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Chapter One

Introduction

1.1 THE EINSTEIN-KLEIN-GORDON COUPLED SYSTEM

The Einstein field equations of General Relativity are a covariant geometric system that connect the Ricci tensor of a Lorentzian metric $\mathbf{g}$ on a manifold M to the energy-momentum tensor of the matter fields in the spacetime, according to the equation

$$\mathbf{G}_{\alpha\beta} = 8\pi T_{\alpha\beta}. \tag{1.1.1}$$

Here $\mathbf{G}_{\alpha\beta} = \mathbf{R}_{\alpha\beta} - (1/2)\mathbf{R}\mathbf{g}_{\alpha\beta}$ is the Einstein tensor, where $\mathbf{R}_{\alpha\beta}$ is the Ricci tensor, $\mathbf{R}$ is the scalar curvature, and $T_{\alpha\beta}$ is the energy-momentum tensor of the matter in the spacetime.

In this monograph we are concerned with the Einstein-Klein-Gordon coupled system, which describes the coupled evolution of an unknown Lorentzian metric $\mathbf{g}$ and a massive scalar field ψ. In this case the associated energy momentum tensor $T_{\alpha\beta}$ is given by

$$T_{\alpha\beta} := \mathbf{D}_\alpha\psi\mathbf{D}_\beta\psi - \frac{1}{2}\mathbf{g}_{\alpha\beta}\big(\mathbf{D}_\mu\psi\mathbf{D}^\mu\psi + \psi^2\big), \tag{1.1.2}$$

where $\mathbf{D}$ denotes covariant derivatives.

Our goal is to prove definitive results on the global stability of the flat space among solutions of the Einstein-Klein-Gordon system. Our main theorems in this monograph include:

(1) A proof of global regularity (in wave coordinates) of solutions of the Einstein-Klein-Gordon coupled system, in the case of small, smooth, and localized perturbations of the stationary Minkowski solution $(\mathbf{g}, \psi) = (m, 0)$;

(2) Precise asymptotics of the metric components and the Klein-Gordon field as the time goes to infinity, including the construction of modified (nonlinear) scattering profiles and quantitative bounds for convergence;

(3) Classical estimates on the solutions at null and timelike infinity, such as bounds on the metric components, weak peeling estimates of the Riemann curvature tensor, ADM and Bondi energy identities and estimates, and asymptotic description of null and timelike geodesics.

The general plan is to work in a standard gauge (the classical wave coordinates) and transform the geometric Einstein-Klein-Gordon system into a coupled system of quasilinear wave and Klein-Gordon equations. We then analyze this system in a framework inspired by the recent advances in the global existence theory for quasilinear dispersive models, such as plasma models and water waves.

More precisely, we rely on a combination of energy estimates and Fourier analysis. At a very general level one should think that energy estimates are used, in combination with vector-fields, to control high regularity norms of the solutions. The Fourier analysis is used, mostly in connection with normal forms, analysis of resonant sets, and a special norm, to prove dispersion and decay in lower regularity norms.

The method we present here incorporates Fourier analysis in a critical way. Its main advantage over the classical physical space methods is the ability to identify clearly resonant and non-resonant nonlinear quadratic interactions. We can then use normal forms to dispose of the non-resonant interactions, and focus our attention on a small number of resonant quadratic interactions. This leads to very precise estimates.

In particular, some of our asymptotic results appear to be new even in the much-studied case of the Einstein-vacuum equations (corresponding to $\psi = 0$) mainly because we allow a large class of non-isotropic perturbations. Indeed, our assumptions on the metric on the initial slice are weak, essentially of the type

$$\mathbf{g}_{\alpha\beta} = m_{\alpha\beta} + \varepsilon_0 O(\langle x\rangle^{-1+}), \qquad \partial\mathbf{g}_{\alpha\beta} = \varepsilon_0 O(\langle x\rangle^{-2+}).$$

These assumptions are consistent with non-isotropic decay, in the sense that we do not assume that the metric has radial decay of the form M/r up to lower order terms. Even with these weaker assumptions we are still able to derive suitable asymptotics of the spacetime, such as weak peeling estimates for the Riemann tensor, and construct a Bondi energy function.

1.1.1 Wave Coordinates and PDE Formulation of the Problem

The system of equations (1.1.1)–(1.1.2) is a geometric system, written in covariant form. To analyze it quantitatively and state our main theorems we need to fix a system of coordinates and reformulate our problem as a PDE problem.

We start by recalling some of the basic definitions and formulas of Lorentzian geometry. At this stage, all the formulas are completely analogous to the Riemannian case, hold in any dimension, and the computations can be performed in local coordinates. A standard reference is the book of Wald [73]. Assume $\mathbf{g}$ is a sufficiently smooth Lorentzian metric in a 4 dimensional open set O. We assume that we are working in a system of coordinates x^0, x^1, x^2, x^3 in O. We define the connection coefficients $\mathbf{\Gamma}$ and the covariant derivative $\mathbf{D}$ by

$$\mathbf{\Gamma}_{\mu\alpha\beta} := \mathbf{g}(\partial_\mu, \mathbf{D}_{\partial_\beta}\partial_\alpha) = \frac{1}{2}(\partial_\alpha\mathbf{g}_{\beta\mu} + \partial_\beta\mathbf{g}_{\alpha\mu} - \partial_\mu\mathbf{g}_{\alpha\beta}), \tag{1.1.3}$$

where $\partial_\mu := \partial_{x^\mu}$, $\mu \in \{0, 1, 2, 3\}$. Thus

$$\mathbf{D}_{\partial_\alpha}\partial_\beta = \mathbf{D}_{\partial_\beta}\partial_\alpha = \mathbf{\Gamma}^\nu{}_{\alpha\beta}\partial_\nu, \qquad \mathbf{\Gamma}^\nu{}_{\alpha\beta} := \mathbf{g}^{\mu\nu}\mathbf{\Gamma}_{\mu\alpha\beta}, \tag{1.1.4}$$

where $\mathbf{g}^{\alpha\beta}$ is the inverse of the matrix $\mathbf{g}_{\alpha\beta}$, i.e., $\mathbf{g}^{\alpha\beta}\mathbf{g}_{\mu\beta} = \delta^\alpha_\mu$. For $\mu, \nu \in$

$\{0,1,2,3\}$ let

$$\mathbf{\Gamma}_\mu := \mathbf{g}^{\alpha\beta}\mathbf{\Gamma}_{\mu\alpha\beta} = \mathbf{g}^{\alpha\beta}\partial_\alpha \mathbf{g}_{\beta\mu} - \frac{1}{2}\mathbf{g}^{\alpha\beta}\partial_\mu \mathbf{g}_{\alpha\beta}, \qquad \mathbf{\Gamma}^\nu := \mathbf{g}^{\mu\nu}\mathbf{\Gamma}_\mu. \tag{1.1.5}$$

We record also the useful general identity

$$\partial_\alpha \mathbf{g}^{\mu\nu} = -\mathbf{g}^{\mu\rho}\mathbf{g}^{\nu\lambda}\partial_\alpha \mathbf{g}_{\rho\lambda}, \tag{1.1.6}$$

and the Jacobi formula

$$\partial_\alpha(\log|\mathbf{g}|) = \mathbf{g}^{\mu\nu}\partial_\alpha \mathbf{g}_{\mu\nu}, \qquad \alpha \in \{0,1,2,3\}, \tag{1.1.7}$$

where $|\mathbf{g}|$ denotes the determinant of the matrix $\mathbf{g}_{\alpha\beta}$ in local coordinates.

Covariant derivatives can be calculated in local coordinates according to the general formula

$$\mathbf{D}_\alpha T_{\beta_1\ldots\beta_n} = \partial_\alpha T_{\beta_1\ldots\beta_n} - \sum_{j=1}^{n} \mathbf{\Gamma}^\nu{}_{\alpha\beta_j} T_{\beta_1\ldots\nu\ldots\beta_n}, \tag{1.1.8}$$

for any covariant tensor T. In particular, for any scalar function f

$$\square_{\mathbf{g}} f = \mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha \mathbf{D}_\beta f = \widetilde{\square}_{\mathbf{g}} f - \mathbf{\Gamma}^\nu \partial_\nu f, \tag{1.1.9}$$

where $\widetilde{\square}_{\mathbf{g}} := \mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta$ denotes the reduced wave operator.

The Riemann curvature tensor measures commutation of covariant derivatives according to the covariant formula

$$\mathbf{D}_\alpha \mathbf{D}_\beta \omega_\mu - \mathbf{D}_\beta \mathbf{D}_\alpha \omega_\mu = \mathbf{R}_{\alpha\beta\mu}{}^\nu \omega_\nu, \tag{1.1.10}$$

for any form ω. The Riemann tensor $\mathbf{R}$ satisfies the symmetry properties

$$\begin{aligned} &\mathbf{R}_{\alpha\beta\mu\nu} = -\mathbf{R}_{\beta\alpha\mu\nu} = -\mathbf{R}_{\alpha\beta\nu\mu} = \mathbf{R}_{\mu\nu\alpha\beta}, \\ &\mathbf{R}_{\alpha\beta\mu\nu} + \mathbf{R}_{\beta\mu\alpha\nu} + \mathbf{R}_{\mu\alpha\beta\nu} = 0, \end{aligned} \tag{1.1.11}$$

and the covariant Bianchi identities

$$\mathbf{D}_\rho \mathbf{R}_{\alpha\beta\mu\nu} + \mathbf{D}_\alpha \mathbf{R}_{\beta\rho\mu\nu} + \mathbf{D}_\beta \mathbf{R}_{\rho\alpha\mu\nu} = 0. \tag{1.1.12}$$

Its components can be calculated in local coordinates in terms of the connection coefficients according to the formula

$$\mathbf{R}_{\alpha\beta\mu}{}^\rho = -\partial_\alpha \mathbf{\Gamma}^\rho{}_{\beta\mu} + \partial_\beta \mathbf{\Gamma}^\rho{}_{\alpha\mu} - \mathbf{\Gamma}^\rho{}_{\alpha\nu}\mathbf{\Gamma}^\nu{}_{\beta\mu} + \mathbf{\Gamma}^\rho{}_{\beta\nu}\mathbf{\Gamma}^\nu{}_{\alpha\mu}. \tag{1.1.13}$$

