Contents

Introduction

This book is intended as an exploration of the moduli space Poly_d of complex polynomials of degree $d \geq 2$ in one variable using tools primarily coming from arithmetic geometry.

The Mandelbrot set in Poly_2 has undoubtedly been the focus of the most comprehensive set of studies, and its local geometry is still an active research field in connection with the Fatou conjecture; see [19] and the references therein. In their seminal work, Branner and Hubbard [30, 31] gave a topological description of the space of cubic polynomials with disconnected Julia sets using combinatorial tools. In any degree, Poly_d is a complex orbifold of dimension $d-1$, and is therefore naturally amenable to complex analysis and in particular to pluripotential theory. This observation has been particularly fruitful to describe the locus of instability, and to investigate the boundary of the connectedness locus. DeMarco [49] constructed a positive closed $(1,1)$ current whose support is precisely the set of unstable parameters. Dujardin and the first author [68] then noticed that the Monge-Ampère measure of this current defines a probability measure μ_{bif} whose support is in a way the right generalization of the boundary of the Mandelbrot set in higher degree, capturing the part of the moduli space where the dynamics is the most unstable (see also [11] for the case of rational maps). The support of μ_{bif} has a very intricate structure: it was proved by Shishikura [152] in degree 2 and later generalized in higher degree by the second author [87] that the Hausdorff dimension of the support of μ_{bif} is maximal equal to $2(d-1)$.

A polynomial is said to be post-critically finite (or PCF) if all its critical points have a finite orbit. The Julia set of a PCF polynomial is connected, of zero measure, and the dynamics on it is hyperbolic off the post-critical set. PCF polynomials form a countable subset of larger classes of polynomials (such as Misiurewicz, or Collet-Eckmann) for which the thermodynamical formalism is well understood [141, 142]. They also play a pivotal role in the study of the connectedness locus of Poly_d: their distribution was described in a series of papers [76, 90, 91] and proved to represent the bifurcation measure μ_{bif}.

Any PCF polynomial is the solution of a system of $d-1$ equations of the form $P^n(c) = P^m(c)$ where c denotes a critical point and n, m are two distinct integers. In the moduli space, these equations are algebraic with integral coefficients, so that any PCF polynomial is in fact defined over a number field. Ingram [109] pushed this remark further and built a natural height $h_{\mathrm{bif}} \colon \mathrm{Poly}_d(\bar{\mathbb{Q}}) \to \mathbb{R}_+$

for which the set of PCF polynomials coincides with $\{h_{\mathrm{bif}} = 0\}$.

Height theory yields interesting new perspectives on the geometry of Poly_d, and more specifically on the distribution of PCF polynomials. We will be mostly interested here in the so-called dynamical André-Oort conjecture, which appeared in [6]; see also [156].

This remarkable conjecture was set out by Baker and DeMarco, who were motivated by deep analogies between PCF dynamics and CM points in Shimura varieties, and more specifically by works by Masser-Zannier [27, 123, 171] on torsion points in elliptic curves. An historical account on the introduction of these ideas in arithmetic dynamics is given in [5, §1.2] and [6, §1.2]; see also [93]. We note that this analogy goes far beyond the problems considered in this book, and applies to various conjectures described in [52, 155]. We refer to the book by Zannier [171] for a beautiful discussion of unlikely intersection problems in arithmetic geometry.

Baker and DeMarco proposed characterizing irreducible subvarieties of Poly_d (or more generally of the moduli space of rational maps) containing a Zariski dense subset of PCF polynomials, and conjectured that such varieties were defined by critical relations. This conjecture was proven for unicritical polynomials in [97] and [98], and in degree 3 in [77] and [103].

It is our aim to give a proof of that conjecture for *curves* in Poly_d for any $d \geq 2$, and based on this result to attempt a classification of these curves in terms of combinatorial data encoding critical relations.

Our proof roughly follows the line of arguments devised in the original paper of Baker and DeMarco, and relies on equidistribution theorems of points of small height by Thuillier [160] and Yuan [168]; on the expansion of the Böttcher coordinates; and on Ritt's theory characterizing equalities of composition of polynomials.

We needed, though, to overcome several important technical difficulties, such as proving the continuity of metrics naturally attached to families of polynomials. We also had to inject new ingredients, most notably some dynamical rigidity results concerning families of polynomials with a marked point whose bifurcation locus is real-analytic.

For the most part in the book, we shall work in the more general context of polynomial dynamical pairs (P, a) parametrized by a complex affine curve C, postponing the proof of the dynamical André-Oort conjecture to the last chapter. We investigate quite generally the problem of unlikely intersection that was promoted in the context of torsion points on elliptic curves by Zannier and his co-authors [123, 171], and later studied by Baker and DeMarco [5, 6] in our context. This problem amounts to understanding when two polynomial dynamical pairs (P, a) and (Q, b) parametrized by the same curve C have an infinite set of common parameters for which the marked points are preperiodic. We obtain quite definite answers for polynomial pairs, and we prove finiteness theorems that we feel are of some interest for further exploration.

