# CONTENTS

# Chapter 1

# The 1800s

As we have seen, Johns Hopkins University was the first American educational establishment to be founded with an aim of encouraging and providing facilities for research, and in the fall of 1875 its first president, Daniel Coit Gilman, traveled to Europe to headhunt the very best researchers to lead its departments. Mathematics was the first faculty to open, with James Joseph Sylvester appointed as its guiding light. Sylvester published many papers, including some that relate to graph theory.[1]

The story of Johns Hopkins and its mathematics during its first few years is essentially that of Sylvester, but also involves other notable figures. Two scholars important to its early history, and to the development of mathematics in America, were William Edward Story, a mathematician with a talent for organization but little luck, and Charles Sanders Peirce, a brilliant but somewhat wayward polymath. Also important to our story is Alfred Kempe, a compatriot of Sylvester's, whose erroneous solution of the four color problem was to have a profound influence on graph theory in America over the ensuing years.

## JAMES JOSEPH SYLVESTER

J. J. Sylvester was born James Joseph on September 3, 1814, in London. His father, a Jewish merchant, was named Abraham Joseph. In his teenage years, James Joseph added the surname Sylvester, as three names were a necessary requirement for possible migration to America, a step being taken by his brother at the time.

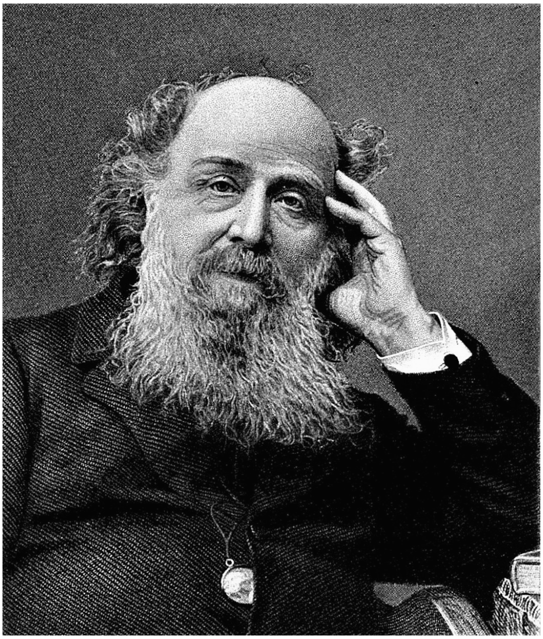
At the age of 14, Sylvester entered University College, London, where he was taught by Augustus De Morgan, the professor of mathematics, but after five months his family decided to withdraw him and send him to study at the Royal Institute School in Liverpool. In 1831, he went to St John's College in the University of Cambridge, but suffered from an illness that caused him to miss most of the academic years 1833–35.

Although a brilliant scholar, coming second in the 1837 Mathematical Tripos examinations, he was not permitted to receive his Cambridge degree because he was Jewish and unwilling to sign the Articles of the Church of England. Indeed, because of his religion, he was unable to gain a university position at either Cambridge or Oxford, even though his undoubted ability deserved such an appointment. However, from 1837 to 1841 he was professor of natural philosophy at University College, London, one of the few non-sectarian institutions, and in 1841 he was awarded bachelor's and master's degrees from Trinity College, Dublin.

In the same year, Sylvester was appointed professor of mathematics at the University of Virginia, but he resigned after only a few months following an unfortunate clash with a student and a lack of support from the university. Unable to obtain another post in America, he reluctantly returned to England where he gained employment as an actuary at a life insurance company in London; he also gave private lessons in mathematics. In 1846, he decided to study law, and during his training as a barrister he met the mathematician Arthur Cayley, whose four-year fellowship at Trinity College, Cambridge, had just ended. Unwilling to take Holy Orders, then a condition of appointment at Trinity, Cayley needed a profession and chose law, studying at Lincoln's Inn in London. Despite their very different personalities, Cayley and Sylvester became lifelong friends and collaborated on many mathematical problems.

In 1855, Sylvester became professor of mathematics at the Royal Military Academy at Woolwich, where he remained until 1870 when War Office regulations required him to retire at age 55. So Sylvester was already retired when in 1876 he received President Gilman's invitation to become the first professor of mathematics at Johns Hopkins University. In September of the previous year, Benjamin Peirce, a friend of Sylvester's, had already written to Gilman to urge him to engage Sylvester:[2]

> Hearing that you are in England, I take the liberty to write you concerning an appointment in your new university, which I think would be greatly for the benefit of our country and of American science if you could make it. It is that of one of the two greatest geometers of England, J. J. Sylvester. If you inquire about him, you will hear his genius universally recognized but his power of teaching will probably be said to be quite deficient. Now there is no man living who is more luminary in his language, to those who have the capacity to comprehend him than Sylvester, provided the hearer is in a lucid



James Joseph Sylvester (1814–97).

> interval. But as the barn yard fowl cannot understand the flight of the eagle, so it is the eaglet only who will be nourished by his instruction . . .
>
> Among your pupils, sooner or later, there must be one, who has a genius for geometry. He will be Sylvester's special pupil—the one pupil who will derive from his master, knowledge and enthusiasm—and that one pupil will give more reputation to your institution than ten thousand, who will complain of the obscurity of Sylvester, and for whom you will provide another class of teachers . . .
>
> I hope that you will find it in your heart to do for Sylvester—what his own country has failed to do—place him where he belongs—and the time will come, when all the world will applaud the wisdom of your selection.

Even though many considered Sylvester to be the finest mathematician in the English-speaking world, he was both surprised and delighted to receive Gilman's invitation to occupy a position from which he would derive considerable enjoyment and success.

On taking up his appointment in May of 1876, at a salary of $5000 per annum (which, at his insistence, was paid in gold),[3] Sylvester set about realizing Gilman's objective by initiating research work in the mathematics department. He selected two graduate fellows, George Bruce Halsted and Thomas Craig, to join the mathematics faculty, and in the fall he recruited William E. Story from Harvard.

Sylvester presented his inaugural lecture on February 22, 1877, the first anniversary of the official opening of the university. His presentation covered many subjects, including how mathematics should be taught and studied, and the role that Johns Hopkins should play in the development of mathematics and of further education in America. He also took the opportunity to attack those English universities that discriminated against all who were not Protestant Christians. Having encountered such prejudice himself, he criticized the damage that had been done to higher education by the exclusion of Jews, Catholics, and others.

