Contents

CHAPTER 1

Why Quantum Field Theory?

Quantum theory began with quanta of the electromagnetic field, with Planck's blackbody spectrum and with Einstein's concept of a photon of energy $E = \hbar\omega$ in the photoelectric effect. The latter process is driven by a photon in the initial state that is absorbed and does not appear in the final state. However, the standard curriculum of quantum mechanics initially bypasses this topic and instead emphasizes wavefunctions and quantum properties of massive particles at low energies, such as those typical of atomic physics. In these cases, the particle number is conserved. To provide a proper quantum treatment of the emission and absorption of the electromagnetic quanta, one needs Quantum Field Theory. We need to transition from thinking about wavefunctions to discussing fields. As we will see, this allows the numbers and identities of the particles to change in reactions, which has wide applicability. In fact, this is a unifying concept. Not only does the electromagnetic field behave like a particle in certain settings, but also the entities that we think of as *particles*, such as electrons, can behave like *waves* in diffraction experiments. Moreover, all types of particles can be created and destroyed, as in the reaction $e^+e^- \to 2\gamma$ with e^- being the electron, e^+ the positron, and γ being the photon. To describe such processes, we need Quantum Field Theory.

Another indication of the need for Quantum Field Theory arose in attempts to marry quantum mechanics and the Theory of Special Relativity. Schrödinger's first attempts to write a differential equation whose solutions would describe de Broglie's matter waves were based on applying the identification $E \leftrightarrow i\hbar\partial_t$, $\mathbf{p} \leftrightarrow -i\hbar\nabla$ to the relativistic energy-momentum relation $E^2 = m^2 c^4 + \mathbf{p}^2 c^2$ for a particle of mass m (an effort that would eventually lead to the Klein-Gordon equation). This construction, however, led to complications (negative probability densities as well as negative energy states), and Schrödinger went ahead with the more modest(!) goal of writing a nonrelativistic equation for the hydrogen atom. Nonrelativistic quantum mechanics was thus born as a plan B, because Schrödinger noticed that its relativistic counterpart was leading to mathematical and physical inconsistencies. Indeed, it took another couple of decades or so to realize that a consistent treatment of relativistic quantum mechanics requires a quantum theory of fields and to formalize this theory. Along the way, people had to deal with the subtleties that we will discuss later in this book. These involved abandoning the concept of wavefunction in favor of a field *operator* acting on a Hilbert space of states.

1.1 A successful framework

Quantum Field Theory has been successful in making predictions throughout all branches of quantum physics. While the framework of this theory was developed to describe electrons and photons, it finds applications in the theory of elementary particles, in macroscopic systems of condensed matter physics, and in the Early Universe. Today, it is impossible to list all the successful applications of the quantum theory of fields. Based on the highly arbitrary choices of the authors, these include:

- one of the most iconic predictions of Quantum Field Theory: the existence of antimatter, which emerged from the formulation of Dirac's equation. Dirac was motivated by the need of making sense of the relativistic relation $E^2 = m^2 c^4 + \mathbf{p}^2 c^2$ without incurring the presence of negative norm states. A major experimental fact (the prediction of a new form of matter, antimatter, four years before the discovery of the positron) stemmed from a strictly mathematical requirement. With that same equation, Dirac also postdicted the ratio (in appropriate units approximately equal to 2) between the magnetic moment of the electron and its spin (the so-called gyromagnetic ratio or g-factor of the electron), which was not justified by any existing theory at that time.
- the exquisite agreement of the value of the electron's g-factor as predicted by Quantum Field Theory with its measured value is one of the quantities most precisely measured in physics, where the quantity $(g-2)/2$ is measured[1] to be 0.00115965218073(28) in very close agreement with the theoretical value 0.00115965218161(23) when using the most precise direct measurement of the fine structure constant. The corrections to the Dirac value $g = 2$ come from loop diagrams, which we will discuss starting in chapter 5.
- the prediction of the Lamb shift—the energy difference between the $^2S_{1/2}$ and $^2P_{1/2}$ energy levels of hydrogen. The levels are degenerate even in the relativistic Dirac theory. A full treatment required the development of Quantum Electrodynamics.
- Landau-Ginzburg's theory of superconductivity, based on Landau's theory of phase transitions, in which the behavior of macroscopic systems is described after coarse-graining their microscopic component.
- the running of coupling constants. In physical processes, the values of the coupling constants depend on the energy or distance scale at which they are measured. This effect is observed both in particle physics and in condensed matter systems near phase transitions, where the running affects the

[1]This value, plus reviews of many tests of Quantum Field Theory predictions, can be found in the *Review of Particle Properties*, which is maintained and updated regularly by P. A. Zyla et al. for the Particle Data Group. See also Particle Data Group et al., "Review of Particle Physics," *Progress of Theoretical and Experimental Physics* 2020, no. 8 (August 2020): 083C01.

value of the *critical exponents*, that is, the way certain quantities evolve as we change the temperature of the system near a phase transition.