We have tried to review all the necessary material for the proof of the dynamical André-Oort conjecture, but we have omitted some technical proofs that are already available in the literature in an optimal form. On the other hand, we have made some efforts to clarify some proofs which we felt are too sketchy in the literature. The group of dynamical symmetries of a polynomial plays a very important role in unlikely intersection problems, and we have thus included a detailed discussion of this notion.

Let us now describe in more detail the content of the book.

Polynomial dynamical pairs

In this paragraph we present the main players of our moograph. The central notion is that of a POLYNOMIAL DYNAMICAL PAIR parametrized by a curve. Such a pair (P, a) is by definition an algebraic family of polynomials P_t parametrized by an irreducible affine curve C defined over a field K, accompanied by a regular function $a \in K[C]$ which defines an algebraically varying marked point. Most of the time, these objects will be defined over the field of complex numbers $K = \mathbb{C}$, but it will also be important to consider polynomial dynamical pairs over other fields such as number fields, p-adic fields, or finite fields.

Any polynomial dynamical pair leaves a "trace" on the parameter space C, which may take different forms. Suppose first that K is an arbitrary field, and let $\bar{K}$ be an algebraic closure of K. The first basic object to consider is the set $\mathrm{Preper}(P, a)$ of (closed) points $t \in C(\bar{K})$ such that $a(t)$ is preperiodic under P_t. This set is either equal to C or at most countable.

A slightly more complicated but equally important object one can attach to (P, a) is the following divisor. Let $\bar{C}$ be the completion of C, that is, the unique projective algebraic curve containing C as a Zariski dense open subset, and smooth at all points $\bar{C} \setminus C$. Points in $\bar{C} \setminus C$ are called branches at infinity of C. Any pair (P, a) induces an effective divisor $\mathsf{D}_{P,a}$ on $\bar{C}$, which is obtained by setting

$$\mathrm{ord}_{\mathfrak{c}}\left(\mathsf{D}_{P,a}\right) := \lim_{n \to \infty} -\frac{1}{d^n} \min\{0, \mathrm{ord}_{\mathfrak{c}}(P^n(a))\}, \tag{1}$$

for any branch $\mathfrak{c}$ at infinity. The limit is known to exist and is always a rational number; see §4.2.2.

When $K = \mathbb{C}$, one can associate various topological objects to a polynomial dynamical pair. One can consider the locus of stability of the pair (P, a) which consists of the open set in which the family of holomorphic maps $\{P^n(a)\}_{n \geq 0}$ is normal. Its complement is the BIFURCATION LOCUS, which we denote by $\mathrm{Bif}(P, a)$. This set can be characterized using potential theory as follows. Recall the definition of the Green function of a polynomial P of degree d:

$$g_P(z) := \lim_{n \to \infty} \frac{1}{d^n} \max\{\log |P^n(z)|, 0\},$$

so that $\{g_P = 0\}$ is the filled-in Julia set of P consisting of those points having

bounded orbits. On the parameter space C, we then define the function

$$g_{P,a}(t) = g_{P_t}(a(t)).$$

It is a non-negative continuous subharmonic function on C, and the support of the measure $\mu_{\text{bif}} = \Delta g_{P,a}$ is precisely equal to $\text{Bif}(P, a)$. Of crucial technical importance is the following result from [78], which relates the function $g_{P,a}$ to the divisor defined above.

Theorem 1. *In a neighborhood of any branch at infinity $\mathfrak{c} \in \bar{C}$, one has the expansion*

$$g_{P,a}(t) = \text{ord}_{\mathfrak{c}} (\mathsf{D}_{P,a}) \log |t|^{-1} + \tilde{g}(t)$$

where t is a local parameter centered at $\mathfrak{c}$ and $\tilde{g}$ is continuous at 0.

This result can be interpreted in the langage of complex geometry by saying that $g_{P,a}$ induces a continuous semi-positive metrization on the $\mathbb{Q}$-line bundle $\mathcal{O}_{\bar{C}}(\mathsf{D}_{P,a})$. This fact is the key to applying techniques from arithmetic geometry.