Therefore, the Ricci tensor $\mathbf{R}_{\alpha\mu} = g^{\beta\rho}\mathbf{R}_{\alpha\beta\mu\rho}$ is given by the formula

$$\mathbf{R}_{\alpha\mu} = -\partial_\alpha \mathbf{\Gamma}^\rho{}_{\rho\mu} + \partial_\rho \mathbf{\Gamma}^\rho{}_{\alpha\mu} - \mathbf{\Gamma}^\rho{}_{\nu\alpha}\mathbf{\Gamma}^\nu{}_{\rho\mu} + \mathbf{\Gamma}^\rho{}_{\rho\nu}\mathbf{\Gamma}^\nu{}_{\alpha\mu}.$$

Simple calculations using (1.1.3) and (1.1.5) show that the Ricci tensor is given by

$$2\mathbf{R}_{\alpha\mu} = -\widetilde{\Box}_{\mathbf{g}} \mathbf{g}_{\alpha\mu} + \partial_\alpha \mathbf{\Gamma}_\mu + \partial_\mu \mathbf{\Gamma}_\alpha + F^{\geq 2}_{\alpha\mu}(g, \partial g), \tag{1.1.14}$$

where $F^{\geq 2}_{\alpha\beta}(g, \partial g)$ is a quadratic semilinear expression,

$$\begin{aligned} F^{\geq 2}_{\alpha\beta}(g, \partial g) = {} & \frac{1}{2}\mathbf{g}^{\rho\mu}\mathbf{g}^{\nu\lambda}\big\{\partial_\nu \mathbf{g}_{\rho\mu}\partial_\beta \mathbf{g}_{\alpha\lambda} + \partial_\nu \mathbf{g}_{\rho\mu}\partial_\alpha \mathbf{g}_{\beta\lambda} - \partial_\nu \mathbf{g}_{\rho\mu}\partial_\lambda \mathbf{g}_{\alpha\beta}\big\} \\ & + \mathbf{g}^{\rho\mu}\mathbf{g}^{\nu\lambda}\big\{ - \partial_\rho \mathbf{g}_{\mu\lambda}\partial_\alpha \mathbf{g}_{\beta\nu} - \partial_\rho \mathbf{g}_{\mu\lambda}\partial_\beta \mathbf{g}_{\alpha\nu} \\ & \qquad + \partial_\rho \mathbf{g}_{\mu\lambda}\partial_\nu \mathbf{g}_{\alpha\beta} + \partial_\alpha \mathbf{g}_{\rho\lambda}\partial_\mu \mathbf{g}_{\beta\nu} + \partial_\beta \mathbf{g}_{\rho\lambda}\partial_\mu \mathbf{g}_{\alpha\nu}\big\} \\ & - \frac{1}{2}\mathbf{g}^{\rho\mu}\mathbf{g}^{\nu\lambda}(\partial_\alpha \mathbf{g}_{\nu\mu} + \partial_\nu \mathbf{g}_{\alpha\mu} - \partial_\mu \mathbf{g}_{\alpha\nu})(\partial_\beta \mathbf{g}_{\rho\lambda} + \partial_\rho \mathbf{g}_{\beta\lambda} - \partial_\lambda \mathbf{g}_{\beta\rho}). \end{aligned} \tag{1.1.15}$$

We consider the Einstein field equations (1.1.1)–(1.1.2) for an unknown spacetime $(M, \mathbf{g})$; for simplicity, we drop the factor of 8π from the energy-momentum tensor. The covariant Bianchi identities $\mathbf{D}^\alpha \mathbf{G}_{\alpha\beta} = 0$ can be used to derive an evolution equation for the massive scalar field ψ. The equation is

$$\Box_{\mathbf{g}}\psi - \psi = 0. \tag{1.1.16}$$

Therefore the main unknowns in the problem are the metric tensor $\mathbf{g}$ and the scalar field ψ, which satisfy the covariant coupled equations (1.1.1) and (1.1.16).

To construct solutions we need to fix a system of coordinates. In this paper we work in *wave coordinates*, which is the condition

$$\mathbf{\Gamma}^\alpha = -\Box_{\mathbf{g}} x^\alpha \equiv 0 \quad \text{for} \quad \alpha \in \{0, 1, 2, 3\}. \tag{1.1.17}$$

Wave coordinates are known to be a good system of coordinates to prove global stability at least in the Einstein-vacuum equations due to the work of Lindblad-Rodnianski [63]. Our construction of global solutions of the Einstein-Klein-Gordon system is based on the following proposition, which can be proved by straightforward calculations.

Proposition 1.1. *Assume* $\mathbf{g}$ *is a Lorentzian metric in a* 4 *dimensional open set* O*, with induced covariant derivative* $\mathbf{D}$ *and Ricci curvature* $\mathbf{R}_{\alpha\beta}$*, and* $\psi : O \to \mathbb{R}$ *is a scalar. Let* x^0, x^1, x^2, x^3 *denote a system of coordinates in* O *and let* $\mathbf{\Gamma}^\nu$ *be defined as in* (1.1.5)*.*

(i) Assume that $(\mathbf{g}, \psi)$ *satisfies the Einstein-Klein-Gordon system*

$$\mathbf{R}_{\alpha\beta} - \mathbf{D}_\alpha\psi\mathbf{D}_\beta\psi - \frac{\psi^2}{2}\mathbf{g}_{\alpha\beta} = 0, \qquad \Box_{\mathbf{g}}\psi - \psi = 0, \tag{1.1.18}$$

in O*. Assume also that* $\mathbf{\Gamma}^\mu \equiv 0$ *in* O*,* $\mu \in \{0, 1, 2, 3\}$ *(the harmonic gauge* *condition). Then*

$$\begin{aligned}&\widetilde{\Box}_{\mathbf{g}}\mathbf{g}_{\alpha\beta} + 2\partial_\alpha\psi\partial_\beta\psi + \psi^2\mathbf{g}_{\alpha\beta} - F^{\geq 2}_{\alpha\beta}(g,\partial g) = 0,\\ &\widetilde{\Box}_{\mathbf{g}}\psi - \psi = 0,\end{aligned} \tag{1.1.19}$$

where the quadratic semilinear terms $F^{\geq 2}_{\alpha\beta}(g,\partial g)$ are defined in (1.1.15) and $\widetilde{\Box}_{\mathbf{g}} := \mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta$ denotes the reduced wave operator .

(ii) Conversely, assume that the equations (1.1.19) (the reduced Einstein-Klein-Gordon system) hold in O. Then

$$\begin{aligned}&\mathbf{R}_{\alpha\beta} - \partial_\alpha\psi\partial_\beta\psi - \frac{\psi^2}{2}\mathbf{g}_{\alpha\beta} - \frac{1}{2}(\partial_\alpha\mathbf{\Gamma}_\beta + \partial_\beta\mathbf{\Gamma}_\alpha) = 0,\\ &\Box_{\mathbf{g}}\psi - \psi + \mathbf{\Gamma}^\mu\partial_\mu\psi = 0,\end{aligned} \tag{1.1.20}$$

and the functions $\mathbf{\Gamma}_\beta = g_{\beta\nu}\mathbf{\Gamma}^\nu$ satisfy the reduced wave equations

$$\begin{aligned}\widetilde{\Box}_{\mathbf{g}}\mathbf{\Gamma}_\beta = {}& 2\mathbf{\Gamma}^\nu\partial_\nu\psi\partial_\beta\psi + \mathbf{g}^{\rho\alpha}[\mathbf{\Gamma}^\nu{}_{\rho\alpha}(\partial_\nu\mathbf{\Gamma}_\beta + \partial_\beta\mathbf{\Gamma}_\nu)\\ &+ \mathbf{\Gamma}^\nu{}_{\rho\beta}(\partial_\alpha\mathbf{\Gamma}_\nu + \partial_\nu\mathbf{\Gamma}_\alpha)] + \partial_\mu\mathbf{\Gamma}_\nu\partial_\beta\mathbf{g}^{\mu\nu}.\end{aligned} \tag{1.1.21}$$

In particular, the pair $(\mathbf{g},\psi)$ solves the Einstein-Klein-Gordon system (1.1.18) if $\mathbf{\Gamma}_\mu \equiv 0$ in O.

Our basic strategy to construct global solutions of the Einstein-Klein-Gordon system is to use Proposition 1.1. We construct first the pair $(\mathbf{g},\psi)$ by solving the reduced Einstein-Klein-Gordon system (1.1.19) (regarded as a quasilinear Wave-Klein-Gordon system) in the domain $\mathbb{R}^3\times[0,\infty)$. In addition, we arrange that $\mathbf{\Gamma}_\mu, \partial_t\mathbf{\Gamma}_\mu$ vanish on the initial hypersurface, so they vanish in the entire open domain, as a consequence of the wave equations (1.1.21). Therefore the pair $(\mathbf{g},\psi)$ solves the Einstein-Klein-Gordon system as desired.

In other words, the problem is reduced to constructing global solutions of the quasilinear system (1.1.19) for initial data compatible with the wave coordinates condition.

1.2 THE GLOBAL REGULARITY THEOREM

To state our global regularity theorem we introduce first several spaces of functions on $\mathbb{R}^3$.

