Contents

Chapter One

Introduction

In this book we construct moduli stacks of étale (φ, Γ)-modules (projective, of some fixed rank, and with coefficients in p-adically complete rings), and establish some of their basic properties. We also present some first applications of this construction to the theory of Galois representations.

1.1 MOTIVATION

Mazur's theory of deformations of Galois representations [Maz89] is modeled on the geometric study of infinitesimal neighborhoods of points in moduli spaces via formal deformation theory. In the mid-2000s, Kisin suggested that some kind of moduli spaces of local Galois representations should exist; that is, there should be formal algebraic stacks over $\mathbf{Z}_p$ whose closed points correspond to representations $\overline{\rho}\colon G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$, and whose versal rings at such points should recover appropriate Galois deformation rings. This expectation is borne out by the results of this book. (In fact, Kisin was motivated by calculations of crystalline deformation rings for $\mathrm{GL}_2(\mathbf{Q}_p)$ that had been carried out by Berger–Breuil using the p-adic local Langlands correspondence, and suggested that the versal rings should give crystalline deformation rings. Thus his suggestion is realized by the stacks $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}}$ of Theorem 1.2.4 below.)

A natural way to construct such a stack would be to consider a literal moduli stack of continuous representations $\rho\colon G_K \to \mathrm{GL}_d(A)$, for K a p-adic field and A a p-adically complete $\mathbf{Z}_p$-algebra; indeed such stacks were constructed by Carl Wang-Erickson [WE18]. However, the stacks constructed in this way are less "global" than one would wish, and in particular the corresponding families of mod p representations $\overline{\rho}\colon G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ have constant semisimplification.

In this book, we instead consider moduli stacks of étale (φ, Γ)-modules. These contain Wang-Erickson's stacks as substacks, and coincide with them on the level of $\overline{\mathbf{F}}_p$-points, but their geometry is quite different; in particular, we see much larger families, exhibiting some unexpected features (for example, irreducible representations arising as limits of reducible representations). The relationship between the theory and constructions that we develop here and the usual formal deformation theory of Galois representations, is the same as that between the theory of moduli spaces of algebraic varieties and the formal deformation theory of algebraic varieties: the latter gives valuable local information about the former, but moduli spaces, when they can be constructed, capture global aspects of the situation inaccessible to the purely infinitesimal tools of formal deformation theory.

1.2 OUR MAIN THEOREMS

Our goal in this book is to construct, and establish, the basic properties of moduli stacks of étale (φ, Γ)-modules. More precisely, if we fix a finite extension K of $\mathbf{Q}_p$, and a non-negative integer d (the rank), then we let $\mathcal{X}_d$ denote the category fibred in groupoids over $\operatorname{Spf} \mathbf{Z}_p$ whose groupoid of A-valued points, for any p-adically complete $\mathbf{Z}_p$-algebra A, is equal to the groupoid of rank d projective étale (φ, Γ)-modules with A-coefficients. (See Section 1.3 below for a definition of these.) Our first main theorem is the following. (See Corollary 5.5.18 and Theorem 6.5.1.)

1.2.1 Theorem. *The category fibred in groupoids $\mathcal{X}_d$ is a Noetherian formal algebraic stack. Its underlying reduced substack $\mathcal{X}_{d,\mathrm{red}}$ (which is an algebraic stack) is of finite type over $\mathbf{F}_p$, and is equidimensional of dimension $[K:\mathbf{Q}_p]d(d-1)/2$. The irreducible components of $\mathcal{X}_{d,\mathrm{red}}$ admit a natural labelling by Serre weights.*

We will elaborate on the labelling of components by Serre weights further below. For now, we mention that, under the usual correspondence between étale (φ, Γ)-modules and Galois representations, the groupoid of $\overline{\mathbf{F}}_p$-points of $\mathcal{X}_d$, which coincides with the groupoid of $\overline{\mathbf{F}}_p$-points of the underlying reduced substack $\mathcal{X}_{d,\mathrm{red}}$, is naturally equivalent to the groupoid of continuous representations $\overline{\rho}\colon G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$. (More generally, if A is any finite $\mathbf{Z}_p$-algebra, then the groupoid $\mathcal{X}_d(A)$ is canonically equivalent to the groupoid of continuous representations $G_K \to \mathrm{GL}_d(A)$.) It is expected that our labelling of the irreducible components can be refined (by adding further labels to some of the components) to give a geometric description of the weight part of Serre's conjecture, so that $\overline{\rho}$ corresponds to a point in a component of $\mathcal{X}_{d,\mathrm{red}}$ which is labeled by the Serre weight $\underline{k}$ if and only if $\overline{\rho}$ admits $\underline{k}$ as a Serre weight; we discuss this expectation, and what is known about it, in Section 1.7 below (and in more detail in Chapter 8).

Again using the correspondence between étale (φ, Γ)-modules and Galois representations, we see that the universal lifting ring of a representation $\overline{\rho}$ as above will provide a versal ring to $\mathcal{X}_d$ at the corresponding $\overline{\mathbf{F}}_p$-valued point. Accordingly we expect that the stacks $\mathcal{X}_d$ will have applications to the study of Galois representations and their deformations. As a first example of this, we prove the following result on the existence of crystalline lifts; although the statement of this theorem involves a fixed $\overline{\rho}$, we do not know how to prove it without using the stacks $\mathcal{X}_d$, over which $\overline{\rho}$ varies.

1.2.2 Theorem (Theorem 6.4.4)**.** *If $\overline{\rho}\colon G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ is a continuous representation, then $\overline{\rho}$ has a lift $\rho^\circ\colon G_K \to \mathrm{GL}_d(\overline{\mathbf{Z}}_p)$ for which the associated p-adic representation $\rho\colon G_K \to \mathrm{GL}_d(\overline{\mathbf{Q}}_p)$ is crystalline of regular Hodge–Tate weights.*

We can furthermore ensure that ρ° is potentially diagonalizable.

(The notion of a potentially diagonalizable representation was introduced in [BLGGT14], and is recalled as Definition 6.4.2 below.) In combination with potential automorphy theorems, this has the following application to the globalization of local Galois representations.

1.2.3 Theorem (Corollary 6.4.7)**.** *Suppose that $p \nmid 2d$, and fix $\overline{\rho}\colon G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$. Then there is an imaginary CM field F and an irreducible conjugate self dual automorphic Galois representation $\overline{r}\colon G_F \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ such that for every $v|p$, we have $F_v \cong K$ and either $\overline{r}|_{G_{F_v}} \cong \overline{\rho}$ or $\overline{r}|_{G_{F_{v^c}}} \cong \overline{\rho}$.*

Another key result of the book is the following theorem, describing moduli stacks of étale (φ, Γ)-modules corresponding to *crystalline* and *semistable* Galois representations.