Let us now suppose that $K = \mathbb{K}$ is a number field. For any place v of $\mathbb{K}$, denote by $\mathbb{K}_v$ the completion of $\mathbb{K}$, and by $\mathbb{C}_v$ the completion of its algebraic closure. It is then possible to mimic the previous constructions at any (finite or infinite) place v of $\mathbb{K}$ to obtain functions $g_{P,a,v} : C_v^{\text{an}} \to \mathbb{R}_+$ on the analytification (in the sense of Berkovich) C_v^{an} of the curve C over $\mathbb{C}_v$. Summing all these functions yields a height function $h_{P,a} : C(\bar{\mathbb{K}}) \to \mathbb{R}_+$. Alternatively, we may start from the standard Weil height $h_{\text{st}} : \mathbb{P}^1(\bar{\mathbb{K}}) \to \mathbb{R}_+$; see e.g. [105]. Then for any polynomial with algebraic coefficients, we define its canonical height [36] to be

$$h_P(z) := \lim_{n \to \infty} \frac{1}{d^n} h_{\text{st}}(P^n(z)),$$

and finally we set $h_{P,a}(t) := h_{P_t}(a(t))$. Using the Northcott theorem, one obtains that $\{h_{P,a} = 0\}$ coincides with the set $\text{Preper}(P, a)$ of parameters $t \in C(\bar{\mathbb{K}})$ for which $a(t)$ is a preperiodic point of P_t.

It is an amazing fact that all the objects attached to a polynomial dynamical pair (P, a) we have seen so far are tightly interrelated, as the next theorem due to DeMarco [51] shows.

An isotrivial pair (P, a) is a pair which is conjugated to a constant polynomial and a constant marked point, possibly after a base change. A marked point is STABLY PREPERIODIC when there exist two integers $n > m$ such that $P_t^n(a(t)) = P_t^m(a(t))$.

Theorem 2. *Let (P, a) be a polynomial dynamical pair of degree $d \geq 2$ which is parametrized by an affine irreducible curve C defined over a number field $\mathbb{K}$. If the pair is not isotrivial, then the following assertions are equivalent:*

(1) *the set $\text{Preper}(P, a)$ is equal to $C(\bar{\mathbb{K}})$;*

(2) *the marked point is stably preperiodic;*

(3) *the divisor* $\mathrm{D}_{P,a}$ *of the pair* (P,a) *vanishes;*

(4) *for any Archimedean place* v, *the bifurcation measure* $\mu_{P,a,v} := \Delta g_{P,a,v}$ *vanishes;*

(5) *the height* $h_{P,a}$ *is identically zero.*

A pair (P,a) which satisfies either one of the previous conditions is said to be passive; otherwise it is called an ACTIVE PAIR. For an active pair, $\mathrm{Preper}(P,a)$ is countable, the bifurcation measure $\mu_{P,a}$ is non-trivial, and the height $h_{P,a}$ is non-zero.

HOLOMORPHIC RIGIDITY FOR POLYNOMIAL DYNAMICAL PAIRS

Rigidity results are pervasive in (holomorphic) dynamics. One of the most famous rigidity results was obtained by Zdunik [172] and asserts that the measure of maximal entropy of a polynomial P is absolutely continuous with respect to the Hausdorff measure of its Julia set iff P is conjugated by an affine transformation to either a monomial map $M_d(z) = z^d$, or to a Chebyshev polynomial $\pm T_d$ where $T_d(z + z^{-1}) = z^d + z^{-d}$. In particular, these two families of examples are the only ones having a smooth Julia set, a theorem due to Fatou [74].

The following analog of Zdunik's result for polynomial dynamical pairs is our first main result.

Theorem A. *Let* (P,a) *be a polynomial dynamical pair of degree* $d \geq 2$ *which is parametrized by a connected Riemann surface* S. *Assume that* $\mathrm{Bif}(P,a)$ *is non-empty and included in a smooth real curve. Then one of the following holds:*

- *either* P_t *is conjugated to* M_d *or* $\pm T_d$ *for all* $t \in S$;
- *or there exists a univalent map* $\imath \colon \mathbb{D} \to S$ *such that* $\imath^{-1}(\mathrm{Bif}(P,a))$ *is a non-empty closed and totally disconnected perfect subset of the real line and the pair* $(P \circ \imath, a \circ \imath)$ *is conjugated to a real family over* $\mathbb{D}$.

We say that a polynomial dynamical pair (P,a) parametrized by the unit disk is a real family whenever the power series defining the coefficients of P and the marked point have all real coefficients.

The previous theorem is a crucial ingredient for handling the unlikely intersection problem that we will describe later. Its proof builds on a transfer principle from the parameter space to the dynamical plane, which can be decomposed into two parts.

The first step is to find a parameter t_0 at which $a(t_0)$ is preperiodic to a repelling orbit of P_{t_0} and such that $t \mapsto a(t)$ is transversal at t_0 to the preperiodic orbit degenerating to $a(t_0)$. This step builds on an argument of Dujardin [67]. The second step relies on Tan Lei's similarity theorem [159], which shows that the bifurcation locus $\mathrm{Bif}(P,a)$ near t_0 is conformally equivalent at small scales to the Julia set of P_{t_0}.

