# Contents

and hence $W$ is positive definite. That $W(0) = 0$ follows immediately from $V(0) = 0$ and $\alpha \in \mathcal{K}_\infty$. Note that if we have an upper bound as discussed in Remark 2.19, the upper bound on $W$ is trivial to derive.

An application of the chain rule allows us to see that, for all $x \in \mathbb{R}^n \backslash \{0\}$,

$$\langle \nabla W(x), f(x) \rangle = \alpha'(V(x)) \langle \nabla V(x), f(x) \rangle \leq -\alpha'(V(x))\rho(x).$$

Since $\alpha'(s) > 0$ for all $s \in \mathbb{R}_{>0}$, as a consequence of the bound $\alpha_1(|x|) \leq V(x)$, for all $x \in \mathbb{R}^n \backslash \{0\}$ we see that $\alpha'(V(x))\rho(x) > 0$. Furthermore, since $\alpha'$ and $\rho$ are both continuous, and since $\rho(0) = 0$, we see that $\alpha'(V(x))\rho(x)$ is positive definite. Therefore, $W$ satisfies the decrease condition

$$\langle \nabla W(x), f(x) \rangle \leq -\hat{\rho}(x) \doteq -\alpha'(V(x))\rho(x), \quad \forall x \in \mathbb{R}^n$$

and, with the upper and lower bounds derived in (2.26), is hence a Lyapunov function for (2.1). □

Recall that a continuously differentiable $\alpha \in \mathcal{K}_\infty$ does not necessarily satisfy $\alpha'(s) > 0$ for all $s > 0$. For example, $\alpha(s) = \sin(s) + s$ is of class-$\mathcal{K}_\infty$ but satisfies $\alpha'(s) = 0$ for infinitely many $s > 0$.

**Theorem 2.21** (Exponentially decreasing Lyapunov functions)**.** *If there exists a Lyapunov function for system* (2.1) *satisfying* (2.17) *and* (2.19)*, then there exists a continuously differentiable function* $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *with* $W(0) = 0$ *and* $\hat{\alpha}_1 \in \mathcal{K}_\infty$ *so that, for all* $x \in \mathbb{R}^n$*,*

$$\hat{\alpha}_1(|x|) \leq W(x) \tag{2.27}$$

*and*

$$\langle \nabla W(x), f(x) \rangle \leq -W(x). \tag{2.28}$$

This theorem indicates that if we have one Lyapunov function, not only can we find an infinite number of Lyapunov functions (via Theorem 2.20), but we can find a Lyapunov function that decreases exponentially fast. To see this, applying the comparison principle (Lemma 2.14) to (2.28), we consider

$$\dot{w} \leq -w,$$

which yields $w(t) \leq w(0)e^{-t}$, and hence the Lyapunov function decreases exponentially.

This exponential decrease can be a very useful property as it is relatively easy to manipulate. However, exponential decrease of the Lyapunov function is not the same as exponential decrease of the solution of (2.1). Indeed, (2.27) yields

$$\hat{\alpha}_1(|x(t)|) \leq W(x(t)) \leq W(x(0))e^{-t},$$

which implies that $|x(t)| \leq \hat{\alpha}_1^{-1}(W(x(0))e^{-t})$. Since $\hat{\alpha}_1 \in \mathcal{K}_\infty$ is, in general, nonlinear, the exponential decrease of $W$ does not translate to an exponential decrease of $|x|$.

*Proof of Theorem 2.21:* As in the analytical proof of Theorem 2.16, there exists

$\hat{\rho} \in \mathcal{P}$ so that

$$\langle \nabla V(x), f(x) \rangle \leq -\hat{\rho}(V(x)), \quad \forall x \in \mathbb{R}^n.$$

Lemma A.8 yields a continuously differentiable $\alpha \in \mathcal{K}_\infty$ satisfying $\alpha'(s) > 0$ for all $s > 0$ and

$$\alpha(s) \leq \alpha'(s)\hat{\rho}(s), \quad \forall s \in \mathbb{R}_{\geq 0}.$$

Defining $W(x) = \alpha(V(x))$ for all $x \in \mathbb{R}^n$, Theorem 2.20 yields that $W$ is a Lyapunov function. We then calculate the decrease condition as

$$\begin{aligned} \langle \nabla W(x), f(x) \rangle &= \alpha'(V(x))\langle \nabla V(x), f(x) \rangle \\ &\leq -\alpha'(V(x))\hat{\rho}(V(x)) \;\leq\; -\alpha(V(x)) = -W(x) \end{aligned}$$

for all $x \in \mathbb{R}^n$. □

### 2.3.1 Time-Varying Systems

Lyapunov theory for time-varying systems is quite a bit more subtle than for time-invariant systems. We limit ourselves to two of the most important sufficient conditions, with an additional condition left to the exercises. Note that, in general, when dealing with time-varying systems (2.9) we also need to allow the associated Lyapunov functions to be time-varying as well.

**Theorem 2.22** (Lyapunov uniform asymptotic stability [86, Theorem 4.8])**.** *Given the time-varying system* (2.9) *with $f(t, 0) = 0$ for all $t \geq t_0 \geq 0$, if there exist a continuously differentiable function $V : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}$ and $\rho \in \mathcal{P}$ such that, for all $x \in \mathcal{D}$ and $t \geq t_0 \geq 0$,*

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|) \qquad \textit{and} \tag{2.29}$$

$$\tfrac{d}{dt}V(t, x) = \nabla_t V(t, x) + \langle \nabla_x V(t, x), f(t, x) \rangle \leq -\rho(|x|), \tag{2.30}$$

*then the origin is uniformly asymptotically stable. If additionally $\mathcal{D} = \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, then the origin is uniformly globally asymptotically stable.*

Comparing the conditions in Theorem 2.22 and Theorem 2.16, other than the function $V$ being time-varying, the major difference is the upper bound in (2.29). As we pointed out above (2.23), for time-invariant Lyapunov functions, this upper bound always exists. However, for time-varying Lyapunov functions, it is necessary to explicitly assume this upper bound to provide a bound that is independent of $t$. The property captured by this upper bound is sometimes called *decrescent*.

Warning: A common error when considering time-varying Lyapunov functions is to forget to take the partial derivative with respect to time; i.e., to leave out the term $\nabla_t V(t, x)$.

Having highlighted the importance of the upper bound in Theorem 2.22, we might ask what happens when this bound is removed. A partial answer is given in the following theorem and an interesting related case is discussed in [59, Chapter VII, §53].

**Theorem 2.23** (Lyapunov equiasymptotic stability theorem)**.** *Given the time-*

*varying system (2.9) with* $f(t,0) = 0$ *for all* $t \geq t_0 \geq 0$*, if there exist a continuously differentiable function* $V : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}_{\geq 0}$*,* $V(t,0) = 0$ *for all* $t \geq 0$*, a function* $\alpha \in \mathcal{K}$ *and* $\lambda > 0$ *such that, for all* $x \in \mathcal{D}$ *and* $t \geq t_0 \geq 0$*,*

$$\alpha(|x|) \leq V(t,x), \tag{2.31}$$

*and*

$$\tfrac{d}{dt}V(t,x) = \nabla_t V(t,x) + \langle \nabla_x V(t,x), f(t,x)\rangle \leq -\lambda V(t,x), \tag{2.32}$$

*then the origin is asymptotically stable. If additionally* $\mathcal{D} = \mathbb{R}^n$ *and* $\alpha \in \mathcal{K}_\infty$*, then the origin is globally asymptotically stable.*

*Proof.* The decrease condition (2.32) implies

$$V(t,x(t)) \leq V(t_0,x(t_0))e^{-\lambda(t-t_0)}.$$

The function $\alpha \in \mathcal{K}$ (or $\mathcal{K}_\infty$) is invertible on its range, which with the above expression yields

$$|x(t)| \leq \alpha^{-1}\left(V(t,x(t))\right) \leq \alpha^{-1}\left(V(t_0,x(t_0))e^{-\lambda(t-t_0)}\right).$$

□

It is important to note that the bound achieved in the proof is dependent not just on the elapsed time, $t - t_0$, but also explicitly on the initial time as seen in the first argument of the function $V$. The reader can verify that replacing (2.31) with (2.29) allows a continuation of the final calculation in the above proof that yields an upper bound that is independent of the initial time.

### 2.3.2 Instability

Based on our development of stability concepts and their relation to decreasing energy, or a generalized energy in the form of a Lyapunov function, it is reasonable to extend this same thinking to the definition of instability. One immediate form of this is to simply change the sign of the decrease condition (2.18) in Theorem 2.15.

**Theorem 2.24** (Lyapunov theorem for instability [59, Theorem 25.4])**.** *Given* (2.1) *with* $f(0) = 0$*, suppose there exist a continuously differentiable positive definite function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *and an* $\varepsilon > 0$ *such that*

$$\langle \nabla V(x), f(x)\rangle > 0 \tag{2.33}$$

*for all* $x \in \mathcal{B}_\varepsilon \backslash \{0\}$*. Then the origin is unstable.*

This is not the most general instability theorem and, in fact, cannot be applied in many cases of practical interest. Recall system (2.4),

$$\dot{x}_1 = x_1, \qquad \dot{x}_2 = -x_2. \tag{2.34}$$

Changing the direction of the inequality in (2.25), we see that (2.33) implies that the angle between the outward-facing normal $\nabla V(x)$ and $f(x)$ must be less than

$\frac{\pi}{2}$. It is clear that this is not possible for all points on the axis $x_2 \neq 0$ and $x_1 = 0$. The phase portrait of (2.34) is shown in Figure 2.4.

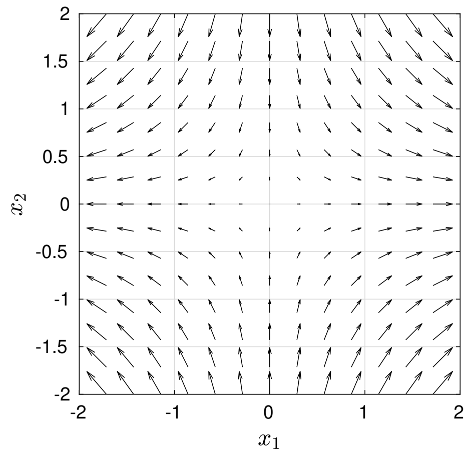


Figure 2.4: Phase portrait of the linear system (2.34) around the origin.

Unfortunately, it is not an unusual situation where a system may exhibit (asymptotically) stable behavior in some directions and unstable behavior in others. Equilibria satisfying Theorem 2.24 are usually called *completely unstable* to distinguish them from unstable equilibria with stable behavior in some directions. Fortunately, there is a more refined energy-like test for instability.

**Theorem 2.25** (Chetaev's theorem [86, Thm. 4.3])**.** *Given* (2.1) *with* $f(0) = 0$, *let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable function with* $V(0) = 0$ *and* $\mathcal{O}_r = \{x \in \mathcal{B}_r(0) |\ V(x) > 0\} \neq \emptyset$ *for all* $r > 0$. *If for certain* $r > 0$,

$$\langle \nabla V(x), f(x) \rangle > 0 \qquad \forall\ x \in \mathcal{O}_r, \tag{2.35}$$

*then the origin is unstable.*

The sets in Theorem 2.25 are indicated in Figure 2.5.