Definition 1.2. *For $a \geq 0$ let H^a denote the usual Sobolev spaces of index a on $\mathbb{R}^3$. We define the Banach spaces $H^{a,b}_\Omega$, $a,b\in\mathbb{Z}_+$, by the norms*

$$\|f\|_{H^{a,b}_\Omega} := \sum_{|\alpha|\leq b}\|\Omega^\alpha f\|_{H^a}, \tag{1.2.1}$$

where $\Omega^\alpha = \Omega_{23}^{\alpha_1}\Omega_{31}^{\alpha_2}\Omega_{12}^{\alpha_3}$ *and* $\Omega_{jk} = x_j\partial_k - x_k\partial_j$ *are the rotation vector-fields of* $\mathbb{R}^3$*. We also define the weighted Sobolev spaces* $H^{a,b}_{S,wa}$ *and* $H^{a,b}_{S,kg}$ *by the norms*

$$\|f\|_{H^{a,b}_{S,wa}} := \sum_{|\beta'|\le|\beta|\le b} \|x^{\beta'}\partial^\beta f\|_{H^a}, \qquad \|f\|_{H^{a,b}_{S,kg}} := \sum_{|\beta|,|\beta'|\le b} \|x^{\beta'}\partial^\beta f\|_{H^a}, \tag{1.2.2}$$

where $x^{\beta'} = x_1^{\beta_1'}x_2^{\beta_2'}x_3^{\beta_3'}$ *and* $\partial^\beta := \partial_1^{\beta_1}\partial_2^{\beta_2}\partial_3^{\beta_3}$*. Notice that* $H^{a,b}_{S,kg} \hookrightarrow H^{a,b}_{S,wa} \hookrightarrow H^{a,b}_\Omega \hookrightarrow H^a$*.*

To implement the strategy described above and use Proposition 1.1 we need to prescribe suitable initial data. Let $\Sigma_0 = \{(x,t) \in \mathbb{R}^3 \times [0,\infty) : t = x^0 = 0\}$. We assume that $\overline{g}, k$ are given symmetric tensors on Σ_0, such that $\overline{g}$ is a Riemannian metric on Σ_0. We assume also that $\psi_0, \psi_1 : \Sigma_0 \to \mathbb{R}$ are given initial data for the scalar field ψ.

We start by prescribing the metric components on Σ_0,

$$\mathbf{g}_{ij} = \overline{g}_{ij}, \qquad \mathbf{g}_{0i} = 0, \qquad \mathbf{g}_{00} = -1.$$

The conditions $\mathbf{g}_{00} = -1$ and $\mathbf{g}_{0i} = 0$ hold only on the initial hypersurface and are not propagated by the flow. They are imposed mostly for convenience and do not play a significant role in the analysis. We also prescribe the time derivative of the metric tensor,

$$\partial_t \mathbf{g}_{ij} = -2k_{ij},$$

in such a way that k is the second fundamental form of the surface Σ_0, $k(X,Y) = -\mathbf{g}(\mathbf{D}_X n, Y)$, where $n = \partial_0$ is the future-oriented unit normal vector-field on Σ_0. The conditions $\mathbf{\Gamma}_\alpha = 0$, $\alpha \in \{0,1,2,3\}$, can be used to determine the other components of the initial data for the pair $(\mathbf{g},\psi)$ on the hypersurface Σ_0, which are

$$\begin{aligned}
&\mathbf{g}_{ij} = \overline{g}_{ij}, \qquad \mathbf{g}_{0i} = \mathbf{g}_{i0} = 0, \qquad \mathbf{g}_{00} = -1,\\
&\partial_t\mathbf{g}_{ij} = -2k_{ij}, \qquad \partial_t\mathbf{g}_{00} = 2\overline{g}^{ij}k_{ij}, \qquad \partial_t\mathbf{g}_{n0} = \overline{g}^{ij}\partial_i\overline{g}_{jn} - \frac{1}{2}\overline{g}^{ij}\partial_n\overline{g}_{ij},\\
&\psi = \psi_0, \qquad \partial_t\psi = \psi_1.
\end{aligned} \tag{1.2.3}$$

The remaining restrictions $\partial_t\mathbf{\Gamma}_\alpha = 0$ lead to the constraint equations. In view of (1.1.20) the constraint equations are equivalent to the conditions $\mathbf{R}_{\alpha 0} - (1/2)\mathbf{R}\mathbf{g}_{\alpha 0} = T_{\alpha 0}$, $\alpha \in \{0,1,2,3\}$, where $T_{\alpha\beta}$ is as in (1.1.2). This leads to four constraint equations

$$\begin{aligned}
\overline{D}_n(\overline{g}^{ij}k_{ij}) - \overline{g}^{ij}\overline{D}_j k_{in} &= \psi_1\overline{D}_n\psi_0, \qquad n \in \{1,2,3\},\\
\overline{R} + \overline{g}^{ij}\overline{g}^{mn}(k_{ij}k_{mn} - k_{im}k_{jn}) &= \psi_1^2 + \overline{g}^{ij}\overline{D}_i\psi_0\overline{D}_j\psi_0 + \psi_0^2,
\end{aligned} \tag{1.2.4}$$

where $\overline{D}$ denotes the covariant derivative induced by the metric $\overline{g}$ on Σ_0, and $\overline{R}$

is the scalar curvature of the metric $\overline{g}$ on Σ_0.

We are now ready to state our first main theorem, which concerns global regularity of the system (1.1.19) for small initial data $(\overline{g}_{ij}, k_{ij}, \psi_0, \psi_1)$.

Theorem 1.3. *Let $\Sigma_0 := \{(x,t) \in \mathbb{R}^4 : t = 0\}$ and assume that $(\overline{g}_{ij}, k_{ij}, \psi_0, \psi_1)$ is an initial data set on Σ_0, satisfying the constraint equations (1.2.4) and the smallness conditions*

$$\begin{aligned}\sum_{n=0}^{3}\sum_{i,j=1}^{3}\{\| \, |\nabla|^{1/2+\delta/4}(\overline{g}_{ij}-\delta_{ij})\|_{H^{N(n),n}_{S,wa}} + \| \, |\nabla|^{-1/2+\delta/4}k_{ij}\|_{H^{N(n),n}_{S,wa}}\} \\ + \sum_{n=0}^{3}\{\|\langle\nabla\rangle\psi_0\|_{H^{N(n),n}_{S,kg}} + \|\psi_1\|_{H^{N(n),n}_{S,kg}}\} \leq \varepsilon_0 \leq \overline{\varepsilon}.\end{aligned} \tag{1.2.5}$$

Here $N_0 := 40$, $d := 10$, $\delta := 10^{-10}$, $N(0) := N_0 + 16d$, $N(n) = N_0 - nd$ for $n \geq 1$, $\overline{\varepsilon}$ is a small constant, and the operators $|\nabla|$ and $\langle\nabla\rangle$ are defined by the multipliers $|\xi|$ and $\langle\xi\rangle$.

(i) Then the reduced Einstein-Klein-Gordon system

$$\begin{aligned}\widetilde{\square}_{\mathbf{g}}\mathbf{g}_{\alpha\beta} + 2\partial_\alpha\psi\partial_\beta\psi + \psi^2\mathbf{g}_{\alpha\beta} - F^{\geq 2}_{\alpha\beta}(g,\partial g) = 0, \\ \widetilde{\square}_{\mathbf{g}}\psi - \psi = 0,\end{aligned} \tag{1.2.6}$$

admits a unique global solution $(\mathbf{g},\psi)$ in $M := \{(x,t) \in \mathbb{R}^4 : t \geq 0\}$, with initial data $(\overline{g}_{ij}, k_{ij}, \psi_0, \psi_1)$ on Σ_0 (as described in (1.2.3)). Here $F^{\geq 2}_{\alpha\beta}(g,\partial g)$ are as in (1.1.15) and $\widetilde{\square}_{\mathbf{g}} = \mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu$. The solution satisfies the harmonic gauge conditions

$$0 = \mathbf{\Gamma}_\mu = \mathbf{g}^{\alpha\beta}\partial_\alpha\mathbf{g}_{\beta\mu} - \frac{1}{2}\mathbf{g}^{\alpha\beta}\partial_\mu\mathbf{g}_{\alpha\beta}, \qquad \mu \in \{0,1,2,3\} \tag{1.2.7}$$

in M. Moreover, the metric $\mathbf{g}$ stays close and converges to the Minkowski metric and ψ stays small and converges to 0 as $t \to \infty$ (in suitable norms).

(ii) In view of Proposition 1.1, the pair $(\mathbf{g},\psi)$ is a global solution in M of the Einstein-Klein-Gordon coupled system

$$\mathbf{R}_{\alpha\beta} - \mathbf{D}_\alpha\psi\mathbf{D}_\beta\psi - \frac{\psi^2}{2}\mathbf{g}_{\alpha\beta} = 0, \qquad \square_{\mathbf{g}}\psi - \psi = 0, \tag{1.2.8}$$

with the prescribed initial data $(\overline{g}_{ij}, k_{ij}, \psi_0, \psi_1)$ on Σ_0. In our geometric context, globality means that all future directed timelike and null geodesics starting from points in M extend forever with respect to their affine parametrization.

The proof of Theorem 1.3 is based on a complex bootstrap argument, involving energy estimates, vector-fields, Fourier analysis, and nonlinear scattering. We summarize some of its main elements in subsection 1.3.1 below, and then provide a more extensive outline of its proof in section 2.2.

The global regularity conclusion of Theorem 1.3 is essentially a qualitative statement, which can only be proved by a precise quantitative analysis of the spacetime. In Chapter 7 we state and prove more precise theorems describing our spacetime. These theorems include global quantitative control and nonlinear scattering of the metric tensor and the Klein-Gordon field (Theorem 7.1), pointwise decay estimates in the physical space (Theorem 7.2 and Lemma 7.4), global control of timelike and null geodesics (Theorem 7.6), weak peeling estimates for the Riemann curvature tensor (Theorem 7.7 and Proposition 7.9), and ADM and Bondi energy formulas (Proposition 7.11, Proposition 7.13, Theorem 7.23, and Proposition 7.24). We will discuss some of these more precise conclusions in section 1.3 below.

In the rest of this section we discuss previous related work and motivate some of the assumptions on the initial data.

1.2.1 Global Stability Results in General Relativity

Global stability of physical solutions is an important topic in General Relativity. For example, the global nonlinear stability of the Minkowski spacetime among solutions of the Einstein-vacuum equation is a central theorem in the field, due to Christodoulou-Klainerman [12]. See also the more recent extensions of Klainerman-Nicolò [52], Lindblad-Rodnianski [62], Bieri and Zipser [6], Speck [72], and Hintz-Vasy [33].