Combining these two ingredients, we see that if $\mathrm{Bif}(P, a)$ is connected, then Zdunik's theorem implies that the family is isotrivial and P_t conjugated to M_d or $\pm T_d$ for all $t \in C$. When $\mathrm{Bif}(P, a)$ is disconnected, then we prove that all multipliers of P_{t_0} are real and we conclude that P_t is real for all nearby parameters using an argument of Eremenko and Van Strien [73].

In many results that we present below, we shall exclude all polynomials that are affinely conjugated to either M_d or $\pm T_d$. These dynamical systems carry different names in the literature: Zdunik [172] names them maps with parabolic orbifolds; they are called special in [55, 136]; and Medvedev and Scanlon call them non-disintegrated polynomials; see the discussion on [126, p.16]. We shall refer them to as INTEGRABLE polynomials by analogy with the notion of integrable system in Hamiltonian dynamics (see [40, 164]). A family of polynomials $\{P_t\}_{t \in C}$ will be called non-integrable whenever there exists a dense open set $U \subset C$ such that P_t is not integrable for any $t \in U$.

UNLIKELY INTERSECTIONS FOR POLYNOMIAL DYNAMICAL PAIRS

Our next objective is to investigate the problem of characterizing when two polynomial dynamical pairs (P, a) and (Q, b) parametrized by the same algebraic curve C leave the same "trace" on C.

Analogies with arithmetic geometry suggested that the quite weak condition of $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(Q, b)$ being infinite in fact implies very strong relations between the two pairs. This phenomenon was first observed for Lattès maps by Masser and Zannier [123], and later for unicritical polynomials by Baker and DeMarco [5], and for more general families of polynomials parametrized by the affine line by Ghioca, Hsia, and Tucker [95]. We refer to the surveys [93], [52], and [14] where this problem is also addressed.

A precise conjecture was formulated by DeMarco in [53, Conjecture 4.8]: up to symmetries and taking iterates, the two families P and Q are actually equal, and the marked points belong to the same grand orbit. In other words, the existence of unlikely intersections forces some algebraic rigidity between the dynamical pairs.

We prove here DeMarco's conjecture for polynomial dynamical pairs defined over a number field.

Theorem B. *Let (P, a) and (Q, b) be active non-integrable polynomial dynamical pairs parametrized by an irreducible algebraic curve C of respective degree $d, \delta \geq 2$. Assume that the two pairs are defined over a number field $\mathbb{K}$. Then, the following are equivalent:*

(1) *the set $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(Q, b)$ is an infinite subset of $C(\bar{\mathbb{K}})$;*

(2) *the two height functions $h_{P,a}, h_{Q,b} \colon C(\bar{\mathbb{K}}) \to \mathbb{R}_+$ are proportional;*

(3) *there exist integers $N, M \geq 1$, $r, s \geq 0$, and families R, τ, and π of polyno-*

mials of degree ≥ 1 parametrized by C such that

$$\tau \circ P^N = R \circ \tau \quad and \quad \pi \circ Q^M = R \circ \pi, \tag{†}$$

and $\tau(P^r(a)) = \pi(Q^s(b))$.

It is not difficult to see that $(3) \Rightarrow (2) \Rightarrow (1)$ so that the main content of the theorem are the implications $(1) \Rightarrow (2) \Rightarrow (3)$. To obtain $(1) \Rightarrow (2)$, we first apply Yuan-Thuillier's equidistribution result [160, 168] for points of small height: it is precisely at this step that the continuity of $\tilde{g}$ in Theorem 1 is crucial. This allows one to prove that the bifurcation measures $\mu_{P,a,v}$ and $\mu_{Q,b,v}$ are proportional at any place v of $\mathbb{K}$. From there, one infers the proportionality of height functions, i.e., (2), using our above rigidity result (Theorem A).

The implication $(2) \Rightarrow (3)$ is more involved. We first prove that $\deg(P)$ and $\deg(Q)$ are multiplicatively dependent using an argument taken from [69] which relies on computing the Hölder constants of continuity of the potentials of the bifurcation measures at a complex place. From this, we obtain (3) by combining in a quite subtle way several ingredients including:

- a precise understanding of the expansion at infinity of the Böttcher coordinate;
- an algebraization result of germs of curves defined by adelic series due to Xie [165]; and
- the classification of invariant curves by product maps $(z, w) \mapsto (R(z), R(w))$.