*Example 2.26.* Consider again system (2.4),

$$\dot{x}_1 = x_1, \qquad \dot{x}_2 = -x_2,$$

and take

$$V(x) = \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2. \tag{2.36}$$

We see that $V(x) > 0$ for all $|x_1| > |x_2|$. In particular, taking $x_0 = [x_1, x_2]^T$, we see that as long as $|x_1| > |x_2|$, we have $V(x_0) > 0$ even for $|x_0|$ arbitrarily small.

Then the expression (2.35) is

$$\langle \nabla V(x), f(x) \rangle = [x_1 \quad -x_2] \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = x_1^2 + x_2^2 > 0 \tag{2.37}$$

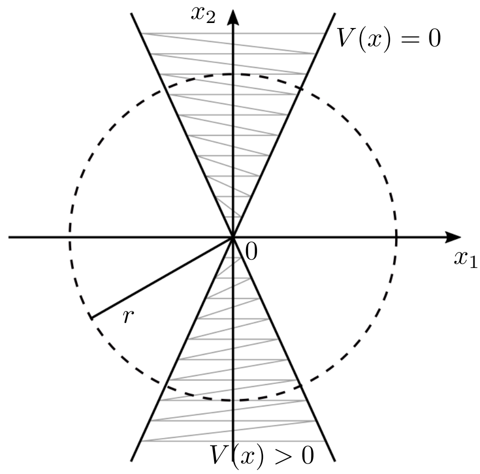


Figure 2.5: The sets involved in Theorem 2.25.

for all $x \neq 0$ so that, in particular, this expression is strictly positive where $V(x) > 0$.

### 2.3.3 Partial Convergence and the LaSalle-Yoshizawa Theorem

Based on the results so far, there is a gap between Theorem 2.15 guaranteeing stability, i.e., boundedness of solutions, and Theorem 2.16 guaranteeing asymptotic stability, i.e., convergence of all state variables $x_i$, $i \in \{1, \ldots, n\}$, to an equilibrium. In particular, if a subset of the states $x_i$, $i \in \{1, \ldots, n\}$ is converging to a stable equilibrium, neither Theorem 2.15 nor Theorem 2.16 can be used to capture this asymptotic behavior. The gap between these two results is occupied by the *LaSalle-Yoshizawa theorem* which we present in its general form for time-varying systems (2.9) here.

**Theorem 2.27** (LaSalle-Yoshizawa)**.** *Consider the time-varying system* (2.9) *with* $f(t, 0) = 0$ *for all* $t \geq t_0 \geq 0$*. Additionally, assume that* $f$ *is locally Lipschitz in* $x$ *uniformly in* $t$*, i.e., for all* $\mathcal{D} \subset \mathbb{R}^n$ *compact, there exists* $L > 0$ *such that*

$$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2| \qquad \forall\, x \in \mathcal{D},\ \forall t \geq t_0.$$

*If there exist a continuously differentiable function* $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$*, functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$*, and* $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *continuous such that, for all* $x \in \mathbb{R}^n$ *and* $t \geq t_0 \geq 0$*,*

$$\begin{aligned} \alpha_1(|x|) \leq & V(t, x) \leq \alpha_2(|x|) \qquad \textit{and} \\ \tfrac{d}{dt}V(t, x) = \nabla_t V(t, x) &+ \langle \nabla_x V(t, x), f(t, x)\rangle \leq -W(x), \end{aligned} \tag{2.38}$$

*then all solutions of* (2.9) *are globally uniformly bounded and satisfy*

$$\lim_{t \to \infty} W(x(t)) = 0. \tag{2.39}$$

In the case that $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is positive definite, there exists $\rho \in \mathcal{P}$ so that $\rho(|x|) \leq W(x)$ for all $x \in \mathbb{R}^n$, i.e., Theorem 2.27 reduces to Theorem 2.22. However, if $W(x)$ is positive semidefinite but not positive definite, then convergence to the set $\{x \in \mathbb{R}^n | \; W(x) = 0\}$ is guaranteed by (2.39), while Theorem 2.22 is not applicable. Thus, for example, if the assumptions of Theorem 2.27 are satisfied for a time-invariant system (2.1) and with

$$W(x) = x^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x \geq 0 \qquad \forall \, x \in \mathbb{R}^2,$$

then Theorem 2.15 only guarantees stability of the origin, while Theorem 2.27 additionally implies that $x_1(t) \to 0$. In particular, it is guaranteed that a subset of the states (i.e., $x_1$) is converging to the origin while the remaining states (i.e., $x_2$) stay bounded.

For a proof of Theorem 2.27 we follow the exposition in [93, Theorem A.8].

*Proof of Theorem 2.27:* Since $\dot{V}(t,x) \leq -W(x) \leq 0$, the function $V(\cdot, x(\cdot))$ is nonincreasing along solutions. Thus, it follows from $|x(t)| \leq \alpha^{-1}(V(t,x(t))) \leq \alpha^{-1}(V(t_0,x(t_0)))$ for all $t \geq t_0$ and for all $x(t_0) \in \mathbb{R}^n$ that $x(\cdot)$ is uniformly globally bounded, i.e., $|x(t)| \leq r \in \mathbb{R}$ for all $t \geq t_0$.

Since $|x(t)| \leq r$ and since $V(t,x(t))) \geq 0$ is nonincreasing, we can conclude that the limit $V_\infty = \lim_{t\to\infty} V(t,x(t)) \in \mathbb{R}$ exists. Integrating the decrease condition (2.38), it holds that

$$\lim_{t\to\infty} \int_{t_0}^{t} W(x(\tau)) \; d\tau \leq -\lim_{t\to\infty} \int_{t_0}^{t} \dot{V}(\tau, x(\tau)))d\tau \leq -\left[V_\infty - V(t_0, x(t_0))\right]$$

and thus the limit $\lim_{t\to\infty} \int_{t_0}^{t} W(x(\tau)) \; d\tau$ exists and is finite. (The existence follows from the monotonicity $0 \leq \int_{t_0}^{t_1} W(x(\tau)) \; d\tau \leq \int_{t_0}^{t_2} W(x(\tau)) \; d\tau$ for all $t_0 \leq t_1 \leq t_2$.)

Since $|x(t)| \leq r$ and $f$ is locally Lipschitz in $x$ uniformly continuous in $t$, for all $t \geq t_0 \geq 0$, it holds that

$$|x(t) - x(t_0)| = \left| \int_{t_0}^{t} f(x(\tau), \tau) \; d\tau \right| \leq \int_{t_0}^{t} |x(\tau)| \; d\tau \leq Lr|t - t_0|,$$

where $L$ denotes the Lipschitz constant of $f$ on the set $x \in \overline{\mathcal{B}}_r$. Choosing $\delta(\varepsilon) = \frac{\varepsilon}{Lr}$, it holds that

$$|x(t) - x(t_0)| \leq \varepsilon, \qquad \forall |t - t_0| \leq \delta(\varepsilon),$$

which implies that $x(\cdot)$ is uniformly continuous. Moreover, since $W$ is continuous, it is uniformly continuous on compact sets. From the uniform continuity of $x(\cdot)$ and $W(\cdot)$ we can thus conclude the uniform continuity of $W(x(\cdot))$. We can thus apply Barbalat's lemma (see Lemma A.4) showing that $W(x(t)) \to 0$ for $t \to \infty$. □

## 2.4 REGION OF ATTRACTION

We have seen in the previous sections that stability and attractivity are in general *local* properties of equilibria $x^e$ of differential equation (2.1). Asymptotic stability (defined through stability and attractivity) of an equilibrium requires the existence of a domain around the equilibrium such that all solutions of (2.1) starting in this domain converge to the equilibrium. However, we have not addressed how large the *region of attraction* (or *domain* or *basin of attraction*) of an equilibrium is.

**Definition 2.28** (Region of attraction)**.** *Consider* (2.1) *with an asymptotically stable equilibrium* $f(x^e) = 0$, $x^e \in \mathbb{R}^n$. *The region of attraction of* $x^e$ *is defined as*

$$\mathcal{R}_f(x^e) = \{x \in \mathbb{R}^n : x(t) \to x^e \text{ as } t \to \infty, x(0) = x\}. \tag{2.40}$$

For $x^e = 0$, we will use the shorthand notation $\mathcal{R}_f = \mathcal{R}_f(0)$. The region of attraction is an open, connected, invariant set.

The computation of the region of attraction is far from trivial. We illustrate two methods to estimate the region of attraction based on an example. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2 \end{aligned} \tag{2.41}$$

with a unique equilibrium at the origin. Note that the origin is locally asymptotically stable. Before proceeding, the reader should attempt to verify this using the common quadratic Lyapunov function candidate $V(x) = x_1^2 + x_2^2$. Why does this fail?

*Example 2.29.* We start by illustrating how Lyapunov's second method (or direct method) can be used to obtain an approximation of the region of attraction around the origin.

Let $\lambda_{\min}$ and $\lambda_{\max}$ denote the minimum and maximum eigenvalues, respectively, of the positive definite symmetric matrix $P$. We leave it to the reader to verify that the function $V(x) = x^T P x$ defined through the matrix

$$P = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

satisfies the inequality

$$\lambda_{\min}|x|^2 \leq V(x) \leq \lambda_{\max}|x|^2 \tag{2.42}$$

for $\lambda_{\min} = 0.69$ and $\lambda_{\max} = 1.81$ and is a Lyapunov function for the system (2.41) with respect to the origin.