In 1861, Yale College became the first American institution to confer doctoral degrees. The first mathematics doctorate there was in 1862, and later degrees were awarded for dissertations on "The Daily Motion of a Brick Tower Caused by Solar Heat" and "On Three-Bar Motion". In the 1870s, doctorates in mathematics were awarded four times at Yale, once at Cornell and at Dartmouth, and twice at Harvard and at Johns Hopkins. Sylvester had been quick to take on postgraduate students, and while in Baltimore he supervised eight of them:[4]

1878: Thomas Craig, *The Representation of One Surface upon Another, and Some Points in the Theory of the Curvature of Surfaces*

1879: George Bruce Halsted, *Basis for a Dual Logic*

1880: Fabian Franklin, *Bipunctual Coordinates*

1880: Washington Irving Stringham, *Regular Figures in n-Dimensional Space*

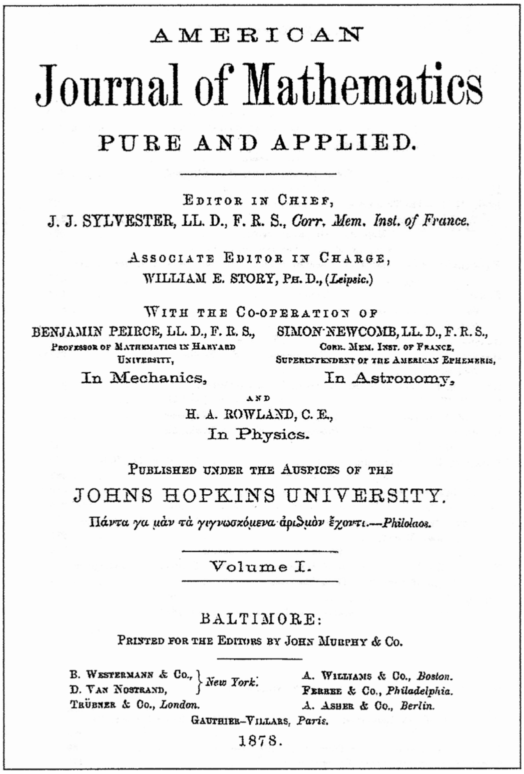
1882: Oscar Howard Mitchell, *Some Theorems in Numbers*

1883: William Pitt Durfee, *Symmetric Functions*

1883: George Stetson Ely, *Bernoulli's Numbers*

1884: Ellery William Davis, *Parametric Representations of Curves*

Another of Sylvester's preoccupations was the *American Journal of Mathematics*, the oldest mathematics journal in continuous publication in North America, and still being published today. Sylvester is usually credited as its founder, and with the help of William Story he published the first issue in 1878. The *Journal* was intended to be a vehicle for dia-


AMERICAN

Journal of Mathematics

PURE AND APPLIED.

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PUBLISHED UNDER THE AUSPICES OF THE
JOHNS HOPKINS UNIVERSITY.
Πάντα γα μὰν τὰ γιγνωσκόμενα ἀριθμὸν ἔχοντι.—*Philolaos.*

Volume I.

BALTIMORE:
PRINTED FOR THE EDITORS BY JOHN MURPHY & Co.

B. WESTERMANN & Co., D. VAN NOSTRAND, *New York.*
TRÜBNER & Co., *London.*
A. WILLIAMS & Co., *Boston.*
FERREE & Co., *Philadelphia.*
A. ASHER & Co., *Berlin.*
GAUTHIER-VILLARS, *Paris.*
1878.

The first issue of the *American Journal of Mathematics*, 1878.

log between American mathematicians, although space was also made available for foreign contributions. Indeed, the first issue included contributions from the Americans Simon Newcomb, C. S. Peirce, William Story, Thomas Craig, George Halsted, and Fabian Franklin, while other contributing authors were the Englishmen Arthur Cayley, William Kingdon Clifford, Edward Frankland, and Sylvester himself.

The first six volumes of the *Journal*, which covered 1878–83 and for which Sylvester was responsible, contained nearly two hundred articles. Papers by Sylvester featured in each volume, with thirty-two entries in total, and Cayley contributed to five of these volumes. Another early European contributor was the Danish mathematician Julius Petersen (see Interlude A), while other Americans included Benjamin Peirce and the rest of Sylvester's doctoral students. Moreover, Sylvester had been successful in promoting the new publication, with its "List of Subscribers" on July 1, 1878, totaling nearly 150; thirty-six of these were institutions, some of which took multiple copies, with three addresses in Paris, six in England, and two in Canada.

Sylvester was happy at Johns Hopkins University. For the first time in his life, he was able to teach and carry out research based on his own ideas and on chosen topics within a university environment. His *Mathematical Seminarium*, as he called his school of mathematics, was soon recognized in American mathematical circles and in Europe, while papers published by this group, most of which appeared in the *American Journal of Mathematics*, were widely read at home and abroad. The American mathematician George Andrews has commented that the collective output during these years amounted to a "monumental" contribution to combinatorics,[5] and it was widely accepted that Sylvester and his school were succeeding in putting America on the mathematical map.

In December of 1879, the university issued the first of its *Johns Hopkins University Circulars*. This publication was initially intended to communicate the full scope of the research being undertaken throughout the university; indeed, Sylvester published some of his notes, papers, and lectures there. It also included correspondence between (and information about) members of the various faculties, and in a letter to Cayley in 1883, Sylvester observed that the *Circulars* acted as "a sort of record of progress in connection with the work and personality of the Johns Hopkins".[6]

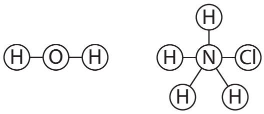
## Chemistry and Algebra

William Kingdon Clifford, a graduate of Trinity College, Cambridge, was one of the major British mathematicians of his time before his untimely death at the age of 33. Clifford believed, as did Sylvester, that there were direct connections between chemistry and the algebra of invariants.

Edward Frankland was a British chemist who held appointments in Britain and in continental Europe, and who for many years was responsible for the continuous analysis of London water supplies; he also served on a Royal Commission on water pollution. In 1866, based on the chemical theory of valency that had recently been introduced by August Kekulé and others, Frankland published his introductory *Lecture Notes for Chemical Students*,[7] in which he explained how atoms and bonds could be depicted graphically with circles and connecting lines. Beginning with water, he also listed the "symbolic formulae" and "graphic notations" for several chemical compounds. His symbolic formulae were expres-

Edward Frankland (1825–99) and William Kingdon Clifford (1845–79).

sions of the atoms and their quantities which combine to form chemical compounds, and for his graphic notation he represented each atom by a letter enclosed in a circle, with all single and multiple bonds identified by lines joining the appropriate circles. For example, he gave water the symbolic formula $\mathbf{O}H_2$ to indicate an oxygen atom (with valency 2) linked to two hydrogen atoms, and his symbolic formula for "ammonic chloride" was $\mathbf{N}H_4Cl$, with a nitrogen atom (with valency 5) linked to a chlorine atom and four hydrogen atoms.



Frankland's graphic notations for water ($H_2O$) and ammonium chloride ($NH_4Cl$).