The latter result is due to Medvedev and Scanlon [126], whose proof elaborates on Ritt's theory [144]. This theory aims at describing all possible ways a polynomial can be written as the composition of lower degree polynomials. It is very combinatorial in nature and was treated by several authors including Zannier [170] and Müller-Zieve [175]; see also the references therein. Of particular relevance for us are the series of papers by Pakovich [134, 135, 136], and by Ghioca, Nguyen, and Ye [99, 101].

As mentioned above, the line of arguments for proving Theorem B is mostly taken from the seminal paper of Baker and DeMarco, but with considerably more technical issues. The core of the proof takes about eight pages and is the content of §5.4.

It would be desirable to extend Theorem B to families defined over an arbitrary field of characteristic zero. Reducing to a family defined over a number field typically uses a specialization argument. We faced an essential difficulty in the course of this argument, and thus had to require an additional assumption.

Theorem C. *Pick any irreducible algebraic curve C defined over a field of characteristic 0. Let (P, a) and (Q, b) be active non-integrable polynomial dynamical pairs parametrized by C of respective degree $d, \delta \geq 2$. Assume that*

any branch at infinity $\mathfrak{c}$ of C belongs to the support of the divisor $\mathsf{D}_{P,a}$. ($\triangle$)

Then, the following are equivalent:

(1) *the set* $\mathrm{Preper}(P,a) \cap \mathrm{Preper}(Q,b)$ *is an infinite subset of* C;

(2) *there exist integers* $N, M \geq 1$, $r, s \geq 0$, *and families* R, τ, *and* π *of polynomials parametrized by* C *such that*

$$\tau \circ P^N = R \circ \tau \quad and \quad \pi \circ Q^M = R \circ \pi,$$

and $\tau(P^r(a)) = \pi(Q^s(b))$.

Note that although $(\triangle)$ may not hold in general, it is always satisfied when C admits a unique branch at infinity, e.g., when C is the affine line. In particular, our result yields a far-reaching generalization of [5, Theorem 1.1].

In the sequel, we call two active polynomial dynamical pairs (P,a) and (Q,b) ENTANGLED when $\mathrm{Preper}(P,a) \cap \mathrm{Preper}(Q,b)$ is Zariski dense. This terminology inspired by quantum theory reflects the fact the two pairs are dynamically strongly correlated.

DESCRIPTION OF ALL PAIRS ENTANGLED TO A FIXED PAIR

Let us fix a polynomial dynamical pair (P,a) parametrized by an algebraic curve C and for which the previous theorems apply (i.e., either the field of definition of the pair is a number field, or condition $(\triangle)$ holds). We would like now to determine *all* pairs that are entangled to (P,a).

In principle this problem is solvable by Ritt's theory. Given a polynomial P, it is, however, very delicate to describe all polynomials Q for which (†) holds, in particular because there is no a priori bounds on the degrees of τ and π. Much progress has been made by Pakovich [136] but it remains unclear whether one can design an algorithm to solve this problem.

To get around this, we consider a more restrictive question, which is to determine all pairs (P,b) that are entangled with (P,a). In this problem, the notion of symmetries of a polynomial plays a crucial role, and most of Chapter 3 is devoted to the study of this notion from the algebraic, topological, and arithmetic perspectives. The group $\Sigma(P)$ of symmetries of a complex polynomial P is the group of affine transformations preserving its Julia set. Over an arbitrary field, the definition is less satisfactory. Any monic centered polynomial can be written under the form $P(z) = z^\mu Q(z^m)$ with $\deg(Q)$ minimal, and when P is not integrable we set $\Sigma(P)$ to be the cyclic group of order m. It is also the maximal finite group of affine transformations such that $P(g \cdot z) = \rho(g) \cdot P(z)$ for some morphism $\rho \colon \Sigma(P) \to \Sigma(P)$.

We then prove the following more intrinsic characterization of the symmetry group:

Theorem 3. *For any field K of characteristic zero and any $P \in K[z]$ of degree $d \geq 2$, the group $\Sigma(P)$ coincides with the set of $g \in \mathrm{Aff}(K)$ such that*

$g(\mathrm{Preper}(P, \bar{K})) \cap \mathrm{Preper}(P, \bar{K})$ *is infinite.*

Of importance in the latter discussion is the subgroup $\Sigma_0(P)$ of affine maps $g \in \Sigma(P)$ such that $P^n(g \cdot z) = P^n(z)$ for some $n \in \mathbb{N}^*$.

We also introduce the notion of PRIMITIVE polynomials. A polynomial P is primitive if any equality $P = g \cdot Q^n$ with $g \in \Sigma(P)$ implies $n = 1$.

These notions of symmetries and primitivity allow us to obtain the following neat statement.

