$ is called a *constant-coefficient subequation constraint set* (or simply *subequation*, or *constraint set*) if $\mathcal{F}$ is not $\emptyset$ or $\mathcal{S}(n)$, and

$$\mathcal{F}+\mathcal{P}_0\subset\mathcal{F},\quad \mathcal{F}+\mathcal{N}_0\subset\mathcal{F},\quad \text{and}\quad \mathcal{F}=\overline{\operatorname{Int}\mathcal{F}}, \tag{1.1}$$

where $\mathcal{P}_0 := \{0\}\times\{0\}\times\mathcal{P}$ and $\mathcal{N}_0 := \mathcal{N}\times\{0\}\times\{0\}$ in $\mathcal{J}^2=\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}(n)$, with

$$\mathcal{P} := \big\{A\in\mathcal{S}(n) : A\geq 0\big\}\quad \text{and}\quad \mathcal{N} := \big\{r\in\mathbb{R} : r\leq 0\big\}. \tag{1.2}$$

Associated to a constraint set $\mathcal{F}$ is its *dual* constraint set[2]

$$\widetilde{\mathcal{F}} := \sim\big\{-\operatorname{Int}\mathcal{F}\big\} = -\big\{\sim\operatorname{Int}\mathcal{F}\big\}. \tag{1.3}$$

Now, each constraint set $\mathcal{F}$ determines a potential theory of $\mathcal{F}$-*subharmonic functions*. A C^2-function u on an open subset $X\subset\mathbb{R}^n$ is $\mathcal{F}$-*subharmonic* on X if

$$J^2_{x_0}u := (u(x_0), Du(x_0), D^2u(x_0))\in\mathcal{F}\quad \text{for all } x_0\in X. \tag{1.4}$$

Using viscosity theory, this condition can be transferred pointwise from the 2-jet $J^2_{x_0}u$ to the set of *upper test jets* (see Definition 1.3) by requiring

$$J^2_{x_0}\varphi\in\mathcal{F}\quad \text{for all upper test functions } \varphi \text{ for } u \text{ at } x_0\in X, \tag{1.5}$$

[2] Throughout this book, $\operatorname{Int}\mathcal{F}$ is the interior of $\mathcal{F}$ and $\sim\mathcal{F}=\mathcal{J}^2\setminus\mathcal{F}$ the complement of $\mathcal{F}$ with respect to $\mathcal{J}^2$.

thereby extending the notion of $\mathcal{F}$-subharmonic from C^2-functions to the space $\mathrm{USC}(X)$ of all upper-semicontinuous, $[-\infty, \infty)$-valued functions on X.

In addition to the notion of duality (1.3), the other fundamental concept for this work is monotonicity.

Definition 1.1. A *monotonicity cone* for a subequation $\mathcal{F}$ is a cone $\mathcal{M} \subset \mathcal{J}^2$ (with vertex at the origin) such that

$$\mathcal{F} + \mathcal{M} \subset \mathcal{F}, \tag{1.6}$$

and in this case we say that *$\mathcal{F}$ is $\mathcal{M}$-monotone.*

Note that since $\mathcal{M}$ contains the origin, the inclusion (1.6) is an equality $\mathcal{F} + \mathcal{M} = \mathcal{F}$.

Since $\mathcal{F}$ is a subequation, one can always enlarge a monotonicity cone to one where

$$\mathcal{M} \supset \mathcal{N} \times \{0\} \times \mathcal{P} \quad \text{and} \quad \mathcal{M} \text{ is a closed convex cone.} \tag{1.7}$$

In fact, the closed convex cone hull of a monotonicity cone is also a monotonicity cone. For each $\mathcal{F}$ there is a *maximal* monotonicity cone. Moreover, in this work we are interested in subequations $\mathcal{F}$ which have monotonicity cones $\mathcal{M}$ which satisfy (1.7) and

$$\mathrm{Int}\, \mathcal{M} \neq \emptyset, \tag{1.8}$$

so that $\mathcal{M}$ is itself a subequation, called a *monotonicity cone subequation.* To see this, note that for a closed convex cone $\mathcal{M}$, we have

$$\mathrm{Int}\, \mathcal{M} \neq \emptyset \quad \Leftrightarrow \quad \mathcal{M} = \overline{\mathrm{Int}\, \mathcal{M}}.$$

From this assumption (1.8), which holds for many constraint sets (including all second-order, in fact, all gradient-free subequations), many important things follow:

- the correspondence principle;
- comparison theorems;
- the existence of canonical operators;
- the existence of unique solutions to the Dirichlet problem;
- and much more; see below.

1.2 THE DIFFERENTIAL OPERATOR SETTING

We now address the companion setting of differential operators. There are two cases: the unconstrained case and the constrained case.

Definition 1.2. A *compatible operator–subequation pair* $(F, \mathcal{F})$ consists of either

the unconstrained case: $\mathcal{F} = \mathcal{J}^2$ and $F \in C(\mathcal{F})$

or

the constrained case: $\mathcal{F} \subset \mathcal{J}^2$ is a subequation and $F \in C(\mathcal{F})$, with the properties

$$\inf_{\mathcal{F}} F > -\infty \quad \text{and} \quad \partial\mathcal{F} = \{J \in \mathcal{F} : F(J) = c_0\}, \tag{1.9}$$

where $c_0 := \inf_{\mathcal{F}} F \in \mathbb{R}$.

In both cases, $F(\mathcal{F})$ is called the set of *admissible levels* of the pair.

Classical examples in the constrained case are the real and complex Monge–Ampère operators, where one assumes $A \geq 0$ and (respectively) $A_{\mathbb{C}} \geq 0$; that is, one restricts to the subequations $\mathcal{P}$ and $\mathcal{P}_{\mathbb{C}}$. More generally, the constrained case is well illustrated by *Gårding–Dirichlet operators*, as discussed in Section 1.6. On the other hand, the unconstrained case is well illustrated by the so-called *canonical operators* as discussed in Section 1.4.

Let $\mathcal{M} \subset \mathcal{J}^2$ be a convex cone with vertex at the origin. We say that a compatible operator–subequation pair $(F, \mathcal{F})$ is *$\mathcal{M}$-monotone* if $\mathcal{F}$ is $\mathcal{M}$-monotone and

$$F(J + J') \geq F(J) \quad \forall\, J \in \mathcal{F},\ \forall\, J' \in \mathcal{M}. \tag{1.10}$$

If $\mathcal{M} \supset \mathcal{N} \times \{0\} \times \mathcal{P}$, then $(F, \mathcal{F})$ is called *proper elliptic*.[3] These are the only operators we consider, because of our focus on comparison.

Next, these proper elliptic operators are divided into two classes: those which are *topologically pathological*, meaning that, as a function on the 2-jet space, the operator has a level set with interior, and those which are referred to as *topologically tame*. The topologically pathological case is discarded here because uniqueness of solutions (and hence comparison) is trivially impossible. Therefore,

all proper elliptic operators in this book will be assumed to be tame.

Various equivalent formulations of topological tameness appear in Theorem 11.10. For this topologically tame case, we will establish a rigorous *correspondence principle* between potential theory and PDEs.

[3]Often in the viscosity literature (such as [30]), one uses the simpler term *proper* for the $\mathcal{N} \times \{0\} \times \mathcal{P}$-monotonicity, but we prefer the phrase *proper elliptic* to recall both the $\mathcal{P}$-monotonicity (degenerate ellipticity or positivity) and $\mathcal{N}$-monotonicity (properness or negativity) for us.

1.3 THE CORRESPONDENCE PRINCIPLE

This result builds a bridge between nonlinear potential theory (subharmonics for a subequation and its dual) and nonlinear PDEs (admissible viscosity sub/supersolutions of PDEs); in particular, it represents the part of the theory using monotonicity and duality for which the two approaches are equivalent.

The main tool will come from *viscosity theory*, which was developed by Crandall, Ishii, Lions, Jensen, Evans, and others (see [30] and the references therein). For what we have to say in this work, one should note that it was Jensen [69] who made the initial breakthrough on comparison principles for viscosity solutions of fully nonlinear PDEs. Additional remarks on the development of viscosity solution techniques will be given in Section 1.9.

One of the fundamental definitions from viscosity theory is the following.

Definition 1.3. Let $x_0 \in X \subset \mathbb{R}^n$ (an open subset) and $u \in \mathrm{USC}(X)$. An *upper test function* for u at x_0 is a C^2-function φ defined near x_0 with

$$u(x) \leq \varphi(x) \quad \text{and} \quad u(x_0) = \varphi(x_0).$$

A *lower test function* for u at x_0 is a C^2-function φ such that $-\varphi$ is an upper test function for $-u$ at x_0. We will denote by $J^{2,\pm}_{x_0} u \subset \mathcal{J}^2$ the spaces of (*upper/lower*) *test jets for u at x_0*, that is, the set of all $J = J^2_{x_0}\varphi$ where φ is a C^2 (upper/lower) test function for u at x_0.

For compatible pairs $(F, \mathcal{F})$ which admit a monotonicity cone subequation, there is a potential theory at each admissible level $c \in F(\mathcal{F})$.

Definition 1.4. Let $(F, \mathcal{F})$ be a compatible operator–subequation pair, which admits a monotonicity subequation $\mathcal{M}$, and let $c \in F(\mathcal{F})$ be an admissible level. Consider the subequation $\mathcal{F}_c \equiv \{J \in \mathcal{F} : F(J) \geq c\}$. Let $u \in \mathrm{USC}(X)$, where $X \subset \mathbb{R}^n$ is an open subset. Then u is said to be $\mathcal{F}_c$-*subharmonic* on X if

$$J^{2,+}_{x_0} u \subset \mathcal{F}_c \quad \text{for all } x_0 \in X. \tag{1.11}$$

Let $\mathrm{LSC}(X)$ denote the space of lower-semicontinuous $(-\infty, +\infty]$-valued functions on X. A function $v \in \mathrm{LSC}(X)$ is said to be $\mathcal{F}_c$-*superharmonic* on X if $-v$ is $\widetilde{\mathcal{F}}_c$-subharmonic on X. By duality, this is equivalent to asking that $J^{2,-}_{x_0} v \subset {\sim}\,\mathrm{Int}\,\mathcal{F}_c$ for each $x_0 \in X$.

We now present an essential notion for the constrained case, that is, of admissible viscosity subsolutions and supersolutions where the subequation $\mathcal{F}$ places a constraint on the upper/lower test jets that compete in the definition. In particular, part (b) in the definition below makes systematic what is often done in an ad hoc way in the literature.