- the origin of structure in the Early Universe. While on large scales the Universe is largely uniform, on small scales we see clumping of matter as well as voids. There are strong indications that this is the result of the amplification of quantum fluctuations in the Early Universe, which can be described by Quantum Field Theory.

1.2 A universal framework

The point of view that informs this book is that the main reason why we need Quantum Field Theory is because it is *universal*. As we will see in chapter 2, any system governed by quantum mechanics in which we ignore the ultimate microscopic behavior is controlled, at sufficiently low energies/long wavelengths, by the rules of Quantum Field Theory. This is true for the description of the sound waves that propagate in your desk when you hit it. It could even be true for our "elementary" particles, as our understanding of what is elementary has changed over time. (For example, the proton and neutron were once considered elementary, but now we have a more fundamental description in terms of quarks and gluons.) At whatever scale we are working, Quantum Field Theory can be an appropriate description.

In the end, Quantum Field Theory provides an elegant and understandable treatment of *all* particles. All fields are treated on the same footing. Its rules theory can handle all types of transitions with powerful techniques. Quantum Field Theory is conceptually unified and clear once one learns how to think appropriately about the subject.

CHAPTER 2

Quanta

To start down the path to Quantum Field Theory, we have to first head back to 1905, when Einstein first postulated that photons carry energy in quanta of $\hbar\omega$, and uncover the quantum of a field. The same transition from classical physics to quantum physics that works in ordinary quantum mechanics will also work here. This leads to the quantization rules for fields and to the concepts of field operators and particle quanta.

2.1 From classical particle mechanics to classical waves: Phonons

Mathematically, fields are functions of space and time, such as a function $\phi(t, x)$. For us as physicists, this also means that they satisfy wave equations, that they carry energy, that they ultimately have interactions, etc. Let us start by constructing a field in a way that also allows us to quantize it.

Consider a one-dimensional array of particles of mass m with coordinates $y_j(t)$ interacting with their neighbors, as in figure 2.1. Near the equilibrium configuration, the potential can be approximated by a harmonic oscillator, that is, by a set of springs. We will denote by a the rest length of the springs and by k the spring constant, so that the interaction term between the $(j+1)$-th and the j-th particle is $\frac{k}{2}(y_{j+1} - y_j - a)^2$. Denoting $\delta y_j(t) \equiv y_j(t) - a\,j$ as the deviation of the j-th particle from its equilibrium position, the system is described by a Lagrangian

$$L(\delta y_j, \delta\dot{y}_j) = \sum_j \left[\frac{m}{2}\,\delta\dot{y}_j^2 - \frac{k}{2}\,(\delta y_{j+1} - \delta y_j)^2 \right]. \tag{2.1}$$

(We will assume here that the string is infinitely long, so that we do not have to worry about boundary conditions.) Physically we know that if the spacing is very small compared to the wavelength, $a \ll \lambda$, the system will be described by wavelike solutions, like sound waves propagating through a solid. When quantized this becomes a one-dimensional model for phonons in a solid.

The techniques of Lagrangian mechanics instruct us to define a canonical momentum

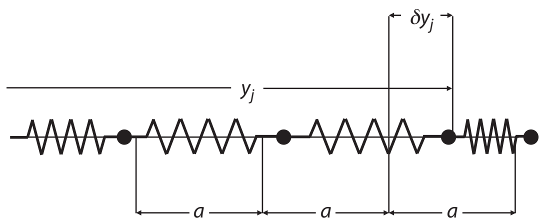

Figure 2.1. The system with many particles that provides our starting point. The Lagrangian for this system is given in equation (2.1).