Theorem D. *Let* (P, a) *be any active, primitive, non-integrable polynomial dynamical pair parametrized by an algebraic curve defined over a field K of characteristic 0. Assume that K is a number field, or that $(\triangle)$ is satisfied.*

For any marked point $b \in K[C]$ *such that* (P, b) *is active, the following assertions are equivalent:*

(1) *the set* $\mathrm{Preper}(P, a) \cap \mathrm{Preper}(P, b)$ *is infinite (i.e., a and b are entangled),*

(2) *there exist* $g \in \Sigma(P)$ *and integers* $r, s \geq 0$ *such that* $P^r(b) = g \cdot P^s(a)$.

Note that this gives a positive answer to [95, Question 1.3] for polynomials.

Suppose that $s = 0$ and r is sufficiently large. Then solutions b to the equation $P^r(b) = a$ are not necessarily regular functions on C: they belong to a finite extension of $K(C)$, and their degree is expected to tend to infinity as $r \to \infty$. The next result gives a more detailed description on all marked points parametrized by C which are entangled with (P, a).

Theorem E. *Let* (P, a) *be any active primitive non-integrable polynomial dynamical pair parametrized by an irreducible affine curve C defined over $\bar{\mathbb{Q}}$.*

The set of marked points in $\bar{\mathbb{Q}}[C]$ *that are entangled with a is the union of* $\{g \cdot P^n(a) \,;\, n \geq 0 \text{ and } g \in \Sigma_0(P)\}$ *and a finite set.*

This result seems to be new even for the unicritical family.[1]

It would obviously be more natural to assume the pair to be defined over an algebraically closed field of characteristic 0, but we use at a crucial step the assumption that (P, a) is defined over $\bar{\mathbb{Q}}$.

Interestingly enough, the proof of this finiteness theorem relies on the same ingredients as Theorem C, namely the expansion of the Böttcher coordinate, an algebraization result of adelic curves, and Ritt's theory. The proof in fact shows that one may only suppose $b \in \bar{\mathbb{Q}}(C)$.

UNICRITICAL POLYNOMIALS.

In the short Chapter 7, we discuss in more depth some aspects of unlikely intersection problems for unicritical polynomials.

Recall that in their seminal paper, Baker and DeMarco obtained the following striking result: for any $d \geq 2$, and any $a, b \in \mathbb{C}$, the pairs $\mathrm{Preper}(z^d + t, a)$, $\mathrm{Preper}(z^d + t, b)$ are entangled iff $a^d = b^d$. This result was further expanded

by Ghioca, Hsia, and Tucker to more general families of polynomials and not necessarily constant marked points; see [95, Theorem 2.3].

Building on our previous results, we obtain the following statement, which slightly generalizes op. cit.

Theorem F. *Let $d, \delta \geq 2$. If a, b are polynomials of the same degree and $\mathrm{Preper}(z^d + t, a(t)) \cap \mathrm{Preper}(z^\delta + t, b(t))$ is infinite, then $d = \delta$ and $a(t)^d = b(t)^d$.*

After proving this theorem, we make some preliminary exploration of the set $\mathbb{M}$ of complex numbers $\lambda \in \mathbb{C}^*$ such that the bifurcation locus ∂M_λ is connected, where we have set $M_\lambda := \{t, \lambda^{-1} t \in K(z^d + t)\}$, $K(z^d + t)$ being the filled-in Julia set of $z^d + t$. We observe that $\lambda \in \mathbb{M}$ iff $M_\lambda \subset \mathcal{M}(d, 0)$, and prove that $\mathbb{M}$ is the union of finitely isolated points with a closed set of $\mathbb{C}^*$ included in the unit disk, and containing the punctured disk $\mathbb{D}^*(0, 1/8)$. We also include a series of pictures obtained by A. Chéritat suggesting that the core of $\mathbb{M}$ is a topological punctured disk.

Special curves in the parameter space of polynomials

We finally come back in Chapter 8 to our original objective, namely the classification of curves in Poly_d which contain an infinite subset of PCF polynomials, and the proof of Baker and DeMarco's conjecture claiming that these curves are cut out by critical relations.

A first answer to Baker and DeMarco's question is given by the next result.

Theorem G. *Pick any non-isotrivial complex family P of polynomials of degree $d \geq 2$ with marked critical points, parametrized by an algebraic curve C, and containing infinitely many PCF parameters.*

If the family is primitive, then possibly after a base change, there exists a subset A of the set of critical points of P such that for any pair $c_i, c_j \in \mathsf{A}$, there exists a symmetry $\sigma \in \Sigma(P)$ and integers $n, m \geq 0$ such that

$$P^n(c_i) = \sigma \cdot P^m(c_j) \; ; \tag{2}$$

and for any $c_i \notin \mathsf{A}$ there exist integers $n_i > m_i \geq 0$ such that

$$P^{n_i}(c_i) = P^{m_i}(c_i) \; . \tag{3}$$

When the family is not primitive the statement is not true because the family may exhibit symmetries of degree ≥ 2, as exemplified by Baker and Demarco [6, Example 4]. After a base change, we may write $P = \sigma \cdot P_0^n$ with $\sigma \in \Sigma(P)$ and P_0 primitive and apply our result.