1.2.4 Theorem (Theorem 4.8.12)**.** *If $\underline{\lambda}$ is a collection of labeled Hodge–Tate weights, and if $\mathcal{O}$ denotes the ring of integers in a finite extension E of $\mathbf{Q}_p$ containing the Galois closure of K (which will serve as the ring of coefficients), then there is a closed substack $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}}$ of $(\mathcal{X}_d)_{\mathcal{O}}$ which is a p-adic formal algebraic stack and is flat over $\mathcal{O}$, and which is characterized as being the unique closed substack of $(\mathcal{X}_d)_{\mathcal{O}}$ which is flat over $\mathcal{O}$ and whose groupoid of A-valued points, for any finite flat $\mathcal{O}$-algebra A, is equivalent (under the equivalence between étale (φ, Γ)-modules and continuous G_K-representations) to the groupoid of continuous representations $G_K \to \mathrm{GL}_d(A)$ which become* crystalline *after extension of scalars to $A \otimes_{\mathcal{O}} E$, and whose labeled Hodge–Tate weights are equal to $\underline{\lambda}$.*

Similarly, there is a closed substack $\mathcal{X}_d^{\mathrm{ss},\underline{\lambda}}$ of $(\mathcal{X}_d)_{\mathcal{O}}$ which is a p-adic formal algebraic stack and is flat over $\mathcal{O}$, and which is characterized as being the unique closed substack of $(\mathcal{X}_d)_{\mathcal{O}}$ which is flat over $\mathcal{O}$ and whose groupoid of A-valued points, for any finite flat $\mathcal{O}$-algebra A, is equivalent to the groupoid of continuous representations $G_K \to \mathrm{GL}_d(A)$ which become semistable *after extension of scalars to $A \otimes_{\mathcal{O}} E$, and whose labeled Hodge–Tate weights are equal to $\underline{\lambda}$.*

1.2.5 Remark. In fact, Theorem 4.8.12 also proves the analogous result for potentially crystalline and potentially semistable representations of arbitrary inertial type, but for simplicity of exposition we restrict ourselves to the crystalline and semistable cases in this introduction.

A crucial distinction between the stacks $\mathcal{X}_d$ and their closed substacks $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}}$ and $\mathcal{X}_d^{\mathrm{ss},\underline{\lambda}}$ is that while $\mathcal{X}_d$ is a formal algebraic stack lying over $\mathrm{Spf}\,\mathbf{Z}_p$, it is not actually a p-adic formal algebraic stack (in the sense of Definition A.7);

see Proposition 6.5.2. On the other hand, the stacks $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}}$ and $\mathcal{X}_d^{\mathrm{ss},\underline{\lambda}}$ *are* p-adic formal algebraic stacks, which implies that their mod p^a reductions are in fact algebraic stacks. This gives in particular a strong interplay between the structure of the mod p fibres of crystalline and semistable lifting rings and the geometry of the underlying reduced substack $\mathcal{X}_{d,\mathrm{red}}$. This plays an important role in determining the structure of this reduced substack, and also in the proof of Theorem 1.2.2. As we explain in more detail in Section 1.7 below, it also allows us to reinterpret the Breuil–Mézard conjecture in terms of the interaction between the structure of the mod p fibres of the stacks $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}}$ and $\mathcal{X}_d^{\mathrm{ss},\underline{\lambda}}$ and the geometry of $\mathcal{X}_{d,\mathrm{red}}$.

1.3 (φ, Γ)-MODULES WITH COEFFICIENTS

There is quite a lot of evidence, for example from Colmez's work on the p-adic local Langlands correspondence [Col10], and work of Kedlaya–Liu [KL15], that rather than considering families of representations of G_K, it is more natural to consider families of étale (φ, Γ)-modules.

The theory of étale (φ, Γ)-modules for $\mathbf{Z}_p$-representations was introduced by Fontaine in [Fon90]. There are various possible definitions that can be made, with perfect, imperfect, or overconvergent coefficient rings, and different choices of Γ; we discuss the various variants that we use, and the relationships between them, at some length in the body of the book. For the purpose of this introduction we simply let $\mathbf{A}_K = W(k)((T))^\wedge$, where k is a finite extension of $\mathbf{F}_p$ (depending on K), and the hat denotes the p-adic completion. This ring is endowed with a Frobenius φ and an action of a profinite group Γ (an open subgroup of $\mathbf{Z}_p^\times$) that commutes with φ; the formulae for φ and for this action can be rather complicated for general K, although they admit a simple description if $K/\mathbf{Q}_p$ is abelian. (See Definition 2.1.12 and the surrounding material.)

An étale (φ, Γ)-module is then, by definition, a finite $\mathbf{A}_K$-module endowed with commuting semi-linear actions of φ and Γ, with the property that the linearized action of φ is an isomorphism. There is a natural equivalence of categories between the category of étale (φ, Γ)-modules and the category of continuous representations of G_K on finite $\mathbf{Z}_p$-modules.

Let A be a p-adically complete $\mathbf{Z}_p$-algebra. We let $\mathbf{A}_{K,A} := A \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p} \mathbf{A}_K$ (where the completed tensor product is taken with respect to the p-adic topology on A and the so-called weak topology on $\mathbf{A}_K$), and define an étale (φ, Γ)-module with A-coefficients just as in the case $A = \mathbf{Z}_p$ described above, but now using the ring $\mathbf{A}_{K,A}$. In the case that A is finite as a $\mathbf{Z}_p$-module there is again an equivalence of categories with the category of continuous representations of G_K on finite A-modules, but for more general A no such equivalence exists. Our moduli stack $\mathcal{X}_d$ is defined to be the stack over $\operatorname{Spf} \mathbf{Z}_p$ with the property that $\mathcal{X}_d(A)$ is the groupoid of projective étale (φ, Γ)-modules of rank d with A-coefficients. (That this is indeed an étale stack, indeed even an *fpqc* stack, follows from results of Drinfeld.) Using the machinery of our paper [EG21] we are able to show that $\mathcal{X}_d$ is an Ind-algebraic stack, but to prove Theorem 1.2.1 we need to go further and make a detailed study of its special fiber and of the underlying reduced substack. This study is guided by ideas coming from Galois deformation theory and the weight part of Serre's conjecture, in a manner that we now describe.

1.4 FAMILIES OF EXTENSIONS

As we have already explained, over a general base A there is no longer an equivalence between (φ, Γ)-modules and representations of G_K. Perhaps surprisingly, from the point of view of applications of our stacks to the study of p-adic Galois representations, this is a feature rather than a bug. For example, an examination of the known results on the reductions modulo p of two-dimensional crystalline representations of $G_{\mathbf{Q}_p}$ (see for example [Ber11, Thm. 5.2.1]) suggests that any moduli space of mod p representations of G_K should have the feature that the representations are generically reducible, but can specialize to irreducible representations. A literal moduli space of representations of a group cannot behave in this way (essentially because Grassmannians are proper), but it turns out that the underlying reduced substack $\mathcal{X}_{d,\mathrm{red}}$ of $\mathcal{X}_d$ does have this property. (See also Section 6.7 and Remark 7.2.19 for further discussions of the relationship between our stacks of (φ, Γ)-modules and stacks of representations of a Galois or Weil–Deligne group.)

More precisely, the results of [Ber11, Thm. 5.2.1], together with the weight part of Serre's conjecture, suggest that each irreducible component of $\mathcal{X}_{d,\mathrm{red}}$ should contain a dense set of $\overline{\mathbf{F}}_p$-points which are successive extensions of characters $G_K \to \overline{\mathbf{F}}_p$, with the extensions being as nonsplit as possible. This turns out to be the case. The restrictions of these characters to the inertia subgroup I_K are constant on the irreducible components, and the discrete data of these characters, together with some further information about peu- and très ramifiée extensions, determines the components. This discrete data can be conveniently and naturally organized in terms of "Serre weights" $\underline{k}$, which are tuples of integers which biject with the isomorphism classes of the irreducible $\overline{\mathbf{F}}_p$-representations of $\mathrm{GL}_d(\mathcal{O}_K)$. The relationship between Serre weights and Galois representations is important in the p-adic Langlands program, and in proving automorphy lifting theorems, and we discuss it further in Section 1.7.

