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Introduction

Mathematics, because of its language and notation and its odd-looking special symbols, is closed off from the surrounding world as by a high wall. What goes on behind that wall is, for the most part, a secret to the layman. He thinks of dull uninspiring numbers, of a lifeless mechanism which functions according to laws of inescapable necessity. On the other hand, that wall very often limits the view of him who stays within. He is prone to measure all mathematical things with a special yardstick and he prides himself that nothing profane shall enter his realm.

Is it possible to breach this wall, to present mathematics in such a way that the spectator may enjoy it? Cannot the enjoyment of mathematics be extended beyond the small circle of those who are "mathematically gifted"? Indeed, only a few are mathematically gifted in the sense that they are endowed with the talent to discover new mathematical facts. But by the same token, only very few are musically gifted in that they are able to compose music. Nevertheless there are many who can understand and perhaps reproduce music, or who at least enjoy it. We believe that the number of people who can understand simple mathematical ideas is not relatively smaller than the number of those who are commonly called musical, and that their interest will be stimulated if only we can eliminate the aversion toward mathematics that so many have acquired from childhood experiences.

It is the aim of these pages to show that the aversion toward mathematics vanishes if only truly mathematical, essential ideas are presented. This book is intended to give samples of the diversified phenomena which comprise mathematics, of mathematics for its own sake, and of the *intrinsic* values which it possesses.

The attempt to present mathematics to nonmathematicians has often been made, but this has usually been done by emphasizing the usefulness of mathematics in other fields of human endeavor in an effort to secure the comprehension and interest of the reader. Frequently the advantages which it offers in technological and other applications have been described and these advantages have been illustrated by numerous examples. On the other hand, many books have been written on mathematical games and pastimes. Although these books contain much interesting material, they give at best a very distorted picture of what mathematics really is. Finally, other books have discussed the foundations of mathematics with regard to their general philosophical validity. A reader of the following pages who is primarily interested in the pure, the absolute mathematics will naturally direct his attention toward just such an epistemological evaluation of mathematics. But this seems to us to be attaching an extraneous value to mathematics, to be judging its value according to measures outside itself.

In the following pages we will not be able to demonstrate the effects of the ideas to be presented on the domain of mathematics itself. We cannot consider the interior applications, so to speak, of mathematics, the use of the ideas and results of one field in other fields of mathematics. This means that we must omit something that is quite essential in the nature of the mathematical edifice, the great and surprising cross-connections that permeate this edifice in all directions. This omission is quite involuntary on our part, for the greatest mathematical discoveries are those which have revealed just such far-reaching interrelations. In order to present these interrelations, however, we would need long and comprehensive preparations and would have to assume a thorough training on the part of the reader. This is not our intention here.

In other words, our presentation will emphasize not the *facts* as other sciences can disclose them to the outsider but the *types of phenomena*, the *method of proposing problems*, and the *method of solving problems*. Indeed, in order to understand the great mathematical events, the comprehensive theories, long schooling and persistent application would be required. But this is also true with music. On going to a concert for the first time one is not able to appreciate fully Bach's "The Art of Fugue," nor can one immediately visualize the structure of a symphony. But besides the great works of music there are the smaller pieces which have something of true sublimity and whose spirit reveals itself to everyone. We plan to select such "smaller pieces" from the huge realm of mathematics: a sequence of subjects each one complete in itself, none requiring more than an hour to read and understand. The subjects are independent, so that one need not remember what has gone before when reading any chapter. Also, the reader is not required to remember what he may have been compelled to learn in his younger years. No use is made of logarithms or trigonometry. No mention is made of differential or integral calculus. The theorems of geometrical congruence and the multiplication of algebraic sums will gradually be brought back to the memory of the reader; that will be all.

In the case of a small work of music it may not only be the line of melody with which it opens that makes it beautiful. A little variation of the theme, a surprising modulation may well be the climax of the whole. Only he who has listened attentively to the basic theme will fully perceive and understand this climax. In a similar sense our reader will have to "listen" readily and attentively to the basic motive of a problem, to its development, to the first few examples which illustrate each theme, before the decisive modulation to the cardinal thought takes place. He will have to follow the reasoning with a little more active attention than is usually required in reading. If he does this, he will find no difficulty in grasping the essential ideas of each subject. He will then get a glimpse of what a few great thinkers have created when they have occasionally left the realm of their comprehensive theoretical production and have built, from simple beginnings, a small self-contained piece of art, a fragment of the prototype of mathematics.