Definition 1.5. Let $(F, \mathcal{F})$ be a compatible operator–subequation pair as above. Let Ω be a domain in $\mathbb{R}^n$ and let $c \in F(\mathcal{F})$ be an admissible level.

(a) A function $u \in \mathrm{USC}(\Omega)$ is said to be an *$\mathcal{F}$-admissible viscosity subsolution of $F(u, Du, D^2u) = c$ in Ω* if for every $x_0 \in \Omega$ one has

$$J \in J^{2,+}_{x_0} u \;\Rightarrow\; J \in \mathcal{F} \text{ and } F(J) \geq c. \tag{1.12}$$

(b) A function $w \in \mathrm{LSC}(\Omega)$ is said to be an *$\mathcal{F}$-admissible viscosity supersolution of $F(u, Du, D^2u) = c$ in Ω* if

$$J \in J^{2,-}_{x_0} w \;\Rightarrow\; \text{either } [J \in \mathcal{F} \text{ and } F(J) \leq c] \text{ or } J \notin \mathcal{F}. \tag{1.13}$$

The following theorem is a main result of this monograph.

Theorem 11.13 (Correspondence principle for compatible pairs). *Let $(F, \mathcal{F})$ be a compatible proper elliptic operator–subequation pair, which is $\mathcal{M}$-monotone for a convex cone subequation $\mathcal{M}$. Suppose also that F is topologically tame. Let $c \in F(\mathcal{F})$ be an admissible value, and set $\mathcal{F}_c \equiv \{J \in \mathcal{F} : F(J) \geq c\}$ as above. Fix a domain $\Omega \subset \mathbb{R}^n$. Then,*

(a) $u \in \mathrm{USC}(\Omega)$ *is an $\mathcal{F}$-admissible viscosity subsolution of $F(u, Du, D^2u) = c$ in Ω if and only if u is $\mathcal{F}_c$-subharmonic on Ω;*
(b) $u \in \mathrm{LSC}(\Omega)$ *is an $\mathcal{F}$-admissible viscosity supersolution of $F(u, Du, D^2u) = c$ in Ω if and only if u is $\mathcal{F}_c$-superharmonic on Ω;*
(c) *comparison for the subequation $\mathcal{F}_c$ on a domain Ω is valid if and only if comparison for the equation $F(u, Du, D^2u) = c$ on Ω is valid.*

The correspondence principle is a very general and powerful tool, which needs to be "unpacked" in order to fully appreciate it. First, there is an important *dichotomy* between the *unconstrained case* $(F, \mathcal{J}^2)$ in which the operator F is proper elliptic on all of $\mathcal{J}^2$ and the *constrained case* $(F, \mathcal{F})$ where F is proper elliptic only when restricted to some compatible subequation $\mathcal{F}$. Note that in the constrained case, the constraint set $\mathcal{F}$ on the domain of the operator F is used in Definition 1.5 of *$\mathcal{F}$-admissible sub/supersolutions*, while in the unconstrained case, sub/supersolutions are in the standard viscosity sense.

Second, using this principle, one can reduce PDE comparison to potential-theoretic comparison, in order to free the operator from its particular form, retaining only the need to analyze its maximal monotonicity cone $\mathcal{M}$. This is done in Chapter 12 for many classes of operators.

A final (important) remark concerning the correspondence principle is in order.

Remark 1.6 (Correspondence and the Dirichlet problem). One important (and immediate) consequence of the correspondence principle is that there are two equivalent formulations of the Dirichlet problem; namely, given a bounded domain Ω in $\mathbb{R}^n$ and given $\varphi \in C(\partial\Omega)$, for each admissible value $c \in F(\mathcal{F})$ find $h \in C(\overline{\Omega})$ such that

$$h \text{ is } \mathcal{F}_c\text{-harmonic on } \Omega \quad \text{and} \quad h = \varphi \text{ on } \partial\Omega \tag{DP}$$

or

$$h \text{ is an } \mathcal{F}\text{-admissible solution of } F(J^2 h) = c \text{ on } \Omega \quad \text{and} \quad h = \varphi \text{ on } \partial\Omega. \tag{DP$'$}$$

The *potential-theoretic formulation* (DP) is equivalent to the *operator-theoretic formulation* (DP$'$) since the notion of "solution" in both formulations is that the corresponding (and equivalent) conditions (a) and (b) in Theorem 11.13 hold. For a more extensive discussion on this equivalence in a more general setting, see [65, Section 1.3].

In both formulations, when the comparison principle holds, the existence of a (unique) solution on Ω for each fixed φ can be obtained by *Perron's method* (see Theorem A.2). In general, the domain will need to be *boundary pseudoconvex* in a suitable strict sense. A new feature of the potential-theoretic approach begun in [46] and extended in [49] is that the required *boundary geometry* is determined by the subequation $\mathcal{F}$ which defines the potential theory. Each subequation $\mathcal{F}$ determines an *asymptotic interior* $\overrightarrow{\mathcal{F}}$, which is a cone. A domain is said to be *$\mathcal{F}$-pseudoconvex* if $\partial\Omega$ admits a local defining function which is C^2-strictly $\overrightarrow{\mathcal{F}}$-subharmonic. This is a local question. This boundary geometry can be characterized in terms of the second fundamental form. These local defining functions serve as the needed barriers in Perron's method and existence holds provided that Ω is both $\mathcal{F}$-pseudoconvex and $\widetilde{\mathcal{F}}$-pseudoconvex. In many situations, there is no boundary geometry; that is, there are no geometric restrictions to impose on the boundary in order to have existence. For example, this happens in the *uniformly elliptic* case. On the other hand, when there is boundary geometry, the computations are interesting (see [50, Section 7]). Many subequations (and hence many operators) have the same boundary geometry.

Finally, when the correspondence principle holds, this boundary geometry analysis carries over to all operators F which are compatible with $\mathcal{F}$. This has been successfully exploited for variable coefficient proper elliptic operators in [23, 24].

1.4 CANONICAL OPERATORS

This collection of operators gives some of the best illustrations of the unconstrained case. The construction starts with a subequation $\mathcal{F}$ which admits a monotonicity subequation $\mathcal{M}$. One then chooses an element $J_0 \in \operatorname{Int} \mathcal{M}$. Associated to this is a *canonical operator* $F \in C(\mathcal{J}^2)$, defined on all of $\mathcal{J}^2$, with very nice properties. It is canonically defined via the structure theorem

(Theorem 11.14), which says that for each $J \in \mathcal{J}^2$, the set

$$I_J := \{t \in \mathbb{R} : J + tJ_0 \in \mathcal{F}\} \tag{1.14}$$

is a closed interval of the form $[t_J, +\infty)$ with $t_J \in \mathbb{R}$ (finite). Moreover,

(a) $J + tJ_0 \notin \mathcal{F} \;\Leftrightarrow\; t < t_J$;
(b) $J + t_J J_0 \in \partial\mathcal{F}$;
(c) $J + tJ_0 \in \operatorname{Int}\mathcal{F} \;\Leftrightarrow\; t > t_J$.

The canonical operator $\mathcal{F}\colon \mathcal{J}^2 \to \mathbb{R}$ is then defined by

$$F(J) = -t_J \tag{1.15}$$

and it has the following properties: F decomposes $\mathcal{J}^2$ into three disjoint pieces

$$\partial\mathcal{F} = \{F(J) = 0\}, \quad \operatorname{Int}\mathcal{F} = \{F(J) > 0\}, \quad \text{and} \quad \mathcal{J}^2 \setminus \mathcal{F} = \{F(J) < 0\}, \tag{1.16}$$

and F is strictly increasing in the direction J_0. In fact,

$$F(J + tJ_0) = F(J) + t \quad \forall\, J \in \mathcal{J}^2,\ \forall\, t \in \mathbb{R}. \tag{1.17}$$

Furthermore, F is proper elliptic on $\mathcal{J}^2$ and, in fact, it is $\mathcal{M}$-monotone. It is also Lipschitz (see Propositions 11.17, 11.19, and 11.25).

Interestingly, there exist important cases where this construction is quite useful, that is, because there exist cases where there are probably no polynomial operators. Examples come from the geometrical potential theories for the special Lagrangian, G(2), and Spin(7). A different general construction of a natural operator to any subequation, namely the signed distance operator, is examined in Remark 1.10 below.

Concerning canonical operators, we have the following two results.

Theorem 11.20 (Canonical operators and compatible pairs). *Suppose that a subequation $\mathcal{F}$ admits a monotonicity cone subequation $\mathcal{M}$. Let $F \in C(\mathcal{J}^2)$ be the canonical operator for $\mathcal{F}$ determined by any fixed $J_0 \in \operatorname{Int}\mathcal{M}$. Then,*

(a) *$(F, \mathcal{J}^2)$ is an unconstrained proper elliptic operator–subequation pair;*
(b) *$F(\mathcal{J}^2) = \mathbb{R}$ and the operator F is topologically tame;*
(c) *for each $c \in \mathbb{R}$, the set $\mathcal{F}_c := \{J \in \mathcal{J}^2 : F(J) \geq c\}$ is a subequation constraint set with $\mathcal{F}_0 = \mathcal{F}$ and the pair $(F, \mathcal{F}_c)$ satisfies the compatibility conditions*

$$\inf_{\mathcal{F}_c} F = c \quad \text{and} \quad \partial\mathcal{F}_c = \{J \in \mathcal{F}_c : F(J) = c\}.$$

In addition, the canonical operator (determined by $J_0 \in \operatorname{Int} \mathcal{M}$) for the dual subequation $\widetilde{\mathcal{F}}$ is given by

$$\widetilde{F}(J) := -F(-J) \quad \text{for all } J \in \mathcal{J}^2.$$

Also note that

$$\widetilde{\mathcal{F}_c} = \widetilde{\mathcal{F}}_{-c}.$$

Statements analogous to (a), (b), *and* (c) *hold for* $(\widetilde{F}, \mathcal{J}^2)$ *and* $(\widetilde{F}, \widetilde{\mathcal{F}_c})$.