Sylvester was already convinced of the connection between chemistry and algebra and was much taken with Frankland's *Lecture Notes*. In 1878, while at Johns Hopkins, Sylvester wrote a short note that was published in *Nature*.[8] Its opening paragraph shows his enthusiasm for the subject, and the extent to which he had been energized by Frankland:

> It may not be wholly without interest to some of the readers of *Nature* to be made acquainted with an analogy that has recently forcibly impressed

> me between branches of human knowledge apparently so dissimilar as modern chemistry and modern algebra. I have found it of great utility in explaining to non-mathematicians the nature of the investigations which algebraists are at present busily at work upon to make out the so-called *Grundformen* or irreducible forms appurtenant to binary quantics taken singly or in systems, and I have also found that it may be used as an instrument of investigation in purely algebraical inquiries. So much is this the case that I hardly ever take up Dr. Frankland's exceedingly valuable *Notes for Chemical Students*, which are drawn up exclusively on the basis of Kekulé's exquisite conception of *valence*, without deriving suggestions for new researches in the theory of algebraical forms. I will confine myself to a statement of the grounds of the analogy, referring those who may feel an interest in the subject and are desirous for further information about it to a memoir which I have written upon it for the new *American Journal of Pure and Applied Mathematics*, the first number of which will appear early in February.

This note was typical of Sylvester's writing style—scholarly, but verging on the flowery. As promised, he then expanded on this note in a paper in Volume I of his *American Journal of Mathematics*.[9]

### J. J. Sylvester: *On an application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics,—with three appendices* (1878)

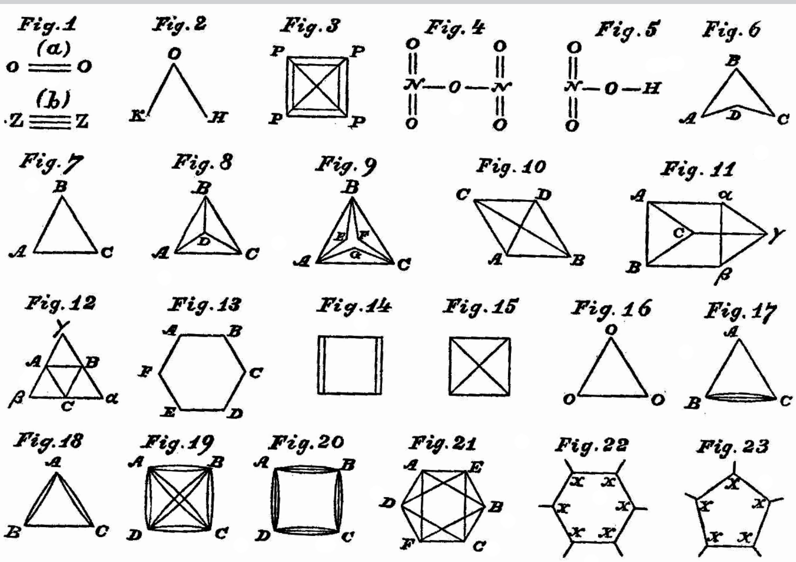
In this lengthy paper, Sylvester described in detail his reasons for believing in a close connection between the chemistry of organic molecules and the algebraic study of invariant theory. Its first two paragraphs give a flavor of his prose, in language that one now rarely encounters in academic papers:

> By the *new* Atomic Theory I mean that sublime invention of Kekulé which stands to the *old* in a somewhat similar relation as the Astronomy of Kepler to Ptolemy's, or the System of Nature of Darwin to that of Linnaeus;—like the latter it lies outside of the immediate sphere of energetic, basing its laws on pure relations of form, and like the former as perfected by Newton, these laws admit of exact arithmetic definitions.
>
> Casting about, as I lay awake in bed one night, to discover some means of conveying an intelligible conception of the objects of modern

> algebra to a mixed society, mainly composed of physicists, chemists and biologists, interspersed only with a few mathematicians, to which I stood engaged to give some account of my recent researches in this subject of my predilection, and impressed as I had long been with a feeling of affinity if not identity of object between the inquiry into compound radicals and the search for "Grundformen" or irreducible invariants, I was agreeably surprised to find, of a sudden, distinctly pictured on my mental retina a chemico-graphical image serving to embody and illustrate the relations of these derived algebraical forms to their primitives and to each other which would perfectly accomplish the object I had in view, as I will now proceed to explain.

In this paper he again heaped praise on Frankland's *Lecture Notes*:

> The more I study Dr Frankland's wonderfully beautiful little treatise the more deeply I become impressed with the harmony or homology (I might call it, rather than analogy) which exists between the chemical and algebraical theories.




Some chemical diagrams from Sylvester's paper.

Later in the same work he became even more eloquent, enthusing that "I feel as Aladdin might have done in walking in the garden where every tree was laden with precious stones", and continuing:

> Chemistry has the same quickening and suggestive influence upon the algebraist as a visit to the Royal Academy, or the old masters may be supposed to have on a Browning or a Tennyson. Indeed it seems to me that an exact homology exists between painting and poetry on the one hand and modern chemistry and modern algebra on the other. In poetry and algebra we have the pure idea elaborated and expressed through the vehicle of language, in painting and chemistry the idea enveloped in matter, depending in part on manual processes and the resources of art for its due manifestation.

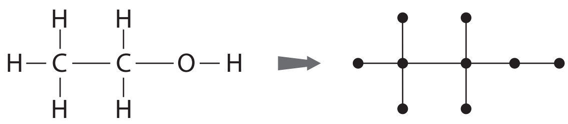
The analogy that Sylvester was trying to make was between "binary quantics" in algebra and atoms in chemistry. A *binary quantic* is a homogeneous expression in two variables, such as

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

and an *invariant* is a function of the coefficients $a$, $b$, $c$, and $d$ that remains essentially unaltered under linear transformations of the variables $x$ and $y$. Sylvester explained that this analogy evolved from his diagrammatic representations of chemical compounds, and in his 1878 note in *Nature*, he provided the following explanation of the connections between atoms and binary quantics. It is here that the word *graph* (in our modern sense) made its first appearance.

> The analogy is between atoms and *binary* quantics exclusively.
>
> I compare every binary quantic with a chemical atom. The number of factors (or rays, as they may be regarded by an obvious geometrical interpretation) in a binary quantic is the analogue of the number of *bonds*, or the *valence*, as it is termed, of a chemical atom.
>
> Thus a linear form may be regarded as a monad atom, a quadratic form as a duad, a cubic form as a triad, and so on.
>
> An invariant of a system of binary quantics of various degrees is the analogue of a chemical substance composed of atoms of corresponding *valences*.

> The order of such [an] invariant in each set of coefficients is the same as the number of atoms of the corresponding *valence* in the chemical compound . . .
>
> The weight of an invariant is identical with the number of the bonds in the chemicograph of the analogous chemical substance, and the weight of the leading term (or basic differentiant) of a co-variant is the same as the number of bonds in the chemicograph of the analogous compound radical. Every invariant and covariant thus becomes expressible by a *graph* precisely identical with a Kekuléan diagram or chemicograph . . . I give a rule for the geometrical multiplication of graphs, that is, for constructing a *graph* to the product of in- or co-variants whose separate graphs are given.