$$\delta p_j \equiv \frac{\partial \mathcal{L}}{\partial(\delta \dot{y}_j)} = m\,\delta \dot{y}_j \tag{2.2}$$

and a Hamiltonian

$$H = \sum_j \delta p_j\,\delta\dot{y}_j - L(\delta y_j, \delta\dot{y}_j) = \sum_j \left[\frac{\delta p_j^2}{2\,m} + \frac{k}{2}\,(\delta y_{j+1} - \delta y_j)^2\right]. \tag{2.3}$$

Now let us look at this over such large distances that the continuum limit is a good approximation. Mathematically, we obtain this by sending the distance $a \to 0$ and the number of sites to infinity, so that $a \times j$ stays finite. We will thus describe the position by a continuous variable

$$x = a\,j, \quad \text{so that} \quad \sum_j \to \int dj = \int \frac{dx}{a}. \tag{2.4}$$

It then makes sense to describe the displacement $\delta y_j(t)$ by a continuous function $\phi(t,x)$

$$\delta y_j(t) = \sqrt{\frac{1}{k\,a}}\,\phi(t,x). \tag{2.5}$$

The normalization constant shown does not change the physics in the end,[1] but is chosen to make the intermediate steps look cleaner. To proceed, we note that in the continuum limit $\delta y_{j+1}(t) - \delta y_j(t) \to a\,\partial_x\phi(t,x)/\sqrt{ka}$. We are now in position to derive the expression of the field-theoretical Lagrangian. The "potential" term takes the form

$$\sum_j \frac{k}{2}\,(\delta y_{j+1} - \delta y_j)^2 \to \int \frac{dx}{a} \times \frac{k}{2} \times \left(\frac{a}{\sqrt{k\,a}}\frac{\partial\phi}{\partial x}\right)^2 = \int dx\,\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2, \tag{2.6}$$

while the kinetic energy term reads

[1] To verify this statement, one has to work through the steps leading up to the quantized Hamiltonian, equation (2.18). The equations of motion will be unchanged, and the canonical momentum will have a compensating change in normalization.

$$\sum_j \frac{m}{2}\,\delta\dot{y}_j^2 \to \int \frac{dx}{a} \times \frac{m}{2} \times \frac{1}{k\,a}\left(\frac{\partial\phi}{\partial t}\right)^2 \equiv \int dx \frac{1}{2\,v^2}\left(\frac{\partial\phi}{\partial t}\right)^2 , \tag{2.7}$$

where we have defined the quantity

$$v = \sqrt{\frac{k}{m}}\, a\,, \tag{2.8}$$

that has the dimensions of a velocity.

The end result is an action

$$\boxed{S = \int dt\, dx\, \mathcal{L}(\partial_t\phi, \partial_x\phi)} = \int dt\, dx \left[\frac{1}{2\,v^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2\right] , \tag{2.9}$$

so that the action is written as the integral over the entire *spacetime* of a *Lagrangian density* $\mathcal{L}$.

We can derive the wave equation either by varying the original Lagrangian and taking the continuum limit or by directly varying the continuum action. To do the latter, we define the small variation $\delta\phi$

$$\phi(t,x) = \bar{\phi}(t,x) + \delta\phi(t,x)\,, \tag{2.10}$$

with the variation vanishing at the endpoints and set the first variation of the action to 0

$$\begin{aligned}\delta S = 0 &= \int dt\, dx \left[\frac{1}{v^2}\frac{\partial\bar{\phi}}{\partial t}\frac{\partial\delta\phi}{\partial t} - \frac{\partial\bar{\phi}}{\partial x}\frac{\partial\delta\phi}{\partial x}\right] \\ &= \int dt\, dx \left[-\frac{1}{v^2}\frac{\partial^2\bar{\phi}}{\partial t^2} + \frac{\partial^2\bar{\phi}}{\partial x^2}\right]\delta\phi(t,x)\,.\end{aligned} \tag{2.11}$$

The second line is obtained by integrating by parts, with the surface term vanishing because we required $\delta\phi$ to vanish at the boundaries of the system. By requiring that the variation of the action vanishes for any $\delta\phi(t,x)$, we get the wave equation

$$\left[\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right]\bar{\phi}(t,x) = 0\,. \tag{2.12}$$

This is the *Euler-Lagrange equation of motion* for this field.