The time derivative of $V(x(t))$ satisfies the equation

$$\tfrac{d}{dt}V(x) = -x_1^2 - x_2^2 - x_1^3 x_2 + 2x_2^2 x_1^2.$$

Young's inequality (Lemma A.4) provides the estimate

$$\tfrac{d}{dt}V(x) \leq -x_1^2 - x_2^2 + x_1^6 + \tfrac{1}{4}x_2^2 + x_1^4 + x_2^4 = -x_1^2\left(1 - x_1^2 - x_1^4\right) - x_2^2\left(\tfrac{3}{4} - x_2^2\right),$$

which implies that $\dot{V}(x) < 0$ whenever

$$1 - x_1^2 - x_1^4 > 0 \qquad \text{and} \qquad \tfrac{3}{4} - x_2^2 > 0.$$

These inequality constraints can be translated into the box constraints

$$\mathcal{C} = \{x \in \mathbb{R}^2 : -0.79 < x_1 < 0.79,\ -0.89 < x_2 < 0.89\}, \tag{2.43}$$

which are shown in Figure 2.6 as the black rectangle. Even though $V$ is a Lyapunov

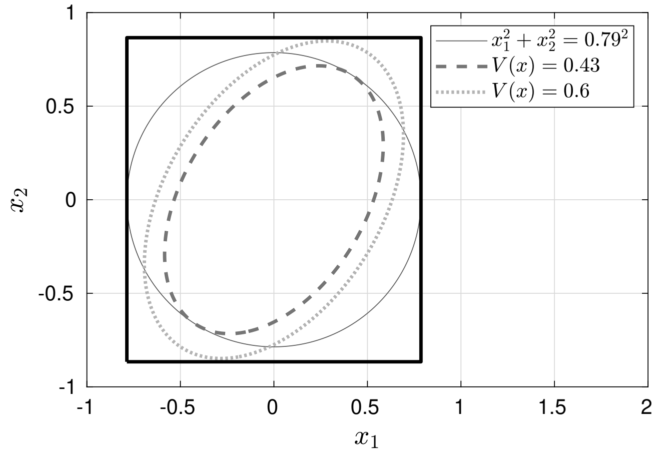


Figure 2.6: Estimate of the region of attraction through level sets of Lyapunov functions.

function with $\dot{V}(x) < 0$ in the set $\mathcal{C}$ defined in (2.43), $\mathcal{C}$ cannot be used as an estimate of the region of attraction since it is not a forward-invariant set. To this end, we need to define a sublevel set of the function $V$ contained in $\mathcal{C}$. Observe that the inclusions

$$\{x \in \mathbb{R}^2 : x^T P x \leq \lambda_{\min}\} \subset \{x \in \mathbb{R}^2 : x^T x \leq 1\},$$
$$\{x \in \mathbb{R}^2 : x^T x \leq 0.79^2\} \subset \mathcal{C}$$

are satisfied, which can be combined to obtain

$$\{x \in \mathbb{R}^2 : x^T P x \leq 0.79^2 \lambda_{\min}\} \subset \{x \in \mathbb{R}^2 : x^T x \leq 0.79^2\} \subset \mathcal{C}. \tag{2.44}$$

Thus, the forward-invariant sublevel set $\{x \in \mathbb{R}^2 : V(x) \leq 0.43\}$ is contained in $\mathcal{C}$ and hence can be used as an estimate for the region of attraction, i.e.,

$$\{x \in \mathbb{R}^2 : V(x) \leq 0.43\} \subset \mathcal{R}_f.$$

The corresponding sets are shown in Figure 2.6. As visualized through the dotted line, in the two-dimensional setting, a better estimate of the region of attraction can be obtained by increasing the level set $V(x) = c$, $c > 0$ until $\{x \in \mathbb{R}^2 : V(x) \leq c\}$ is no longer contained in $\mathcal{C}$. This is, however, in general only possible for systems of dimension $n \leq 2$, while the estimate (2.44) may be applicable regardless of the dimension of the system.

Since the estimate of the region of attraction is based on a level set of a Lyapunov function, the estimate automatically depends on the particular choice of $V$.

This example shows how an estimate of the region of attraction can be obtained from a quadratic Lyapunov function. If it is possible to visualize the Lyapunov function and the decrease, the estimate can be improved in general. However, the estimate derived in Example 2.29 is very conservative, as we will show in the next example. Here we show how simulating the system in backward time can be used to estimate the region of attraction. However, this approach will in general be limited to systems in $\mathbb{R}^2$.

*Example 2.30.* We consider again the asymptotically stable origin of the system (2.41) and look at the solution $x(\cdot)$ directly. However, rather than considering $t \to \infty$, consider simulating backwards in time; i.e., take $t \to -\infty$. To see the effect of this, let $\tau = -t$, which implies $d\tau = -dt$ and

$$\tfrac{d}{d\tau}x(\tau) = -\tfrac{d}{dt}x(-t) = -f(x(-t)) = -f(x(\tau)). \tag{2.45}$$

In other words, simulating the system backwards in time merely requires changing the sign of the vector field. Choosing an initial condition close to the origin and simulating backwards in time provides a continuum of initial conditions that converge to the origin in forward time. In $\mathbb{R}^2$ when the region of attraction is bounded, the backwards-in-time simulated trajectory converges to the boundary of the region of attraction.

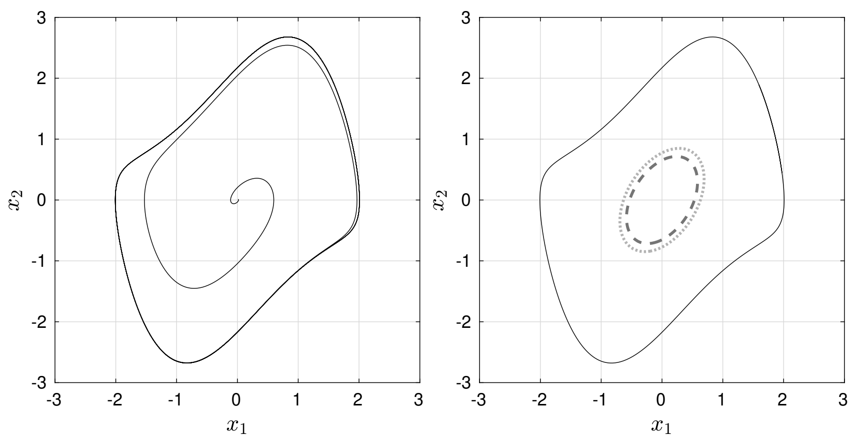


Figure 2.7: Backward simulation to estimate the region of attraction. On the right, additionally the level sets from Figure 2.6 are visualized for comparison.

In Figure 2.7 a solution of the system (2.41) in backward time, starting close to the equilibrium, is visualized on the left. On the right only the tail of the solution is shown, together with the level sets from Figure 2.6. In this example the tail of the solution provides an arbitrarily good approximation of the region of attraction of the origin $\mathcal{R}_f$.

## 2.5 CONVERSE THEOREMS

We have indicated that once we have one Lyapunov function we can construct infinitely many more Lyapunov functions (Theorem 2.20) and even Lyapunov functions which decrease exponentially (Theorem 2.21). However, the question remains: how do we find a first Lyapunov function that then leads us to all these others?

Unfortunately, this is a difficult problem. While there are some frequently used functions, such as quadratic forms, looking for a Lyapunov function remains something of a mysterious art. We can, though, assert that if a stability property holds, then we are guaranteed the existence of a Lyapunov function. Such theorems are referred to as *Converse Lyapunov Theorems.*

**Theorem 2.31** (Converse Lyapunov theorem [59, Theorem 49.4])**.** *If the origin is uniformly globally asymptotically stable for* (2.9)*, then there exist a (smooth) function* $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$*, functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$*, and a function* $\rho \in \mathcal{P}$ *such that, for all* $x \in \mathbb{R}^n$ *and all* $t \geq t_0 \geq 0$*,*

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|) \tag{2.46}$$

*and*

$$\nabla_t V(t,x) + \langle \nabla_x V(t,x), f(t,x) \rangle \leq -\rho(|x|). \tag{2.47}$$

*If* $f(t,x)$ *is periodic in* $t$*, then there exists* $V(t,x)$ *periodic in* $t$*. If* $f(t,x)$ *is time-invariant, then there exists* $V(t,x)$ *independent of* $t$*.*

A similar result can be given with respect to local uniform asymptotic stability for a function $V$ defined on a neighborhood around the origin.

While the above theorem does not tell us how to construct a Lyapunov function, it is nonetheless reassuring that the search for a Lyapunov function is not futile. Additionally, the above result allows us to pursue a certain form of modular feedback design whereby we assume a stabilizing feedback is available for some portion of the system of interest. Theorem 2.31 then guarantees that a Lyapunov function is available that can be used for subsequent design.

It is beyond the scope of our discussions here to prove Theorem 2.31, but it is reasonable to wonder how one proves Theorem 2.31 and yet does not end up with a usable Lyapunov function. The reason for this is that the constructed Lyapunov function relies on solutions of the system (2.1). For example, a building block in a standard converse Lyapunov theorem for exponential stability is the function

$$V(x) = \int_0^\infty |x(\tau)| e^\tau \, d\tau, \qquad x = x(0) \in \mathbb{R}^n,$$

which requires knowledge of the solutions of (2.1) from every initial condition $x \in \mathbb{R}^n$. However, solving (2.1) is precisely what we are trying to avoid by using a Lyapunov function. Hence, we see that the proof of Theorem 2.31 does not really provide a starting point for construction of a Lyapunov function.

The assumption of exponential stability, along with an assumption on the vector field, allows us to derive a Lyapunov function with a few extra properties.

**Theorem 2.32** (Converse Lyapunov theorem [86, Theorem 4.14])**.** *Suppose the origin is globally exponentially stable for* (2.1)*. Furthermore, assume* $f(\cdot)$ *is continuously differentiable and the Jacobian matrix* $[\partial f/\partial x]$ *is bounded. Then there exist constants* $a_1, a_2, a_3, a_4 > 0$ *and a continuously differentiable function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$

*such that, for all* $x \in \mathbb{R}^n$*,*

$$a_1|x|^2 \le V(x) \le a_2|x|^2, \tag{2.48}$$
$$\langle \nabla V(x), f(x)\rangle \le -a_3|x|^2, \tag{2.49}$$

*and*

$$|\nabla V(x)| \le a_4|x|. \tag{2.50}$$

### 2.5.1 Stability

Converse theorems are also available for Theorem 2.15 (stability) and Theorem 2.25 (instability). Interestingly, in contrast to Theorem 2.31, where time-invariant systems with an asymptotically stable origin always admit time-invariant Lyapunov functions, this is not the case for time-invariant systems with merely a stable origin.

Consider the second-order system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \sin^2\left(\frac{\pi}{x_1^2 + x_2^2}\right) x_2 - x_1 \tag{2.51}$$

which has *periodic orbits* given by

$$\Gamma_n = \left\{x \in \mathbb{R}^2 |\; x_1^2 + x_2^2 = \tfrac{1}{n}\right\}, \qquad n = 1, 2, \ldots$$

To see this, note that (2.51) reduces to the oscillator

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1$$

on $\Gamma_n$. Furthermore, trajectories spiral outward between periodic orbits; i.e., for initial conditions between periodic orbits $\Gamma_n$ and $\Gamma_{n+1}$, solutions converge to the outer periodic orbit $\Gamma_n$.

Suppose there exists a continuous function $V(x)$ that decreases on any periodic orbit. For any initial condition on a periodic orbit $\Gamma_n$, call it $x_n(0)$, there exists a time $T > 0$ so that $x_n(0) = x_n(T)$. The fact that $V(x)$ is decreasing then implies

$$V(x_n(T)) < V(x_n(0)) = V(x_n(T))$$

yielding a contradiction. Hence, $V(x)$ must be constant on any periodic orbit.

Since $V(x)$ is nonincreasing along trajectories and satisfies $V(0) = 0$, we have

$$V|_{\Gamma_1} \le V|_{\Gamma_2} \le \cdots \le V|_{\Gamma_n} \le \cdots \le V(0) = 0,$$

which contradicts the requirement that $V(x)$ be positive definite. Consequently, despite the fact that the origin is a stable equilibrium point and (2.51) is time-invariant, the system does not admit a time-invariant Lyapunov function.