The graph of a chemical molecule.

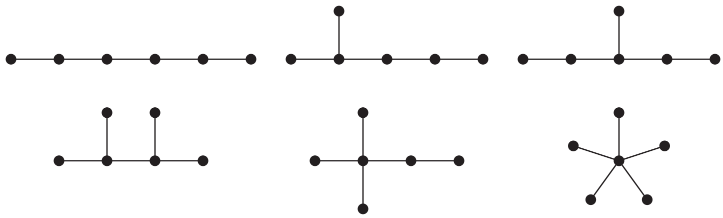
In spite of his enthusiasm for his analogy between chemistry and algebra, Sylvester was somewhat apprehensive that it might not meet with universal acceptance. Perhaps he suspected that it would be rejected, as he wrote to Simon Newcomb, a mathematician and astronomer at the Naval Observatory in Washington, who in 1884 became professor of mathematics and astronomy at Johns Hopkins:[10]

> I feel anxious as to how it will be received as it will be thought by many strained and over-fanciful. It is more a "reverie" than a regular mathematical paper. I have however added some supplementary mathematical matter which will I hope serve to rescue the chemical portion from absolute contempt. It may at the worst serve to suggest to chemists and Algebraists that they may have something to learn from each other.

Although there was some academic debate on the theory, it soon ran its course as it became apparent that the only link between chemistry and algebra was "the use of a similar notation".[11] Despite the detailed descriptions in Sylvester's note, the associated paper, and his subsequent correspondence with chemists and mathematicians, his ideas were generally considered to have only a passing connection between Kekulé's notation for chemical compositions and the theory of trees developed by Arthur Cayley.

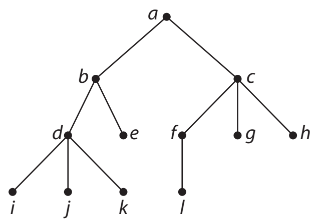
## Trees

A *tree* is a connected graph without cycles. In any tree the number of edges is one less than the number of vertices, and any connected graph with this property is a tree.

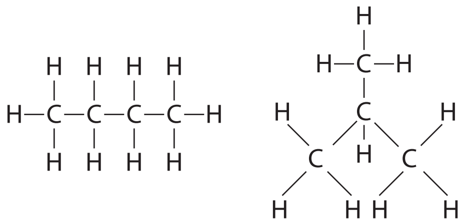


The trees with six vertices.

As we have seen, Cayley had met Sylvester during their years in London, and they remained lifelong friends and collaborators on mathematical matters. Between 1857 and 1889, Cayley produced a number of publications on trees. His first paper of 1857 was the earliest to use the word "tree" in our sense,[12] although both Gustav Kirchhoff (in connection with his work on electrical networks) and Karl Georg Christian von Staudt had used the idea around ten years earlier. Cayley's interest in trees originally "arose . . . from the study of operators in the differential calculus", being inspired by some of Sylvester's work on "differential transformation and the reversion of serieses". His earliest papers dealt with *rooted trees* only, in which one particular vertex is designated as the "root", usually placed at the top, as follows:



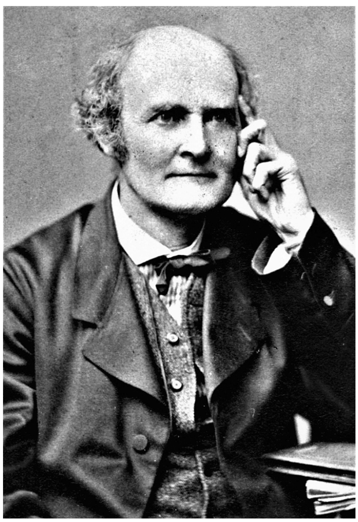

*Isomers* are chemical compounds with the same chemical formula but different atomic configurations; the next figure shows two molecules with the formula $C_4H_{10}$ (n-butane and 2-methyl propane, formerly called butane and isobutane). Cayley wrote several papers in which he related work on chemical compositions to his studies of trees, and in 1874 he published the short paper "On the mathematical theory of isomers",[13] in which his work on trees was used in the recognition and enumeration of chemical isomers. Two further papers, in 1875 and 1877, also dealt with the connections between trees and chemical composition.[14]




Two chemical isomers.

Sylvester wrote two short papers on trees while at Johns Hopkins. The first of these, "On the mathematical question, what is a tree?", was published in 1879 in the *Mathematical Questions with Their Solutions, from the "Educational Times"*. The second, on "ramifications" (his name for trees), appeared in the same year in the first issue of the *Johns Hopkins University Circulars*.[15]

Sylvester undoubtedly felt the lack of mathematical peers at Johns Hopkins University and in the United States generally, especially after the death of Benjamin Peirce in 1880, and wished that Cayley could visit him. Deprived of their frequent meetings of earlier days, Sylvester sent



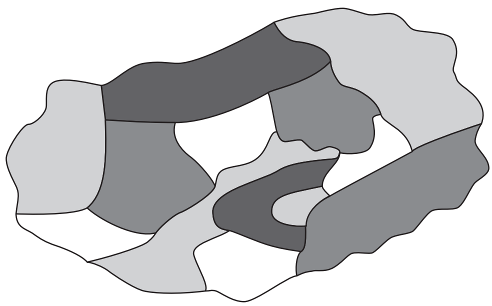
Arthur Cayley (1821–95).

him a number of letters in early 1881, inviting him to teach for a period at Johns Hopkins. Sylvester painted an encouraging picture of the social and academic life in Baltimore, and promised that Cayley would be rewarded both academically and financially. His letters, and a visit to Cayley in Cambridge in August 1881, finally persuaded Cayley to visit Johns Hopkins for six months during the spring semester of 1882, and to present a series of lectures during his visit. While there, Cayley also published papers in the *Johns Hopkins University Circulars* and the *American Journal of Mathematics*.

## ALFRED KEMPE

In 1852, Francis Guthrie, a former student of Augustus De Morgan's at University College, London, was coloring the counties of a map of England. Finding that just four colors were sufficient for this task, he asked the following more general question, which would become known as the *four color problem*:

> Can the countries of every map be colored with at most four colors so that no two neighboring countries are colored the same?



A map that requires four colors.