Following the terminology of [6, §1.4] inspired from arithmetic geometry, we call SPECIAL any curve in Poly_d containing infinitely many PCF polynomials.

Our theorem says that a special curve in the moduli space of polynomials of degree d either arises as the image under the composition map of a special

curve in a lower degree moduli space, or is defined by critical relations (including symmetries) such that all active critical points are entangled.

This result opens up the possibility to give a combinatorial classification of all special curves in the moduli space of polynomials of a fixed degree Poly_d. Recall that a combinatorial classification of PCF polynomials in terms of Hubbard trees has been developed by Douady and Hubbard [61, 62] and Bielefeld-Fisher-Hubbard [25] and further expanded by Poirier [140] and Kiwi [112]. We make here a first step toward the ambitious goal of classification of special curves using a combinatorial gadget: THE CRITICALLY MARKED DYNAMICAL GRAPH.

We refer to §8.2 for a precise definition of a critically marked dynamical graph. It is a (possibly infinite) graph $\Gamma(P)$ together with a dynamics that encodes precisely all dynamical critical relations (up to symmetry) of a given polynomial P. We show that to any irreducible curve C in the moduli space of (critically marked) polynomials, one can attach a marked dynamical graph $\Gamma(C)$ such that $\Gamma(P) = \Gamma(C)$ for all but countably many $P \in C$. We then identify a class of marked graphs that we call special which arise from special curves. Under the assumptions that the special graph Γ has no symmetry and that its marked points are not periodic, we conversely prove that the set of polynomials such that $\Gamma(P) = \Gamma$ defines a (possibly reducible) special curve.

Our precise statement is quite technical; see Theorem 8.30. To give a sample of the results we obtain, let us describe the situation for cubic polynomials, in which case the picture is quite complete. Recall first that the space of cubic polynomials with marked critical points $\mathrm{MPoly}^3_{\mathrm{crit}}$ is two-dimensional and that any cubic polynomial has two critical points (counted with multiplicity). Cubic polynomials having a non-trivial symmetry group are either unicritical ($P_t(z) = z^3 + t$, $\Sigma(P_t) = \mathbb{U}_3$), or of the form $P_t(z) = z(z^2 + t)$ with $\Sigma(P_t) = \mathbb{U}_3$. We obtain our first two special curves in $\mathrm{MPoly}^3_{\mathrm{crit}}$ that we denote by $\Sigma(3,3,0)$ and $\Sigma(3,2,1)$. Let C be any special curve different from these two curves. By Theorem G, either one critical point c_1 is persistently preperiodic on C, or there is a persistent collision between the two critical points c_1 and c_2. In the former case, the graph $\Gamma(C)$ is a union of a straight half-line and a finite connected graph having a cycle with n vertices together with a segment with m vertices attached which encodes the fact that m is the smallest integer such that $P_t^m(c_1)$ is periodic of exact period n for all $t \in C$ (this graph is depicted in the upper left of Figure 8.4). Denote by $\Gamma_1(n, m)$ this graph. In the latter case, $\Gamma(C)$ is an infinite tree obtained by attaching by their extremities two segments with n_1 and n_2 vertices respectively to the origin of a half-line (see the upper right of Figure 8.4). Here n_1 and n_2 correspond to the least integers such that $P_t^{n_1}(c_1) = P_t^{n_2}(c_2)$ for all $t \in C$ (note that in this case we cannot have $n_1 = n_2 = 1$ for degree issues). Denote by $\Gamma_2(n_1, n_2)$ this graph.

Theorem 4. *For any pair of integers (n, m) with $n > m \geq 0$ (resp. $(n_1, n_2) \neq (1, 1)$), there exists a special curve C in $\mathrm{MPoly}^3_{\mathrm{crit}}$ such that $\Gamma(C) = \Gamma_1(n, m)$ (resp. $= \Gamma_2(n_1, n_2)$).*

We do not know whether the curve C is unique (see our question (SC3) on p. 212).

The proof of Theorem 8.30 (which in turn implies the previous statement) builds on two constructions of polynomials with special combinatorics, one by Floyd, Kim, Koch, Parry, and Saenz on the realization of PCF combinatorics [84], and one by McMullen and DeMarco on dynamical trees [57]. Binding together these two results was quite challenging. In arbitrary degree, we have been able to prove only a partial correspondence under simplifying additional assumptions (e.g., the family should have no symmetry).