More recently, small data global regularity theorems have also been proved for other coupled Einstein field equations. The Einstein-Klein-Gordon system (the same system we analyze here) was considered recently by LeFloch-Ma [58], who proved small data global regularity for restricted data, which agrees with a Schwarzschild solution with small mass outside a compact set. A similar result was announced by Wang [74].

Our main goals in this monograph are (1) to prove global nonlinear stability for general unrestricted small initial data, and (2) to develop the full asymptotic analysis of the spacetime. In particular, we answer the natural question, raised in the physics literature by Okawa-Cardoso-Pani [66], of whether the Minkowski solution is stable or unstable for small massive scalar field perturbations. A similar global regularity result for general small data was announced recently by LeFloch-Ma [59].

We also refer to the work by Fajman-Joudioux-Smulevici [19], Lindblad-Taylor [64], and, more recently, Bigorgne-Fajman-Joudioux-Smulevici-Thalleron [7] on the global stability of Einstein-Vlasov systems.

In a different direction, one can also raise the question of linear and nonlinear stability of other physical solutions of the Einstein equations. Stability of the Kerr family of solutions has been under intense study in recent years, first at the linearized level (see, for example, [13, 30] and the references therein) and more recently at the full nonlinear level (see [26, 32, 54]).

The stability of Kerr in the presence of a massive scalar field seems interesting as well. Solutions to the Klein-Gordon equation in Kerr can grow exponentially even from smooth initial data, as shown in [70], and this phenomenon was used by Chodosh and Shlapentokh-Rothman [10] to construct a curve of time-periodic solutions of the Einstein-Klein-Gordon system bifurcating from (empty) Kerr (see [31] for a prior numerical construction). Therefore a result on stability of Kerr similar to our main theorem could only be possible, if at all, in a stronger topology where this curve is not continuous (see also the discussion on the mini-bosons in subsection 1.2.5 below).

1.2.1.1 Restricted initial data

One can often simplify considerably the global analysis of wave and Klein-Gordon equations by considering initial data of compact support. The point is that the solutions have the finite speed of propagation, thus remain supported inside a light cone, and one can use the hyperbolic foliation method and its refinements (see [56] for a recent account) to analyze the evolution.

However, to implement this method one needs to first control the solution on an initial hyperboloid (the "initial data"), so the method is restricted to the case when one can establish such control. Due to the finite speed of propagation, this is possible for compactly supported data (for systems of wave or Klein-Gordon equations), or data that agrees with the Schwarzschild solution outside a compact set (in the case of the Einstein equations).

The use of "restricted initial data" coupled with the hyperbolic foliation method leads to significant simplifications of the global analysis, particularly at the level of proving decay. In the context of the Einstein equations these ideas have been used by many authors, such as Friedrich [21], Lindblad-Rodnianski [62], Fajman-Joudioux-Smulevici [19], Lindblad-Taylor [64], LeFloch-Ma [58], and Wang [74].

1.2.2 Simplified Wave-Klein-Gordon Models

Our system (1.2.6) is complicated, but one can gain intuition by looking at simpler models. For example, one can consider the simplified system

$$\begin{aligned} -\Box u &= A^{\alpha\beta}\partial_\alpha v\partial_\beta v + Dv^2, \\ (-\Box + 1)v &= uB^{\alpha\beta}\partial_\alpha\partial_\beta v + Euv, \end{aligned} \tag{1.2.9}$$

where u, v are real-valued functions, and $A^{\alpha\beta}$, $B^{\alpha\beta}$, D, and E are real constants. This system was introduced by LeFloch-Ma [57] as a model for the full Einstein-Klein-Gordon system (1.2.6). Intuitively, the deviation of the Lorentzian metric $\mathbf{g}$ from the Minkowski metric is replaced by a scalar function u, and the massive scalar field ψ is replaced by v. The system (1.2.9) has the same linear structure as the Einstein-Klein-Gordon system (1.2.6), but only keeps, schematically, quadratic interactions that involve the massive scalar field; for simplicity, all the quadratic interactions of the wave component with itself are neglected in this model.

Small data global regularity for the system (1.2.9) was proved by LeFloch-Ma [57] in the case of compactly supported initial data (the restricted data case), using the hyperbolic foliation method. For general small initial data, global regularity was proved by the authors [38].

Global regularity of Wave-Klein-Gordon coupled systems in 3 dimensions is a natural topic, motivated by physical models such as the Dirac-Klein-Gordon equations, and had been investigated earlier by Georgiev [22] and Katayama [44]. A similar system, the massive Maxwell-Klein-Gordon system, was analyzed recently by Klainerman-Wang-Yang [55], who also proved global regularity for general small initial data.

Coupled Wave-Klein-Gordon systems have also been considered in 2 dimensions, where the decay is slower and the global analysis requires nonlinearities with much more favorable structure (see, for example, Ifrim-Stingo [34] and the references therein).

1.2.3 Small Data Global Regularity Results

The system (1.2.6) can be easily transformed into a quasilinear coupled system of wave and Klein-Gordon equations. Indeed, let m denote the Minkowski metric and write

$$\mathbf{g}_{\alpha\beta} = m_{\alpha\beta} + h_{\alpha\beta}, \qquad \mathbf{g}^{\alpha\beta} = m^{\alpha\beta} + g^{\alpha\beta}_{\geq 1}, \qquad \alpha, \beta \in \{0, 1, 2, 3\}.$$

It follows from (1.2.6) that the metric components $h_{\alpha\beta}$ satisfy the nonlinear wave equations

$$(\partial_0^2 - \Delta)h_{\alpha\beta} = \mathcal{N}^h_{\alpha\beta} := \mathcal{KG}_{\alpha\beta} + g^{\mu\nu}_{\geq 1}\partial_\mu\partial_\nu h_{\alpha\beta} - F^{\geq 2}_{\alpha\beta}(g, \partial g) \tag{1.2.10}$$

where $F^{\geq 2}_{\alpha\beta}(g, \partial g)$ are the semilinear terms in (1.1.15) and $\mathcal{KG}_{\alpha\beta} := 2\partial_\alpha\psi\partial_\beta\psi + \psi^2(m_{\alpha\beta} + h_{\alpha\beta})$. Moreover, the field ψ satisfies the quasilinear Klein-Gordon equation

$$(\partial_0^2 - \Delta + 1)\psi = \mathcal{N}^\psi := g^{\mu\nu}_{\geq 1}\partial_\mu\partial_\nu\psi. \tag{1.2.11}$$

Therefore Theorem 1.3 can be regarded as a small data global regularity result for a quasilinear evolution system. Several important techniques have been developed over the years in the study of such problems, starting with seminal contributions of John, Klainerman, Shatah, Simon, Christodoulou, Alinhac, and Delort [1, 2, 11, 12, 14, 15, 42, 43, 48, 49, 50, 51, 69, 71]. These include the vector-field method, normal forms, and the isolation of null structures.

In the case of Einstein equations and other hyperbolic systems, most global results have been proved mostly using the "physical space" framework, based on pointwise spacetime estimates. This is well adapted to geometric backgrounds with non-constant coefficients. The analysis is naturally carried out through weighted estimates and relies heavily on the presence of symmetries (vector-fields) that can be used to extract information about solutions. This is the main framework for many works on General Relativity, especially away from Minkowski space and in vacuum or with electromagnetic and massless scalar-fields, such as [6, 7, 12, 13, 19, 26, 52, 54, 58, 59, 62, 63, 64, 72].

1.2.3.1 Fourier analysis and the Z-norm method

In the last few years new ideas have emerged in the study of global solutions of quasilinear evolutions, inspired mainly by the advances in semilinear theory. The basic goal is to combine the classical energy and vector-fields methods with refined analysis of the Duhamel formula, using the Fourier transform. This starts by decomposing an unknown U into a superposition of elementary waves

$$U(x,t) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{V}(\xi,t) e^{i[\langle x,\xi\rangle - t\Lambda(\xi)]} d\xi, \tag{1.2.12}$$

for some appropriate dispersion relation Λ. The main objective is then to understand quantitatively properties of the "linear profile" V during the evolution.

The main advantage of the Fourier transform method over physical space methods is the ability to identify clearly resonant and non-resonant nonlinear interactions, by decomposing the various waves as in (1.2.12) and examining their interactions. One can then dispose of the non-resonant interactions (using, for example, normal forms), and concentrate on a small number of resonant interactions. This is particularly important in low dimensions (like 1 or 2 dimension), when decay by itself cannot be enough to lead to global control of solutions.

In semilinear dispersive and hyperbolic equations Fourier analysis is a central tool that has led to major progress in the entire field. On the other hand, in the context of quasilinear evolutions, Fourier analysis has only been used more recently, starting essentially with the "method of spacetime resonances" of Germain-Masmoudi-Shatah [24, 25] and Gustafson-Nakanishi-Tsai [29]. The main difficulty in the quasilinear case is that the Duhamel formula cannot be used exclusively to study the evolution, due to derivative loss, and one has to rely also on energy estimates.

Our general philosophy, which we use in this monograph to prove Theorem 1.3, is to work both in the physical space, mainly to prove energy estimates (including vector-fields), and in the Fourier space, mainly to investigate resonances using the Duhamel formula and prove decay of the solutions in time. At the implementation level, the analysis in the Fourier space is based on a choice of a "Z-norm" to measure the size of the linear profiles dynamically in time. This choice is very important, and one should think of it as analogous to the choice of the "resolution norm" in the case of semilinear evolutions (the classical choices being Strichartz norms or $X^{s,b}$ norms). The key point is that the Z-norm has to complement well the information coming from energy estimates.

The Z-norm method, with different choices of the norm itself, depending on the problem, was used recently by the authors and their collaborators in several small data global regularity problems, for water waves and plasmas, such as [16, 17, 27, 28, 35, 36, 37, 39, 40, 41, 46]. It is particularly well suited to the study of systems with multiple characteristics, in which different components of the system evolve according to different dispersion relations and have different speeds of propagation, such as plasma models or the Einstein-Klein-Gordon system (1.2.10)–(1.2.11). The point is that such systems tend to have fewer joint symmetries, which complicates significantly the analysis in the physical space, but the Fourier analysis method is much less sensitive to the presence of symmetries.