Theorem 11.21 (Comparison for canonical operators). *Let $\mathcal{F} \subset \mathcal{J}^2$ be a subequation constraint set which admits a monotonicity cone subequation $\mathcal{M}$. Further, suppose that $\mathcal{M}$ admits a strict approximator ψ on a bounded domain Ω, that is, $\psi \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that $J^2_x \psi \in \operatorname{Int} \mathcal{M}$ for each $x \in \Omega$. Then, for each $J_0 \in \operatorname{Int} \mathcal{M}$ fixed, the canonical operator F for $\mathcal{F}$ determined by J_0 satisfies the comparison principle at every level $c \in \mathbb{R}$, that is,*

$$u \leq w \text{ on } \partial\Omega \;\Rightarrow\; u \leq w \text{ on } \Omega$$

for $u \in \mathrm{USC}(\overline{\Omega})$ and $w \in \mathrm{LSC}(\overline{\Omega})$, which are respectively viscosity subsolutions and supersolutions to $F(u, Du, D^2u) = c$ on Ω.

This gives rise to many beautiful operator–subequation pairs, starting simply with just the subequation $\mathcal{F}$ itself.

Example 1.7 (Minimal eigenvalue operator). For the simplest example of a canonical operator, let $\mathcal{F} = \mathbb{R} \times \mathbb{R}^n \times \mathcal{P}$ (the real convexity subequation), and take $J_0 = (0, 0, \frac{1}{n} I)$. Then

$$F(J) = F(r, p, A) = \lambda_1(A) \quad \text{(the smallest eigenvalue of } A\text{)}.$$

Of course, there are many other operators which are compatible with $\mathcal{F}$ and are zero on $\partial\mathcal{F}$, such as $\det(A)$ or $\det(A)^{\frac{1}{n}}$. However, for any such operator we know from Theorem 11.21 above that *comparison always holds.*

It is interesting to note that all linear operators are canonical (see Lemma 12.18). In addition, the concave operator F, which is the infimum over a suitably renormalized *pointed family* $\mathfrak{F} = \{F_\sigma\}_{\sigma \in \Sigma}$ of linear operators, is also the canonical operator for the convex cone subequation $\mathcal{F}$ which is the intersection of the associated half-space constraint sets $\{\mathcal{F}_\sigma\}_{\sigma \in \Sigma}$ (see Theorem 12.21). Similar considerations hold for the canonical supremum operator associated to the closure of the union of the $\mathcal{F}_\sigma$ (see Remark 12.31). The precise notion of being pointed is given in Definition 12.20 and is a geometrical hypothesis (see Remark 12.22) on the set of coefficient vectors

$$S = \left\{ J_\sigma = (a_\sigma, b_\sigma, E_\sigma) \right\}_{\sigma \in \Sigma} \subset \mathcal{J}^2$$

defining the operators in the family by

$$F_\sigma(J) = F_\sigma(r, p, A) := \operatorname{tr}(E_\sigma A) + \langle b_\sigma, p\rangle + a_\sigma r = \langle J_\sigma, J\rangle, \quad J \in \mathcal{J}^2. \tag{1.18}$$

In the proper elliptic case, where each $(a_\sigma, E_\sigma) \in \mathcal{N} \times \mathcal{P}$, one also has the validity comparison principle (see Theorem 12.26) which depends on the interesting fact that a necessary and sufficient condition for the canonical operator for $\mathcal{F}$ to be $\mathcal{M}$-monotone is that S is contained in the convex polar cone $\mathcal{M}^\circ$ of $\mathcal{M}$. To facilitate the application of Theorem 12.26, the polars of many monotonicity cones are listed in Proposition 12.27. The following example application comes from optimal control and is discussed in Example 12.29.

Example 1.8 (Hamilton–Jacobi–Bellman operators with directed drift). In optimal control, one problem concerns an agent who seeks to minimize an infinite-horizon discounted cost functional by acting on its drift and volatility parameters. The relevant operator to consider is the infimum over a family of linear operators like (1.18), where we will specialize to

$$F_\sigma(J) = F_\sigma(r, p, A) = \operatorname{tr}(E_\sigma A) + \langle b_\sigma, p\rangle + cr = \langle J_\sigma, J\rangle, \quad \sigma \in \Sigma, \tag{1.19}$$

where $\delta := -c > 0$ is the *discount factor*, b_σ is the *drift term*, and E_σ is the (*squared*) *volatility*. Under the assumptions that E_σ is allowed to vary in bounded sets and the set of drifts $S_d := \{b_\sigma\}_{\sigma\in\Sigma}$ share a "preferred" direction b_0 (the family is pointed with axis $b_0 \in \mathbb{R}^n \setminus \{0\}$), Theorem 12.26 shows that the comparison principle holds on arbitrary bounded domains for the equation $F(u, Du, D^2u) = c$ for each $c \in \mathbb{R}$.

We now consider an important example of an unconstrained operator that is *not* a canonical operator. This particular equation has received much attention in recent years from quite varied points of view. There is some history in [62].

Example 1.9 (Special Lagrangian potential operator). This pure second-order operator was introduced along with special Lagrangian geometry in [44]. It takes the form

$$F(A) := \sum_{k=1}^{n} \arctan(\lambda_k(A))$$

and is $\mathcal{P}$-monotone on all of $\mathcal{J}^2$. Comparison on arbitrary bounded domains holds for the equation $F(D^2u) = c$ at all admissible levels $c \in (-n\pi/2, n\pi/2)$, as shown in [46], which was a major motivation for that work. The constant c represents a (constant) *phase* (as described in [62]) and the problem is also interesting for nonconstant phases. The comparison principle in the nonconstant phase case has been completely settled in [13, 24].

We end this section with another class of operators naturally associated to any subequation.

Remark 1.10. An additional construction of a natural operator associated to any subequation constraint set $\mathcal{F} \subset \mathcal{J}^2$ is suggested by Krylov [77, Theorem 3.2] (in the pure second-order case). We like to call it the (*signed*) *distance operator*, which is defined by

$$\widehat{F}(J) := \begin{cases} \operatorname{dist}(J, \partial\mathcal{F}), & J \in \mathcal{F}, \\ -\operatorname{dist}(J, \partial\mathcal{F}), & J \in \mathcal{J}^2 \setminus \mathcal{F}. \end{cases} \tag{1.20}$$

The upper level sets of $\widehat{F}$ play a key role in the method for proving comparison in [49].

The signed distance operator $\widehat{F}$ is clearly Lipschitz and topologically tame (as is the canonical operator F) but is also well defined for any subequation $\mathcal{F}$ (while the canonical operator construction requires the $\mathcal{M}$-monotonicity of $\mathcal{F}$). On the other hand, when $\mathcal{F}$ is $\mathcal{M}$-monotone, the signed distance operator is (*linearly*) *tame* in the sense that it satisfies an inequality similar to (but weaker than) identity (1.17) satisfied by the canonical operator F. The tameness condition on an operator plays an important role in treating comparison for inhomogeneous equations, as will be discussed in Section 1.8. Given the importance of tameness, we record that fact in the following lemma. For clarity and simplicity, we restrict attention to a constant-coefficient pure second-order subequation $\mathcal{F} \subset \mathcal{S}(n)$.

Lemma 1.11 (Tameness of the signed distance operator). *Let $\mathcal{F} \subset \mathcal{S}(n)$ be a constant-coefficient pure second-order subequation. Then the signed distance operator*

$$\widehat{F}(A) := \begin{cases} \operatorname{dist}(A, \partial\mathcal{F}), & A \in \mathcal{F}, \\ -\operatorname{dist}(A, \partial\mathcal{F}), & A \in \mathcal{S}(n) \setminus \mathcal{F}, \end{cases} \tag{1.21}$$

satisfies

$$\widehat{F}(A + tI) - \widehat{F}(A) \geq t \quad \forall\, A \in \mathcal{S}(n),\ \forall\, t \in \mathbb{R}, \tag{1.22}$$

or equivalently,

$$\widehat{F}(A + P) - \widehat{F}(A) \geq t \quad \forall\, A \in \mathcal{S}(n),\ \forall\, t \in \mathbb{R},\ \forall\, P \geq tI. \tag{1.23}$$

Proof. Clearly, (1.23) implies (1.22) by using $P = tI \geq tI$. The reverse implication follows by writing $P = tI + P'$ with $P' \geq 0$ and using the degenerate ellipticity of $\widehat{F}$.

Notice that (1.22) is trivial for $t = 0$ and it suffices to prove (1.22) for $A \in \mathcal{F}$ and $t > 0$ by exploiting duality where the signed distance operator $\widehat{\widehat{F}}$ of the dual subequation $\widetilde{\mathcal{F}}$ satisfies

$$\widehat{\widetilde{F}}(A) = -\widehat{F}(-A) \quad \forall\, A \in \mathcal{S}(n), \tag{1.24}$$

and both vanish on their common boundary.

Condition (1.22) for $A \in \mathcal{F}$ and $t > 0$ can be restated in terms of metric balls in $\mathcal{S}(n)$ as

$$\text{for all } r > 0, \quad \text{if } B_r(A) \subset \mathcal{F} \text{ then } B_{r+t}(A + tI) \subset \mathcal{F}. \tag{1.25}$$

Now $B_{r+t}(A + tI) = B_{r+t}(tI) + A = B_r(tI) + A + B_t(tI) = B_r(A + tI) + B_t(tI)$. Hence, if $B_r(A) \subset \mathcal{F}$ then

$$B_{r+t}(A + tI) = (B_r(A) + tI) + B_t(tI) \in \mathcal{F} + \mathcal{P} \subset \mathcal{F},$$

which gives (1.25) by the positivity of $\mathcal{F}$. □

Notice that inequality (1.22) for the signed distance operator $\widehat{F}$ is weaker than identity (1.17), $F(A + tI) = F(A) + t$, satisfied by the canonical operator.

1.5 GRADIENT-FREE OPERATORS

Given a subequation $\mathcal{F}$, our results apply if the maximal monotonicity cone of $\mathcal{F}$ has interior. However, notice that *this is true for every pure second-order subequation* $\mathcal{F} = \mathbb{R} \times \mathbb{R}^n \times \mathcal{F}_0$ since $\mathbb{R} \times \mathbb{R}^n \times \mathcal{P}$ is always a monotonicity subequation for $\mathcal{F}$. In fact, *this is true for every pure gradient-free subequation* $\mathcal{F}$, since $\mathcal{N} \times \mathbb{R}^n \times \mathcal{P}$ is always a monotonicity subequation for $\mathcal{F}$ by Definition 10.2 of *gradient-free*. We have the following result.