## 2.6 INVARIANCE THEOREMS

In our original, energy-based analysis of the mass-spring system (Example 1.2) and the pendulum (Example 1.4), we were unable to definitively prove that the system came to rest at the origin. Recast in the theory of Lyapunov functions from Section 2.3, the problem was that the time derivative of the Lyapunov function was only negative *semidefinite* rather than negative *definite*. Our intuition, though, is that these systems do come to rest at the origin. Invariance theorems provide a tool that allows us to draw conclusions about asymptotic stability from Lyapunov functions that have a negative semidefinite time derivative.

### 2.6.1 Krasovskii-LaSalle Invariance Theorem

The following theorem was developed independently in the Soviet Union by Krasovskii and in the West by LaSalle. Hence, in the English language literature it is sometimes referred to as LaSalle's Invariance Theorem.

**Theorem 2.33** (Krasovskii-LaSalle invariance theorem [158, Thm. 5.3.77])**.** *Suppose there exists a positive definite and continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that, in an open domain containing the origin $0 \in \mathcal{D} \subset \mathbb{R}^n$, it holds that*

$$\langle \nabla V(x), f(x) \rangle \leq 0. \tag{2.52}$$

*Choose a constant $c > 0$ such that the level set $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ is bounded and contained in $\mathcal{D}$. Let $S = \{x \in \Omega_c : \langle \nabla V(x), f(x) \rangle = 0\}$ and suppose no solution other than the origin can stay identically in $S$. Then the origin is asymptotically stable.*

We illustrate the use of Theorem 2.33 on the pendulum and a mass-spring-damper example.

*Example 2.34.* Recall the pendulum system given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{g}{\ell} \sin x_1 - \tfrac{k}{m} x_2 \end{aligned}$$

and defined on the domain $\mathcal{D} = (-\pi, \pi) \times \mathbb{R}$. The total energy of the pendulum is given by

$$V(x) = mg\ell(1 - \cos x_1) + \tfrac{1}{2} m\ell^2 x_2^2,$$

which satisfies

$$\langle \nabla V(x), f(x) \rangle = -k\ell^2 x_2^2.$$

Observe that this expression is only negative semidefinite since the right-hand side is equal to zero for $x_2 = 0$ regardless of the value of $x_1$.

We see that $\langle \nabla V(x), f(x) \rangle = 0$ implies $x_2 = 0$, so in Theorem 2.33,

$$S = \{x \in \mathcal{D} : x_2 = 0\}.$$

In order for $x_2$ to remain at 0, $\dot{x}_2$ also needs to be zero, which implies $x_1 = 0$. Also, $x_2 = 0$ implies $\dot{x}_1 = 0$. Therefore, the only solution that can remain in $S$ is $x_1(t) = 0$, $x_2(t) = 0$. Hence, consistent with our intuition, in the presence of friction ($k > 0$), the downward equilibrium is asymptotically stable.

*Example 2.35.* Consider the mass-spring-damper system, shown in Figure 2.8, with a nonlinear spring

$$m\ddot{y} + b\dot{y}|\dot{y}| + k_0 y + k_1 y^3 = 0.$$

With $x_1 = y$ and $x_2 = \dot{y}$ we obtain the state space model

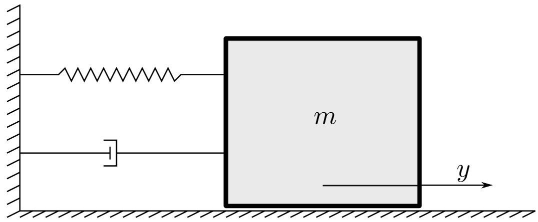


Figure 2.8: Mass-spring-damper system.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tfrac{1}{m}\left(-k_0 x_1 - k_1 x_1^3 - b x_2 |x_2|\right). \end{aligned}$$

Consider the candidate Lyapunov function

$$V(x) = \tfrac{k_0}{2m} x_1^2 + \tfrac{k_1}{4m} x_1^4 + \tfrac{1}{2} x_2^2. \tag{2.53}$$

Then

$$\begin{aligned} \langle \nabla V(x), f(x) \rangle &= \tfrac{k_0}{m} x_1 x_2 + \tfrac{k_1}{m} x_1^3 x_2 - \tfrac{k_0}{m} x_1 x_2 - \tfrac{k_1}{m} x_1^3 x_2 - \tfrac{b}{m} x_2^2 |x_2| \\ &= -\tfrac{b}{m} x_2^2 |x_2| \le 0. \end{aligned}$$

As in the pendulum example, $\langle \nabla V(x), f(x) \rangle = 0$ implies $x_2 = 0$ and hence

$$S = \{x \in \mathbb{R}^2 : x_2 = 0\}.$$

In $S$, $\dot{x}_1 = 0$ and in order to stay at $x_2 = 0$, we require $\dot{x}_2 = 0$. This implies

$$0 = -\tfrac{1}{m}(k_0 x_1 + k_1 x_1^3) \quad \Rightarrow \quad \left[ x_1 = 0 \;\; \textit{or} \;\; x_1 = \pm j\sqrt{\tfrac{k_0}{k_1}} \right].$$

Therefore, $x = 0$ is asymptotically stable.

### 2.6.2 Matrosov's Theorem

Theorem 2.33 only applies to time-invariant systems and does not appear to be directly generalizable to time-varying systems. However, with the use of a second function, the following result was provided by Matrosov. Similarly to what is done in Theorem 2.33, denote the set where the time derivative of the time-varying Lyapunov function $V$ is zero by

$$S = \{x \in \mathbb{R}^n : \nabla_t V(t,x) + \langle \nabla_x V(t,x), f(t,x) \rangle = 0\}. \tag{2.54}$$

**Theorem 2.36** (Matrosov invariance theorem [59, Theorem 55.3]). *Given continuously differentiable functions $V, W : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$, suppose that*

1. *$V$ is positive definite and decrescent; that is, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that for all $x \in \mathbb{R}^n$ and all $t \geq 0$,*

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|);$$

2. *the time derivative of $V$ along solutions of (2.9) is negative semidefinite; that is,*

$$\nabla_t V(t, x) + \langle \nabla_x V(t, x), f(t, x) \rangle \leq 0;$$

3. *$W$ is bounded; that is, there exists $h \geq 0$ so that for all $x \in \mathbb{R}^n$ and all $t \geq 0$*

$$|W(t, x)| \leq h;$$

4. *the time derivative of $W$ along solutions of (2.9) is bounded away from zero on $S$ in the following sense: for every $a > 0$ there exist $r, b > 0$ so that*

$$|\nabla_t W(t, x) + \langle \nabla_x W(t, x), f(t, x) \rangle| > b$$

*for all $t \geq 0$ and all $x$ in the set*

$$\{x \in \mathbb{R}^n : |x| > a \text{ and } |x|_S < r\}.$$

*Then the origin is uniformly globally asymptotically stable.*

Here, $|x|_S = \inf_{y \in S} |x - y|$ denotes the distance to the set $S$. The intuition behind Theorem 2.36 is as follows. The negative semidefinite time derivative of $V$ indicates that solutions of (2.9) converge toward the set $S$ where the time derivative is zero. In a neighborhood of the set $S \backslash \{0\}$ as well as in $S$, however, the function $W$ necessarily grows (either positive or negative) because its time derivative is bounded away from zero. Furthermore, $W$ is bounded, which implies that $W$ cannot grow indefinitely. The conclusion, then, is that eventually every solution needs to approach the origin, where the time derivative of both $V$ and $W$ are zero.

## 2.7 EXERCISES

*Exercise 2.1.* Prove that stability as phrased in Definition 2.1 is equivalent to the existence of $\alpha \in \mathcal{K}$ satisfying (2.3).

*Hint:* The fact that for any continuous and positive function $\rho \in \mathcal{P}$ there exists a function $\alpha \in \mathcal{K}$ such that $\rho(s) \leq \alpha(s)$ for all $s \in \mathbb{R}_{\geq 0}$ might be useful, [81, Lemma 1]. Moreover, you can assume that $\delta$ in Definition 2.1 depends continuously on $\varepsilon$.

*Exercise 2.2.* Consider the differential equation (2.6) with unique equilibrium at the origin. In this exercise we numerically investigate attractivity and stability of the origin.

1. Write a MATLAB function

```
dx = odeVinograd(t,x)
```

capturing the dynamics of the ordinary differential equation (2.6).

2. Solve the ordinary differential equation (2.6) for different initial values $x_0$ using `ode45.m` and visualize the solutions $(x_1(t), x_2(t))$ in the $x_1$-$x_2$-plane. For the numerical solutions, select the time span long enough so that the behavior $x(t) \to [0,0]^T$ can be observed.
3. Visualize the phase portrait of the ordinary differential equation (2.6) in a neighborhood around the origin by using the function `quiver.m`.
4. Explain based on your visualizations and based on the $\varepsilon$-$\delta$-stability criterion (Definition 2.1) why the origin of the ordinary differential equation is unstable.

*Exercise 2.3.* Modify the analytical proof of Theorem 2.16 to obtain the proof of Theorem 2.17.

*Exercise 2.4.* Consider the dynamics of the pendulum (1.11) together with the Lyapunov function

$$V(x) = \tfrac{1}{2}x^T \begin{bmatrix} \frac{1}{2}\left(\frac{k}{m}\right)^2 & \frac{1}{2}\frac{k}{m} \\ \frac{1}{2}\frac{k}{m} & 1 \end{bmatrix} x + \tfrac{g}{\ell}(1-\cos(x_1)) \tag{2.55}$$

derived in Example 2.18.

Construct the sets in the geometric proof of asymptotic stability visualized in Figure 2.3 for the Lyapunov function (2.55). In particular, using the parameters $g = 9.81$, $m = 1$, $\ell = 4$, and $k = 0.1$, for $\varepsilon = 3$ numerically find $c > 0$ and $\delta > 0$ such that

$$\overline{\mathcal{B}}_\delta(0) \subset \{x \in [-\pi,\pi]\times\mathbb{R} : \ V(x) \le c\} \subset \overline{\mathcal{B}}_\varepsilon(0)$$

is satisfied. Visualize the sets (as in Figure 2.3) together with the phase portrait of the ordinary differential equation (1.11).

*Exercise 2.5.* Consider the one-dimensional differential equations

$$\dot v = v, \qquad \dot w = 0, \qquad \dot x = -x^3, \qquad \dot y = -y,$$

which all have the origin as unique equilibrium.

Investigate the stability properties of the origin of the differential equations through Lyapunov or Lyapunov-like functions. In particular, what is the difference between instability, stability, asymptotic stability and exponential stability?

*Exercise 2.6.* Consider the ordinary differential equation

$$\dot x_1 = x_1 - x_1x_2, \qquad \dot x_2 = -x_2 + x_1x_2. \tag{2.56}$$

Use Theorem 2.25 and the function $V(x) = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2$ to show that the origin of the differential equation is unstable (2.56).

*Exercise 2.7.* Consider the functions

$$\begin{aligned} V_1(t,x) &= x_1^2(1+\sin^2(t)) + x_2^2(1+\cos^2(t)) \\ V_2(t,x) &= x_1^2 + x_2^2(1+t). \end{aligned}$$

Show that $V_1$ is decrescent while $V_2$ is not decrescent.