1.2.4 Assumptions on the Initial Data

The precise form of the smallness assumptions (1.2.5) on the metric initial data $\overline{g}_{ij}$ and k_{ij} is important. Indeed, in view of the positive mass theorem of Schoen-Yau [68], one expects the metric components $\overline{g}_{ij} - \delta_{ij}$ to decay no faster than $M/\langle x\rangle$ and the second fundamental form k to decay no faster than $M/\langle x\rangle^2$, where $M \ll 1$ is the mass. Capturing this type of decay, using L^2-based norms, is precisely the role of the homogeneous multipliers $|\nabla|^{1/2+\delta/4}$ and $|\nabla|^{-1/2+\delta/4}$ in (1.2.5). Notice that these multipliers are sharp, up to the $\delta/4$ power.

Our assumptions on the metric are essentially of the type

$$g_{ij} = \delta_{ij} + \varepsilon_0 O(\langle x\rangle^{-1+\delta/4}), \qquad k_{ij} = \varepsilon_0 O(\langle x\rangle^{-2+\delta/4}) \tag{1.2.13}$$

at time $t = 0$. These are less restrictive than the assumptions used sometimes even in the vacuum case $\psi \equiv 0$—see, for example, [12, 52, 63]—in the sense that the initial data is not assumed to agree with the Schwarzschild initial data up to lower order terms. For maximal time foliations, our assumptions are, however, more restrictive than the ones in Bieri's work [6], but we are able to prove more precise asymptotic bounds on the metric and the Riemann curvature tensor; see section 1.3 below.

We remark also that our assumptions (1.2.5) allow for non-isotropic initial data, possibly with different "masses" in different directions. For the vacuum case, initial data of this type, satisfying the constraint equations, have been constructed recently by Carlotto-Schoen [9].

1.2.5 The Mini-bosons

A serious potential obstruction to small data global stability theorems is the presence of non-decaying "small" solutions, such as small solitons. A remarkable fact is that there are such small non-decaying solutions for the Einstein-Klein-Gordon system, namely the so-called mini-boson stars. These are time-periodic (therefore non-decaying) and spherically symmetric exact solutions of the Einstein-Klein-Gordon system. They were discovered numerically by physicists, such as Kaup [47], Friedberg-Lee-Pang [20] (see also [60]), and then constructed rigorously by Bizon-Wasserman [8].

These mini-bosons can be thought of as arbitrarily small (hence the name) in certain topologies, as explained in [8]. However, the mini-bosons (in particular the Klein-Gordon component) are not small in the stronger topology we use here, as described by (1.2.5), so we can thankfully avoid them in our analysis.

1.3 MAIN IDEAS AND FURTHER ASYMPTOTIC RESULTS

In this section we provide first a brief summary of some of the main ingredients in the proof of the global nonlinear stability result in Theorem 1.3. Then, in subsections 1.3.2–1.3.6 we present some of the additional theorems we prove in Chapter 7, concerning the global geometry of our spacetime.

1.3.1 Global Nonlinear Stability

The classical mechanism to establish small data global regularity for quasilinear dispersive and hyperbolic systems has two main components:

(1) Propagate control of energy functionals (high order Sobolev norms and vector-fields);

(2) Prove dispersion/decay of the solution over time.

These are our basic goals here as well, as we investigate solutions of the coupled Wave-Klein-Gordon system (1.2.10)–(1.2.11) in the variables $h_{\alpha\beta}$ and ψ. As expected, our analysis also involves vector-fields, corresponding to the natural symmetries of the linearized equations, namely the Lorentz vector-fields Γ_a and the rotation vector-fields Ω_{ab},

$$\Gamma_a := x_a\partial_t + t\partial_a, \qquad \Omega_{ab} := x_a\partial_b - x_b\partial_a, \tag{1.3.1}$$

for $a, b \in \{1, 2, 3\}$. These vector-fields commute with both the wave operator and the Klein-Gordon operator in the flat Minkowski space. We note that the scaling vector-field $S = t\partial_t + x \cdot \nabla_x$ does not satisfy nice commutation properties with the linearized system (due to the Klein-Gordon field), so we cannot use it in our analysis.

The main objects we analyze in the proof of nonlinear stability are the normalized solutions $U^{\mathcal{L}h_{\alpha\beta}}$ and $U^{\mathcal{L}\psi}$ and the associated *linear profiles* $V^{\mathcal{L}h_{\alpha\beta}}$ and $V^{\mathcal{L}\psi}$, defined by

$$\begin{aligned} U^{\mathcal{L}h_{\alpha\beta}}(t) &:= \partial_t(\mathcal{L}h_{\alpha\beta})(t) - i\Lambda_{wa}(\mathcal{L}h_{\alpha\beta})(t), \qquad & V^{\mathcal{L}h_{\alpha\beta}}(t) &:= e^{it\Lambda_{wa}}U^{\mathcal{L}h_{\alpha\beta}}(t), \\ U^{\mathcal{L}\psi}(t) &:= \partial_t(\mathcal{L}\psi)(t) - i\Lambda_{kg}(\mathcal{L}\psi)(t), \qquad & V^{\mathcal{L}\psi}(t) &:= e^{it\Lambda_{kg}}U^{\mathcal{L}\psi}(t), \end{aligned} \tag{1.3.2}$$

where $\Lambda_{wa} = |\nabla|$, $\Lambda_{kg} = \langle\nabla\rangle = \sqrt{|\nabla|^2 + 1}$. Here $\mathcal{L}$ denotes differential operators obtained by applying up to three vector-fields Γ_a or Ω_{ab}, and these operators are applied to the metric components $h_{\alpha\beta}$ and the field ψ.

The complex-valued normalized solutions $U^{\mathcal{L}h_{\alpha\beta}}$ and $U^{\mathcal{L}\psi}$ capture both the time derivatives (as the real part) and the spatial derivatives (as the imaginary part) of the variables $h_{\alpha\beta}$ and ψ. The linear profiles $V^{\mathcal{L}h_{\alpha\beta}}$ and $V^{\mathcal{L}\psi}$, which are constructed by going forward in time along the nonlinear evolution, and then going backwards in time along the linear flow, capture the cumulative effect of the nonlinearity over time.

Our proof of global stability relies on controlling simultaneously three types of norms, as part of a bootstrap argument:

(1) High order energy norms, involving Sobolev derivatives and the vector-fields Γ_a and Ω_{ab}, with slow growth in time;

(2) Matching weighted estimates on the profiles $V^{\mathcal{L}h_{\alpha\beta}}$ and $V^{\mathcal{L}\psi}$ in Sobolev spaces, again with slow growth in time;

(3) Sharp uniform in time estimates on the Klein-Gordon profile V^{ψ} and on some parts of the metric profiles $V^{h_{\alpha\beta}}$, in a suitable Z-norm to be defined.

We discuss these estimates in more detail in the rest of this subsection.

1.3.1.1 Energy estimates and weighted estimates on the profiles

The main energy estimates we prove as part of our bootstrap argument are

$$\big\|(\langle t\rangle|\nabla|_{\leq 1})^{\delta/4}|\nabla|^{-1/2}U^{\mathcal{L}h_{\alpha\beta}}(t)\big\|_{H^{n(\mathcal{L})}} + \big\|U^{\mathcal{L}\psi}(t)\big\|_{H^{n(\mathcal{L})}} \lesssim \varepsilon_0\langle t\rangle^{H(\mathcal{L})\delta}, \quad (1.3.3)$$

for a suitable hierarchy of parameters $n(\mathcal{L})$ and $H(\mathcal{L})$ that depend on the differential operator $\mathcal{L}$. We remark that the energy estimates we prove for the metric variables $U^{\mathcal{L}h_{\alpha\beta}}$ also contain significant information at low frequencies, due to the operators $|\nabla|^{-1/2}$ and $|\nabla|_{\leq 1}$, which are connected to the natural $|x|^{-1+}$ decay of the metric components $h_{\alpha\beta}$. The nonlinear propagation of the low-frequency energy bounds is, in fact, the more subtle part of the argument.

The second component of our bootstrap argument consists of compatible weighted estimates on the profiles $V^{\mathcal{L}h_{\alpha\beta}}$ and $V^{\mathcal{L}\psi}$, of the form

$$\begin{aligned}2^{k/2}(2^{k^-}\langle t\rangle)^{\delta/4}\|P_k(x_l V^{\mathcal{L}h_{\alpha\beta}})(t)\|_{L^2}\\ + 2^{k^+}\|P_k(x_l V^{\mathcal{L}\psi})(t)\|_{L^2} \lesssim \varepsilon_0\langle t\rangle^{H'(\mathcal{L})\delta}2^{-n'(\mathcal{L})k^+},\end{aligned} \quad (1.3.4)$$

for any $k \in \mathbb{Z}$, $l \in \{1, 2, 3\}$, and differential operator $\mathcal{L}$ containing at most two vector-fields Γ_a or Ω_{ab}. Here P_k denote Littlewood-Paley projections to frequencies $\approx 2^k$ and $x^+ = \max(x, 0)$ and $x^- = \min(x, 0)$ for any $x \in \mathbb{R}$.

The energy estimates (1.3.3) and the weighted estimates (1.3.4) are compatible, at the level of the important parameters $H(\mathcal{L})$, $n(\mathcal{L})$, $H'(\mathcal{L})$, and $n'(\mathcal{L})$ that measure the slow growth in time and the Sobolev smoothness of the various components.

The weighted estimates (1.3.4) imply almost optimal pointwise decay estimates on the metric components and the Klein-Gordon field, with improved decay at low and high frequencies, due to Lemma 3.9. We emphasize, however, that weighted estimates on linear profiles are a lot stronger than pointwise decay estimates on solutions, and serve many other purposes. For example, space localization of the linear profiles allows us to decompose the main variables both in frequency and space, which leads to precise control in nonlinear estimates.