The canonical momentum for the field ϕ can also be constructed in analogy with the usual coordinate construction

$$\delta p_j \equiv \frac{\partial\mathcal{L}}{\partial(\delta\dot{y}_j)} \Rightarrow \pi(t,x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{1}{v^2}\frac{\partial\phi}{\partial t}\,. \tag{2.13}$$

For the present calculation, it is useful to display the exact relation

$$\delta p_j \equiv \frac{\partial \mathcal{L}}{\partial(\delta \dot{y}_j)} = m\, \delta \dot{y}_j = \frac{m}{\sqrt{k\,a}}\, \dot{\phi} = \frac{a\sqrt{k\,a}}{v^2}\, \dot{\phi} = a\sqrt{k\,a}\, \pi(t,x)\,. \tag{2.14}$$

This, in particular, tells us that

$$\sum_j \delta p_j\, \delta \dot{y}_j = \int \frac{dx}{a} \times \frac{m}{\sqrt{k\,a}} \dot{\phi} \times \frac{1}{\sqrt{k\,a}} \dot{\phi} = \int dx\, \frac{1}{v^2} \left(\frac{\partial \phi}{\partial t}\right)^2 = \int dx\, \pi(t,x)\, \dot{\phi}(t,x)\,. \tag{2.15}$$

The Hamiltonian is then easy to construct either through the continuum limit

$$H = \sum_j \left[\frac{\delta p_j^2}{2\,m} + \frac{k}{2}\,(\delta y_{j+1} - \delta y_j)^2\right] = \int dx\, \left[\frac{1}{2\,v^2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2\right], \tag{2.16}$$

or through the field-theoretical *Hamiltonian density* $\mathcal{H}$

$$H = \int dx\, \mathcal{H} \tag{2.17}$$

defined by

$$\mathcal{H} = \pi\, \dot{\phi} - \mathcal{L}\,, \tag{2.18}$$

which in this case is equal to

$$\mathcal{H} = \frac{v^2}{2}\pi^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 = \frac{1}{2\,v^2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2\,. \tag{2.19}$$

This simply describes the continuum limit of classical mechanics. The result of which is waves propagating in this system.

2.2 From quantum mechanics to Quantum Field Theory

In taking the continuum limit, we have found that

$$\delta y_j(t) \to \frac{1}{\sqrt{k\,a}}\, \phi(t,\, x)\,, \qquad \delta p_j(t) \to a\,\sqrt{k\,a}\, \pi(t,\, x)\,, \tag{2.20}$$

so that we can quantize the field ϕ starting from the canonical commutation relation

$$[\delta y_j(t),\, \delta p_{j'}(t)] = i\hbar\, \delta_{j,j'}\,, \tag{2.21}$$

obtaining

$$[\phi(t,x),\, \pi(t,x')] = \frac{1}{a}\, [\delta y_j(t), \delta p_{j'}(t)] = i\hbar\, \frac{\delta_{j,j'}}{a}\,. \tag{2.22}$$

The correct continuum identification turns the right-hand side into a Dirac delta function. In fact, from the continuum limit of the sum, $\sum_j \to \int \frac{dx}{a}$, we get

$$\sum_j \delta_{j,j'} = 1 \to \int \frac{dx}{a} \delta_{j,j'} = \int dx\, \delta(x - x') = 1\,. \tag{2.23}$$

The end result is the commutator for field quantization

$$\boxed{[\phi(t,x), \pi(t,x')] = i\hbar\, \delta(x - x')}\,, \tag{2.24}$$

which is the starting point of what is referred to as *canonical quantization* in Quantum Field Theory.

Like all things quantum, this rule takes some getting used to. It is saying that the field, which we can visualize classically as a wave propagating in front of us, is no longer just a function, but is an operator. There are a few things to say about this. We note that as we progressed in quantum mechanics we have become used to coordinates and momenta—also things that we have a good picture for classically—being operators as in equation (2.21). So at this moment, we will take a deep breath and just wait and see where this leads. In practice, it leads to a final result that is even easier to come to grips with than the usual quantum-mechanical formalism is. We will see that the field-operator formalism is really a bookkeeping device for keeping track of the creation and annihilation of particles.[2] That is actually much easier than the usual statement that position x is an operator. The coordinate x that appears in the quantum field–theoretical treatment of the system, for instance in equation (2.24), is *not* an operator, as it descends from the index j in the discrete system in equation (2.1). Finally, we should note that in the path integral formalism (see chapter 8), the fields again are treated as functions, and there is not an operator in sight. For now, you are counselled to be patient.[3]

2.3 Creation operators and the Hamiltonian

Now let us figure out how to solve the commutation rule in equation (2.24). To do this, we need to establish the general solutions to the wave equation. We propose doing this using the *box normalization*, in which the system is taken to be finite but very large, with length L. The exact boundary conditions are not important, but you can think of periodic boundary conditions for definiteness. What is useful about this choice is that energy levels are discrete, with a label n running on all the integers. This avoids having to simultaneously introduce the continuum momenta notation on top of the other ideas discussed in this section.