SOME TECHNICAL DETAILS THAT WE HAVE WORKED OUT AND HOPEFULLY CLARIFIED!

Besides presenting a set of new results, we have made special efforts to clarify some technical aspects of the standard approach to the unlikely intersection problem for polynomials. We emphasize some of them below.

- We include a self-contained proof by J. Xie of his algebraization result for adelic curves (Theorem 1.17).
- We give the full expansion of the Böttcher coordinates for polynomials over a field of characteristic 0 without assuming it to be centered or monic (§2.5).
- We study over an arbitrary field the group of symmetries of a polynomial. In particular, we give a purely algebraic characterization of this group (Theorem 3.18).
- We introduce the notion of primitivity in §3.4, which seems appropriate to exclude tricky examples of entangled pairs.
- We give a detailed proof of the fact that the height $h_{P,a}(t) = h_{P_t}(a(t))$ attached to any polynomial dynamical pair is adelic (Proposition 4.35).
- For a family of polynomials $\{P_t\}$ parametrized by an algebraic variety Λ, we consider the preperiodic locus in $\Lambda \times \mathbb{A}^1$ which is a union of countably many algebraic subvarieties. We study the set of points which are included in an infinite collection of irreducible components of the preperiodic locus (Theorem 2.35). This result is crucial to our specialization argument to obtain Theorem C and clarifies some arguments used in [100].

OPEN QUESTIONS AND PERSPECTIVES.

There are many directions in which our results could find natural generalizations.

Let us indicate first why the restriction to families of polynomials is crucial for us. Given a family of rational maps R_t parametrized by an algebraic curve C, and given any marked point a, one can attach to the pair (R, a) a natural height by setting $h_{R,a}(t) = h_{R_t}(a(t))$ and a divisor at infinity $\mathsf{D}_{R,a}$ generalizing the definition (1) above. It is not completely clear, however, whether $\mathsf{D}_{R,a}$ has

rational coefficients. Some cases have been worked out by DeMarco and Ghioca
[54] but the general case remains elusive. It is not completely clear either if
this height is a Weil height associated to $\mathsf{D}_{R,a}$ (in the sense of Moriwaki [131]).
There are instances (see [60]) where this height is not adelic, but a recent result
by Demarco and Mavraki [58] proves $h_{R,a}(t)$ to be a height associated to a
continuous semi-positive metrization under quite general assumptions.

Yuan-Zhang [169] and the second author [89] have, however, singled out a
class of height functions on quasi-projective varieties for which equidistribution
of small points holds unconditionnally. It turns out that all height functions of
the form $h_{R,a}$ fall into these classes. In particular, the first step of the general
strategy developed in the current text can be now adapted for general families
of rational maps.

We also note that Ritt's theory is much less powerful for rational maps
leading to weaker classification of curves left invariant by product maps (see
[137]). We also refer to [166] for a characterization of rational maps having the
same maximal entropy measure; and to [130] for a version of Theorem B for
constant families of rational maps (but varying marked point).

It would be extremely interesting to look at polynomial dynamical tuples
parametrized by higher dimensional algebraic variety Λ and prove unlikely in-
tersection statements. The obstacles to surmount are also formidable. It is
unclear whether the canonical height is a Weil height on a suitable compact-
ification of Λ. Also in this case, the bifurcation measure is naturally defined
as a Monge-Ampère measure of some psh function on Λ, and dealing with a
non-linear operator makes things much more intricate. We refer, though, to the
papers by Ghioca, Hsia, and Tucker [96] and Ghioca, Hsia, and Nguyen [94] for
attempts to handle higher dimensional parameter spaces using one-dimensional
slices.

Let us list a couple of questions that are directly connected to our work.

(Q1) Prove the following purely Archimedean rigidity statement: two *complex*
polynomial dynamical pairs (P,a) and (Q,b) having identical bifurcation
measures are necessarily entangled. One of the problems to arriving at
such a statement is proving the multiplicative dependence of the degrees
in this context. Observe that Theorem 5.10 yields this dependence but at
the cost of a much stronger assumption.

(Q2) Is it possible to remove condition ($\triangle$) and obtain Theorem C over any
field of characteristic 0?

(Q3) Can one extend Theorem E to any field of characteristic 0?

(Q4) Give a classification of special (irreducible) curves in the moduli space
of critically marked polynomials in terms of suitable combinatorial data.
Ideally, one would like to attach to each special irreducible curve a com-
binatorial object (such as a family of decorated graphs) and prove a one-
to-one correspondence between special curves and these objects. It would

also be interesting to study the distribution (as currents) of special curves whose associated combinatorics has complexity increasing to infinity.

Further, more specific open problems can be found in the three sections §3.7 (related to Ritt's theory), §5.6 (on extensions of Theorem B) and §8.8 (on special curves).

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