Having guessed that the $\overline{\mathbf{F}}_p$-points of $\mathcal{X}_d$ should be arranged in irreducible components in this way, an inductive strategy to prove this suggests itself. It is easy to see that irreducible representations of G_K are "rigid", in that there are up to twist by unramified characters only finitely many in each dimension; furthermore, it is at least intuitively clear that each such family of unramified twists of a d-dimensional irreducible representation should give rise to a zero-dimensional substack of $\mathcal{X}_d$ (there is a $\mathbf{G}_m$ of twists, but also a $\mathbf{G}_m$ of automorphisms). On the other hand, given characters $\overline{\chi}_1, \dots, \overline{\chi}_d \colon G_K \to \overline{\mathbf{F}}_p^\times$, a Galois cohomology calculation suggests that there should be a substack of $\mathcal{X}_{d,\mathrm{red}}$

of dimension $[K:\mathbf{Q}_p]d(d-1)/2$ given by successive extensions of unramified twists of the $\overline{\chi}_i$. Accordingly, one could hope to construct the stacks corresponding to the Serre weights $\underline{k}$ by inductively constructing families of extensions of representations.

To confirm this expectation, we use the machinery originally developed by Herr [Her98, Her01], who gave an explicit complex which is defined in terms of (φ,Γ)-modules and computes Galois cohomology. This definition goes over unchanged to the case with coefficients, and with some effort we are able to adapt Herr's arguments to our setting, and to prove finiteness and base change properties (following Pottharst [Pot13], we in fact find it helpful to think of the Herr complex of a (φ,Γ)-module with A-coefficients as a perfect complex of A-modules). Using the Herr complex, we can inductively construct irreducible closed substacks $\mathcal{X}^{\underline{k}}_{d,\mathrm{red}}$ of $\mathcal{X}_{d,\mathrm{red}}$ of dimension $[K:\mathbf{Q}_p]d(d-1)/2$ whose generic $\overline{\mathbf{F}}_p$-points correspond to successive extensions of characters as described above (the restrictions of these characters to I_K being determined by $\underline{k}$). Furthermore, by a rather involved induction, we can show that the union of the $\mathcal{X}^{\underline{k}}_{d,\mathrm{red}}$, together possibly with a closed substack of $\mathcal{X}_{d,\mathrm{red}}$ of dimension strictly less than $[K:\mathbf{Q}_p]d(d-1)/2$, exhausts $\mathcal{X}_{d,\mathrm{red}}$. In particular, each $\mathcal{X}^{\underline{k}}_{d,\mathrm{red}}$ is an irreducible component of $\mathcal{X}_{d,\mathrm{red}}$, and any irreducible component that is not one of the $\mathcal{X}^{\underline{k}}_{d,\mathrm{red}}$ is of strictly smaller dimension than these components.

One way to show that the $\mathcal{X}^{\underline{k}}_{d,\mathrm{red}}$ exhaust the irreducible components of $\mathcal{X}_{d,\mathrm{red}}$ would be to show that every representation $G_K\to\mathrm{GL}_d(\overline{\mathbf{F}}_p)$ occurs as an $\overline{\mathbf{F}}_p$-valued point of some $\mathcal{X}^{\underline{k}}_{d,\mathrm{red}}$. We expect this to be difficult to show directly; indeed, already for $d=2$ the paper [CEGS19] shows that the closed points of $\mathcal{X}^{\underline{k}}_{d,\mathrm{red}}$ are governed by the weight part of Serre's conjecture, and the explicit description of this conjecture is complicated (see, e.g., [BDJ10, DDR16]). Furthermore it seems hard to explicitly understand the way in which families of reducible (φ,Γ)-modules degenerate to irreducible ones, or to reducible representations with different restrictions to I_K (phenomena which are implied by the weight part of Serre's conjecture).

Instead, our approach is to show by a consideration of versal rings that $\mathcal{X}_{d,\mathrm{red}}$ is equidimensional of dimension $[K:\mathbf{Q}_p]d(d-1)/2$; this suffices, since our inductive construction showed that any other irreducible component would necessarily have dimension strictly less than $[K:\mathbf{Q}_p]d(d-1)/2$. Our proof of this equidimensionality relies on Theorems 1.2.2 and 1.2.4, as we explain in Remark 1.5.4 below.

1.5 CRYSTALLINE LIFTS

Theorem 1.2.2 solves a problem that has been considered by various authors, in particular [Mul13, GHLS17]. It admits a well-known inductive approach (which is taken in [Mul13, GHLS17]): one writes $\overline{\rho}$ as a successive extension of irreducible representations, lifts each of these irreducible representations to a crystalline representation, and then attempts to lift the various extension classes. The difficulty that arises in this approach (which has proved an obstacle to obtaining general statements along the lines of Theorem 1.2.2 until now) is showing that the mod p extension classes that appear in this description of $\overline{\rho}$ can actually be lifted to crystalline extension classes in characteristic 0. The basic source of the difficulty is that the local Galois H^2 can be nonzero, and nonzero classes in H^2 obstruct the lifting of extension classes (which can be interpreted as classes lying in H^1). In fact, the difficulty is not so much in obtaining crystalline extension classes, as in lifting to any classes in characteristic 0; indeed, it was not previously known that an arbitrary $\overline{\rho}$ had *any* lift to characteristic 0 at all. (Subsequently a different proof of the existence of such a lift has been found by Böckle–Iyengar–Paškūnas [BIP21].)

Our proof of Theorem 1.2.2 relies on the inductive strategy described in the preceding paragraph, but we are able to prove the following key result, which controls the obstructions that can be presented by H^2, and is a consequence of Theorems 5.5.12 and 6.5.1 (see also Remark 1.5.4).

1.5.1 Proposition. *The locus of points $\overline{\rho} \in \mathcal{X}_{d,\mathrm{red}}(\overline{\mathbf{F}}_p)$ at which*

$$\dim H^2(G_K, \overline{\rho}) \geq r$$

is Zariski closed in $\mathcal{X}_{d,\mathrm{red}}(\overline{\mathbf{F}}_p)$, and is of codimension $\geq r$.

Let $R^{\square}_{\overline{\rho}}$ denote the universal lifting ring of $\overline{\rho}$, with universal lifting ρ^{univ}. For each regular tuple of labeled Hodge–Tate weights $\underline{\lambda}$, we let $R^{\mathrm{crys},\underline{\lambda}}_{\overline{\rho}}$ denote the quotient of $R^{\square}_{\overline{\rho}}$ corresponding to crystalline lifts of $\overline{\rho}$ with Hodge–Tate weights $\underline{\lambda}$ (of course, this quotient is zero unless $\overline{\rho}$ admits such a crystalline lift). Then $H^2(G_K, \rho^{\mathrm{univ}})$ is an $R^{\square}_{\overline{\rho}}$-module, and Proposition 1.5.1 implies the following corollary.

1.5.2 Corollary. *For any regular tuple of labeled Hodge–Tate weights $\underline{\lambda}$ the locus of points $x \in \operatorname{Spec} R^{\mathrm{crys},\underline{\lambda}}_{\overline{\rho}}/p$ for which*

$$\dim_{\kappa(x)} H^2(G_K, \rho^{\mathrm{univ}}) \otimes_{R^{\square}_{\overline{\rho}}} \kappa(x) \geq r$$

has codimension $\geq r$.