*Exercise 2.8.* Consider the differential equation

$$\dot{x}_1 = x_2^3, \qquad \dot{x}_2 = -x_1^3 - x_2. \tag{2.57}$$

Use Theorem 2.33 together with the function $V(x) = \frac{1}{4}x_1^4 + \frac{1}{4}x_2^4$ to show that the origin of (2.57) is asymptotically stable.

*Exercise 2.9.* In Example 2.18 we have derived the Lyapunov function

$$V(x) = \frac{1}{2}x^T P x + \frac{g}{\ell}(1 - \cos(x_1)), \qquad P = \begin{bmatrix} \frac{1}{2}\left(\frac{k}{m}\right)^2 & \frac{1}{2}\frac{k}{m} \\ \frac{1}{2}\frac{k}{m} & 1 \end{bmatrix},$$

with parameters $g, \ell, k, m \in \mathbb{R}_{>0}$, with respect to the origin $x^e = 0$ of the differential equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{g}{\ell}\sin(x_1) - \frac{k}{m}x_2. \tag{2.58}$$

1. Write MATLAB functions

   ```
   dx = odePendulum(t,x,parameters)
   ```

   and

   ```
   Vx = LyapunovPendulum(x,parameters)
   ```

   capturing the dynamics. The Lyapunov function and the parameters $g, \ell, k, m$ are stored in `parameters`.
2. Visualize the Lyapunov function on the domain $[-\pi, \pi]^2$ by using the command `surf.m`. Solve the differential equation with respect to the initial condition $x(0) = [1, 1]^T$ and the parameters $g = 9.81$, $k = 0.1$, and $\ell = m = 1$. Visualize the solution $(x_1(t), x_2(t), V(x(t)))$ using `plot3.m` in the same figure as the Lyapunov function.
   *Hint:* The additional option `'linestyle','none'` in `surf.m` might improve the plot. To ensure that the solution is visible, use the option `'linewidth',2` and `'color','red'` in `plot3.m`.
3. Use `plot.m` to visualize $(t, (x_1(t))$ and $(t, x_2(t))$ (for $x(0) = [1, 1]^T$, $t \in [0, 50]$, and the parameters $g = 9.81$, $k = 0.1$, and $\ell = m = 1$).
4. Use `plot.m` to visualize $(t, V(x(t)))$ and $(t, |x(t)|^2)$ (for $x(0) = [1, 1]^T$, $t \in [0, 50]$, and the parameters $g = 9.81$, $k = 0.1$, and $\ell = m = 1$). Is $\tilde{V}(x) = |x|^2$ a Lyapunov function for the pendulum?

## 2.8 BIBLIOGRAPHICAL NOTES AND FURTHER READING

Aleksandr Mikhailovich Lyapunov published both his first and second methods for stability analysis in his doctoral dissertation in 1892 [102]. We have focused on his second method (also sometimes referred to as the direct method) in this chapter.

This chapter has largely focused on the *analysis* of nonlinear systems as opposed to the *synthesis* or *design* of feedback systems. Standard texts with a more comprehensive coverage of analysis topics include [86] and [158]. Despite its age, [59] remains an excellent text for topics in stability theory.

The short monograph [31] uses the same notation as this book and covers Lyapunov and control Lyapunov results for differential inclusions, a more general class of systems than the ones discussed here. Control Lyapunov functions are introduced later in this book in Chapter 9.

The time-invariant system in Section 2.5.1 with a stable origin that requires a time-varying Lyapunov function is from [22, Example 4.11]. The system of Exercise 2.2 possessing an attractive but unstable origin is from [59, Sec. 40], where attribution is given to a 1957 paper (in Russian) by R. E. Vinograd.

In the context of time-varying systems, we have used the term *asymptotic stability* to implicitly include uniformity in the initial state and *uniform asymptotic stability* to cover uniformity in both the initial state and initial time. We have done so because non-uniformity in the initial state appears to be extremely rare. However, what we have termed asymptotic stability is sometimes referred to as equiasymptotic or non-uniform in time stability in order to reserve the term asymptotic stability for a stability and convergence property that is uniform neither in time nor in the initial state (see [59]). Additional texts on nonlinear systems analysis with a significant coverage of time-varying systems include [131] and [167].

The history of converse theorems (Section 2.5) captures much of the history of state space methods, particularly in relation to initial developments in the Soviet Union and the West trying to rapidly catch up following the launch of Sputnik. See [82].

More general versions of Theorem 2.33 are possible whereby $V$ need not be positive definite. Furthermore, rather than requiring the negative semidefinite decrease on all of $\mathbb{R}^n$, attention can be restricted to an invariant set and convergence is guaranteed to a (smaller) invariant set. See [86, Theorem 4.4].

# Chapter Three

## Linear Systems and Linearization

All real-world systems are nonlinear. The clearest example of this is that constraints are inherently nonlinear. However, a linear model in many cases provides a very good approximation, particularly when restricted to some region of the state space. This is advantageous because linear systems provide a significant amount of structure that can be exploited in both analysis and design. For example, we can derive closed-form solutions for linear ordinary differential equations and there are several constructive and algebraic methods for analysis and design.

Indeed, many books have been written on linear systems theory, and some references are provided in the bibliographical notes in Section 3.7. Here, we present only those results necessary for our subsequent development of nonlinear topics.

In addition to linear systems, Lyapunov functions for polynomial systems obtained through sum of squares programming are discussed in Section 3.4.

### 3.1 LINEAR SYSTEMS REVIEW

As the simplest possible example, consider a one-dimensional system

$$\dot{x} = ax,$$

with initial state $x(0) \in \mathbb{R}$ and constant $a \in \mathbb{R}$. It is easy to verify that the solution is given by

$$x(t) = x(0)e^{at}, \qquad t \geq 0,$$

since $\frac{d}{dt}x(t) = ax(0)e^{at} = ax(t)$. Furthermore, the origin is:

- (uniformly) globally exponentially stable if and only if $a < 0$;
- globally stable but not exponentially stable if and only if $a = 0$; and
- unstable if and only if $a > 0$.

Finally, it is trivial to see that, when $a < 0$, $V(x) = x^2$ is a Lyapunov function that guarantees global exponential stability (recall Theorem 2.17). While quite simple, this example contains many of the core elements of linear systems theory, particularly in terms of stability theory and Lyapunov functions.

Consider now the linear system

$$\dot{x} = Ax \tag{3.1}$$

with initial condition $x(0) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ (that is, $A$ is an $n \times n$ matrix consisting of real elements). The solution of (3.1) depends on the matrix exponential

$$x(t) = e^{At}x(0) = \left(\sum_{k=0}^{\infty} \frac{1}{k!}(At)^k\right) x(0).$$

### 3.1.1 Stability Properties for Linear Systems

Suppose that the matrix $A \in \mathbb{R}^{n\times n}$ is diagonalizable. Then there exists an invertible matrix $T \in \mathbb{C}^{n\times n}$ so that $\Lambda = T^{-1}AT$, where $\Lambda \in \mathbb{C}^{n\times n}$ is a diagonal matrix with the eigenvalues of $A$, denoted by $\lambda_i$, on the diagonal and the columns of $T$ contain the corresponding eigenvectors of the matrix $A$.

We first observe that

$$A^k = (T\Lambda T^{-1})(T\Lambda T^{-1})\cdots(T\Lambda T^{-1}) = T\Lambda^k T^{-1}.$$

Furthermore, since $\Lambda$ is diagonal, raising it to a power is the same as raising each diagonal element of $\Lambda$ to the same power. Therefore,

$$\begin{aligned} e^{At} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = T\left(\sum_{k=0}^{\infty} \frac{t^k}{k!}\Lambda^k\right) T^{-1} \\ &= T \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} T^{-1}. \end{aligned} \tag{3.2}$$

It is immediate that in this case (i.e., $A$ diagonalizable) stability properties of the origin can be characterized based on the location of the eigenvalues in the complex plane. For example, if all the eigenvalues have strictly negative real parts, then the matrix-vector product $e^{At}x(0)$ converges to the zero vector exponentially quickly.

Before stating a general result, we must consider what happens when matrices are not completely diagonalizable. In this case, we rely on the Jordan normal form and, for discussion purposes, we restrict attention to the $2 \times 2$ Jordan block

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

and examine the matrix exponential $e^{Jt}$. It is not difficult to see that

$$J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}$$

and therefore, in the infinite sum defining the matrix exponential, the diagonal elements sum to $e^{\lambda t}$ as in (3.2). A little manipulation of the upper diagonal element yields

$$\sum_{k=0}^{\infty} \frac{kt^k}{k!}\lambda^{k-1} = t\sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!}\lambda^{k-1} = t\sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!}\lambda^\ell = te^{\lambda t}.$$

Therefore, the matrix exponential of the Jordan block $J$ yields

$$e^{Jt} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Here, we see that if the real part of $\lambda$ is strictly negative, we would still obtain convergence to the origin since $e^{\lambda t}$ will converge faster than $t$ will diverge. However, if $\lambda$ has zero real part (i.e., is purely imaginary), then $e^{\lambda t}$ is oscillatory, rather than converging, and hence $te^{\lambda t}$ will grow to infinity.

Higher order Jordan blocks yield a similar structure (see Exercise 3.2) and lead us to the following result.

**Theorem 3.1** (Stability of linear systems [63, Theorem 8.1])**.** *For the linear system* (3.1)*, the origin is*

1. *stable if and only if the eigenvalues of* $A$ *have negative or zero real parts and all the Jordan blocks corresponding to eigenvalues with zero real parts are* $1 \times 1$*;*
2. *unstable if and only if at least one eigenvalue of* $A$ *has a positive real part or zero real part with the corresponding Jordan block larger than* $1 \times 1$*;*
3. *exponentially stable if and only if all the eigenvalues of* $A$ *have strictly negative real parts.*

Note that stability is a property of an equilibrium point. This is particularly important to keep in mind for nonlinear systems, which may well have multiple equilibrium points, some of which may be asymptotically stable or unstable or any combination of stability properties. By contrast, linear systems cannot have isolated equilibrium points other than the origin. Consequently, in an abuse of terminology, it is not uncommon for the system itself to be referred to as "exponentially stable" or "unstable."

A matrix $A$ is said to be *Hurwitz* if all the eigenvalues of $A$ have strictly negative real parts. Therefore, based on Theorem 3.1 item 3, referring to the exponentially stable origin of a linear system (3.1) is often done by simply referring to a Hurwitz matrix $A$.

Moreover, note that Theorem 3.1 does not mention asymptotic stability and does not distinguish between local and global stability properties. The reason for this is that for linear systems asymptotic stability is equivalent to exponential stability and local (exponential) stability implies global (exponential) stability of the origin of the linear system (3.1). The former statement follows from the fact that solutions are combinations of exponential functions and hence the obtained bounds for an asymptotically stable origin will be exponential. The latter statement follows from the fact that a linear system with a locally exponentially stable origin can only have an isolated equilibrium point at the origin and hence if the origin is locally exponentially stable it is also globally exponentially stable.