De Morgan became intrigued by the problem and wrote to the Irish mathematician William Rowan Hamilton and others asking whether four colors always suffice. He also mentioned it in a review of a book by William Whewell in *The Athenaeum*,[16] but died in 1871 without knowing the answer. The *four color theorem*, that all maps on the plane or a sphere can indeed be so colored, was not proved until 1976—by Kenneth Appel and Wolfgang Haken, two mathematicians at the University of Illinois at Urbana–Champaign (see Chapter 6).[17]

Arthur Cayley also became interested in the four color problem, and on June 13, 1878, at a meeting of the London Mathematical Society, he asked whether it had been solved; his query was recorded in the society's *Proceedings* and in a report of the meeting in *Nature*.[18] In a short note in the *Proceedings of the Royal Geographical Society* in April 1879,[19] he described some of the difficulties inherent in tackling the problem. He also made the useful suggestion that certain restrictions can be imposed on the maps under consideration without any loss of generality; in particular, he proved that when tackling the four color problem, we may assume that they are *cubic maps*, with exactly three countries at each meeting point. From now on, when desirable, we shall assume that the maps we are considering are cubic maps.

Alfred Bray Kempe (1849–1922).

Also attending the London Mathematical Society's meeting was Alfred Kempe (pronounced "kemp"), a former student of Cayley's at Cambridge, and yet another English mathematician who then became a barrister. Most of Kempe's early mathematical work was associated with the geometry of mechanical linkages. He later became treasurer of the Royal Society of London and was knighted in 1912.

Kempe was intrigued by Cayley's query on the four color problem and believed that he could solve it. On July 17, 1879, he announced a "solution" in *Nature*.[20] His attempted proof of the four color theorem was "On the geographical problem of the four colours" and—presumably at Cayley's suggestion—he submitted it to the newly founded *American Journal of Mathematics,* which was seeking papers from European authors. Kempe outlined the inherent challenge as follows:[21]

> Some inkling of the nature of the difficulty of the question, unless its weak point be discovered and attacked, may be derived from the fact that a very small alteration in one part of a map may render it necessary to recolour it throughout. After a somewhat arduous search, I have succeeded, suddenly, as might be expected, in hitting upon the weak point, which proved an easy one to attack. The result is, that the experience of the map-makers has not deceived them, the maps they had to deal with, viz: those drawn on simply connected surfaces, can, in every case, be painted with four colours. How this can be done I will endeavour—at the request of the Editor-in-Chief—to explain.

As we have seen, the editor in chief was J. J. Sylvester.

Kempe's paper was published later in the year in Volume 2 of the *American Journal of Mathematics*. Unfortunately, it contained a fatal error which was not discovered until eleven years later, during which time his proof had become generally accepted. In 1890 Percy Heawood exposed Kempe's error (see Interlude A).

### A. B. Kempe: *On the geographical problem of the four colours* (1879)

In 1750, Leonhard Euler observed that if a polyhedron has $F$ faces, $E$ edges, and $V$ vertices, then $F+V=E+2$. Using this result, Kempe deduced that if a map has $D$ districts or countries (not counting the external region), $B$ boundaries between countries, and $P$ "points of concourse" where at least three districts meet, then

$$P+D-B-1=0.$$

He then used a counting argument to show that, for a general map,

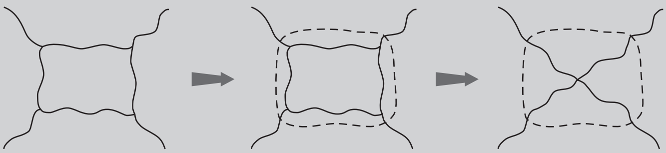
$$5d_1+4d_2+3d_3+2d_4+d_5-\text{etc.}=0,$$

where, for each $k$, $d_k$ is the number of districts of the map with $k$ boundaries, and the term "etc." is a collection of terms whose sum is positive. It follows that the sum of the first five terms is also positive, and so not all of $d_1$ to $d_5$ can be 0—that is:

> Every map drawn on a simply connected surface must have a district with less than six boundaries.

(A surface is *simply connected* if it is in one piece and has no "holes", so a plane or sphere is simply connected but a torus is not.)

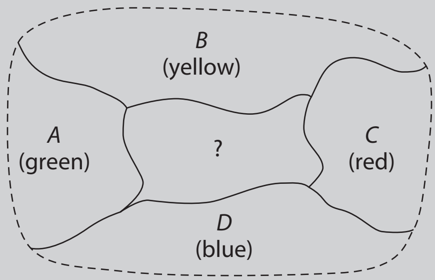
From this remarkable result, Kempe developed an algorithm for coloring any map by using a system of what he called "patches". This process involved selecting a district with five or fewer neighbors and covering it with a slightly larger blank piece of paper, or patch. He then joined all the boundaries that touch the edge of the patch to a single point within the patch; this has the effect of reducing the number of districts by 1, as shown below. The process is then repeated until only one district remains—as Kempe put it, "The whole map is patched out"—and this remaining district is then colored with any of the four colors.



Kempe's patching process.

Kempe then reversed the patching process, removing one patch at a time and successively coloring the uncovered districts with any available color until the original map was colored with four colors. Unfortunately, his explanation of this final step was incomplete. His patching procedure works as long as each restored district has at most three boundaries, but if it has four or five boundaries, then it may be surrounded by districts that require all four colors.

To overcome this difficulty, Kempe developed a strategy for coloring maps that is now called the *method of Kempe chains* or a *Kempe-chain argument*. In this method, we interchange two colors in order to enable the coloring of two neighboring districts that could not previously be colored. His argument was based on the fact that if we are given a map in which all the districts except one are colored, and if the districts that surround the uncolored one are assigned all four colors, then such an interchange of colors can enable the uncolored district to be colored also. This important line of argument was later to become one of the standard tools in the coloring of maps.



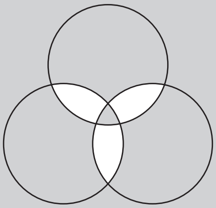

As an example of his method, consider the above map in which the uncolored district is surrounded by four districts that have been assigned different colors. The districts *A* and *C* that have been assigned the colors *green* and *red* are either connected by a continuous chain of *green* and *red* districts, or are not so connected. In the latter case, we may exchange the colors *green* and *red* in the chain of *green–red* districts connected to district *A* without altering the color of district *C*; this exchange of colors results in districts *A* and *C* both being colored *red*, so that the uncolored district can be colored *green*. However, if there is a continuous chain of *green–red* districts that joins *A* and *C*, then there is no advantage to making such an interchange of colors. But in this case there can be no continuous chain of *yellow* and *blue* districts joining districts *B* and *D*, and so we can recolor either of the *yellow–blue* chains connected to *B* or *D*. The districts *B* and *D* are then either both *yellow* or both *blue*, thereby allowing the four surrounding districts to be colored with three colors, and leaving the fourth color for the central one.

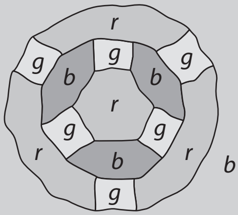
Kempe then considered maps containing an uncolored district with five sides, and the incorrect application of his method in this case gave rise to his famous error. His mistake was to attempt two color interchanges at the same time; either interchange by itself would have been valid, but to apply them simultaneously could result in two neighboring districts receiving the same new color.