1.3.1.2 Weak null structure and decomposition of the metric tensor

The proof of the global stability theorem is involved, mainly because the nonlinearities $\mathcal{N}^h_{\alpha\beta}$ and $\mathcal{N}^\psi$ have complicated structure, both at the semilinear level (for $\mathcal{N}^h_{\alpha\beta}$) and at the quasilinear level.

In particular, it is well known that the semilinear terms $F^{\geq 2}_{\alpha\beta}(g, \partial g)$ do not have the classical null structure. They have, however, a remarkable weak null structure in harmonic coordinates, which is still suitable for global analysis as discovered by Lindblad-Rodnianski [62]. To identify and use this weak null structure we need to decompose the tensor $h_{\alpha\beta}$.

The standard way to decompose the metric tensor in General Relativity is based on null frames (see, for instance, [12] or [62]). Here we use a different decomposition of the metric tensor, reminiscent of the div-curl decomposition of vector-fields in fluid models, which is connected to the classical work of Arnowitt-Deser-Misner [3] on the Hamiltonian formulation of General Relativity. For us, this decomposition has the advantage of being more compatible with the Fourier transform and the vector-fields Ω_{ab} and Γ_a.

More precisely, let $R_j = |\nabla|^{-1}\partial_j$, $j \in \{1, 2, 3\}$, denote the Riesz transforms on $\mathbb{R}^3$, and let

$$\begin{aligned}
&F := (1/2)[h_{00} + R_j R_k h_{jk}], \qquad \underline{F} := (1/2)[h_{00} - R_j R_k h_{jk}],\\
&\rho := R_j h_{0j}, \qquad \omega_j := \in_{jkl} R_k h_{0l},\\
&\Omega_j := \in_{jkl} R_k R_m h_{lm}, \qquad \vartheta_{jk} := \in_{jmp} \in_{knq} R_m R_n h_{pq}.
\end{aligned} \tag{1.3.5}$$

Geometrically, the variables $F + \underline{F}$, ρ, and ω are linked to the lapse and the shift vector, $F - \underline{F}$ and Ω are gauge components associated to spatial coordinates, while ϑ corresponds to the (linearized) coordinate-free component of the spatial metric (see Proposition 7.14). The metric tensor h can be recovered linearly from the components $F, \underline{F}, \rho, \omega_j, \Omega_j, \vartheta_{jk}$.

Our analysis shows that the components $F, \omega_j, \Omega_j, \vartheta_{jk}$ satisfy good wave equations, with all the quadratic semilinear terms having suitable null structure. On the other hand, the components $\underline{F}$ and ρ (which are related elliptically due to the harmonic gauge conditions) satisfy wave equations with some quadratic semilinear terms with no null structure. However, these non-null quadratic semilinear terms have the redeeming feature that they can be expressed only in terms of the good components ϑ_{jk}.

This algebraic structure suggests that we should aim to prove that the good components $F, \omega_j, \Omega_j, \vartheta_{jk}$ do not grow during the evolution, in suitable norms to be made precise. On the other hand, the components $\underline{F}, \rho$, as well as all the components $\mathcal{L}h_{\alpha\beta}$ and $\mathcal{L}\psi$ which contain some weighted vector-fields Ω_{ab} or Γ_a, should be allowed to grow in time slowly, at suitable rates to be determined. We note that our vector-fields are adapted to the Minkowski geometry, containing the coordinate functions x_a and t, not to the true geometry of the spacetime; thus it is expected that they can only be useful only up to $\langle t\rangle^{0+}$ losses. At a qualitative level, this is precisely what our final conclusions are.

1.3.1.3 Uniform bounds and the Z-norm

To prove uniform control on the good metric components $F, \omega_j, \Omega_j, \vartheta_{jk}$ and the field ψ we use what we call *the Z-norm method*: we define the special norms

$$\begin{aligned} \|f\|_{Z_{wa}} &:= \sup_{k\in\mathbb{Z}} 2^{N_0 k^+} 2^{k^-(1+\kappa)} \|\widehat{P_k f}\|_{L^\infty}, \\ \|f\|_{Z_{kg}} &:= \sup_{k\in\mathbb{Z}} 2^{N_0 k^+} 2^{k^-(1/2-\kappa)} \|\widehat{P_k f}\|_{L^\infty}, \end{aligned} \tag{1.3.6}$$

where $N_0 = 40$ and $\kappa = 10^{-3}$. The last component of our bootstrap construction involves uniform bounds of the form

$$\|V^F(t)\|_{Z_{wa}} + \|V^{\omega_a}(t)\|_{Z_{wa}} + \|V^{\vartheta_{ab}}(t)\|_{Z_{wa}} + \|V^\psi(t)\|_{Z_{kg}} \lesssim \varepsilon_0, \tag{1.3.7}$$

for any $t \in [0, \infty)$ and $a, b \in \{1, 2, 3\}$, where the profiles V^G are defined in as (1.3.2),

$$U^G(t) := \partial_t G(t) - i\Lambda_{wa} G(t), \qquad V^G(t) := e^{it\Lambda_{wa}} U^G(t), \tag{1.3.8}$$

for $G \in \{F, \omega_a, \vartheta_{ab}\}$. The main point of the estimates (1.3.7) is the uniformity in time, in particular allowing us to prove sharp $\varepsilon_0\langle t\rangle^{-1}$ pointwise decay on some components of the metric tensor.

The Z-norms defined in (1.3.6) measure the L^∞ norm of solutions in the Fourier space, with weights that are particularly important at low frequencies. They cannot be propagated using energy estimates, since they are not L^2-based norms. We use instead the Duhamel formula, in the Fourier space, which leads to derivative loss. Because of this the Z-norm bounds (1.3.7) are weaker than the energy bounds (1.3.3) at very high frequencies. One should think of the Z-norm bounds as effective at middle frequencies, say $\langle t\rangle^{-1/2} \lesssim 2^k \lesssim \langle t\rangle^{1/2}$.

1.3.2 Nonlinear Scattering

The global dynamics of solutions is complicated mainly because they do not scatter linearly as $t \to \infty$. This is due to the low frequencies of the metric tensor in the quasilinear terms $g^{\mu\nu}_{\geq 1}\partial_\mu\partial_\nu h_{\alpha\beta}$ and $g^{\mu\nu}_{\geq 1}\partial_\mu\partial_\nu\psi$, which create a long-range perturbation.

To understand the asymptotic behavior of our spacetime we need to renormalize the profiles. More precisely, we define the wave phase correction (related to optical functions)

$$\begin{aligned}\Theta_{wa}(\xi,t) := \int_0^t \Big\{ & h_{00}^{low}(s\xi/\Lambda_{wa}(\xi),s)\frac{\Lambda_{wa}(\xi)}{2} \\ & + h_{0j}^{low}(s\xi/\Lambda_{wa}(\xi),s)\xi_j + h_{jk}^{low}(s\xi/\Lambda_{wa}(\xi),s)\frac{\xi_j\xi_k}{2\Lambda_{wa}(\xi)}\Big\}\,ds\end{aligned} \tag{1.3.9}$$

and the Klein-Gordon phase correction

$$\begin{aligned}\Theta_{kg}(\xi,t) := \int_0^t \Big\{ & h_{00}^{low}(s\xi/\Lambda_{kg}(\xi),s)\frac{\Lambda_{kg}(\xi)}{2} \\ & + h_{0j}^{low}(s\xi/\Lambda_{kg}(\xi),s)\xi_j + h_{jk}^{low}(s\xi/\Lambda_{kg}(\xi),s)\frac{\xi_j\xi_k}{2\Lambda_{kg}(\xi)}\Big\}\,ds,\end{aligned} \tag{1.3.10}$$

where $h_{\alpha\beta}^{low}$ are low frequency components of the metric tensor,

$$\widehat{h_{\alpha\beta}^{low}}(\rho,s) := \varphi_{\leq 0}(\langle s\rangle^{p_0}\rho)\widehat{h_{\alpha\beta}}(\rho,s), \qquad p_0 := 0.68. \tag{1.3.11}$$

The choice of p_0, slightly bigger than $2/3$, is important in the proof to justify the correction. Geometrically, the two phase corrections Θ_{wa} and Θ_{kg} are obtained by integrating suitable low frequency components of the metric tensor along the characteristics of the wave and the Klein-Gordon linear flows.

The nonlinear profiles are obtained by multiplication in the Fourier space,

$$\widehat{V_*^G}(\xi,t) := e^{-i\Theta_{wa}(\xi,t)}\widehat{V^G}(\xi,t), \qquad \widehat{V_*^\psi}(\xi,t) := e^{-i\Theta_{kg}(\xi,t)}\widehat{V^\psi}(\xi,t), \tag{1.3.12}$$

for $G \in \{F,\omega_a,\vartheta_{ab}\}$. Notice that $\|V_*^G\|_{Z_{wa}} = \|V^G\|_{Z_{wa}}$ and $\|V_*^\psi\|_{Z_{kg}} = \|V^\psi\|_{Z_{kg}}$, since the phases Θ_{wa} and Θ_{kg} are real-valued. The point of this construction is that the new nonlinear profiles V_*^F, $V_*^{\omega_a}$, $V_*^{\vartheta_{ab}}$, and V_*^ψ converge as the time goes to infinity, i.e.,

$$\begin{aligned}\|V_*^F(t)-V_\infty^F\|_{Z_{wa}} + \|V_*^{\omega_a}(t)-V_\infty^{\omega_a}\|_{Z_{wa}} + \|V_*^{\vartheta_{ab}}(t)-V_\infty^{\vartheta_{ab}}\|_{Z_{wa}} &\lesssim \varepsilon_0\langle t\rangle^{-\delta/2}, \\ \|V_*^\psi(t)-V_\infty^\psi\|_{Z_{kg}} &\lesssim \varepsilon_0\langle t\rangle^{-\delta/2},\end{aligned} \tag{1.3.13}$$

where $V_\infty^F, V_\infty^{\omega_a}, V_\infty^{\vartheta_{ab}} \in Z_{wa}$ and $V_\infty^\psi \in Z_{kg}$ are the *nonlinear scattering data*. These functions, in particular the components $V_\infty^{\vartheta_{ab}}$ and V_∞^ψ, are important in the asymptotic analysis of our spacetime. Chapter 5 is mainly concerned with the proofs of the bounds (1.3.13).