1.5.3 Remark. Tate local duality, together with the compatibility of H^2 with base change, shows that

$$\begin{aligned}\dim_{\kappa(x)} \big(H^2(G_K, \rho^{\mathrm{univ}}) \otimes_{R^{\square}_{\overline{\rho}}} \kappa(x)\big) &= \dim_{\kappa(x)} H^2\big(G_K, \rho^{\mathrm{univ}} \otimes_{R^{\square}_{\overline{\rho}}} \kappa(x)\big) \\ &= \dim_{\kappa(x)} \operatorname{Hom}_{G_K}\big((\rho^{\mathrm{univ}})^{\vee} \otimes_{R^{\square}_{\overline{\rho}}} \kappa(x), \overline{\epsilon}\big)\end{aligned}$$

(where $\overline{\epsilon}$ denotes the mod p cyclotomic character, thought of as taking values in $\kappa(x)^\times$). Thus the statement of Corollary 1.5.2 is related to the way in which $\operatorname{Spec} R_{\overline{\rho}}^{\mathrm{crys},\underline{\lambda}}/p$ intersects the reducibility locus in $\operatorname{Spec} R_{\overline{\rho}}^{\square}$.

Given Corollary 1.5.2, we prove Theorem 1.2.2 by working purely within the context of formal lifting rings. However we don't know how to prove the corollary while staying within that context. Indeed, as Remark 1.5.3 indicates, this corollary is related to the way in which the special fiber of a potentially crystalline deformation ring intersects another natural locus in $\operatorname{Spec} R_{\overline{\rho}}^{\square}$ (namely, the reducibility locus). Since the special fiber of a potentially crystalline lifting ring is not directly defined in deformation-theoretic terms, such questions are notoriously difficult to study directly. Our proof of the corollary proceeds differently, by replacing a computation on the special fiber of the potentially crystalline deformation ring by a computation on $\mathcal{X}_{d,\mathrm{red}}$; this latter space has a concrete description in terms of families of varying $\overline{\rho}$, whose H^2 we are able to compute, as a result of the inductive construction of families of extensions described in Section 1.4.

In order to deduce Corollary 1.5.2 from Proposition 1.5.1, it is crucial that we know that the natural morphism $\operatorname{Spf} R^{\mathrm{crys},\underline{\lambda}}/p \to \mathcal{X}_d$ is effective, in the sense that it arises from a morphism $\operatorname{Spec} R^{\mathrm{crys},\underline{\lambda}}/p \to \mathcal{X}_d$. More concretely, the universal representation ρ^{univ} gives an étale (φ,Γ)-module over each Artinian quotient of $R_{\overline{\rho}}^{\square}$. By passing to the limit over these quotients, we obtain a "universal formal étale (φ,Γ)-module" over the completion of $(k \otimes_{\mathbf{Z}_p} R_{\overline{\rho}}^{\square}/p)((T))$ with respect to the maximal ideal $\mathfrak{m}$ of $R_{\overline{\rho}}^{\square}$. Since the special fiber of $\mathcal{X}_d$ is formal algebraic but not algebraic (see Section 1.8 below), there is no corresponding (φ,Γ)-module with $R_{\overline{\rho}}^{\square}/p$-coefficients; the φ and Γ actions on the universal formal étale (φ,Γ)-module involve Laurent tails of unbounded degree (with the coefficients of T^{-n} tending to zero $\mathfrak{m}$-adically as $n \to \infty$).

The assertion that $\operatorname{Spf} R^{\mathrm{crys},\underline{\lambda}}/p \to \mathcal{X}_d$ is effective is equivalent to showing that the base change of the universal formal étale (φ,Γ)-module to $R^{\mathrm{crys},\underline{\lambda}}/p$ arises from a genuine (φ,Γ)-module, i.e., from one that involves only Laurent tails of bounded degree. We deduce this from Theorem 1.2.4. Indeed, the ring $R^{\mathrm{crys},\underline{\lambda}}/p$ is a versal ring for the special fiber of the p-adic formal algebraic stack $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}}$, and (by the very definition of a p-adic formal algebraic stack) this special fiber is an algebraic stack; and the versal rings for algebraic stacks are always effective.

1.5.4 Remark. As our citation of both Theorems 5.5.12 and 6.5.1 for the proof of Proposition 1.5.1 may indicate, our proof of Proposition 1.5.1 is somewhat intricate. Indeed, in Theorem 5.5.12, we show that $\mathcal{X}_{d,\mathrm{red}}$ has dimension at most $[K:\mathbf{Q}_p]d(d-1)/2$, and that the locus considered in Proposition 1.5.1 has dimension at most $[K:\mathbf{Q}_p]d(d-1)/2 - r$. This is in fact enough to deduce Corollary 1.5.2, as $\operatorname{Spec} R^{\mathrm{crys},\underline{\lambda}}/p$ is known to be equidimensional.

Given Corollary 1.5.2, we prove Theorem 1.2.2. In combination with the effective versality of the crystalline deformation rings discussed above we are then able to deduce the equidimensionality of $\mathcal{X}_{d,\mathrm{red}}$, and then also prove Proposition 1.5.1 as stated.

1.6 CRYSTALLINE AND SEMISTABLE MODULI STACKS

We now explain the proof of Theorem 1.2.4; the proof is essentially identical in the crystalline and semistable cases, so we concentrate on the crystalline case. To prove the theorem, it is necessary to have a criterion for a (φ,Γ)-module to come from a crystalline Galois representation. In the case that $K/\mathbf{Q}_p$ is unramified, it is possible to give an explicit criterion in terms of Wach modules [Wac96], but no such direct description is known for general K. Instead, following Kisin's construction of the crystalline deformation rings $R_{\overline{\rho}}^{\mathrm{crys},\lambda}$ in [Kis08], we use the theory of Breuil–Kisin modules. More precisely, Kisin shows that crystalline representations of G_K have finite height over the (non-Galois) Kummer extension K_∞/K obtained by adjoining a compatible system of p-power roots of a uniformizer of K; here being of finite height means that the corresponding étale φ-modules admit certain φ-stable lattices, called Breuil–Kisin modules.

While not every representation of finite height over K_∞ comes from a crystalline representation, we are able to show in Appendix F (jointly written by T. G. and Tong Liu) that a representation $G_K \to \mathrm{GL}_d(\overline{\mathbf{Z}}_p)$ is crystalline if and only if it is of finite height for *every* choice of K_∞, and if the corresponding Breuil–Kisin modules satisfy certain natural compatibilities. (These compatibilities are best expressed in terms of Bhatt–Scholze's prismatic site, as in [BS21], but we do not make use of that perspective in this book. Instead, we write down explicit conditions on the corresponding Breuil–Kisin–Fargues modules; recall that Breuil–Kisin–Fargues modules are a variant of Breuil–Kisin modules introduced by Fargues; see, e.g., [BMS18, §4].)