Kempe then noted the following two special cases of interest:

If there is an even number of boundary lines at each point, two colors suffice to color the map.

If every district of a cubic map has an even number of boundary lines, three colors suffice.






Before leaving his paper, we mention that Kempe was the first to introduce the idea of *duality* and to pose the dual formulation of the four color problem:

> If we lay a sheet of tracing paper over a map and mark a point on it over each district and connect the points corresponding to districts which have a common boundary, we have on the tracing paper a diagram of a "linkage," and we have as the exact analogue of the question we have been considering, that of lettering the points in the linkage with as few letters as possible, so that no two directly connected points shall be lettered with the same letter.


Coloring a map and its dual graph.

So the four color problem can be equivalently stated in terms of coloring the points of a related linkage or graph, as just explained. This idea will reappear in Chapter 4, where we investigate duality, and Chapter 6, where we describe the eventual resolution of the problem.

In the subsequent months, Kempe issued two revisions of his proof.[22] The first, an untitled abstract that is "simpler, and is free from some errors which appeared in the former" was published in the *Proceedings of the London Mathematical Society* in 1879. There, Kempe mentioned his proof in the *American Journal of Mathematics*, and provided a streamlined description of his reduction and patching methods, with instructions for interchanging colors within chains. His second follow-up paper, "How to colour a map with four colours", published in *Nature* on February 26, 1880, was similar in content to the untitled abstract. It was again offered as a simplification, and in Kempe's own words,

> I have succeeded in obtaining the following simple solution in which mathematical formulae are conspicuous by their absence.

Neither of these revised versions indicated any recognition of his fundamental error.

On reading these two papers today, we cannot help arriving at the conclusion that Kempe was not trumpeting his claimed achievement, but was modestly confident that he had found the solution to a problem that had vexed and entertained a considerable number of mathematicians, both professional and amateur.

## WILLIAM STORY

William Edward Story was born on April 29, 1850, in Boston, Massachusetts. One ancestor, the Englishman Elisha Story, had arrived in America around 1700 and settled in Boston, while another took part in the Boston Tea Party.

William Story entered Harvard University in 1867, and was one of the first students to be awarded the newly created honors degree in mathematics. He then became one of the earliest American mathematicians to attend a German university, gaining a doctorate from Leipzig University

William Story
(1859–1930).

in 1875 for his dissertation, *On the Algebraic Relations Existing Between the Polars of a Binary Quantic*.

On returning to the United States, Story became a tutor at Harvard University. He is known to have impressed Benjamin Peirce while an undergraduate at Harvard, and this view increased as Story carried out his tutorial duties. Indeed, so convinced was Peirce of Story's merits, that when Sylvester solicited suggestions of suitable mathematicians to join the newly founded department of mathematics at Johns Hopkins University, Peirce recommended Story.

In the hot summer months of 1876, Sylvester decided to return to England, and it was left to the Johns Hopkins president, Daniel Gilman, to interview Story and to make any decision on his employment as Sylvester's assistant. Gilman's initial terse approach was not enthusiastically received by Story, who found it a little patronizing, and his reply was perhaps a trifle sharp. But he did ask for an interview, during which he outlined his ideas for the creation of a learned mathematical journal and a student society. Story was duly offered the Johns Hopkins position, but not before he had tried unsuccessfully to improve his status at Harvard. In the autumn, Story moved to Baltimore as an "associate" (equivalent to an assistant professor at some other universities). Later, in 1883, when the university introduced the title of associate professor, Story was promoted to that position.

Initially, life at Johns Hopkins went well for Story. He set about helping to develop the mathematics department, and his preference was to model it on the example he had experienced while in Germany. He assisted Sylvester in setting up the *American Journal of Mathematics* and was intimately involved in the founding of a mathematical society within the university. As Roger Cooke and V. Frederick Rickey have observed:[23]

> There is evidence that Story succeeded in founding his student mathematical society. *The Johns Hopkins University Circulars*, which are a rich source of information about the university, contain titles and reports of the talks given at the monthly meeting of the "Mathematical Society." From one of these we learn that when Lord Kelvin lectured at Hopkins in 1884, he spoke to a group of mathematicians who called themselves "the coefficients".

Because Sylvester was not good with either finance or management, he appointed Story as associate editor in charge of the *Journal*, and soon praised his second-in-command in a letter to Benjamin Peirce:[24]

> Story is a most careful managing editor and a most valuable man to the University in all respects and an honor to the University and its teachers from whom he received his initiation.

However, the way in which the *Journal* was run was soon to cause friction between Story and Sylvester. This was not a personal difference, but a dissimilarity in the ways that they believed the journal should be edited. During his time at Johns Hopkins, it was Sylvester's custom to spend each summer in England, leaving America in the late spring and returning for the start of the next academic year, while Story was left in charge for the duration of Sylvester's annual leave. The situation came to a head during Sylvester's absence from America through the publication of Kempe's paper on the four color problem.

Story had reviewed Kempe's paper, and on November 5, 1879, he presented the salient points of the "proof" to an audience of eighteen at a meeting of the Johns Hopkins Scientific Association. He then offered "a number of minor improvements", which he put in the form of a note that "was intended to make the proof absolutely rigorous". Story's "Note on the preceding paper" was then published in the *American Journal of Mathematics*, immediately following Kempe's.[25] By presenting it, Story was to incur the wrath of Sylvester, as we shall see.

In his note, Story addressed special cases that Kempe had not covered in his paper. He used both Euler's formula and the patch method, as Kempe had done, but endeavored to be more precise in his use of the formulas contained in Kempe's paper. Story's opening paragraph set out his intention, saying:

> it seems desirable, to make the proof absolutely rigorous, that certain cases which are liable to occur, and whose occurrence will render a change in the formulae, as well as some modification of the method of proof, necessary, should be considered separately.

It is disappointing that Story was not able to identify the major flaw in Kempe's argument in his review of Kempe's paper and in developing his own contribution.

**W. E. Story: *Note on the preceding paper* (1879)**

Story concentrated on two parts of Kempe's paper. The first of these expanded on the patching method, as applied to three of Kempe's figures, and the second dealt with the cases in which more than three districts meet at a point.

At each stage of the patching, Kempe had denoted the number of districts by $D$, the number of boundaries by $B$, and the number of points by $P$, and had used the corresponding symbols $D'$, $B'$, and

$P'$ after the next patch was removed. Story took up the argument that if the next patch had no point or boundary on it when it was removed, then an island would appear. Following Kempe, he concluded that, in this case,

$$P' = P, \quad D' = D + 1, \quad \text{and} \quad B' = B + 1.$$

However, if the patch had no point but only a single boundary, so that a peninsula or a district with two boundaries appeared when the patch was removed, then for the peninsula,

$$P' = P + 1, \quad D' = D + 1, \quad \text{and} \quad B' = B + 2,$$

and for the district with two boundaries,

$$P' = P + 2, \quad D' = D + 1, \quad \text{and} \quad B' = B + 3.$$

In the second case, Story referred to Kempe's Figure 15.