1.3.3 Asymptotic Bounds and Causal Geodesics

Our core bootstrap argument relies on controlling the solution both in the physical space and in the Fourier space, as summarized above. However, after closing the main bootstrap argument, we can derive classical bounds on the solutions in the physical space, without explicit use of the Fourier transform.

We start with decay estimates in the physical space. Let

$$L := \partial_t + \partial_r, \qquad \underline{L} := \partial_t - \partial_r, \tag{1.3.14}$$

where $r := |x|$ and $\partial_r := |x|^{-1}x^j\partial_j$. Let

$$\mathcal{T} := \{L, r^{-1}\Omega_{12}, r^{-1}\Omega_{23}, r^{-1}\Omega_{31}\} \tag{1.3.15}$$

denote the set of "good" vector-fields, tangential to the (Minkowski) light cones.

In Theorem 7.2 we prove that the metric components satisfy the bounds

$$|h(x,t)| + \langle t+r\rangle|\partial_V h(x,t)| + \langle t-r\rangle|\partial_{\underline{L}} h(x,t)| \lesssim \varepsilon_0\langle t+r\rangle^{2\delta'-1}, \tag{1.3.16}$$

in the manifold $M := \{(x,t) \in \mathbb{R}^3 \times [0,\infty)\}$, where $r = |x|$, $V \in \mathcal{T}$, $h \in \{h_{\alpha\beta}\}$, $\partial_W := W^\alpha\partial_\alpha$, and $\delta' = 2000\delta$. The scalar field decays faster but with no derivative improvement,

$$\begin{aligned} |\psi(x,t)| + |\partial_0\psi(x,t)| &\lesssim \varepsilon_0\langle t+r\rangle^{\delta'/2-1}\langle r\rangle^{-1/2}, \\ |\partial_b\psi(x,t)| &\lesssim \varepsilon_0\langle t+r\rangle^{\delta'/2-3/2}, \qquad b \in \{1,2,3\}. \end{aligned} \tag{1.3.17}$$

Also, in Lemma 7.4 we show that the second order derivatives to the metric satisfy the bounds

$$\begin{aligned} \langle r\rangle^2|\partial_{V_1}\partial_{V_2}h(x,t)| &+ \langle t-r\rangle^2|\partial_{\underline{L}}^2 h(x,t)| \\ &+ \langle t-r\rangle\langle r\rangle|\partial_{\underline{L}}\partial_{V_1}h(x,t)| \lesssim \varepsilon_0\langle r\rangle^{3\delta'-1}, \end{aligned} \tag{1.3.18}$$

in the region $M' := \{(x,t) \in M : t \geq 1, |x| \geq 2^{-8}t\}$, where $V_1, V_2 \in \mathcal{T}$ are good vector-fields.

The pointwise bounds (1.3.16)–(1.3.18) are as expected, including the small δ' losses that are due to our weak assumptions (1.2.13) on the initial data. These bounds follow mainly from the profile bounds (1.3.4) and linear estimates.

As an application, we can describe precisely the future-directed causal geodesics in our spacetime M. Indeed, in Theorem 7.6 we show that if $p = (p^0, p^1, p^2, p^3)$ is a point in M and $v = v^\alpha\partial_\alpha$ is a null or timelike vector at p, normalized with $v^0 = 1$, then there is a unique affinely parametrized global geodesic curve $\gamma : [0,\infty) \to M$ with

$$\gamma(0) = p = (p^0, p^1, p^2, p^3), \qquad \dot\gamma(0) = v = (v^0, v^1, v^2, v^3).$$

Moreover, the geodesic curve γ becomes asymptotically parallel to a geodesic line of the Minkowski space, i.e., there is a vector $v_\infty = (v^0_\infty, v^1_\infty, v^2_\infty, v^3_\infty)$ such that, for any $s \in [0,\infty)$,

$$|\dot\gamma(s) - v_\infty| \lesssim \varepsilon_0(1+s)^{-1+6\delta'} \quad \text{and} \quad |\gamma(s) - v_\infty s - p| \lesssim \varepsilon_0(1+s)^{6\delta'}.$$

1.3.4 Weak Peeling Estimates

These are classical estimates on asymptotically flat spacetimes, which assert, essentially, that certain components of the Riemann curvature tensor have improved decay compared to the general estimate $|\mathbf{R}| \lesssim \varepsilon_0 \langle t+r\rangle^{-1+}\langle t-r\rangle^{-2}$. The rate of decay is mainly determined by the *signature* of the component.

More precisely, we use the Minkowski frames $(L, \underline{L}, e_a)$, where $L, \underline{L}$ are as in (1.3.14) and $e_a \in \mathcal{T}_h := \{r^{-1}\Omega_{12}, r^{-1}\Omega_{23}, r^{-1}\Omega_{31}\}$, and assign signature $+1$ to the vector-field L, -1 to the vector-field $\underline{L}$, and 0 to the horizontal vector-fields in $\mathcal{T}_h$. With $e_1, e_2, e_3, e_4 \in \mathcal{T}_h$, we define $\mathrm{Sig}(a)$ as the set of components of the Riemann tensor of total signature a, so

$$\begin{aligned}
\mathrm{Sig}(-2) &:= \{\mathbf{R}(\underline{L}, e_1, \underline{L}, e_2)\},\\
\mathrm{Sig}(2) &:= \{\mathbf{R}(L, e_1, L, e_2)\},\\
\mathrm{Sig}(-1) &:= \{\mathbf{R}(\underline{L}, e_1, e_2, e_3), \mathbf{R}(\underline{L}, L, \underline{L}, e_1)\},\\
\mathrm{Sig}(1) &:= \{\mathbf{R}(L, e_1, e_2, e_3), \mathbf{R}(L, \underline{L}, L, e_1)\},\\
\mathrm{Sig}(0) &:= \{\mathbf{R}(e_1, e_2, e_3, e_4), \mathbf{R}(L, \underline{L}, e_1, e_2), \mathbf{R}(L, e_1, \underline{L}, e_2), \mathbf{R}(L, \underline{L}, L, \underline{L})\}.
\end{aligned} \tag{1.3.19}$$

These components capture the entire curvature tensor, due to the symmetries (1.1.11).

In Theorem 7.7 we prove that if $\Psi_{(a)} \in \mathrm{Sig}(a)$, $a \in \{-2, -1, 1, 2\}$, then

$$\begin{aligned}
|\Psi_{(-2)}(x,t)| &\lesssim \varepsilon_0 \langle r\rangle^{7\delta'-1}\langle t-r\rangle^{-2},\\
|\Psi_{(-1)}(x,t)| &\lesssim \varepsilon_0 \langle r\rangle^{7\delta'-2}\langle t-r\rangle^{-1},\\
|\Psi_{(2)}(x,t)| + |\Psi_{(1)}(x,t)| + |\Psi_{(0)}(x,t)| &\lesssim \varepsilon_0 \langle r\rangle^{7\delta'-3},
\end{aligned} \tag{1.3.20}$$

in the region $M' = \{(x,t) \in M : t \geq 1 \text{ and } |x| \geq 2^{-8}t\}$. This holds in all cases except if $\Psi_{(0)}$ is of the form $\mathbf{R}(L, e_1, \underline{L}, e_2) \in \mathrm{Sig}(0)$, in which case we can only prove the weaker bounds

$$|\mathbf{R}(L, e_1, \underline{L}, e_2)(x,t)| \lesssim \varepsilon_0 \langle r\rangle^{7\delta'-2}\langle t-r\rangle^{-1}. \tag{1.3.21}$$

Notice that we define our decomposition in terms of the Minkowski null pair $(L, \underline{L})$ instead of more canonical null frames (or tetrads) adapted to the metric $\mathbf{g}$ (see, for example, [12], [52], [53]). This is not important however, since the weak peeling estimates are invariant under natural changes of the frame of the form $(L, \underline{L}, e_a) \to (L', \underline{L}', e'_a)$, satisfying

$$|(L-L')(x,t)| + |(\underline{L}-\underline{L}')(x,t)| + |(e_a - e'_a)(x,t)| \lesssim r^{-1+2\delta'} \qquad \text{in } M'.$$

As we show in Proposition 7.9, one can in fact restore the full $\varepsilon_0\langle r\rangle^{7\delta'-3}$ decay of the component $\mathbf{R}(L', e'_1, \underline{L}', e'_2)$, provided that L' is almost null, i.e., $|\mathbf{g}(L', L')(x,t)| \lesssim \langle r\rangle^{-2+4\delta'}$ in M'.

The almost cubic decay we prove in (1.3.20)–(1.3.21) seems optimal in our problem, for two reasons. First, the Ricci components themselves involve squares of the massive field, and cannot decay better than $\langle r\rangle^{-3+}$ in M'. Moreover, the almost cubic decay is also formally consistent with the weak peeling estimates of Klainerman-Nicolò [53, Theorem 1.2 (b)] in the setting of our more general metrics (formally, one would take $\gamma = -1/2-$ and $\delta = 2+$ with the notation in [53], to match our decay assumptions (1.2.13) on the initial data; this range of parameters is not allowed, however, in [53] as δ is assumed to be $< 3/2$).