Theorem 12.2 (Comparison in the gradient-free case). *Suppose that* $(F, \mathcal{F})$ *is a compatible, gradient-free pair. Then for every bounded domain* Ω *and every* $c \in F(\mathcal{F})$*, one has the comparison principle*

$$u \le w \text{ on } \partial\Omega \;\Rightarrow\; u \le w \text{ on } \Omega$$

for $u \in \mathrm{USC}(\overline{\Omega})$ *and* $w \in \mathrm{LSC}(\overline{\Omega})$*, where* u *is* $\mathcal{F}_c$*-subharmonic and* w *is* $\mathcal{F}_c$*-superharmonic (that is,* $-w$ *is* $\widetilde{\mathcal{F}}_c$*-subharmonic).*

We have seen that the unconstrained case is best illustrated by canonical operators. The constrained case is best illustrated by operators involving Gårding hyperbolic polynomials, which we examine in the next section.

1.6 OPERATORS INVOLVING GÅRDING–DIRICHLET POLYNOMIALS

Gårding's theory [40] provides a unified approach to studying many of the most important subequations. The reader should look at Section 11.6 for more details and at [48, 51] for a modern self-contained treatment. An important study of fully nonlinear PDEs exploiting Gårding's theory is contained in the paper of Caffarelli–Nirenberg–Spruck [15]. A *Gårding–Dirichlet polynomial* is a homogeneous polynomial $\mathfrak{g}$ of degree m on $\mathcal{S}(n)$ with the following properties:

(1) (I-hyperbolicity). For each $A \in \mathcal{S}(n)$, the polynomial $p_A(t) \equiv \mathfrak{g}(tI + A)$ has all real roots. The negatives of these real roots are called the *Gårding I-eigenvalues* of A and up to permutation can be written in increasing order as $\lambda_1^{\mathfrak{g}}(A) \le \lambda_2^{\mathfrak{g}}(A) \le \cdots \le \lambda_m^{\mathfrak{g}}(A)$.

(2) (Positivity). We assume $\mathfrak{g}(I) > 0$ and define the *Gårding cone* Γ to be the connected component of $\mathcal{S}(n) \setminus \{\mathfrak{g} = 0\}$ which contains the identity I. This is a convex cone (see Theorem 11.30), given by those A with $\lambda_1^{\mathfrak{g}}(A) > 0$. We assume the positivity property

$$\overline{\Gamma} + \mathcal{P} \subset \overline{\Gamma}, \tag{1.26}$$

which is equivalent to either $\overline{\Gamma} + \mathcal{P} = \overline{\Gamma}$ or $\mathcal{P} \subset \overline{\Gamma}$, since $\mathcal{P}$ contains the origin and $\overline{\Gamma}$ is a convex cone.

We normalize so that $\mathfrak{g}(I) = 1$. Then we have

$$\mathfrak{g}(tI + A) = \prod_{k=1}^{m} (t + \lambda_k^{\mathfrak{g}}(A)), \tag{1.27}$$

which when evaluated at $t = 0$ shows that each Gårding–Dirichlet operator is a *generalized Monge–Ampère operator*, where the Gårding I-eigenvalues of A take the place of the standard eigenvalues of A in the special case $\mathfrak{g} = \det$.

If $\mathfrak{g}(A)$ is a Gårding–Dirichlet polynomial on $\mathcal{S}(n)$ with closed Gårding cone $\overline{\Gamma}$, then this gives rise to a pure second-order polynomial operator $\mathfrak{g}(D^2 u)$ constrained to the pure second-order subequation $\mathbb{R} \times \mathbb{R}^n \times \overline{\Gamma}$. Such operators are the subject of Section 11.6.

Simple examples of Gårding–Dirichlet polynomials $\mathfrak{g}(A)$ are given by the kth elementary symmetric function (with $k = 1, \ldots, n$) of the I-eigenvalues of A (the so-called *Hessian equations*) as discussed in Examples 11.33(2). The case $k = n$ corresponds to the Monge–Ampère operator and the case $k = 1$ corresponds to the Laplacian. There are many more interesting examples, including the Lagrangian Monge–Ampère operator (see Examples 11.33(4)) and the *geometric k-convexity operator* (see Example 1.13 below). Moreover, each of these *universal examples* (defined in terms of the standard eigenvalues $\lambda_k(A)$) gives rise to a huge family of examples by simply replacing the standard eigenvalues $\lambda_k(A)$ by the Gårding I-eigenvalues $\lambda_k^{\mathfrak{g}}(A)$ of A for any Gårding I-hyperbolic polynomial $\mathfrak{g}$ on $\mathcal{S}(n)$. More precisely, we recall the following result. For the proof, see [51, Proposition 7.7].

Proposition 1.12 (Universally defined Gårding–Dirichlet operators). *Suppose that* $\mathfrak{p} = \mathfrak{p}(\lambda_1, \ldots, \lambda_m)$ *is a symmetric homogeneous polynomial of degree N in m variables with nonnegative coefficients, which is e-hyperbolic for* $e := (1, \ldots, 1) \in \mathbb{R}^m$ *and satisfies* $\mathfrak{p}(e) > 0$. *Denote its Gårding cone by* $E \subset \mathbb{R}^m$ *and its Gårding e-eigenvalues by* $\Lambda_1^{\mathfrak{p}}(\lambda), \ldots, \Lambda_N^{\mathfrak{p}}(\lambda)$. *Then $\mathfrak{p}$ induces a Gårding–Dirichlet operator F on $\mathcal{S}(n)$ of degree N for each Gårding–Dirichlet operator $\mathfrak{g}$ on $\mathcal{S}(n)$ of degree m defined by*

$$F(A) := \mathfrak{p}(\lambda_1^{\mathfrak{g}}(A), \ldots, \lambda_m^{\mathfrak{g}}(A)), \tag{1.28}$$

with Gårding cone $\Gamma_F = (\lambda^{\mathfrak{g}})^{-1}(E)$. *The eigenvalues of F are*

$$\Lambda_k^F(A) = \Lambda_k^{\mathfrak{p}}(\lambda_1^{\mathfrak{g}}(A), \ldots, \lambda_m^{\mathfrak{g}}(A)) = \Lambda_k^{\mathfrak{p}}(\lambda^{\mathfrak{g}}(A)), \quad k = 1, \ldots, N.$$

The reader is encouraged to consult [51] for more details.

Such polynomials $\mathfrak{p} = \mathfrak{p}(\lambda)$ are referred to as the "universal eigenvalue operators," and the cone $\overline{E}$ is called a "universal eigenvalue subequation." One can show that the coefficients of $\mathfrak{p}$ being nonnegative is equivalent to E being $\mathbb{R}^m_{>0}$-monotone.

Iterating the construction in Proposition 1.12, together with taking products (since products of Gårding polynomials are again Gårding), allows one to show that the family of Gårding–Dirichlet operators is huge.

There are many interesting equations which involve Gårding–Dirichlet polynomials $\mathfrak{g}(A)$. We now look at some of the examples.

Example 1.13 (k-plurisubharmonicity, the truncated Laplacian, and the geometric k-convexity operator). These examples were introduced in [46, subsection of Section 10] as part of the *geometric p-plurisubharmonic Dirichlet problem*. Here they illustrate the general fact that, given a Gårding polynomial, there are two natural operators: the Gårding operator defined directly by $\mathfrak{g}$ and the canonical operator for the Gårding cone $\overline{\Gamma}$ determined by $\mathfrak{g}$. We discuss an interpolation of operators between them. First, we define the potential theory, which is quite interesting. Fix an integer k, $1 \le k \le n$. A *k-plurisubharmonic function* is defined by requiring that its restriction to every affine k-plane is classically Laplacian subharmonic (or $\equiv -\infty$). The subequation $\mathcal{P}(k)$ is defined by requiring that $A \in \mathcal{S}(n)$ restricts, as a quadratic form, to have a positive trace on all affine k-planes. The k-plurisubharmonic functions are exactly the $\mathcal{P}(k)$-subharmonics.

The canonical operator is the *truncated Laplacian*

$$\Delta_{\mathcal{P}(k)}(A) \equiv \lambda_1(A) + \cdots + \lambda_k(A), \tag{1.29}$$

where $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ are the ordered eigenvalues of A. There is also a polynomial Gårding–Dirichlet operator

$$T_k(A) = \prod_{i_1 < \cdots < i_k} (\lambda_{i_1}(A) + \cdots + \lambda_{i_k}(A)), \tag{1.30}$$

which we call the *geometric k-convexity operator*. This yields two compatible operator–subequation pairs using the canonical operator and a Gårding–Dirichlet operator, namely

$$(\Delta_{\mathcal{P}(k)}, \mathcal{P}(k)) \quad \text{and} \quad (T_k, \mathcal{P}(k)),$$

and yields a new interpolated sequence between the pairs $(\lambda_1, \mathcal{P})$ and $(\det, \mathcal{P})$ at the $k = 1$ end, and the identical pairs $(\Delta, \{\mathrm{tr} \geq 0\})$ and $(\Delta, \{\mathrm{tr} \geq 0\})$ at the $k = n$ end. The canonical operator has been studied in [7] where the terminology *truncated Laplacian* was introduced.

We point out that $\mathcal{P}(k)$-subharmonic functions restrict to be subharmonic on all k-dimensional minimal submanifolds [53].

Example 1.14. The reader might enjoy the article [58] where one has a full-blown Lagrangian plurisubharmonic potential theory, complete with an operator of "Monge–Ampère type" in Lagrangian geometry.

Example 1.15 (Branches of a Gårding–Dirichlet operator). In Section 11.7 we discuss the general notion of branches. A *branch* is a closed subset of $\mathcal{J}^2$ which is the boundary of a subequation. Given a Gårding–Dirichlet polynomial $\mathfrak{g}$ of degree m, there are m distinct branches

$$\Lambda_1^{\mathfrak{g}} \subset \Lambda_2^{\mathfrak{g}} \subset \cdots \subset \Lambda_m^{\mathfrak{g}}, \quad \text{where } \Lambda_k^{\mathfrak{g}} = \{\lambda_k^{\mathfrak{g}} \geq 0\}.$$

Our theory applies to all of these branches, because they are pure second order. (The only natural operator for these branches is the canonical operator $\lambda_k^{\mathfrak{g}}$ unless $k = 1$.)

Example 1.16 (Gradient-free operators with a Gårding–Dirichlet factor). Let $\mathfrak{g}(A)$ be a Gårding–Dirichlet polynomial as above, and let $h \in C((-\infty, 0])$ be nonnegative, nonincreasing, and with $h(r) = 0 \;\Leftrightarrow\; r = 0$. Consider the operator

$$F(r, p, A) = h(r)\mathfrak{g}(A). \tag{1.31}$$

Restricting F to the subequation $\mathcal{F} = \mathcal{N} \times \mathbb{R}^n \times \overline{\Gamma}$ gives a compatible gradient-free pair, and hence comparison holds at every admissible level on every bounded domain.