*Fig. 1.*

*Fig. 15.*

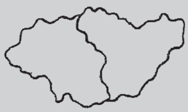


*Fig. 16.*

He went on to assert that

> These formulae hold only if the boundaries joined by the line on the patch counted as *two* (and not *one*, as in Figs. 16 and 1) before the patch was put on.

Story then considered a point where boundaries met, and where a district with $\beta$ boundaries appeared, when the patch was removed. This gave

$$P' = P + \beta - 1, \quad D' = D + 1, \quad \text{and} \quad B' = B + \beta.$$

Story deduced that these equations were identical to those of Kempe (although Kempe had used $\sigma$, rather than $\beta$),

> only when three and no more boundaries meet in each point of concourse about the district patched out,

giving

$$P'+D'-B'-1=P+D-B-1.$$

Story continued by detailing the alternative situation where the patch has no point of concourse, but only a single line that formed part of the boundary of a district or an island. Removing the patch then revealed Kempe's Figure 16 or Figure 1.

For the district,

$$P'=P+1, \quad D'=D+1, \quad \text{and} \quad B'=B+1,$$

and for the island,

$$P'=P+2, \quad D'=D+1, \quad \text{and} \quad B'=B+2,$$

and so in both cases,

$$P'+D'-B'-1=P+D-B.$$

Story next defined a *contour* as an aggregate of boundaries, with the contour being simple or complex, according to whether it contained one, or more than one, district. He asserted that one could improve Kempe's theorem by including contours in the patching procedure. In particular, where Kempe had stated that:

> in every map drawn on a simply connected surface the number of points of concourse and number of districts are together one greater than the number of boundaries,

Story's theorem read:

> in every map drawn on a simply connected surface the number of points of concourse and number of districts are together one greater than the number of boundaries and number of complex-contours together.

As he explained:

> If then $x$ of the contours formed by the boundaries of any map are complex, for that map
>
> $$P + D - B - 1 = x.$$

In the second half of his paper, Story questioned one of Kempe's claims that

> if we develop a map so patched out, since each patch, when taken off, discloses a district with less than six boundaries, not more than five boundaries meet at the point of concourse on the patch.

He asserted that this is valid only when the number of boundaries meeting in each point does not exceed 3, and detailed a procedure to overcome this restriction. His solution was to use an auxiliary patch whenever more than three boundaries met, thereby reducing to 3 the number of boundaries at a point; one could then continue the method of patching as described by Kempe. On completing the patching and arriving at a map with just one district and no boundary, coloring could then commence as the map was developed by removing patches (including auxiliary patches) in reverse order. By this method, he maintained, "the map will be coloured with four colours", as required.

## Sylvester versus Story

Sylvester believed that there had been an undue delay in the publication of Volume 2 of the *American Journal of Mathematics* during his absence in England. He also complained that previously agreed editorial decisions had been changed, and that Story should not have published his note. Sylvester went on to call this "unprofessional", and the relationship between the two colleagues became strained.

In 1880, Sylvester wrote to President Gilman, protesting Story's "conduct" and his "disobeying my directions". In June, he wrote again asking why Story had not sent him an acknowledgment regarding a paper that Sylvester had sent from England. Then, still aggrieved, Sylvester sent a further letter of eight pages to Gilman on July 22—indeed, such was his annoyance that his haste made parts of the letter even more illegible than usual.[26] In this letter, Sylvester complained that he was not told whether the *Journal* had been published and, if so, when. He also objected to his treatment by Story and questioned whether other contributors had been dealt with in an equally poor manner. Sylvester no longer had confidence in Story and was so incensed that he formally requested that Story should have no further involvement with the *Journal*. He also made it clear that Story could be made aware of his opinion and the contents of his letter.

Gilman mediated between the two, but Story's name did not appear on later issues of the *Journal*. Story resigned from the editorial board and began to seek a new position, a task that took him several years to accomplish. As with most disagreements, it would be wrong to put all of the blame on one party. Sylvester had certainly contributed to the delay in publication by making late changes to his own paper and rearranging the order of its contents. However, a letter from C. S. Peirce to Gilman, dated August 7, 1880, included the following comment:[27]

> I have received from Sylvester an account of his difficulty with Story. I have written what I could of a mollifying kind, but it really seems to me that Sylvester's complaint is just. I don't think Story appreciates the greatness of Sylvester, and I think he has undertaken to get the *Journal* into his own control in an unjustifiable degree . . . It is no pleasure to me to intermeddle in any dispute but I feel bound to say that Sylvester has done so much for the University that no one ought to dispute his authority in the management of his department.

By this time, Sylvester was well past what we now think of as normal retirement age. In February 1883, Henry Smith, Oxford University's Savilian Professor of Geometry, died unexpectedly, thereby prompting a search for a successor, preferably an Oxford man. News reached Sylvester, and on March 16 he wrote to Cayley indicating that he would probably offer himself as an applicant, as religious barriers had by then been removed. Sylvester submitted his resignation to Johns Hopkins in the fall of 1883 and returned to Britain on December 21. In January 1884, he wrote to Felix Klein in Germany giving further reasons for leaving America:[28]

> I resigned my position in Baltimore
>
> 1° Because I was anxious to return to my native country
>
> 2° Because I had reasons of a strictly individual and personal nature for wishing to quit Baltimore

> 3° (and *paramountly*) because I did not consider that my mathematical erudition was sufficiently extensive nor the vigor of my mental constitution adequate to keep me abreast of the continually advancing tide of mathematical progress to that extent which ought to be expected from one on whom practically rests the responsibility of directing and moulding the mathematical education of 55 millions of one of the most intellectual races of men upon the face of the earth.

There has been some discussion as to who really founded the *American Journal of Mathematics*. From the beginning, Gilman had desired all departments of his new university to found research-level journals, and the idea of one in mathematics had independently occurred to Story. But most commentators acknowledge Sylvester as the founder, and at his farewell banquet, on December 20, 1883, Gilman indeed gave him the credit. However, Sylvester's response indicated otherwise:[29]

> You have spoken about our *Mathematical Journal*. Who is the founder? Mr Gilman is continually telling people that I founded it. That is one of my claims to recognition which I strongly deny. I assert that he is the founder. Almost the first day that I landed in Baltimore . . . he began to plague me to found a *Mathematical Journal* on this side of the water—something similar to the *Quarterly Journal of Pure and Applied Mathematics* [of Oxford] . . . Again and again he returned to the charge, and again and again I threw all the cold water I could on the scheme, and nothing but the most obstinate persistence and perseverance brought his views to prevail. To him and to him alone, therefore, is really due whatever importance attaches to the foundation of the *American Journal of Mathematics*.