1.3.5 The ADM Energy and the Linear Momentum

The ADM energy (or the ADM mass) measures the total deviation of our space-time from the Minkowski solution. It is calculated according to the standard formula (see, for example, [4])

$$E_{ADM}(t) := \frac{1}{16\pi} \lim_{R\to\infty} \int_{S_{R,t}} (\partial_j \mathbf{g}_{nj} - \partial_n \mathbf{g}_{jj}) \frac{x^n}{|x|}\, dx, \tag{1.3.22}$$

where the integration is over large (Euclidean) spheres $S_{R,t} \subseteq \Sigma_t = \{(x,t) : x \in \mathbb{R}^3\}$ of radius R. In our case we show in Proposition 7.11 that the energy $E_{ADM}(t) = E_{ADM}$ is well defined and constant in time. Moreover, it is non-negative and can be expressed in terms of the scattering profiles V_∞^ψ and $V_\infty^{\vartheta_{mn}}$ (see (1.3.13)) according to the formula

$$E_{ADM} = \frac{1}{16\pi} \|V_\infty^\psi\|_{L^2}^2 + \frac{1}{64\pi} \sum_{m,n\in\{1,2,3\}} \|V_\infty^{\vartheta_{mn}}\|_{L^2}^2. \tag{1.3.23}$$

We can also prove conservation of one other natural quantity, namely the linear momentum. Let N denote the future unit normal vector-field to the hypersurface Σ_t, let $\overline{g}_{ab} = \mathbf{g}_{ab}$ denote the induced (Riemannian) metric on Σ_t, and define the second fundamental form

$$k_{ab} := -\mathbf{g}(\mathbf{D}_{\partial_a} N, \partial_b) = \mathbf{g}(N, \mathbf{D}_{\partial_a}\partial_b) = N^\alpha \mathbf{\Gamma}_{\alpha ab}, \qquad a, b \in \{1,2,3\}.$$

Then we define the linear momentum $\mathbf{p}_a$, $a \in \{1,2,3\}$,

$$\mathbf{p}_a(t) := \frac{1}{8\pi} \lim_{R\to\infty} \int_{S_{R,t}} \pi_{ab} \frac{x^b}{|x|}\, dx, \qquad \pi_{ab} := k_{ab} - (\mathrm{tr}k)\overline{g}_{ab},$$

In Proposition 7.13 we prove that the functions $\mathbf{p}_a$ are well defined and constant in time. Moreover, we show that $\sum_{a\in\{1,2,3\}} \mathbf{p}_a^2 \leq E_{ADM}^2$, so the ADM mass $M_{ADM} := \big(E_{ADM}^2 - \sum_{a\in\{1,2,3\}} \mathbf{p}_a^2\big)^{1/2} \geq 0$ is well defined.

We remark that the momentum $\mathbf{p}_a$ vanishes in the case of metrics $\mathbf{g}$ that agree with the Schwarzschild metric (including time derivatives) up to lower order terms. In particular, it vanishes in the case of metrics considered in earlier work on the stability for the Einstein vacuum equations, such as [12, 52, 62]. However, in our non-isotropic case the linear momentum does not necessarily vanish, and the quantities $\mathbf{p}_a$ defined above are natural conserved quantities of the evolution.

1.3.6 The Bondi Energy

To define a Bondi energy we have to be more careful. We would like to compute integrals over large spheres as in (1.3.22), and then take the limit along outgoing null cones towards null infinity. But the limit exists only if we account properly for the geometry of the problem.

First we need to understand the bending of the light cones caused by the long-range effect of the nonlinearity (i.e., the modified scattering). For this we construct (in Lemma 7.19) an *almost optical function* $u : M' \to \mathbb{R}$, satisfying the properties

$$u(x,t) = |x| - t + u^{cor}(x,t), \qquad \mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = O(\varepsilon_0 \langle r\rangle^{-2+6\delta'}). \tag{1.3.24}$$

In addition, the correction $u^{cor} = O(\varepsilon_0\langle r\rangle^{3\delta'})$ is close to $\Theta_{wa}/|x|$ (see (1.3.9)) near the light cone,

$$\Big| u^{cor}(x,t) - \frac{\Theta_{wa}(x,t)}{|x|}\Big| \lesssim \varepsilon_0 \langle r\rangle^{-1+3\delta'}(\langle r\rangle^{0.68} + \langle t - |x|\rangle), \tag{1.3.25}$$

if $(x,t) \in M'$, $\big|t - |x|\big| \leq t/10$. Notice that we work with an approximate optical condition $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = O(\varepsilon_0\langle r\rangle^{-2+6\delta'})$ instead of the classical optical condition $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$. This is mostly for convenience, since the weaker condition is still good enough for our analysis and almost optical functions are much easier to construct than exact optical functions.

For any $t \geq 1$ we define the hypersurface $\Sigma_t := \{(x,t) \in M : x \in \mathbb{R}^3\}$, and let $\overline{g}_{jk} = \mathbf{g}_{jk}$ denote the induced (Riemannian) metric on Σ_t. With u as above, we define the modified spheres $S^u_{R,t} := \{x \in \Sigma_t : u(x,t) = R\}$ and let $\mathbf{n}_j := \partial_j u(\overline{g}^{ab}\partial_a u \partial_b u)^{-1/2}$ denote the unit vector-field normal to the spheres $S^u_{R,t}$. For $R \in \mathbb{R}$ and t large (say $t \geq 2|R| + 10$) we define

$$E_{Bondi}(R) := \frac{1}{16\pi}\lim_{t\to\infty}\int_{S^u_{R,t}} \overline{g}^{ab}(\partial_a h_{jb} - \partial_j h_{ab})\mathbf{n}^j\, d\sigma, \tag{1.3.26}$$

where $d\sigma = d\sigma(\overline{g})$ is the surface measure induced by the metric $\overline{g}$. Notice that this definition is a more geometric version of the definition (1.3.22), in the sense that the integration is with respect to the metric $\overline{g}$. Geometrically, we fix R and integrate on surfaces $S^u_{R,t}$ that live on the “light cone” $\{u(x,t) = R\}$

In Theorem 7.23 we prove our main result: the limit in (1.3.26) exists, and $E_{Bondi} : \mathbb{R} \to \mathbb{R}$ is a well-defined increasing and continuous function on $\mathbb{R}$, which increases from the Klein-Gordon energy E_{KG} to the ADM energy E_{ADM}, i.e.,

$$\lim_{R\to-\infty} E_{Bondi}(R) = E_{KG} := \frac{1}{16\pi}\|V_\infty^\psi\|_{L^2}^2, \qquad \lim_{R\to\infty} E_{Bondi}(R) = E_{ADM}. \tag{1.3.27}$$

The definition (1.3.26) of the Bondi energy is consistent with the general heuristics in [73, Chapter 11] and with the definition in [67, Section 4.3.4]. It also has expected properties, like monotonicity, continuity, and satisfies the limits (1.3.27).

However, it is not clear to us if this definition is identical to the definition used by Klainerman-Nicolò [52, Chapter 8.5], starting from the Hawking mass. In fact, at the level of generality of our metrics (1.2.13), it not even clear that one can prove sharp r^{-3} pointwise decay on some of the signature 0 components of the curvature tensor, which is one of the ingredients of the argument in [52].

We notice that the Klein-Gordon energy E_{KG} is part of $E_{Bondi}(R)$, for all $R \in \mathbb{R}$. This is consistent with the geometric intuition, since the matter travels at speeds lower than the speed of light and accumulates at timelike infinity, not at null infinity. We can further measure its radiation by taking limits along timelike cones. Indeed, for $\alpha \in (0, 1)$ let

$$E_{i^+}(\alpha) := \frac{1}{16\pi}\lim_{t\to\infty}\int_{S_{\alpha t,t}} (\partial_j h_{nj} - \partial_n h_{jj})\frac{x^n}{|x|}\,dx, \tag{1.3.28}$$

where the integration is over the Euclidean spheres $S_{\alpha t,t} \subseteq \Sigma_t$ of radius αt. In Proposition 7.24 we prove that the limit in (1.3.28) exists, and $E_{i^+} : (0, 1) \to \mathbb{R}$ is a well-defined continuous and increasing function, satisfying

$$\lim_{\alpha\to 0} E_{i^+}(\alpha) = 0, \qquad \lim_{\alpha\to 1} E_{i^+}(\alpha) = E_{KG}. \tag{1.3.29}$$

1.3.7 Organization

The rest of this monograph is organized as follows:

In Chapter 2 we introduce our main notations and definitions and state precisely our main bootstrap Proposition 2.3. This proposition is the key quantitative result leading to global nonlinear stability, and its proof covers Chapters 3, 4, 5, and 6. Then we provide a detailed outline of the proof of this proposition, describing at a conceptual level the entire construction and the main ingredients of the proof.

In Chapter 3 we prove several important lemmas that are being used in the rest of the analysis, such as Lemmas 3.4 and 3.6 on the structure and bounds on quadratic resonances, Lemma 3.9 concerning linear estimates for wave and Klein-Gordon evolutions, and Lemmas 3.10–3.12 concerning bilinear estimates. Finally, we use these lemmas and the bootstrap hypothesis to prove linear estimates on the solutions and the profiles, such as Lemmas 3.15 (localized L^2 bounds) and Lemma 3.16 (pointwise decay).

In Chapter 4 we analyze our main nonlinearities $\mathcal{LN}^h_{\alpha\beta}$ and $\mathcal{LN}^\psi$ at a fixed time t. The main results in this chapter are Proposition 4.7 (localized L^2, L^∞, and weighted L^2 bounds on these nonlinearities), Lemmas 4.19–4.20 (identification of the energy disposable nonlinear components), and Proposition 4.22 (decomposition of the main nonlinearities).

In Chapter 5 we prove the main bootstrap bounds (2.1.50) on the energy functionals. We start from the decomposition in Proposition 4.22, perform energy estimates, and prove bounds on all the resulting spacetime integrals. The main spacetime bounds are stated in Proposition 5.2, and are proved in the rest of the chapter, using normal forms, null structures, angular decompositions, and paradifferential calculus in some of the harder cases.

In Chapter 6 we first prove the main bootstrap bounds (2.1.51) (weighted estimates on profiles) in Proposition 6.2, as a consequence of the improved energy estimates and the nonlinear bounds in Proposition 4.7. Then we prove the main bootstrap bounds (2.1.52) (the Z-norm estimates). This proof has several steps, such as the renormalization procedures in (6.2.4)–(6.2.6) and (6.3.3)–(6.3.4), and the estimates (6.2.14) and (6.3.16) showing boundedness and convergence of the nonlinear profiles in suitable norms in the Fourier space.

In Chapter 7 we prove a full, quantitative version of our main global regularity result (Theorem 7.1) as well as all the other consequences on the asymptotic structure of our spacetimes, as described in detail in subsections 1.3.3–1.3.6 above.

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