An interesting special case comes from affine hyperbolic geometry, as presented in Example 12.6.

Example 1.17 (The hyperbolic affine sphere equation). The partial differential equation

$$\det(D^2u) = \left(\frac{L}{u}\right)^{n+2}, \quad L \le 0, \quad \text{that is, } (-r)^{n+2}\det(A) = (-L)^{n+2}, \tag{1.32}$$

arises in the study of *hyperbolic affine spheres* with mean curvature L, where $u < 0$ is convex and vanishes on the boundary of $\Omega \subset \mathbb{R}^n$ convex (see Cheng–Yau [22]). This equation is covered by the example above if one takes $\mathfrak{g}(A) = \det(A)$, and $h(r) = (-r)^{n+2}$ and $c = (-L)^{n+2} \ge 0$ are the admissible levels.

The next example illustrates a new construction in Section 11.6 (see Lemma 11.35) which produces a gradient-free Gårding–Dirichlet operator from a pure second-order Gårding–Dirichlet operator.

Example 1.18. For each Gårding–Dirichlet polynomial $\mathfrak{g}$ of degree m on $\mathcal{S}(n)$ with Gårding I-eigenvalues of A given by $\lambda_k^{\mathfrak{g}}(A)$, $k = 1, \dots, m$, the operator

$$\mathfrak{h}(r, A) = \prod_{k=1}^{m} (\lambda_k^{\mathfrak{g}}(A) - r) = \mathfrak{g}(A - rI) \tag{1.33}$$

is a $(-\frac{1}{2}, \frac{1}{2}I)$-hyperbolic Gårding–Dirichlet polynomial of degree m on $\mathbb{R} \times \mathcal{S}(n)$ (normalized to have $\mathfrak{h}(-\frac{1}{2}, \frac{1}{2}I) = 1$) with Gårding eigenvalues $\lambda_k^{\mathfrak{h}}(A) = \lambda_k^{\mathfrak{g}}(A) - r$.

Now we consider an example with gradient dependence which requires an additional *directionality property* (D) with respect to a *directional cone* $\mathcal{D}$ (see Definition 2.2).

Example 1.19 (Example 1.16 with a directional cone). Let $\mathfrak{g}$ and h be as in Example 1.16 above, and consider a continuous $d\colon \mathcal{D} \to \mathbb{R}$, where $\mathcal{D} \subsetneq \mathbb{R}^n$ is a directional cone, with

$$d \ge 0 \text{ and } d(p) = 0 \;\Leftrightarrow\; p \in \partial\mathcal{D}, \tag{1.34}$$

$$d(p+q) \ge d(p) \quad \text{for each } p, q \in \mathcal{D}. \tag{1.35}$$

Then the operator

$$F(r, p, A) = h(r)d(p)\mathfrak{g}(A) \tag{1.36}$$

with restricted domain

$$\mathcal{F} = \mathcal{N} \times \mathcal{D} \times \overline{\Gamma} \tag{1.37}$$

defines a compatible $\mathcal{N} \times \mathcal{D} \times \mathcal{P}$-monotone operator–subequation pair $(F, \mathcal{F})$. Hence, comparison holds on arbitrary bounded domains at every admissible level of F.

Note: Examples of such pairs $(d, \mathcal{D})$ include

$$d(p) = p_n \text{ and } \mathcal{D} = \left\{(p', p_n) \in \mathbb{R}^n : p_n \ge 0\right\} \quad \text{(a half-space)}, \tag{1.38}$$

and, for $k \in \{1, \ldots, n\}$,

$$d(p) = \prod_{j=1}^{k} p_j \quad \text{and} \quad \mathcal{D} = \big\{(p_1, \ldots, p_n) \in \mathbb{R}^n : p_j \geq 0 \text{ for each } j = 1, \ldots, k\big\}. \quad (1.39)$$

An interesting special case of Example 1.19 concerns parabolic operators, which are discussed in Section 12.6 in both the constrained (Theorem 12.38) and the unconstrained cases (Theorem 12.37).

Example 1.20 (Parabolic operators). In the case where the gradient pair $(d, \mathcal{D})$ is defined by (1.38), $h \equiv 1$, and $\mathfrak{g}(A)$ is replaced by $G(A')$ which depends only on $A' \in \mathcal{S}(n-1)$ (second-order derivatives only in the *spatial variables* $x' \in \mathbb{R}^{n-1}$), one has a fully nonlinear *parabolic* operator

$$F(r, p, A) := p_n G(r, A') \quad (1.40)$$

of the kind considered by Krylov in his extension of Alexandroff's methods to parabolic equations in [75]. The compatible subequation is described in formula (12.144) of this book.

Another interesting special case of Example 1.19 comes from a very particular form of optimal transport with quadratic cost, as presented in Example 12.34.

Example 1.21 (Potential equation for optimal transport with uniform source density and directed target density). Equations of the form

$$d(Du)\det(D^2 u) = c, \qquad c \geq 0 \quad (1.41)$$

arise in the theory of optimal transport, under some restrictive assumptions. In general, there would be a function $f = f(x)$ in place of the constant c, where f represents the mass density in the source configuration and d represents the mass density of the target configuration (with the mass balance $\|f\|_{L^1} = \|d\|_{L^1}$). One seeks to transport the mass with density f onto the mass with density d at minimal transportation cost (which is quadratic with respect to transport distance). The solution of this minimization problem is given by the gradient of a convex function u, which turns out to be a generalized solution of equation (1.41). In the special case of uniform source density $f \equiv c$ and with target density d having some directionality, comparison principles can be obtained as a special case of Example 1.19 with $h(r) := 1$ and $\mathfrak{g}(A) := \det A$.

Thus we see that seemingly diverse equations can be established from a surprisingly *unified point of view*. It frees the theory from any particular form of the operator. Given a potential theory, that is, given a subequation constraint set $\mathcal{F}$, there are many natural choices for an associated operator. If $\mathcal{F}$ has *sufficient monotonicity*, that is, if $\mathcal{F}$ admits a monotonicity cone subequation $\mathcal{M}$ (that is, the maximal monotonicity cone has interior), there is always one choice that is "canonical," but for proving useful estimates, other choices may be better. For instance, a polynomial operator, if there is one, may be preferable. Restricting attention, as we do here, to the continuous version of the Dirichlet problem (DP), the correspondence principle enables a single potential theory/subequation result to be applied to all of the many compatible operators F associated to the subequation $\mathcal{F}$.

1.7 GENERAL POTENTIAL-THEORETIC COMPARISON THEOREMS

One of the important parts of this monograph is understanding convex cone subequations $\mathcal{M} \subset \mathcal{J}^2$ and the comparison results for subequations $\mathcal{F}$ which are $\mathcal{M}$-monotone.

By comparison results, we mean the validity of the comparison principle on bounded domains $\Omega \subset \mathbb{R}^n$, that is,

$$u \leq w \text{ on } \partial\Omega \ \Rightarrow \ u \leq w \text{ on } \Omega \tag{1.42}$$

for all $u \in \text{USC}(\overline{\Omega})$ and $w \in \text{LSC}(\overline{\Omega})$, which are respectively $\mathcal{F}$-subharmonic and $\mathcal{F}$-superharmonic on Ω. By duality, this is equivalent to showing

$$u + v \leq 0 \text{ on } \partial\Omega \ \Rightarrow \ u + v \leq 0 \text{ on } \Omega \tag{1.43}$$

for all $u, v \in \text{USC}(\overline{\Omega})$, which are respectively $\mathcal{F}$-subharmonic and $\widetilde{\mathcal{F}}$-subharmonic on Ω. Our method of proof for $\mathcal{M}$-monotone subequations $\mathcal{F}$ makes use of this second formulation.

Here is a guide to the method. There are four steps.

Step 1: Jet addition. We have the following elementary but important fact concerning constraint sets, monotonicity, and duality:

$$\mathcal{F} + \mathcal{M} \subset \mathcal{F} \ \Leftrightarrow \ \mathcal{F} + \widetilde{\mathcal{F}} \subset \widetilde{\mathcal{M}}.$$

So the monotonicity condition on the left is equivalent to the condition on the right, which is perfect for comparison, as one sees from (1.43).

Showing that this infinitesimal statement passes to a potential-theoretic statement is the hard analysis step in the method.

Step 2: Subharmonic addition. We prove the following potential-theoretic result.

Theorem 7.4 (Subharmonic addition, monotonicity, and duality). *Suppose that $\mathcal{M} \subset \mathcal{J}^2$ is a monotonicity cone subequation and that $\mathcal{F} \subset \mathcal{J}^2$ is an $\mathcal{M}$-monotone* *subequation constraint set. Then for every open set* $X \subset \mathbb{R}^n$*, one has*

$$\mathcal{F}(X) + \widetilde{\mathcal{F}}(X) \subset \widetilde{\mathcal{M}}(X)$$

(where $\mathcal{F}(X)$ *is the set of* $u \in \mathrm{USC}(X)$ *which are* $\mathcal{F}$*-subharmonic on* X*).*

Step 3: Reduce comparison to the *zero maximum principle* (ZMP) for $\widetilde{\mathcal{M}}$. Armed with Theorem 7.4, it is clear from (1.43) that comparison for $\mathcal{F}$ on Ω will hold if we can prove the (ZMP) for $\widetilde{\mathcal{M}}$ on a bounded domain $\Omega \subset \mathbb{R}^n$, that is,

$$z \le 0 \text{ on } \partial\Omega \;\Rightarrow\; z \le 0 \text{ on } \Omega \tag{1.44}$$

for all $z \in \mathrm{USC}(\overline{\Omega})$ which are $\widetilde{\mathcal{M}}$-subharmonic on Ω.

Step 4: Prove the (ZMP) for $\widetilde{\mathcal{M}}$. A key concept in the proof is the following.

Definition 1.22. Suppose that $\mathcal{M}$ is a convex cone subequation. Given a domain $\Omega \subset\subset \mathbb{R}^n$, we say that $\mathcal{M}$ *admits a strict approximator on* Ω if there exists ψ with

$$\psi \in C(\overline{\Omega}) \cap C^2(\Omega) \quad \text{and} \quad J^2_x\psi \in \mathrm{Int}\,\mathcal{M} \text{ for each } x \in \Omega. \tag{1.45}$$

This important notion gives a sufficient condition for proving the (ZMP) for $\widetilde{\mathcal{M}}$ and hence comparison for $\mathcal{F}$.