We use this description of the crystalline representations to prove the existence of the stacks $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}}$. The proof that $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}}$ is a p-adic formal algebraic stack relies on an analogue of results of Caruso–Liu [CL11] on extensions of the Galois action on Breuil–Kisin modules, which roughly speaking says that the action of G_{K_∞} determines the action of G_K up to a finite amount of ambiguity. More precisely, given a Breuil–Kisin module over a $\mathbf{Z}/p^a$-algebra for some $a \geq 1$, there is a finite subextension K_s/K of K_∞/K depending only on a, K and the height of the Breuil–Kisin module, such that there is a canonical action of G_{K_s} on the corresponding Breuil–Kisin–Fargues module. This canonical action is constructed by Frobenius amplification, and in the case that the Breuil–Kisin module arises from the reduction modulo p^a of a crystalline representation of G_K, the canonical action coincides with the restriction to G_{K_s} of the G_K-action on the Breuil–Kisin–Fargues module. (In [CL11] a version of this canonical action is used to prove ramification bounds on the reductions modulo p^a of crystalline representations; in Chapter 7, we use analogous arguments in the setting of (φ,Γ)-modules to relate our stacks to stacks of Weil group representations in the rank 1 case.)

There is one significant technical difficulty, which is that we need to define morphisms of stacks that correspond to the restriction of Galois representations from K to K_∞. In order to do this we have to compare (φ,Γ)-modules with A-coefficients (which are defined via the cyclotomic extension $K(\zeta_{p^\infty})/K$) to φ-modules with A-coefficients defined via the extension K_∞/K. We do not know of a direct way to do this; we proceed by proving a correspondence between φ-modules over Laurent series rings with φ-modules over the perfections of these Laurent series rings and proving the following descent result which may be of independent interest; in the statement, $\mathbf{C}$ denotes the completion of an algebraic closure of $\mathbf{Q}_p$.

1.6.1 Theorem (Theorem 2.4.1)**.** *Let A be a finite type $\mathbf{Z}/p^a$-algebra, for some $a \geq 1$. Let F be a closed perfectoid subfield of $\mathbf{C}$, with tilt $F^\flat$, a closed perfectoid subfield of $\mathbf{C}^\flat$. Write $W(F^\flat)_A := W(F^\flat) \otimes_{\mathbf{Z}_p} A$.*

Then the inclusion $W(F^\flat)_A \to W(\mathbf{C}^\flat)_A$ is a faithfully flat morphism of Noetherian rings, and the functor $M \mapsto W(\mathbf{C}^\flat)_A \otimes_{W(F^\flat)_A} M$ induces an equivalence between the category of finitely generated projective $W(F^\flat)_A$-modules and the category of finitely generated projective $W(\mathbf{C}^\flat)_A$-modules endowed with a continuous semi-linear G_F-action.

The existence of the required morphism of stacks follows easily from two applications of Theorem 1.6.1, applied with F equal to respectively the completion of K_∞ and the completion of $K(\zeta_{p^\infty})$. Furthermore, this construction gives an alternative description of our stacks, as moduli spaces of $W(\mathbf{C}^\flat)_A$-modules endowed with commuting semi-linear actions of G_K and φ. It seems plausible that this description will be useful in future work, as it connects naturally to the theory of Breuil–Kisin–Fargues modules (and indeed we use this connection in our construction of the potentially semistable moduli stacks). Note though that the description in terms of (φ,Γ)-modules is important (at least in our approach) for establishing the basic finiteness properties of our stacks.

1.7 THE GEOMETRIC BREUIL–MÉZARD CONJECTURE AND THE WEIGHT PART OF SERRE'S CONJECTURE

We will now briefly explain our results and conjectures relating our stacks to the Breuil–Mézard conjecture and the weight part of Serre's conjecture. Further explanation and motivation can be found throughout Chapter 8. Some of these results were previewed in [GHS18, §6], and the earlier sections of that paper (in particular the introduction) provide an overview of the weight part of Serre's conjecture and its connections to the Breuil–Mézard conjecture that may be helpful to the reader who is not already familiar with them. As in the rest of this introduction, we ignore the possibility of inertial types, and we also restrict to crystalline representations for the purpose of exposition. Everything in this section extends to the more general setting of potentially semistable representations, and indeed as we explain in Section 8.6 when discussing the papers [CEGS19] and [GK14], the additional information provided by non-trivial inertial types is very important.

Let $\overline{\rho}\colon G_K \to \mathrm{GL}_d(\mathbf{F})$ be a continuous representation (for some finite extension $\mathbf{F}$ of $\mathbf{F}_p$), and let $R^{\square}_{\overline{\rho}}$ be the corresponding universal lifting ring. The corresponding formal scheme $\operatorname{Spf} R^{\square}_{\overline{\rho}}$ doesn't carry a lot of evident geometry in and of itself; for example, its underlying reduced subscheme is simply the closed point $\operatorname{Spec} \mathbf{F}$, corresponding to $\overline{\rho}$ itself. On the other hand, $\mathcal{X}_d$ has a quite non-trivial underlying reduced substack $\mathcal{X}_{d,\mathrm{red}}$, which parameterizes all the d-dimensional residual representations of G_K. It is natural to ask whether this underlying reduced substack has any significance in formal deformation theory. More precisely, we could ask for the meaning of the fiber product $\operatorname{Spf} R^{\square}_{\overline{\rho}} \times_{\mathcal{X}_d} \mathcal{X}_{d,\mathrm{red}}$.

This fiber product is a reduced closed formal subscheme of $\operatorname{Spf} R^{\square}_{\overline{\rho}}$ of dimension $d^2 + [K:\mathbf{Q}_p]d(d-1)/2$. It arises (via completion at the closed point) from a closed subscheme of $\operatorname{Spec} R^{\square}_{\overline{\rho}}$ (as does any closed formal subscheme of the Spf of a complete Noetherian local ring), whose irreducible components, when thought of as cycles on $\operatorname{Spec} R^{\square}_{\overline{\rho}}$, are precisely the cycles that (conjecturally) appear in the geometric Breuil–Mézard conjecture of [EG14]. More precisely, we obtain the following qualitative version of the geometric Breuil–Mézard conjecture [EG14, Conj. 4.2.1].

1.7.1 Theorem (Theorem 8.1.4)**.** *If $\overline{\rho}\colon G_K \to \mathrm{GL}_d(\mathbf{F})$ is a continuous representation, then there are finitely many cycles of dimension $d^2 + [K:\mathbf{Q}_p]d(d-1)/2$ in $\operatorname{Spec} R^{\square}_{\overline{\rho}}/p$, such that for any regular tuple of labeled Hodge–Tate weights $\underline{\lambda}$, the special fiber $\operatorname{Spec} R^{\mathrm{crys},\underline{\lambda}}_{\overline{\rho}}/p$ is set-theoretically supported on the union of some number of these cycles.*

The cycles in the statement of the theorem are precisely those arising from the fiber products $\operatorname{Spf} R^{\square}_{\overline{\rho}} \times_{\mathcal{X}_d} \mathcal{X}^{\underline{k}}_{d,\mathrm{red}}$, where $\underline{k}$ runs over the Serre weights. While Theorem 1.7.1 is a purely local statement, we do not know how to prove it without using the stacks $\mathcal{X}_d$.