### 3.1.2 Quadratic Lyapunov Functions

To analyze the stability properties of linear systems (3.1) through Lyapunov methods, we can rely on a special class of Lyapunov functions given by

$$V(x) = x^T P x,$$

i.e., quadratic Lyapunov functions described through *symmetric positive definite matrices* $P \in \mathcal{S}^n$. Here

$$\mathcal{S}^n = \{P \in \mathbb{R}^{n\times n} : P = P^T\}$$

denotes the vector space of real symmetric matrices.

A symmetric matrix $P \in \mathcal{S}^n$ is positive definite if

$$x^T P x > 0, \quad \forall x \neq 0.$$

Similarly, the matrix $P$ is *positive semidefinite* if $x^T P x \geq 0$, *negative definite* if $x^T P x < 0$, or *negative semidefinite* if $x^T P x \leq 0$. The set of positive definite matrices is denoted by $\mathcal{S}^n_{>0}$. The validation that a symmetric matrix is positive definite can be checked through different criteria (see, for example, [64, Sections 7.1–7.2]).

**Lemma 3.2.** *The following are equivalent:*

1. $P \in \mathcal{S}^n$ *is positive definite;*
2. *All the eigenvalues of* $P$ *are positive;*
3. *The determinants of all the upper left submatrices (the so-called leading principal minors) of* $P$ *are positive;*
4. *There exists a nonsingular matrix* $H \in \mathbb{R}^{n\times n}$ *such that* $P = H^T H$.

In addition to Lemma 3.2 item 2, a positive definite matrix $P \in \mathcal{S}^n_{>0}$ satisfies

$$0 < \lambda_{\min} x^T x \leq x^T P x \leq \lambda_{\max} x^T x, \quad \forall\ x \neq 0, \tag{3.3}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ denote the minimum and maximum eigenvalues of $P$, respectively. Note that the eigenvalues of symmetric matrices are real valued and thus the minimum and maximum eigenvalue are well defined. Also, particularly with reference to the notation in Theorem 2.17, note that $x^T x = |x|^2$.

Recall from the previous chapter that a characterization of positive definite functions is the existence of a lower bound given by a $\mathcal{K}$ function of the norm of the state (see (2.16)). We see that a positive definite matrix $P$ corresponds to the quadratic function $x^T P x$ being a positive definite function where the desired lower bound $\alpha \in \mathcal{K}$ is $\alpha(s) = \lambda_{\min} s^2$ for $s \geq 0$.

**Theorem 3.3.** *For the linear system* (3.1)*, the following are equivalent:*

1. *The origin is exponentially stable;*
2. *All eigenvalues of* $A$ *have strictly negative real parts;*
3. *For every symmetric positive definite* $Q \in \mathcal{S}^n_{>0}$ *there exists a unique symmetric positive definite* $P \in \mathcal{S}^n_{>0}$*, satisfying*

$$A^T P + PA = -Q. \tag{3.4}$$

*Proof.* The equivalence of items 1 and 2 is simply part of Theorem 3.1, which leaves us to prove the equivalence of items 1 and 3.

'3 ⇒ 1': For simplicity take $Q = I$ (the interested reader may consider the changes needed below for an arbitrary symmetric positive definite $Q$) and the Lyapunov function candidate $V(x) = x^T Px$. Then

$$x^T Px \le \lambda_{\max} x^T x \quad \Rightarrow \quad -x^T x \le -\tfrac{1}{\lambda_{\max}} x^T Px$$

and, applying the chain rule,

$$\begin{aligned}\tfrac{d}{dt}V(x) &= \dot{x}^T Px + x^T P\dot{x} = x^T A^T Px + x^T PAx \\ &= x^T(A^T P + PA)x = -x^T x \le -\tfrac{1}{\lambda_{\max}} x^T Px \\ &= -\tfrac{1}{\lambda_{\max}} V(x).\end{aligned} \tag{3.5}$$

An alternate derivation using the previous (equivalent) gradient notation, keeping in mind that $P = P^T$, uses $\nabla V(x) = \left(x^T P\right)^T + Px = 2Px$, which gives

$$\langle \nabla V(x), Ax\rangle = 2x^T PAx = x^T PAx + \left(x^T PAx\right)^T = x^T PAx + x^T A^T Px.$$

Continuing from (3.5), the comparison principle (Lemma 2.14) yields

$$V(x(t)) \le V(x(0)) \exp\left(-\tfrac{1}{\lambda_{\max}} t\right),$$

from which we can compute

$$\lambda_{\min}|x(t)|^2 \le V(x(t)) \le V(x(0)) \exp\left(-\tfrac{1}{\lambda_{\max}} t\right) \le \lambda_{\max}|x(0)|^2 \exp\left(-\tfrac{1}{\lambda_{\max}} t\right),$$

and hence

$$|x(t)| \le \sqrt{\tfrac{\lambda_{\max}}{\lambda_{\min}}}\, |x(0)| \exp\left(-\tfrac{1}{2\lambda_{\max}} t\right).$$

Referring to Definition 2.7 we see that with $M = \sqrt{\tfrac{\lambda_{\max}}{\lambda_{\min}}}$ and $\lambda = 1/(2\lambda_{\max})$, the origin is exponentially stable.

'1 ⇒ 3': Given the symmetric positive definite matrix $Q \in \mathcal{S}^n_{>0}$, let

$$P = \int_0^\infty e^{A^T \tau} Q e^{A\tau} d\tau. \tag{3.6}$$

Note that the integral in (3.6) is well defined since $\|e^{A^T t} Q e^{At}\|$ converges to zero exponentially fast.

To see that $P$ solves the Lyapunov equation, first note that

$$\tfrac{d}{dt}\left(e^{A^T t} Q e^{At}\right) = A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A.$$

Then we can directly compute

$$A^TP + PA = \int_0^\infty \left(A^Te^{A^T\tau}Qe^{A\tau} + e^{A^T\tau}Qe^{A\tau}A\right)d\tau = \int_0^\infty \tfrac{d}{d\tau}\left(e^{A^T\tau}Qe^{A\tau}\right)d\tau$$
$$= e^{A^Tt}Qe^{At}\Big|_0^\infty = \left(\lim_{t\to\infty} e^{A^Tt}Qe^{At}\right) - e^{A^T0}Qe^{A0} = -Q.$$

It remains to show that $P$ defined by (3.6) is symmetric, positive definite, and unique. That $P$ is symmetric follows from the fact that $Q = Q^T$ since

$$P^T = \int_0^\infty \left(e^{A^T\tau}Qe^{A\tau}\right)^T d\tau = \int_0^\infty e^{A^T\tau}Q^Te^{A\tau}d\tau$$
$$= \int_0^\infty e^{A^T\tau}Qe^{A\tau}d\tau = P.$$

To show that $P$ is positive definite, let $z \in \mathbb{R}^n$ and consider

$$z^TPz = \int_0^\infty z^Te^{A^T\tau}Qe^{A\tau}z\,d\tau.$$

Note that if $z \neq 0$ then $x(\tau) = e^{A\tau}z \neq 0$ and, since $Q$ is positive definite,

$$z^TPz = \int_0^\infty x(\tau)^TQx(\tau)d\tau > 0.$$

Additionally, if $z = 0$ then $x(\tau) = 0$, and so $P$ is indeed positive definite.

Finally, that $P$ is unique can be proved by contradiction and is left to the interested reader. □

Recognizing that the matrices $P$ and $Q$ in (3.4) are used in constructing a Lyapunov function as seen in the proof above, (3.4) is referred to as the *Lyapunov equation.* Note that this provides a constructive method to find a Lyapunov function for linear systems. In fact, there is a MATLAB command that computes $P$ given a positive definite symmetric matrix $Q$. See Exercise 3.3.

## 3.2 LINEARIZATION

We now return to the nonlinear system (1.1) given by

$$\dot{x} = f(x) \tag{3.7}$$

with $f(0) = 0$ and assume that $f$ is continuously differentiable. Recall that, as argued for (1.9), we can shift any equilibrium point of interest to the origin so that, for the translated system, $f(0) = 0$.

Define the matrix $A$ by the Jacobian of $f$ evaluated at the origin,

$$A = \left[\frac{\partial f(x)}{\partial x}\right]_{x=0},$$

and let $f_1(x) = f(x) - Ax$. Note that

$$\lim_{|x|\to 0} \frac{|f_1(x)|}{|x|} = \lim_{|x|\to 0} \frac{|f(x) - Ax|}{|x|} = 0, \tag{3.8}$$

where the last equality follows from an application of L'Hôpital's rule.

A slightly different point of view is to take the Taylor expansion of $f$ at 0, $f(x) = Ax + f_1(x)$, where all the higher order terms are collapsed into $f_1$.

The system

$$\dot{z}(t) = Az(t) \tag{3.9}$$

is called the linearization of (3.7) at the origin. Note that, in an abuse of notation, (3.9) is almost always written as $\dot{x}(t) = Ax(t)$, blurring the distinction between the actual state $x$ of the nonlinear system (3.7) and its linear approximation (3.9).

**Theorem 3.4.** *Consider the nonlinear system* (3.7) *with continuously differentiable right-hand side $f$ and its linearization* (3.9). *If the origin of the linear system* (3.9) *is globally exponentially stable then the origin of* (3.7) *is locally exponentially stable.*

*Proof.* Let the origin of (3.9) be globally exponentially stable and define $Q = I$. Since the origin is exponentially stable for (3.9), Theorem 3.3 provides a symmetric and positive definite $P$ satisfying (3.4). Take $V(x) = x^T Px$. Then

$$\langle \nabla V(x), f(x)\rangle = -x^T x + 2x^T Pf_1(x). \tag{3.10}$$

As before, denote the maximum eigenvalue of $P$ by $\lambda_{\max}$. Choose $r > 0$ and $\rho < \frac{1}{2}$ such that, for all $x$ satisfying $|x| \le r$,

$$|f_1(x)| \le \tfrac{\rho}{\lambda_{\max}}|x|. \tag{3.11}$$

That this can be done follows from (3.8). Then

$$|2x^T Pf_1(x)| \le 2|Px|\,|f_1(x)| \le 2\,(\lambda_{\max}|x|)\left(\frac{\rho}{\lambda_{\max}}|x|\right) = 2\rho x^T x.$$

Therefore, for $|x| \le r$,

$$\begin{aligned}\langle \nabla V(x), f(x)\rangle &\le -x^T x + 2\rho x^T x = -(1-2\rho)x^T x \\ &\le -\frac{1-2\rho}{\lambda_{\max}} V(x) = -cV(x),\end{aligned}$$

where $c = \frac{1-2\rho}{\lambda_{\max}} > 0$ since $\rho < \frac{1}{2}$. We see that $V$ satisfies all the assumptions of Theorem 2.17 and so the origin of (3.7) is locally exponentially stable. □

We can also use the linearization to ascertain if the origin is unstable.