Theorem 6.2 (Zero maximum principle). *Suppose that* $\mathcal{M}$ *is a convex cone subequation that admits a strict approximator on* Ω*. Then the* (ZMP) *holds for* $\widetilde{\mathcal{M}}$ *on* $\overline{\Omega}$*.*

Putting these four steps together gives the following theorem.

Theorem 7.5 (General comparison theorem). *Suppose that* $\mathcal{M} \subset \mathcal{J}^2$ *is a monotonicity cone subequation and that* $\mathcal{F} \subset \mathcal{J}^2$ *is an* $\mathcal{M}$*-monotone subequation constraint set. Suppose that* $\mathcal{M}$ *is a convex cone subequation that admits a strict approximator on* Ω*. Then comparison holds for* $\mathcal{F}$ *on* $\overline{\Omega}$*. That is, given* $u, v \in \mathrm{USC}(\overline{\Omega})$*, where* u *is* $\mathcal{F}$*-subharmonic on* Ω *and* v *is* $\widetilde{\mathcal{F}}$*-subharmonic on* Ω*, then*

$$u + v \le 0 \;\; on \;\; \partial\Omega \;\Rightarrow\; u + v \le 0 \;\; on \;\; \Omega.$$

The conclusion here can be restated as follows. Given $u \in \mathrm{USC}(\overline{\Omega})$ *and* $w \in \mathrm{LSC}(\overline{\Omega})$*, where* u *is* $\mathcal{F}$*-subharmonic on* Ω *and* w *is* $\mathcal{F}$*-superharmonic on* Ω*, then*

$$u \le w \;\; on \;\; \partial\Omega \;\Rightarrow\; u \le w \;\; on \;\; \Omega.$$

To see this last statement we only need to know that w is $\mathcal{F}$-superharmonic on Ω if and only if $v \equiv -w$ is $\widetilde{\mathcal{F}}$-subharmonic on Ω.

Remark 1.23 (Affine jet equivalence). Theorem 7.5 generalizes from constant-coefficient subequations $\mathcal{F}$ to subequations $\mathcal{F}$ which are *locally affinely jet equivalent* to a constant-coefficient subequation. For any such equation, *local and hence global weak comparison holds.* Furthermore, if there exists a strict approximator (a classical strict $\mathcal{M}$-subharmonic) on Ω, then *comparison holds* on Ω. The reader should see [49] for all the details.

We point out that this concept is very useful in geometry. Any invariant polynomial equation, such as the Monge–Ampère equation or the elementary symmetric function equations, on a Riemannian manifold are always locally jet equivalent to constant-coefficient equations in local coordinates. A similar statement holds on almost complex (and therefore complex) manifolds. The reader should consult [54, Proposition 4.5].

Now, the utility of the general comparison theorem (Theorem 7.5) is greatly enhanced by a detailed study of monotonicity cone subequations, which we present in Chapter 5. There is a three-parameter *fundamental family* of monotonicity cone subequations $\mathcal{M}(\gamma, \mathcal{D}, R)$, where $\mathcal{D} \subset \mathbb{R}^n$ is a directional cone, $\gamma \in [0, \infty)$, and $R \in (0, \infty]$. In the fundamental family theorem (Theorem 5.10), it is shown that

> every monotonicity cone subequation $\mathcal{M}$ contains one of these subequations $\mathcal{M}(\gamma, \mathcal{D}, R)$.

We note that

$$\mathcal{M}(\gamma) := \left\{(r, p, A) \in \mathcal{J}^2 : r \le -\gamma|p|\right\}, \quad \mathcal{M}(R) := \left\{(r, p, A) \in \mathcal{J}^2 : A \ge \tfrac{|p|}{R} I\right\},$$

and

$$\mathcal{M}(\gamma, \mathcal{D}, R) := \mathcal{M}(\gamma) \cap \mathcal{M}(\mathcal{D}) \cap \mathcal{M}(R),$$

where $\mathcal{M}(\mathcal{D}) := \mathbb{R} \times \mathcal{D} \times \mathcal{S}(n)$.

The fundamental nature of this family of monotonicity cones, together with the general comparison principle of Theorem 7.5, leads to a main comparison result Theorem 7.6 (the fundamental family comparison theorem), which depends on the cone $\mathcal{M}(\gamma, \mathcal{D}, R)$. For some of these cones, comparison holds on all bounded domains. For the others, comparison holds only on domains $\Omega \subset \mathbb{R}^n$ which are subsets of a translation of the truncated cone $\mathcal{D} \cap B_R(0)$. This is a semilocal comparison with explicit parameters. Note that by the fundamental families theorem (Theorem 5.10), local comparison always holds (see Theorem 7.8).

Concerning the applicability of the fundamental comparison result of Theorem 7.6, it is worth mentioning that larger monotonicity cones $\mathcal{M}$ for a given subequation $\mathcal{F}$ give a better chance of proving comparison (one more likely to be able to construct a strict approximator) but smaller monotonicity cones $\mathcal{M}$ will apply to larger families of subequations. In particular, if one would like to know whether comparison holds on arbitrary bounded domains, one should search for the largest possible $\mathcal{M}$, which is perhaps not in the list of the fundamental family. For example, in Theorems 8.3 and 8.5 we present enlargements of the cones with R finite for which comparison holds on all bounded domains.

On the other hand, the search for sufficient monotonicity to have comparison on arbitrary bounded domains may be futile. In particular, for $\mathcal{F} := \mathcal{M}(R)$, which is its own maximal monotonicity cone, it is shown in Proposition 6.5 that the (ZMP) fails for $\widetilde{\mathcal{M}}(R)$ on large balls, and hence comparison also fails for $\mathcal{F} = \mathcal{M}(R)$ on large balls. This failure of comparison on large balls is extended to interesting subequations $\mathcal{F}$ with maximal monotonicity cone equal to $\mathcal{M}(R)$ in Proposition 9.2. The situation can be even worse.

Remark 1.24 (Failure of local comparison with insufficient monotonicity). In Theorem 9.8 we show that comparison can fail on arbitrarily small balls (even if both (P) and (N) hold) if there is insufficient monotonicity. In the examples the maximal monotonicity cone $\mathcal{M}_{\mathcal{F}}$ has empty interior, hence no strict approximators on any ball, no matter how small. Moreover, if $\mathcal{M}_{\mathcal{F}}$ has empty interior, then its dual is not a subequation.

Concerning step 3 of our method (in which comparison reduces to the validity of the (ZMP) for the dual $\widetilde{\mathcal{M}}$ of the monotonicity cone), the following observation is of interest.

Remark 1.25 (Strong comparison from the strong (ZMP)). The monotonicity and duality method can be used to prove a strong comparison principle which, by the subharmonic addition theorem, reduces to proving a strong (ZMP) for $\widetilde{\mathcal{M}}$ on $\overline{\Omega}$; that is,

$$z \leq 0 \text{ on } \partial\Omega \;\Rightarrow\; z \equiv 0 \text{ or } z < 0 \text{ on } \Omega \tag{1.46}$$

for all $z \in \mathrm{USC}(\overline{\Omega})$ which are $\widetilde{\mathcal{M}}$-subharmonic on Ω. This method was used in [55] to prove strong comparison for pure second-order subequations. We will not attempt to extend this to the general constant-coefficient case in this book. There is, of course, a rich literature on the strong maximum principle for nonlinear operators, including the important work of Bardi–Da Lio initiated in [4], along with the recent papers by Birindelli–Galise–Ishii [7], Vitolo [90], and Goffi–Pediconi [42].

A few additional potential-theoretic ingredients are worth mentioning. First, an elaboration on the potential theory underlying Example 1.8.

Remark 1.26 (Canonical operators, duality, intersections, and unions). For families $\{\mathcal{F}_\sigma\}_{\sigma\in\Sigma}$ of subequations with a common monotonicity cone subequation $\mathcal{M}$, by using unions, intersections, and duality, four interesting $\mathcal{M}$-monotone subequations are constructed, together with their canonical operators (see Theorem 11.23).

Next, an elaboration on the gradient-free case.

Remark 1.27 (Subaffine-plus functions). Subaffine-plus theory concerns the potential theory of the gradient-free subequation

$$\widetilde{\mathcal{Q}} := \{(r, A) \in \mathbb{R} \times \mathcal{S}(n) : r \leq 0 \text{ or } A \in \mathcal{P}\}, \tag{1.47}$$

where $\widetilde{\mathcal{P}}$ is the pure second-order *subaffine* subequation. This $\widetilde{\mathcal{Q}}$ is the dual of the fundamental gradient-free monotonicity cone $\mathcal{M} = \mathcal{Q} := \mathcal{N} \times \mathcal{P}$ and this potential theory is developed in detail (see Theorems 10.7 and 10.8). In particular, we extend the elegant method of using subaffine functions (the $\widetilde{\mathcal{P}}$-subharmonics) to prove that "comparison always holds" for pure second-order subequations. Subaffine-plus functions (the $\widetilde{\mathcal{Q}}$-subharmonics) are used to prove that "comparison always holds" for the larger family of gradient-free subequations.

1.8 LIMITATIONS OF THE METHOD AND COMPARISON WITH THE LITERATURE

The monotonicity and duality method presented here applies to a vast array of constant-coefficient potential theories and operators, but not all of them. There are many interesting and important examples with insufficient monotonicity to be treated by our method. For example, quasi-linear operators such as the *minimal surface operator*

$$F(p, A) := \operatorname{tr}(A) - \frac{\langle Ap, p\rangle}{1 + |p|^2},$$

the *q-Laplacian* (with $1 < q < 2$ or $2 < q < \infty$)

$$F(p, A) := |p|^{q-2} \operatorname{tr}(A) + (q-2)|p|^{q-4}\langle Ap, p\rangle,$$

and the *infinite Laplacian*

$$F(p, A) := \langle Ap, p\rangle,$$

do not have monotonicity cones $\mathcal{M}$ with interior, which we require. Such examples (and others) have been treated by Barles–Busca [6] by using an ingenious transformation of the dependent variable, which still cries out for a potential-theoretic analogue. On the other hand, for these reduced operators F (no explicit dependence on the jet variable $r \in \mathbb{R}$), Barles and Busca require structural assumptions on F such as their condition (F2): *F is strictly elliptic in the gradient direction.* This condition is *not* satisfied by an operator such as

$$F(p, A) := d(p) \det A,$$

which is Example 1.16 with $h(r) \equiv 1$ and $\mathfrak{g}(A) = \det A$. Such examples can be treated by our method.