The full geometric Breuil–Mézard conjecture of [EG14] makes precise predictions about the multiplicities of the cycles of the special fibres of $\operatorname{Spec} R^{\mathrm{crys},\underline{\lambda}}_{\overline{\rho}}/p$; passing from cycles to Hilbert–Samuel multiplicities then recovers the original Breuil–Mézard conjecture [BM02] (or rather a natural generalization of it to GL_d), which we refer to as the "numerical Breuil–Mézard conjecture". In particular, the multiplicities are expected to be computed in terms of quantities $n^{\mathrm{crys}}_{\underline{k}}(\lambda)$ that are defined as follows: one associates an irreducible algebraic representation $\sigma^{\mathrm{crys}}(\lambda)$ of GL_d/K to $\underline{\lambda}$, defined to have highest weight (a certain shift of) $\underline{\lambda}$. The semisimplification of the reduction mod p of $\sigma^{\mathrm{crys}}(\lambda)$ can be written as a direct sum of irreducible representations of $\mathrm{GL}_d(k)$, and $n^{\mathrm{crys}}_{\underline{k}}(\lambda)$ is defined to be the multiplicity with which the Serre weight $\underline{k}$ occurs.

In Chapter 8 we explain that as we run over all $\overline{\rho}$, the geometric Breuil–Mézard conjecture is equivalent to the following analogous conjecture for the special fibres of our crystalline and semistable stacks. Here by a "cycle" in $\mathcal{X}_{d,\mathrm{red}}$ we mean a formal $\mathbf{Z}$-linear combination of its irreducible components $\mathcal{X}_d^{\underline{k}}$.

1.7.2 Conjecture (Conjecture 8.2.2)**.** *There are cycles $Z_{\underline{k}}$ in $\mathcal{X}_{d,\mathrm{red}}$ with the property that for each regular tuple of labeled Hodge–Tate weights $\underline{\lambda}$, the underlying cycle of the special fiber of $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}}$ is $\sum_{\underline{k}} n_{\underline{k}}^{\mathrm{crys}}(\lambda) \cdot Z_{\underline{k}}$.*

In fact we expect that the cycles $Z_{\underline{k}}$ are effective, i.e., that they are a linear combination of the irreducible components $\mathcal{X}_d^{\underline{k}'}$ with non-negative integer coefficients. Since there are infinitely many possible $\underline{\lambda}$, the cycles $Z_{\underline{k}}$, if they exist, are hugely overdetermined by Conjecture 1.7.2.

As first explained in [Kis09a], the (numerical) Breuil–Mézard conjecture has important consequences for automorphy lifting theorems; indeed, proving the conjecture is closely related to proving automorphy lifting theorems in situations with arbitrarily high weight or ramification at the places dividing p. Conversely, following [Kis10], one can use automorphy lifting theorems to deduce cases of the Breuil–Mézard conjecture. Automorphy lifting theorems involve a fixed $\overline{\rho}$, and in fact we can deduce Conjecture 1.7.2 from the Breuil–Mézard conjecture for a finite set of suitably generic $\overline{\rho}$.

In particular, we are able to combine results in the literature to show that for GL_2 the cycles $Z_{\underline{k}}$ in Conjecture 1.7.2 must have a particularly simple form: we necessarily have $Z_{\underline{k}} = \mathcal{X}_{d,\mathrm{red}}^{\underline{k}}$ unless $\underline{k}$ is a so-called "Steinberg" weight, in which case $Z_{\underline{k}}$ is the sum of $\mathcal{X}_{d,\mathrm{red}}^{\underline{k}}$ and one other irreducible component. (More precisely, what we show, following [CEGS19, GK14], is that with these cycles $Z_{\underline{k}}$, Conjecture 1.7.2 holds for all "potentially Barsotti–Tate" representations.)

The weight part of Serre's conjecture predicts the weights in which particular Galois representations contribute to the mod p cohomology of locally symmetric spaces. Following [GK14], this conjecture is closely related to the Breuil–Mézard conjecture; indeed, if Conjecture 1.7.2 holds, then the set of Serre weights associated to a representation $\overline{\rho}\colon G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ should be precisely the weights $\underline{k}$ for which $Z_{\underline{k}}$ is supported at $\overline{\rho}$. In other words, if we refine our labelling of the irreducible components of $\mathcal{X}_{d,\mathrm{red}}$ by labelling each component by the union of the weights $\underline{k}$ for which that component contributes to $Z_{\underline{k}}$, then we expect the set of Serre weights for $\overline{\rho}$ to be the union of the sets of weights labelling the irreducible components containing $\overline{\rho}$. This expectation holds for GL_2 by the main results of [CEGS19].

1.8 FURTHER QUESTIONS

There are many other questions one could ask about the stacks $\mathcal{X}_d$, which we hope to return to in future papers. For example, we show in Proposition 6.5.2 that $\mathcal{X}_d$ is not a p-adic formal algebraic stack. Indeed, if it were p-adic formal algebraic, then its special fiber would be an algebraic stack, whose dimension would be equal to the dimension of its underlying reduced substack. In turn, this would imply that the versal rings $R_{\overline{\rho}}^{\square}$ would have dimension equal to the dimensions of the crystalline deformation rings $R_{\overline{\rho}}^{\mathrm{crys},\underline{\lambda}}$, and this is known not to be true. In fact, it is a folklore conjecture, recently proved by Böckle–Iyengar–Paškūnas [BIP21], that the lifting rings $R_{\overline{\rho}}^{\square}$ are $\mathbf{Z}_p$-flat local complete intersections of dimension $1+d^2+[K:\mathbf{Q}_p]d^2$, which should imply that the stacks $\mathcal{X}_d$ are $\mathbf{Z}_p$-flat local complete intersections of dimension $1+[K:\mathbf{Q}_p]d^2$ (a notion that we do not attempt to make precise for formal algebraic stacks).

It is natural to ask about the rigid analytic generic fiber of $\mathcal{X}_d$; this should exist as a rigid analytic stack in an appropriate sense. The generic fibres of the substacks $\mathcal{X}_d^{\underline{k}}$ should admit morphisms to the stacks of Hartl and Hellmann [HH20] (although these morphisms won't be isomorphisms, since for any finite extension E of $\mathbf{Q}_p$, the $\mathcal{O}_E$-points of $\mathcal{X}_d^{\underline{k}}$, which would coincide with the E-points of its generic fiber, correspond to lattices in crystalline representations, whereas the stacks of [HH20] parameterize crystalline or semistable representations themselves).

We expect that the $\mathcal{X}_d$ will have a role to play in generalizations of the p-adic local Langlands correspondence. For example, we expect that when $K=\mathbf{Q}_p$ the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ can be extended to give rise to sheaves of $\mathrm{GL}_2(\mathbf{Q}_p)$-representations on $\mathcal{X}_2$. More generally, we expect that there will be a p-adic analogue of the work of Fargues–Scholze on the local Langlands correspondence [FS21] involving the stacks $\mathcal{X}_d$.

1.9 PREVIOUS WORK

The description of local Galois representations in terms of étale (φ,Γ)-modules is due to Fontaine [Fon90]. The importance of "height" as an aspect of the theory was already emphasized in [Fon90], and was further developed by Wach [Wac96], who explored the relationship between the finite height condition and crystallinity of Galois representations in the absolutely unramified context.

The use of what are now called Breuil–Kisin modules as a tool for studying crystalline and semistable representations for general (i.e., not necessarily absolutely unramified) p-adic fields (a study which, apart from its intrinsic importance, is crucial for treating potentially crystalline or semistable representations, even in the absolutely unramified context) was due originally to Breuil [Bre98] and was extensively developed by Kisin [Kis09b, Kis08], who used them to study Galois deformation rings.