**Theorem 3.5.** *Consider the nonlinear system* (3.7) *with continuously differentiable right-hand side $f$ and its linearization* (3.9) *and assume that the eigenvalues of $A$ satisfy $\lambda_i + \lambda_j \ne 0$ for all $i, j$. The equilibrium 0 is unstable for* (3.7) *if $A$ has at least one eigenvalue with positive real part.*

Here, we only prove a special case of the theorem which relies on the two following lemmas presented without a proof. For a proof of the general statement of Theorem 3.5 we refer to [86, Theorem 3.7].

**Lemma 3.6** ([158, Lemma 5.4.35]). *The Lyapunov equation* (3.4) *has a unique (real symmetric) solution $P$ for each (real symmetric) $Q$ if and only if the eigenvalues of $A$ satisfy $\lambda_i + \lambda_j \neq 0$ for all $i, j$.*

**Lemma 3.7** ([158, Lemma 5.4.52]). *Suppose the eigenvalues of $A$ satisfy $\lambda_i + \lambda_j \neq 0$ for all $i, j$. If $Q$ is positive definite, and $P$ solves the Lyapunov equation* (3.4)*, then $P$ has as many negative eigenvalues as there are eigenvalues of $A$ with positive real part.*

*Sketch of the proof of Theorem 3.5:* Let $f(x) = Ax + f_1(x)$ satisfy (3.8). Take $Q = I$, $\widehat{P} = -P$, and $V(x) = x^T \widehat{P} x$. If $A$ has at least one eigenvalue with positive real part then $P$ has at least one negative eigenvalue and $\widehat{P}$ has at least one positive eigenvalue. Therefore, there exists an $x_0$ arbitrarily close to the origin such that $V(x_0) > 0$.

In order to apply Theorem 2.25 (Chetaev's theorem), it remains to show that there is a neighborhood of the origin where $\langle \nabla V(x), f(x) \rangle > 0$. This can be done using arguments similar to those of the proof of Theorem 3.4, where (3.10) can be shown to satisfy

$$\langle \nabla V(x), f(x) \rangle = x^T x + 2x^T \widehat{P} f_1(x) \geq cV(x)$$

on some neighborhood of the origin and for some $c > 0$. The details are left to Exercise 3.4. □

Note that if all eigenvalues of $A$ have non-positive real part but $A$ has any eigenvalues with zero real part, then the linearization is inconclusive.

*Example 3.8.* Consider the nonlinear system

$$\dot{x} = cx^3 \tag{3.12}$$

with parameter $c \in \mathbb{R}$. The function $V(x) = \frac{1}{2}x^2$ satisfies (2.17) and

$$\dot{V}(x) = \langle \nabla V(x), cx^2 \rangle = cx^4.$$

Thus, for $c < 0$, the origin of (3.12) is asymptotically stable according to Theorem 2.16 and for $c > 0$ the origin of (3.12) is unstable according to Theorem 2.24.

However, independently of the parameter $c$, the linearization of the system (3.12) around the origin is given by $\dot{z} = Az = 0 \cdot z$. Hence, it is impossible to conclude stability properties of the origin for the nonlinear system based on its linearization if the matrix $A$ contains eigenvalues with zero real part.

*Example 3.9.* Consider the mass-spring system of Example 1.2 with a hardening spring given by $F_{sp} = k_0 y + k_1 y^3 = k_0 x_1 + k_1 x_1^3$ with $k_0, k_1 > 0$, which yields the state space system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tfrac{1}{m}\left(-k_0 x_1 - k_1 x_1^3 - c x_2\right). \end{aligned} \tag{3.13}$$

The origin is an equilibrium and the matrix defining the linear system is given by

$$A = \left[\frac{\partial f(x)}{\partial x}\right]_{x=0} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m} - 3\frac{k_1}{m}x_1^2 & -\frac{c}{m} \end{bmatrix}_{x=0} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m} & -\frac{c}{m} \end{bmatrix}.$$

We can compute the eigenvalues for $A$ as

$$0 = \det(\lambda I - A) = \lambda\left(\lambda + \tfrac{c}{m}\right) + \tfrac{k_0}{m} = \lambda^2 + \lambda\tfrac{c}{m} + \tfrac{k_0}{m},$$

from which we have $\lambda = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k_0}{m}}$. We can identify three distinct cases for the eigenvalues ($k_0 = \frac{c^2}{4}$, $k_0 < \frac{c^2}{4}$, and $k_0 > \frac{c^2}{4}$), all of which yield eigenvalues with negative real parts. Therefore, Theorem 3.3 tells us that the origin is exponentially stable for $\dot{z} = Az$ and Theorem 3.4 yields that the origin is exponentially stable for (3.13).

*Example 3.10.* Consider the pendulum of Example 1.4 with the origin shifted to the upright equilibrium:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{g}{\ell}\sin(x_1 + \pi) - \tfrac{k}{m}x_2. \end{aligned} \tag{3.14}$$

We compute the matrix describing the linearized system by

$$A = \left[\frac{\partial f(x)}{\partial x}\right]_{x=0} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos(x_1 + \pi) & -\frac{k}{m} \end{bmatrix}_{x=0} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{m} \end{bmatrix}.$$

The eigenvalues of A are given by

$$0 = \det(\lambda I - A) = \lambda\left(\lambda + \tfrac{k}{m}\right) - \tfrac{g}{\ell} = \lambda^2 + \lambda\tfrac{k}{m} - \tfrac{g}{\ell}$$

so that

$$\lambda = -\tfrac{k}{2m} \pm \sqrt{\left(\tfrac{k}{2m}\right)^2 + \tfrac{g}{\ell}},$$

which yields two real eigenvalues, where one eigenvalue is negative and the other is positive. Therefore, from Theorem 3.5, the origin, which is the upright equilibrium, is unstable.

*Example 3.11.* Consider the mass-spring-damper from Example 2.35:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tfrac{1}{m}\left(-k_0x_1 - k_1x_1^3 - bx_2|x_2|\right). \end{aligned} \tag{3.15}$$

The linearized system is described by

$$A = \left[\frac{\partial f(x)}{\partial x}\right]_{x=0} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m} - 3\frac{k_1}{m}x_1^2 & -2\frac{b}{m}x_2 \end{bmatrix}_{x=0} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m} & 0 \end{bmatrix}.$$

The eigenvalues are given by

$$0 = \det(\lambda I - A) = \lambda^2 + \tfrac{k_0}{m},$$

which implies $\lambda = \pm j\sqrt{k_0/m}$. Since the eigenvalues of $A$ have zero real parts, the linearization tells us nothing about stability of the origin for (3.15).

In addition to being used to study stability properties of equilibria, the linearization of a nonlinear system can also be used to construct local Lyapunov functions. In particular, Theorem 3.4 and its proof imply the following result.

**Corollary 3.12.** *Consider the nonlinear system* (3.7) *with continuously differentiable right-hand side $f$ and its linearization* (3.9) *with a locally and globally exponentially stable origin for* (3.7) *and* (3.9), *respectively. Let $P \in \mathcal{S}^n$ be the unique solution of the Lyapunov equation* (3.4) *for an arbitrary positive definite matrix $Q \in \mathcal{S}^n_{>0}$. Then $V(x) = x^T P x$ is a local Lyapunov function of the nonlinear system* (3.7).

This corollary shows that it is straighforward to compute local Lyapunov functions for nonlinear systems (3.7) with continuously differentiable right-hand side and with respect to an exponentially stable equilibrium. However, recalling Theorem 2.17, it is in general nontrivial to obtain the domain $\mathcal{D} \subset \mathbb{R}^n$ where the Lyapunov function satisfies the conditions of Theorem 2.17. Corollary 3.12 only guarantees the existence of a $c > 0$ such that the forward-invariant sublevel set $\{x \in \mathbb{R}^n : V(x) \leq c\}$ is contained in the region of attraction $\mathcal{R}_f(0)$. Note that $V$ and $c$ depend on the selection of the positive definite matrix $Q \in \mathcal{S}^n_{>0}$. The calculation of $c$ is again far from being trivial as outlined in Section 2.4. In Section 3.4 we will present a numerical method to compute Lyapunov functions and estimate the region of attraction for a special class of systems.

## 3.3 TIME-VARYING SYSTEMS

Linear time-varying systems

$$\dot{x}(t) = A(t)x(t) \tag{3.16}$$

represent a special class of time-varying systems (1.2). It is important to note that if the matrix $A(t)$ is time-dependent and not constant, then Theorem 3.1 and Theorem 3.3 are not applicable. Even if all the eigenvalues of $A(t)$ have a negative real part for all $t \in \mathbb{R}_{\geq 0}$, (exponential) stability of the origin cannot be concluded.

*Example 3.13.* The matrix

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2(t) & 1 - 1.5\sin(t)\cos(t) \\ -1 - 1.5\sin(t)\cos(t) & -1 + 1.5\sin^2(t) \end{bmatrix} \tag{3.17}$$

has eigenvalues at $\lambda_{1,2} = -0.25 \pm j0.25\sqrt{7}$. However, the solution of $\dot{x}(t) = A(t)x(t)$ is given by

$$x(t) = \begin{bmatrix} e^{0.5t}\cos(t) & e^{-t}\sin(t) \\ -e^{0.5t}\sin(t) & e^{-t}\cos(t) \end{bmatrix} x(0), \tag{3.18}$$

which clearly has a component that exponentially diverges from zero.

For time-invariant systems, the linearization stability theorem (Theorem 3.4) relied on

$$\lim_{|x|\to 0} \frac{|f(x) - Ax|}{|x|} = 0,$$

which is always true when $f(x)$ is continuously differentiable and $A$ is defined as the Jacobian of $f$ at the origin. In particular, this property is used in equation (3.11) to define the neighborhood of the origin where the linearization behaves similarly to the original nonlinear system.

However, from (1.2) with

$$A(t) = \left[\frac{\partial f(t,x)}{\partial x}\right]_{x=0},$$

it is not necessarily true that

$$\lim_{|x|\to 0} \sup_{t\geq 0} \frac{|f(t,x) - A(t)x|}{|x|} = 0. \tag{3.19}$$

Consequently, in order to obtain a result similar to Theorem 3.4 for time-varying systems, it is necessary to assume that (3.19) holds.

*Example 3.14.* ([158, Chapter 5.5]) As an example we consider the nonlinear system

$$\dot{x} = f(t,x) = \begin{bmatrix} -x_1 + tx_2^2 \\ x_1 - x_2 \end{bmatrix} \tag{3.20}$$

with

$$\left[\frac{\partial f(t,x)}{\partial x}\right]_{x=0} x = A(t)x = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x.$$

We see that

$$\lim_{|x|\to 0} \sup_{t\geq 0} \frac{|f(t,x) - A(t)x|}{|x|} \geq \lim_{|x_2|\to 0} \sup_{t\geq 0} \frac{|tx_2^2|}{|x_2|} \geq \lim_{x_2\to 0} \frac{|\frac{1}{x_2}x_2^2|}{|x_2|} = 1,$$

and thus (3.20) is a time-varying system with continuously differentable right-hand side which does not satisfy (3.19). Hence, to obtain a similar result to Theorem 3.4 for time-varying systems, condition (3.19) needs to be explicitly included in the assumptions, in contrast to condition (3.8) for autonomous systems.