Next we discuss a prototype operator which, surprisingly, creates difficulty for any method. The operator looks particularly attractive for comparison since it is strictly decreasing in the solution variable $r \in \mathbb{R}$ and is increasing in the Hessian variable A when restricted to $\mathcal{P} \subset \mathcal{S}(n)$. Namely, consider the seemingly innocuous operator

$$F(r, A) = \det A - r \tag{1.48}$$

(which is further discussed in Remark 12.7). The operator F is gradient-free and proper elliptic on $\mathbb{R} \times \mathcal{P}$; that is, it is $\mathcal{Q} = \mathcal{N} \times \mathcal{P}$-monotone on $\mathcal{F} := \mathbb{R} \times \mathcal{P}$. However, the potential theory equation $\partial\mathcal{F}$ is not contained in the zero locus $\{(r, A) \in \mathcal{F} : F(r, A) = 0\}$, that is, F and $\mathcal{F}$ are not compatible. This cannot be remedied by another choice of $\mathcal{F}$, creating a major obstacle to the study of this operator. This incompatibility means that $\mathcal{F}$-superharmonics will not correspond to $\mathcal{F}$-admissible supersolutions to the equation $F(u, Du, D^2u) = 0$. In order to formulate a notion of admissible supersolution, one could make use of the *generalized equation* approach initiated in [61] for pure second-order equations in which one looks for a second constraint set $\mathcal{G}$ (different from $\mathcal{F}$) such that

$$\mathcal{F} \cap (-\widetilde{\mathcal{G}}) = \{(r, A) \in \mathcal{F} : F(r, A) = 0\}.$$

The admissible supersolutions are those $w \in \mathrm{LSC}(\Omega)$ which are $-\widetilde{\mathcal{G}}$-subharmonic. We will not pursue this program here.

In addition to the paper [6] discussed above, earlier pioneering work in the constant-coefficient case was done by Jensen [69]. The equations treated by him are all unconstrained in our language, where the monotonicity properties (P) and (N) do *not* require restricting the domain F to a constraint set $\mathcal{F}$. In Sections 12.2 and 12.3, we recover Jensen's results in this unconstrained setting (see Remark 12.10). Of course, we also treat many constrained cases in this monograph, which is an important motivation for us.

Concerning the constrained case and our notion of compatible pairs $(F, \mathcal{F})$, we should mention that the special case of Monge–Ampère-type equations with the convexity constraint $\mathcal{P}$ is given in Ishii–Lions [68], together with a notion of admissible supersolutions in our language. A similar admissibility notion was also given by Trudinger [87] for prescribed curvature equations, and later by Trudinger–Wang for the so-called Hessian equations in a series of papers beginning with [88]. As noted previously, another motivation of ours is to treat constrained cases in a robust and general way. The potential-theoretic approach initiated in [46] was influenced by the important paper of Krylov [77] on the general notion of ellipticity, who championed the idea of freeing a given differential operator F from its particular form by looking instead at the constraint that is imposed on the 2-jets of subsolutions to the equation.

Finally, we wish to comment on our choice to focus on the constant-coefficient case. The most basic reason is that in this situation, monotonicity and duality alone suffice to produce comparison for compatible pairs $(F, \mathcal{F})$. Much more can be said when dependence on spatial coordinates is added into the pair, or one works on manifolds, but additional conditions must be imposed in order to prove the comparison principle. We briefly review three situations in which the monotonicity–duality method has been extended past the constant-coefficient setting.

On open sets X in $\mathbb{R}^n$, one can add x-dependence in two ways. The first is by considering *inhomogeneous equations* associated to a constant-coefficient compatible pair $(F, \mathcal{F})$, that is, equations of the form

$$F(J^2 u) = \psi(x), \quad x \in X. \tag{1.49}$$

Here, one should consider $\psi \in C(X)$ taking values in the range $F(\mathcal{F})$ of the operator $F \in C(\mathcal{F})$.

As shown in [60], if $(F, \mathcal{F})$ is a compatible $\mathcal{M}$-monotone pair for some monotonicity cone subequation $\mathcal{M}$ which admits a classical strict subharmonic function, then the *tameness* condition of F on $\mathcal{F}$,

for each $s, \lambda > 0$ *there exists* $c(s,\lambda) > 0$ *such that*

$$F(J + (-r, 0, P)) - F(J) \geq c(s,\lambda) \quad \forall\, J \in \mathcal{F},\ \forall\, r \geq s,\ \forall\, P \geq \lambda I, \tag{1.50}$$

ensures that the comparison principle holds for $\mathcal{F}$-admissible subsolutions and supersolutions of (1.49). The tameness property is crucial to maintain the $\mathcal{F}$-subharmonicity of sup-convolutions of $\mathcal{F}$-subharmonic functions.

This comparison result is part of the content of [60, Theorem 2.7], which also treats the existence of a unique solution to the Dirichlet problem on bounded domains Ω which are boundary pseudoconvex in a suitable strict sense. See also [60, Theorem 2.7′] for the extension to operators F which are *tamable*. Moreover, [60, Theorem 2.7] extends to manifolds X for pairs $(F, \mathcal{F})$ which are *locally jet-equivalent* to a constant-coefficient pair as above (see [60, Theorem 2.11]).

On open sets X in $\mathbb{R}^n$, a more general way to add in x-dependence is to consider general operators $F \in C(\mathcal{G})$ where either $\mathcal{G} = J^2(X)$ or $\mathcal{G} \subsetneq J^2(X)$ is a subequation constraint set. The candidate for a compatible subequation $\mathcal{F} \subset J^2(X)$ for F is defined fiberwise by the *correspondence relation*

$$\mathcal{F}_x := \left\{ J \in \mathcal{G}_x : F(x, J) \geq 0 \right\}, \quad x \in X. \tag{1.51}$$

In this setting, *fiberegularity* is crucial. This condition means that the fiber map Θ, with values $\Theta(x) := \mathcal{F}_x$ as defined in (1.51), is continuous from $X \subset \mathbb{R}^n$ into the closed subsets of $\mathcal{J}^2$ (equipped with the Hausdorff distance). For $\mathcal{M}$-monotone subequations, fiberegularity has a more useful equivalent formulation:

there exists $J_0 \in \operatorname{Int} \mathcal{M}$ such that, for each fixed $\Omega \subset\subset X$ and $\eta > 0$, there exists $\delta = \delta(\eta, \Omega)$ such that

$$x, y \in \Omega, \ |x - y| < \delta \ \Rightarrow \ \Theta(x) + \eta J_0 \subset \Theta(y). \tag{1.52}$$

One can show that if there exists such a jet J_0, then any element of $\operatorname{Int} \mathcal{M}$ will do, where one can simply take $J_0 = (-1, I) \in \mathbb{R} \times \mathcal{S}(n)$ or $J_0 = I \in \mathcal{S}(n)$ in the gradient-free and pure second-order cases, respectively.

Fiberegularity ensures that if u is $\mathcal{F}$-subharmonic on a bounded domain, then there are small C^2-strictly $\mathcal{F}$-subharmonic perturbations of all small translates of u which belong to $\mathcal{F}(\Omega_\delta)$, where $\Omega_\delta := \{x \in \Omega : d(x, \partial\Omega) > \delta\}$. This property is crucial to maintain $\mathcal{F}$-subharmonicity of sup-convolution approximations of $\mathcal{F}$-subharmonics. Hence, in the general setting, fiberegularity plays the same role as tameness does for inhomogeneous equations (as noted after (1.50)).

Fiberegularity, together with monotonicity and duality, has been shown to be sufficient for comparison in the pure second-order, gradient-free, and $\mathcal{M}$-monotone cases in [23–25]. See also the recent paper of Brustad [12]. In particular, this leads to the proof of the comparison principle [24] for the special Lagrangian potential equation with nonconstant phases (as introduced in Example 1.9) provided that the phase function does not take on a *special phase value*. This result is sharp, as shown in Brustad [13]. As noted in the preface, this potential-theoretic approach has been used to prove the comparison principle for PDEs that do not satisfy a standard structural condition from conventional viscosity theory. The following is a simple pure second-order example from [23]. Replacing det with a different *Gårding–Dirichlet polynomial* $\mathfrak{g}$ (as defined in Section 1.4) yields a huge family of such pure second-order examples that can be further generalized by taking $F(r, p, A) = h(r, p)\mathfrak{g}(A)$ for suitable $h \in C(\mathbb{R} \times \mathbb{R}^n)$.

Example 1.28 (Perturbed Monge–Ampère). With fixed $M \in C(\Omega, \mathcal{S}(n))$ and $f \in C(\Omega)$ nonnegative, consider

$$\det(D^2 u + M(x)) = f(x), \quad x \in \Omega \subset\subset \mathbb{R}^n. \tag{1.53}$$

This is an important test example of Krylov [76, Example 8.2.4] for probabilistic and analytic methods. It fails to satisfy the standard viscosity structural conditions for comparison as given in Crandall–Ishii–Lions [30, condition (3.14)] unless M is the square of a Lipschitz continuous matrix-valued function. In [23], comparison is proved for general continuous M (along with the existence of a unique continuous solution of the Dirichlet problem on strictly convex domains). The potential-theoretic proof makes use of the compatible subequation whose fibers are defined by

$$\mathcal{F}_x := \left\{ A \in \mathcal{S}(n) : A + M(x) \geq 0 \text{ and } F(x, A) := \det(A + M(x)) - f(x) \geq 0 \right\},$$

and give a fiberegular $\mathcal{F}$. The strict convexity is the boundary geometry required by $\mathcal{F}$ as described after Remark 1.6.

Finally, constant-coefficient subequations on Euclidean space generate a rich and interesting class of subequations on manifolds X, as developed in [49]. These subequations on X are those which are *locally jet-equivalent to a constant-coefficient subequation.* Any Riemannian G-subequation on a Riemannian manifold X with topological G-structure is such a subequation. For simple examples, let $\mathfrak{p}\colon \mathcal{S}(n) \to \mathbb{R}$ be a continuous function which is invariant under the action of $\mathrm{O}(n)$ (such as the determinant or the trace). Applying $\mathfrak{p}$ to the Riemannian Hessian gives an operator (real Monge–Ampère or Laplace–Beltrami) on X, which has the jet-equivalence property above. These notions are discussed in [49, pp. 398–402], along with much more.