The algebro-geometric and moduli-theoretic perspectives that already played key roles in Kisin's work were further developed by Pappas and Rapoport [PR09], who introduced moduli stacks of Breuil–Kisin modules and of étale φ-modules; it is this work of Pappas and Rapoport, which can be very roughly thought of as constructing moduli stacks of representations of the absolute Galois groups of certain perfectoid fields, which is the immediate launching point for our work in this book, as well as for our paper [EG21]. (We should also mention Drinfeld's work [Dri06], which underpins the verification of the stack property for the constructions of [PR09], as well as for those of the present book.) Our use of moduli stacks of Breuil–Kisin–Fargues modules (in the construction of the potentially crystalline and semistable substacks) was in part inspired by the work of Fargues and Bhatt–Morrow–Scholze (see in particular [BMS18, §4]), which taught us not to be afraid of $\mathbf{A}_{\mathrm{inf}}$.

Moduli stacks parameterizing crystalline and semistable representations have already been constructed by Hartl and Hellmann [HH20]; as remarked upon above, these stacks should have a relationship to the stacks $\mathcal{X}_d^{\underline{k}}$ that we construct. See also the related papers of Hellmann [Hel16, Hel13].

As far as we are aware, the first construction of moduli stacks of representations of G_K in which the residual representation $\overline{\rho}$ can vary is the work of Carl Wang-Erickson [WE18] mentioned above, which constructs and studies such stacks in the case that $\overline{\rho}$ has fixed semisimplification. These are literally moduli stacks of representations of G_K; they are isomorphic to certain substacks of our stacks $\mathcal{X}_d$, as we explain in Section 6.7.

1.10 AN OUTLINE OF THE BOOK

We finish this introduction with a brief overview of the contents of this book. The reader may also wish to refer to the introductions to each chapter, as well as to the overview of this book provided by the notes [EG20].

In Chapter 2 we recall several of the coefficient rings used in the theories of (φ, Γ)-modules and Breuil–Kisin modules, and introduce versions of these rings with coefficients in a p-adically complete $\mathbf{Z}_p$-algebra. We also prove almost Galois descent results for projective modules, and deduce Theorem 1.6.1.

In Chapter 3 we recall the results of [EG21] on moduli stacks of φ-modules, and use them to define our stacks $\mathcal{X}_d$ of étale (φ, Γ)-modules. With some effort, we prove that $\mathcal{X}_d$ is an Ind-algebraic stack. Chapter 4 defines various moduli stacks of Breuil–Kisin and Breuil–Kisin–Fargues modules, and uses them to construct our moduli stacks of potentially semistable and potentially crystalline representations, and in particular to prove Theorem 1.2.4.

Chapter 5 develops the theory of the Herr complex, proving in particular that it is a perfect complex and is compatible with base change. We show how to use the Herr complex to construct families of extensions of (φ, Γ)-modules, and we use these families to define the irreducible substack $\mathcal{X}_{d,\mathrm{red}}^{\underline{k}}$ corresponding to a Serre weight $\underline{k}$. By induction on d we prove that $\mathcal{X}_d$ is a Noetherian formal algebraic stack, and establish a version of Proposition 1.5.1 (although as discussed in Remark 1.5.4, we do not prove Proposition 1.5.1 as stated at this point in the argument).

It may help the reader for us to point out that Chapters 4 and 5 are essentially independent of one another, and are of rather different flavor. Chapter 4 involves an interleaving of stack-theoretic arguments with ideas from p-adic Hodge theory and the theory of Breuil–Kisin modules, while in Chapter 5, once we complete our analysis of the Herr complex, our perspective begins to shift: although at a technical level we of course continue to work with (φ, Γ)-modules, we begin to think in terms of Galois representations and Galois cohomology, and the more foundational arguments of the preceding chapters recede somewhat into the background.

In Chapter 6 we combine the results of Chapters 4 and 5 with a geometric argument on the local deformation ring to prove Theorem 1.2.2. Having done this, we are then able to improve on the results on $\mathcal{X}_d$ established in the earlier chapters by proving Theorem 1.2.1. We also deduce Theorem 1.2.3, as well as determining the closed points of $\mathcal{X}_d$, and describing the relationship of our stacks with Wang–Erickson's stacks of Galois representations.

Chapter 7 gives explicit descriptions of various of our moduli stacks in the case $d = 1$, relating them to moduli stacks of Weil group representations. Chapter 8 explains our geometric version of the Breuil–Mézard conjecture, and proves some results towards it, particularly in the case $d=2$.

Finally the appendices for the most part establish various technical results used in the body of the book. We highlight in particular Appendix A, which summarizes the theory of formal algebraic stacks developed in [Eme], and Appendix F, which combines the theory of Breuil–Kisin–Fargues modules with Tong Liu's theory of $(\varphi, \widehat{G})$-modules to give a new characterization of integral lattices in potentially semistable representations, of which we make crucial use in Chapter 4.

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1.12 NOTATION AND CONVENTIONS

p-adic Hodge theory

Let $K/\mathbf{Q}_p$ be a finite extension. If ρ is a de Rham representation of G_K on a $\overline{\mathbf{Q}}_p$-vector space W, then we will write $\mathrm{WD}(\rho)$ for the corresponding Weil–Deligne representation of W_K (see, e.g., [CDT99, App. B]), and if $\sigma\colon K\hookrightarrow\overline{\mathbf{Q}}_p$ is a continuous embedding of fields then we will write $\mathrm{HT}_\sigma(\rho)$ for the multiset of Hodge–Tate numbers of ρ with respect to σ, which by definition contains i with multiplicity $\dim_{\overline{\mathbf{Q}}_p}(W\otimes_{\sigma,K}\widehat{\overline{K}}(i))^{G_K}$. Thus, for example, if ϵ denotes the p-adic cyclotomic character, then $\mathrm{HT}_\sigma(\epsilon)=\{-1\}$.

By a *d-tuple of labeled Hodge–Tate weights* $\underline{\lambda}$, we mean a tuple of integers $\{\lambda_{\sigma,i}\}_{\sigma\colon K\hookrightarrow\overline{\mathbf{Q}}_p,1\le i\le d}$ with $\lambda_{\sigma,i}\ge\lambda_{\sigma,i+1}$ for all σ and all $1\le i\le d-1$. We will also refer to $\underline{\lambda}$ as a *Hodge type*. By an *inertial type* τ we mean a representation $\tau\colon I_K\to\mathrm{GL}_d(\overline{\mathbf{Q}}_p)$ which extends to a representation of W_K with open kernel (so in particular, τ has finite image).

Then we say that ρ has Hodge type $\underline{\lambda}$ (or labeled Hodge–Tate weights $\underline{\lambda}$) if for each $\sigma\colon K\hookrightarrow\overline{\mathbf{Q}}_p$ we have $\mathrm{HT}_\sigma(\rho)=\{\lambda_{\sigma,i}\}_{1\le i\le d}$, and we say that ρ has inertial type τ if $\mathrm{WD}(\rho)|_{I_K}\cong\tau$.

We often somewhat abusively write that a representation $\rho\colon G_K\to\mathrm{GL}_d(\mathbf{Z}_p)$ is crystalline (or potentially crystalline, or semistable, or ...) if the corresponding representation $\rho\colon G_K\to\mathrm{GL}_d(\mathbf{Q}_p)$ is crystalline (or potentially crystalline, or semistable, or ...).