The reality is that Sylvester had the international standing, with links in Europe and previous experience of being involved in the creation of Oxford's *Quarterly Journal*, of which he was editor until 1878. Independently, Story had formulated the idea of a learned mathematical publication and wanted to be involved in its creation. However, without Gilman's continual encouragement, direction, and belief that such a journal would be of great benefit to mathematics in America, it probably could not have happened as it did in 1878.

What was Sylvester's legacy in the United States? Apart from the *American Journal of Mathematics*, he successfully established at Johns Hopkins University a successful graduate school that invested time and effort into training future researchers. This in turn had an effect on other educational institutions which then established graduate schools, and the level of mathematical research throughout America gradually improved. As a consequence, it was no longer necessary for graduates to journey abroad for postgraduate study, although some continued to do so.

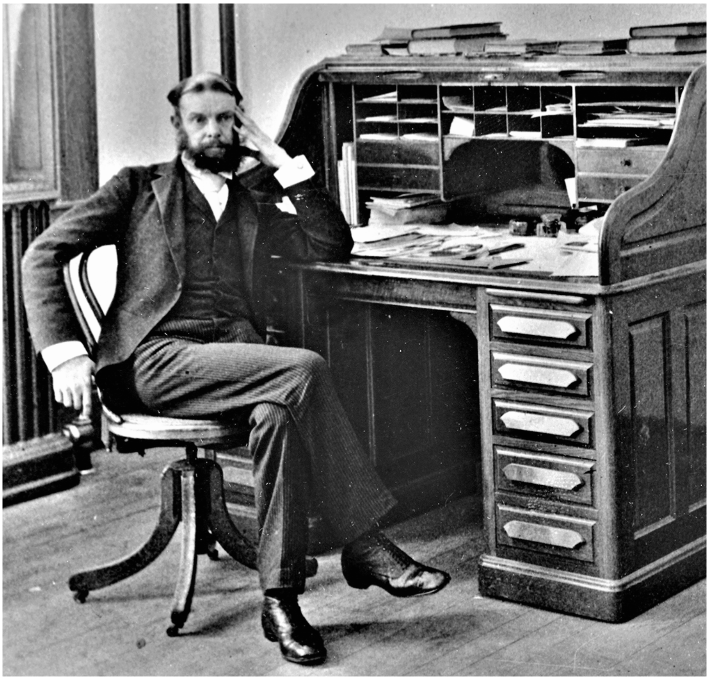
Sylvester was indeed appointed at the age of 69 to the Savilian Chair of Geometry at Oxford University, a position that he held for the rest of his life. In his late 70s, suffering from partial blindness, he returned to London with a deputy appointed to cover his Oxford duties.

An unpredictable, erratic, and flamboyant scholar, Sylvester could be brilliant, quick-tempered, and restless, filled with immense enthusiasm and an insatiable appetite for knowledge. Throughout his life, he had fought for the underdog in society and supported education for the working classes, for women, and for people who were discriminated against. He was awarded many honors and prizes, including his election as a Fellow of the Royal Society in 1839 at the age of 25, and received the Royal Society's Royal Medal in 1861 and the Copley Gold Medal (its highest award) in 1880. The lunar feature Crater Sylvester was named in his honor. He died on March 15, 1897, in London.

As for William Story, Sylvester's departure from Johns Hopkins left him with a similar desire to move to new pastures, and in 1887 he was offered the position of head of mathematics at the newly founded Clark University in Worcester, Massachusetts. His situation is best summed up by Roger Cooke and V. Frederick Rickey:[30]

> There were many reasons why Story might have wanted to leave Hopkins. He was not a full professor there, though he had been there thirteen years. He was not the editor of the *American Journal of Mathematics*, which had been one of his youthful ideas. Finally, he had come to feel that Hopkins was not the wonderful place intellectually that he thought it might and should be . . . But perhaps most importantly of all, he would have the opportunity to develop a department that focused on graduate education and on research. And he could do it the way that he thought best. For all these reasons, it is likely that the opportunity to move to Clark would have attracted Story.

Story did indeed develop a mathematics faculty according to his own ideas—and in particular a doctoral program with twenty-five degrees awarded between 1892 and 1921, nineteen under his direct supervision. Indeed, Story was so successful in his new position that for a time Clark University was considered by some to have the best mathematics department in America. But in spite of all his work, misfortune struck in 1921, when financial problems forced the university to close its graduate



William Story at Clark University.

program, and he was required to resign. In his later years, Story became interested in the history of mathematics and compiled a considerable bibliography of mathematics and mathematicians, which is now in the care of the American Mathematical Society. He died in Worcester, Massachusetts, on April 10, 1930.

## C. S. PEIRCE

Charles Sanders Peirce is usually remembered as a philosopher, mathematician, and logician, and for his controversial and unconventional lifestyle. He was born on September 10, 1839, in Cambridge, Massachusetts. As a young boy, he thrived on the intellectual atmosphere prevailing at the family home, where his father, Benjamin Peirce, entertained academics, politicians, poets, scientists, and mathematicians. Although this provided a scholastic environment, his father avoided discipline, fearing that it might inhibit independence of thought. Such an indulgent attitude provided a platform where the younger Peirce could show off

Charles Sanders Peirce
(1839–1914).

his undoubted genius, but it also left him ignorant of how to behave or interact with people. The lack of parental guidance made it difficult for him to fit in to society and led to problems in later life.

C. S. Peirce enrolled at Harvard College at age 15, but he did not shine in his work, preferring to study on his own with books of his own choosing. He graduated with a bachelor's degree in 1859 and entered the Lawrence Scientific School under the influence of his father, where he met with greater success than in his undergraduate years. He received a master's degree from Harvard in 1862 and a bachelor of science degree from the Lawrence Scientific School in 1863, receiving Harvard's first *summa cum laude* degree in chemistry. He remained at Harvard where he carried out graduate research, and in the spring of 1865 he presented the Harvard Lectures on *The Logic of Science*.

From 1859, for nearly thirty years and in parallel with his academic career, Peirce held a position as a part-time assistant at the Coast Survey; some of this time was under his father as director. In 1876, he produced one of his most notable inventions, the *Quincuncial Map Projection*, which was published in the *American Journal of Mathematics* in 1879; this earned him a reputation as one of the great mapmakers of the time. Although his invention was not taken up at the time, it was used in the mid-20th century to display air routes.

Meanwhile, he was producing seminal work in a wide range of subjects, including probability and statistics, psychophysics (or experimental psychology), and species classification. In addition, he carried out major astronomical research and explored mathematical logic, associative algebra, topology, and set theory. But either through choice or because

(continued...)

# INDEX