1.9 REFLECTIONS ON "POTENTIAL THEORY VERSUS OPERATOR THEORY"

The work of Harvey–Lawson on generalized potential theories began with the realization that every calibrated manifold had an underlying potential theory, which generalized much of the basic pluripotential theory associated to the special case of Kähler manifolds. It soon became clear (but surprising) that the set of p-planes distinguished by the calibration, which gives rise to the pluripotential theory, could be replaced by any closed subset of the p-Grassmannian, yielding an interesting geometric potential theory. Over time it was found that these potential theories have far-reaching generalizations, which give new dimensions to interesting areas of geometry. The third and final step was to bypass the subset of the Grassmannian and go directly to a constraint set on the 2-jets of a function. This investigation has led to a constellation of potential theories as discussed in the survey paper [65, Section 1.1].

Each such potential theory has a basic geometry which gives rise to a large number of differential operators that can be used in the analysis. In fact, in important situations (for example, in G(2) and Spin(7) geometries) there are no historical "polynomial" operators, but one does have the canonical operators and the signed distance operators, which are determined by the geometry of the constraint set. A good example of this is the introduction [46] of the truncated Laplacian for the geometric p-convexity constraint set. This lack of known polynomial operators was discussed in Section 1.4.

On the other hand, if one starts with a nonlinear proper elliptic operator F, one has an associated potential theory coming from the set $\mathcal{F}$ where $F \geq 0$. This potential theory is useful because it gives a different way of thinking; many of the *results* and *techniques* of classical potential theory carry over to these cases, and have led to new results.

One example is the analogue of the Levi problem in several complex variables, established in the new p-geometry, where all p-planes are considered distinguished

[52]. Namely, local p-convexity implies global p-convexity. See the review in [65, Section 1.2] for additional potential-theoretic results suggested by several complex variable theory.

A really important example is given by the sharp L^∞-estimates for the complex Monge–Ampère equation, for which deep results in pluripotential theory were used. The first major advance after Yau's fundamental paper [91] was given by Kolodziej [73]. His work had profound significance and was used for many future developments, and it relied heavily on pluripotential theory! So also did future generalizations of Kolodziej's work by Demailly–Pali [33] and Eyssidieux–Guedj–Zeriahi [38]. This work was crucial in completing the existence of Calabi–Yau metrics in the positive case, a long research program recently finished in [19–21].[4] So in this area, at the center stage of modern research, pluripotential theory was crucial in solving an important differential equation.

Two important advantages of a potential-theoretic approach to nonlinear PDEs are worth repeating. First, as mentioned at the beginning of the introduction, many operators F correspond to the same subequation $\mathcal{F}$ and passing directly to the potential theory "frees" one from any particular form of the differential operator. In the viscosity literature, there is often the need to find an ad hoc reformulation of the operator as a first step, which becomes unnecessary if one passes to the potential theory. This is a major point in the work of Krylov [77]. Second, as mentioned after Remark 1.6, the potential theory approach correctly identifies the needed boundary geometry for existence by Perron's method for any operator F that corresponds to a subequation $\mathcal{F}$ defining the potential theory [46, 49].

The potential theory viewpoint makes PDE concepts purely geometric. It is important in relating different subequations (by containment, intersection, etc.), which leads to understanding things in a new way. It is also important for generalizations of the notion of equation; for example, see [61].

Another accomplishment of the potential theory viewpoint is the transfer of results for important equations to manifolds with Riemannian, symplectic, complex, or almost complex structure. If $\mathcal{F}$ is a subequation, which is invariant under the natural action of $\mathrm{O}(n)$, then $\mathcal{F}$ naturally determines a subequation on every Riemannian manifold. Similarly, if $\mathcal{F}$ is $\mathrm{U}(n)$-invariant, one gets a subequation on every complex (or, even, almost complex) hermitian manifold. If $\mathcal{F}$ is $\mathrm{G}(2)$-invariant, it defines a subequation on any (almost) $\mathrm{G}(2)$ manifold, and so on. This, together with a notion of *affine jet equivalence*, gives wide-ranging results for the Dirichlet problem on domains in such manifolds [49, 50, 60], and more.

There are many recent papers which have picked up the major theme of this book. We mention now several illustrations.

[4] In fact, just recently, in an important paper [43] by Guo–Phong–Tong, these estimates were established by pure PDE methods. The breakthrough enabling [43] was a fundamental new idea of Chen–Cheng [18].

First, the interplay between potential theory and geometry via interesting differential operators on Riemannian manifolds has taken inspiration from [49]. In Mari–Pessoa [81] and Araújo–Mari–Pessoa [2], the potential theory of the infinite Laplacian and the Eikonel operator are used for characterizing various *maximum principles at infinity* and detecting the *forward completeness* of Finsler manifolds, respectively. There is also Goffi–Pediconi [42] on strong maximum principles for operators modeled on Pucci extremal operators, infinite Laplacians, and mean curvature operators.

Second, on hermitian manifolds which are not Kähler, potential-theoretic techniques have been used to construct *Gauduchon metrics* with prescribed volume form in Székelyhidi–Tosatti–Weinkove [86].

Third, on manifolds with an almost complex structure, there are many results that have taken inspiration from [49, 50]. For example, in [54] the Dirichlet problem was solved as well as the closely related *Pali conjecture* stating that distributional plurisubharmonics and classical plurisubharmonics are equivalent. Consequently, the solution to the *obstacle problem* in [49] applies to almost complex manifolds and yields smooth approximations from above of plurisubharmonic functions in [63]. See also [64].

Fourth, following the work of [46] on Euclidean spaces, the interplay between potential theory and (viscosity) operator theory is the major theme of the papers [23–25] treating in wide generality the comparison principle and Perron's method in the variable coefficient fiberegular setting. In addition, this interplay is seen in the solution of the plateau problem for convex hypersurfaces of constant Gaussian curvature in Clark–Smith [26] and in many works dedicated to various forms of the maximum principle as in Amendola–Galise–Vitolo [1] and Birindelli–Galise–Ishii [7, 9], as well as the study of entire subsolutions in Capuzzo Dolcetta–Leoni–Vitolo [16].

Fifth, the many geometric operators with an interesting potential theory developed in [46, 49] (and in subsequent papers) has provided a notable stimulus to many recent investigations. The special Lagrangian potential operator was introduced in calibrated geometry [44], as discussed above in Example 1.9. Providing the proof of comparison for the special Lagrangian potential equation (with constant admissible phases) was one of the major motivations for [46]. This operator is the subject of a long list of papers beginning with Chen–Warren–Yuan [17] and continuing with Nadirashvili–Vlăduţ [82], Rubinstein–Solomon [84], the work of the authors [24, 62], and many others including the very recent paper by Brustad [13]. Also recently, in this direction, has been the fundamental work of Collins–Yau in a series of three papers starting with [27]. The truncated Laplacian operators were introduced in the context of the geometrically p-plurisubharmonic Dirichlet problem [46], as discussed in Example 1.13. These operators have been the object of much recent work including Vitolo [90] and Birindelli–Galise–Ishii [7–9].

Of course, techniques from viscosity theory, in particular the theorem on sums, play a big role in the analysis in [49] and elsewhere. A brief review of some milestones in the history of viscosity methods follows.

The notion of viscosity solutions originates for first-order equations in Crandall–Lions [31], and has its roots in Kruzkov's theory of entropy solutions for conservation laws [74]. The basic idea to put derivatives on test functions by way of the maximum principle originated in the work of Evans [34, 35] on the weak limits in fully nonlinear PDEs by means of Minty's method. One of the major achievements of the viscosity theory has proven to be the treatment of difficult problems by appropriate limiting procedures. There are many illustrations of this achievement when making use of *homogenization* and/or *vanishing viscosity* parameters; for example, see Evans [36, 37]. The notion of viscosity solutions was extended to second-order equations by Lions [79, 80], who first gave a proof of uniqueness using stochastic control arguments when the operator F is convex or concave in (Du, D^2u).

A major breakthrough was the work of Jensen [69, 70] which frees the theory from its dependence on the convexity (or concavity) and makes use of regularizations by way of the sup and inf convolutions. This was refined in Jensen–Lions–Souganidis [71]. See also Remark 12.10, which discusses the relation between two of Jensen's comparison principles and what we obtain by our method.

The analytical underpinnings of the comparison principle were further reformulated making use of the technique of doubling variables and penalization and culminates in the theorem on sums of Crandall–Ishii [29], which in turn had its origins in Ishii [67], Ishii–Lions [68], and Crandall [28]. The theorem on sums was used in the treatment of comparison principles on manifolds in [49].

Another important breakthrough in viscosity theory was the introduction of the comparison principle as a tool for proving existence for the Dirichlet problem via Perron's method. This was first accomplished by Ishii in two landmark papers: first-order equations in [66] and second-order equations in [67]. What is often known as *Ishii's theorem* states that one has the existence of a unique solution provided that the comparison principle holds and provided that there is a good subsolution/supersolution pair. In practice, these two hypotheses may seem unrelated, but the potential theory approach connects the two.

There are at least two major achievements of the conventional viscosity theory which, at least for now, have not been replicated by the potential theory approach. The first is the ability to prove the comparison principle for important operators, such as the minimal surface operator, which lack sufficient monotonicity to use our method (see the discussion in Section 1.8). The second is that the use of limiting procedures via artificial viscosity and/or homogenization parameters has yet to be attempted in the potential-theoretic setting. These two questions provide impetus for further interplay.

We have mentioned the seminal contribution of Jensen [69] to comparison theory. A main result in this work has become known as *Jensen's lemma* (for example, see [30, Lemma A.3]). It is a measure-theoretic result for quasi-convex[5]

[5] Such functions are often referred to as *semiconvex* functions, although the term is a bit misleading. Our use of quasi-convex is consistent with the use of *quasi-plurisubharmonic* functions w in several complex variables.

functions which ensures a wealth of linear perturbations having local maxima in the neighborhood of a strict local maximum of w. It is interesting to note that this lemma is equivalent to a result of Slodkowski [85] (two years earlier) developed in the context of pluripotential theory and can be stated in terms of the measure of the set of *upper contact points* near a strict upper contact point. The equivalence of the Jensen and Slodkowski lemmas is given in [57, Theorem 3.6], which also provides another proof of the Slodkowski lemma using contact paraboloids in place of contact spheres. See also [83], which borrows heavily from [56], for additional reflections on this equivalence and the role of Alexandroff's maximum principle and the area formula of Federer, from which one can prove both lemmas. This equivalence should be more well known than it is and signals an important moment when possible synergy between potential theory and operator theory was missed. It is a major theme of this book to examine the possibly fruitful interplay between potential theory and operator theory.

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