The utility of a linearization is that it provides a reasonable approximation of the behavior of the original system. However, for this example the initial time plays a critical role and, in fact, for initial times different from zero, the linearization is not a good approximation of the nonlinear system, as can be seen in Figure 3.1 and Figure 3.2.

**Theorem 3.15** ([158, Theorem 5.5.15])**.** *Consider the nonlinear time-varying system* (1.2) *and suppose that* $f(t,0) = 0$ *for all* $t \geq t_0$ *and that* $f$ *is locally Lipschitz continuous and continuously differentiable with respect to* $x$*. Assume that* (3.19) *holds and that* $A(\cdot)$ *is bounded. If the origin is an exponentially stable equilibrium for* $\dot{z}(t) = A(t)z(t)$*, then it is also an exponentially stable equilibrium of* (1.2).

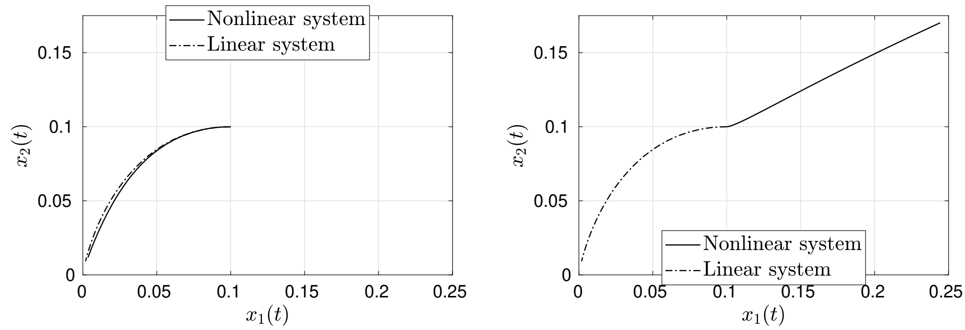


Figure 3.1: Solutions of the dynamics (3.20) and its linearization for $t \in [0, 4]$ (left) and $t \in [10, 14]$ (right) and initial value $x(t_0) = [0.1, 0.1]^T$.

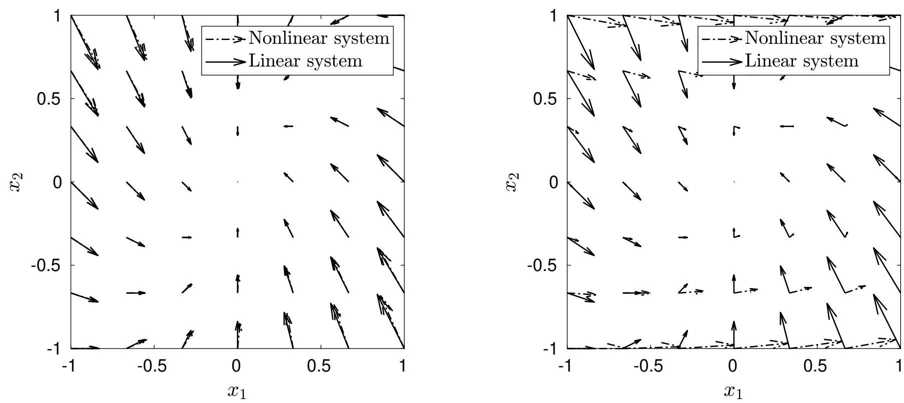


Figure 3.2: Phase portrait of the dynamics (3.20) and its linearization for $t_0 = 0.1$ (left) and $t_0 = 10$ (right).

## 3.4 NUMERICAL CALCULATION OF LYAPUNOV FUNCTIONS

While we can use Lyapunov functions to establish stability properties of equilibria, it is in general difficult to find a Lyapunov function for a given system. In this section we present a method to construct Lyapunov functions for systems $f : \mathbb{R}^n \to \mathbb{R}^n$,

$$\dot{x} = f(x), \tag{3.21}$$

where the right-hand side $f$ is a polynomial function. In particular, we deviate from the main focus of linear systems in this section and discuss a slightly more general class of systems here. For linear systems, i.e., if $f(x)$ is a polynomial of degree 1, the solution of the Lyapunov equation (3.4) provides a quadratic Lyapunov function. Here, we present a method which can be applied to polynomials of higher degree. In particular we rewrite the conditions on $V$ in Theorem 2.16 as a *semidefinite program*, which is a special form of a *convex optimization problem*, The conditions on $V$ are phrased as *linear matrix inequalities* (LMIs), which can be solved through *semidefinite programming.* To this end, candidate Lyapunov functions are defined as *sum of squares* of polynomial functions.

In general it is difficult to validate if a function $W : \mathbb{R}^m \to \mathbb{R}$ satisfies $W(z) \geq 0$ for all $z \in \mathbb{R}^m$. However, if $W$ is of the special form

$$W(z) = |Hz|^2 = z^T H^T Hz$$

for a matrix $H \in \mathbb{R}^{m \times m}$, then $W(z) \geq 0$ for all $z \in \mathbb{R}^m$ follows from the positivity of the norm. Note that for every symmetric positive semidefinite matrix $P \in \mathcal{S}^m_{\geq 0}$ there exists $H \in \mathbb{R}^{m \times m}$ such that $P = H^T H$. We thus focus on candidate Lyapunov functions $W(z) = z^T Pz$ with positive semidefinite matrix $P \in \mathcal{S}^m_{\geq 0}$.

Here $z : \mathbb{R}^n \to \mathbb{R}^m$, $m \in \mathbb{N}$, denotes monomial functions

$$z_j(x) = \textstyle\prod_{i=1}^n x_i^{j_i}$$

for $j_i \in \mathbb{N}$, for all $i \in \{1, \dots, n\}$ for all $j \in \{1, \dots, m\}$. For example, $z : \mathbb{R}^2 \to \mathbb{R}^5$,

$$z(x) \doteq \begin{bmatrix} x_1 & x_2 & x_1^2 & x_2^2 & x_1 x_2 \end{bmatrix}^T, \tag{3.22}$$

captures the monomials of degree less than 3 of a two-dimensional system. Similarly, $y : \mathbb{R}^2 \to \mathbb{R}^9$,

$$y(x) \doteq \begin{bmatrix} x_1 & x_2 & x_1^2 & x_2^2 & x_1 x_2 & x_1^3 & x_2^3 & x_1^2 x_2 & x_1 x_2^2 \end{bmatrix}^T, \tag{3.23}$$

contains the monomials of degree less than 4.

This definition allows us to define candidate Lyapunov functions

$$V(x) = W(z(x)) = z(x)^T Pz(x)$$

of arbitrary polynomial degree. Note that such functions, being quadratic in $z(x)$, will have polynomials that are twice the degree of the monomials in $z(x)$. For the purposes of constructing Lyapunov functions, we will use a weaker formulation of Theorem 2.16.

**Theorem 3.16.** *Consider* (3.21) *with* $f(0) = 0$, *a domain* $\mathcal{D} \subset \mathbb{R}^n$, *and a function* $\kappa : \mathbb{R}^n \to \mathbb{R}$ *such that* $\kappa(x) \leq 0$ *for all* $x \in \mathcal{D}$ *and* $\kappa(x) > 0$ *for all* $x \in \mathbb{R}^n \backslash \mathcal{D}$. *Additionally, suppose we have a continuously differentiable function* $V : \mathbb{R}^n \to \mathbb{R}$, $V(0) = 0$, $\alpha_1, \rho \in \mathcal{K}_\infty$, *and* $\delta_1, \delta_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *satisfying*

$$\alpha_1(|x|) - \delta_1(x)\kappa(x) \leq V(x) \tag{3.24}$$

$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|) + \delta_2(x)\kappa(x) \tag{3.25}$$

*for all* $x \in \mathbb{R}^n$. *Then the origin is locally asymptotically stable. If, additionally,* $\mathcal{D} = \mathbb{R}^n$, *then the origin is globally asymptotically stable.*

For all $x \in \mathcal{D}$, conditions (3.24) and (3.25) satisfy

$$\alpha_1(|x|) \leq \alpha_1(|x|) - \delta_1(x)\kappa(x) \leq V(x)$$
$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|) + \delta_2(x)\kappa(x) \leq -\rho(|x|),$$

and thus if the conditions of Theorem 3.16 are satisfied then so are the conditions of Theorem 2.16. To illustrate how semidefinite programming can be used to compute Lyapunov functions satisfying the assumptions of Theorem 3.16 we will focus on the dynamics

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 + cx_1^3\end{aligned} \tag{3.26}$$

for $c \in \{0, -\frac{1}{4}, \frac{1}{4}\}$. In particular, we will show how the problem of finding a Lyapunov function can be translated into a finite-dimensional convex optimization problem.

Since the expressions become quite lengthy, we start with linear dynamics ($c = 0$), even though we have already seen how to construct Lyapunov functions for linear systems. For $c = -\frac{1}{4}$, the origin is the unique equilibrium of (3.26) and thus a global Lyapunov function will be constructed as a second example. For $c = \frac{1}{4}$, the dynamics (3.26) admit three equilibria $x_1 \in \{0, -2, 2\}$, $x_2 = 0$, and we will construct a local Lyapunov function in a neighborhood of the origin.

### 3.4.1 Linear Matrix Inequalities and Semidefinite Programming

The linear system $\dot{x} = Ax$ describing the dynamics (3.26) for $c = 0$ is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{3.27}$$

In this case, solving the Lyapunov equation (3.4) with $Q = I$ leads to the positive definite matrix

$$P = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}, \tag{3.28}$$

which implies exponential stability of the origin according to Theorem 3.3. Positive definiteness of $P$ can be verified using Lemma 3.2, guaranteeing that $V(x) = x^T Px$ is a Lyapunov function.

While this is a straightforward approach for linear systems to compute a Lyapunov function, we will consider a different approach here to establish asymptotic stability of the origin and to obtain a Lyapunov function which is also applicable to a more general class of systems.

To this end, instead of solving the Lyapunov equation (3.4) we consider the conditions in Theorem 3.16 directly, i.e., we focus on the inequalities

$$\alpha_1(|x|) \leq V(x), \tag{3.29a}$$
$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|). \tag{3.29b}$$

Since we want to find a global Lyapunov function to show global asymptotic stability, we set $\kappa(x) = 0$ for all $x \in \mathbb{R}^2$.

We fix $\varepsilon > 0$, select $\alpha_1(|x|) = \rho(|x|) = \varepsilon |x|^2$, and assume that the Lyapunov function $V$ as well as the left-hand side of (3.29b) can be written as quadratic functions

$$V(x) = x^T Px, \qquad \langle \nabla V(x), f(x) \rangle = -x^T Qx,$$

for symmetric matrices $P, Q \in \mathcal{S}^2$ with unknown parameters (and in particular assume that we do not know $P$ in (3.28)). With these assumptions, rearranging

(continued...)

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