Serre weights and Hodge–Tate weights

By a *Serre weight* $\underline{k}$ we mean a tuple of integers $\{k_{\overline{\sigma},i}\}_{\overline{\sigma}\colon k\hookrightarrow\overline{\mathbf{F}}_p,1\le i\le d}$ with the properties that

- $p-1\ge k_{\overline{\sigma},i}-k_{\overline{\sigma},i+1}\ge 0$ for each $1\le i\le d-1$, and
- $p-1\ge k_{\overline{\sigma},d}\ge 0$, and not every $k_{\overline{\sigma},d}$ is equal to $p-1$.

The set of Serre weights is in bijection with the set of irreducible $\overline{\mathbf{F}}_p$-representations of $\mathrm{GL}_d(k)$, via passage to highest weight vectors (see for example the appendix to [Her09]).

Each embedding $\sigma\colon K \hookrightarrow \overline{\mathbf{Q}}_p$ induces an embedding $\overline{\sigma}\colon k \hookrightarrow \overline{\mathbf{F}}_p$; if $K/\mathbf{Q}_p$ is ramified, then each $\overline{\sigma}$ corresponds to multiple embeddings σ. We say that $\underline{\lambda}$ is a lift of $\underline{k}$ if for each embedding $\overline{\sigma}\colon k \hookrightarrow \overline{\mathbf{F}}_p$, we can choose an embedding $\sigma\colon K \hookrightarrow \overline{\mathbf{Q}}_p$ lifting $\overline{\sigma}$, with the properties that:

- $\lambda_{\sigma,i} = k_{\sigma,i} + d - i$, and
- if $\sigma'\colon K \hookrightarrow \overline{\mathbf{Q}}_p$ is any other lift of $\overline{\sigma}$, then $k_{\sigma',i} = d - i$.

Lifting rings

Let $K/\mathbf{Q}_p$ be a finite extension, and let $\overline{\rho}\colon G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ be a continuous representation. Then the image of $\overline{\rho}$ is contained in $\mathrm{GL}_d(\mathbf{F})$ for any sufficiently large finite extension $\mathbf{F}/\mathbf{F}_p$. Let $\mathcal{O}$ be the ring of integers in some finite extension $E/\mathbf{Q}_p$, and suppose that the residue field of E is $\mathbf{F}$. Let $R_{\overline{\rho}}^{\square,\mathcal{O}}$ be the universal lifting $\mathcal{O}$-algebra of $\overline{\rho}$; by definition, this (pro-)represents the functor given by lifts of $\overline{\rho}$ to representations $\rho\colon G_K \to \mathrm{GL}_d(A)$, for A an Artin local $\mathcal{O}$-algebra with residue field $\mathbf{F}$. The precise choice of E is unimportant, in the sense that if $\mathcal{O}'$ is the ring of integers in a finite extension E'/E, then by [BLGGT14, Lem. 1.2.1] we have $R_{\overline{\rho}}^{\square,\mathcal{O}'} = R_{\overline{\rho}}^{\square,\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}'$.

Fix some Hodge type $\underline{\lambda}$ and inertial type τ. If $\mathcal{O}$ is chosen large enough that the inertial type τ is defined over $E = \mathcal{O}[1/p]$, and large enough that E contains the images of all embeddings $\sigma\colon K \hookrightarrow \overline{\mathbf{Q}}_p$, then we have the usual lifting $\mathcal{O}$-algebras $R_{\overline{\rho}}^{\mathrm{crys},\underline{\lambda},\tau,\mathcal{O}}$ and $R_{\overline{\rho}}^{\mathrm{ss},\underline{\lambda},\tau,\mathcal{O}}$. By definition, these are the unique $\mathcal{O}$-flat quotients of $R_{\overline{\rho}}^{\square,\mathcal{O}}$ with the property that if B is a finite flat E-algebra, then an $\mathcal{O}$-algebra homomorphism $R_{\overline{\rho}}^{\square,\mathcal{O}} \to B$ factors through $R_{\overline{\rho}}^{\mathrm{crys},\underline{\lambda},\tau,\mathcal{O}}$ (resp. through $R_{\overline{\rho}}^{\mathrm{ss},\underline{\lambda},\tau,\mathcal{O}}$) if and only if the corresponding representation of G_K is potentially crystalline (resp. potentially semistable) of Hodge type $\underline{\lambda}$ and inertial type τ. If τ is trivial, we will sometimes omit it from the notation. By the main theorems of [Kis08], these rings are (when they are nonzero) equidimensional of dimension

$$1 + d^2 + \sum_{\sigma} \#\{1 \le i < j \le d | \lambda_{\sigma,i} > \lambda_{\sigma,j}\}.$$

Note that this quantity is at most $1 + d^2 + [K:\mathbf{Q}_p]d(d-1)/2$, with equality if and only if $\underline{\lambda}$ is *regular*, in the sense that $\lambda_{\sigma,i} > \lambda_{\sigma,i+1}$ for all σ and all $1 \le i \le d-1$. As above, we have $R_{\overline{\rho}}^{\mathrm{crys},\underline{\lambda},\tau,\mathcal{O}'} = R_{\overline{\rho}}^{\mathrm{crys},\underline{\lambda},\tau,\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}'$, and similarly for $R_{\overline{\rho}}^{\mathrm{ss},\underline{\lambda},\tau,\mathcal{O}'}$. By [Kis08, Thm. 3.3.8] the localized rings $R_{\overline{\rho}}^{\mathrm{crys},\underline{\lambda},\tau,\mathcal{O}}[1/p]$ are regular, and thus the rings $R_{\overline{\rho}}^{\mathrm{crys},\underline{\lambda},\tau,\mathcal{O}}$ (which embed into their localizations away from p, since they are $\mathcal{O}$-flat) are reduced.

Algebra

Our conventions typically follow [Sta]. In particular, if M is an abelian topological group with a linear topology, then as in [Sta, Tag 07E7] we say that M is *complete* if the natural morphism $M \to \varprojlim_i M/U_i$ is an isomorphism, where

$\{U_i\}_{i \in I}$ is some (equivalently any) fundamental system of neighborhoods of 0 consisting of subgroups. Note that in some other references this would be referred to as being *complete and separated*.

If R is a ring, we write $D(R)$ for the (unbounded) derived category of R-modules. We say that a complex $P^\bullet$ is *good* if it is a bounded complex of finite projective R-modules; then an object $C^\bullet$ of $D(R)$ is called a *perfect complex* if there is a quasi-isomorphism $P^\bullet \to C^\bullet$ where $P^\bullet$ is good. In fact, $C^\bullet$ is perfect if and only if it is isomorphic in $D(R)$ to a good complex $P^\bullet$: if we have another complex $D^\bullet$ and quasi-isomorphisms $P^\bullet \to D^\bullet$, $C^\bullet \to D^\bullet$, then there is a quasi-isomorphism $P^\bullet \to C^\bullet$ ([Sta, Tag 064E]).

Stacks

Our conventions on algebraic stacks and formal algebraic stacks are those of [Sta] and [Eme]. We recall some terminology and results in Appendix A. Throughout the book, if A is a topological ring and $\mathcal{C}$ is a stack we write $\mathcal{C}(A)$ for $\mathcal{C}(\operatorname{Spf} A)$; if A has the discrete topology, this is equal to $\mathcal{C}(\operatorname{Spec} A)$.

Index