[2]For the reader who is not a native English speaker, a bookkeeper is one who keeps track of financial transactions. The use in the present context is that the field operators keep track of the physics transitions.

[3]We are also postponing to section 2.3 the reason that both equations (2.21) and (2.24) are evaluated at equal times even though the positions are different. Right now we are rushing to reach our goal—quanta.

The solutions of the wave equation, equation (2.12), take the form

$$\phi(t, x) = N\, e^{\pm i(\omega_n t - k_n x)} \quad \text{with} \quad \omega_n = |k_n| v, \qquad k_n = \frac{2\pi n}{L}, \tag{2.25}$$

where N is a arbitrary constant, so that the most general solution will be a superposition

$$\phi(t, x) = \sum_n N_n [\hat{a}_n\, e^{-i(\omega_n t - k_n x)} + \hat{a}_n^\dagger\, e^{+i(\omega_n t - k_n x)}], \tag{2.26}$$

where we have used the fact that ϕ is a real field. The coefficients $\hat{a}_n$ are now to be considered operators because ϕ is an operator.[4] The normalization factor N_n will be determined in equation (2.30).

Because $\hat{a}_n$ and $\hat{a}_n^\dagger$ are now operators, they must obey some commutation rules. If the different modes are to be orthogonal, we expect operators with different n values to commute. This leads to a set of rules that up to an overall normalization, reads

$$[\hat{a}_n, \hat{a}_{n'}] = 0, \tag{2.27}$$

$$[\hat{a}_n, \hat{a}_{n'}^\dagger] = \delta_{n,n'}. \tag{2.28}$$

The arbitrary overall normalization can be absorbed in the N_n factor in equation (2.26). This choice in fact does provide a solution to the field commutator rule, as we can readily see

$$\begin{aligned}
[\phi(t,x), \pi(t,x')] &= \sum_{n,n'} N_n N_{n'} \left(-i\frac{\omega_{n'}}{v^2}\right) \\
&\times [\hat{a}_n e^{-i(\omega_n t - k_n x)} + \hat{a}_n^\dagger e^{+i(\omega_n t - k_n x)}, \hat{a}_{n'} e^{-i(\omega_{n'} t - k_{n'} x')} - \hat{a}_{n'}^\dagger e^{+i(\omega_{n'} t - k_{n'} x')}] \\
&= \sum_{n,n'} N_n N_{n'} \left(2i\frac{\omega_{n'}}{v^2}\right) [\hat{a}_n, \hat{a}_{n'}^\dagger]\, e^{i(k_n x - k_{n'} x')} \\
&= \sum_n \frac{i\hbar}{L} e^{ik_n(x - x')} = i\hbar\, \delta(x - x'),
\end{aligned} \tag{2.29}$$

provided that we take the normalization factor to be

$$N_n = \sqrt{\frac{\hbar v^2}{2\omega_n L}}. \tag{2.30}$$

The delta function identity

$$\delta(x - x') = \sum_n \frac{1}{L} e^{ik_n(x - x')} \tag{2.31}$$

follows from the completeness of the Fourier series.

[4]These coefficients are not to be confused with the length a between the mass points.

One could equivalently derive the commutation relation of the $\hat{a}_n$ and $\hat{a}_n^\dagger$ operators directly from the canonical quantization condition in equation (2.24) and invert the relations that give $\phi(t, x)$ and $\pi(t, x)$ as a function of $\hat{a}_n$ and $\hat{a}_n^\dagger$. More explicitly, we can rewrite equation (2.26) as

$$\phi(t, x) = \sum_n N_n\, e^{ik_n x}\, [\hat{a}_n e^{-i\omega_n t} + \hat{a}^\dagger_{-n} e^{+i\omega_n t}], \tag{2.32}$$

where we have used $\omega_{-n} = \omega_n$ and have assumed $N_n = N_{-n}$. Then, by inverting the Fourier series we obtain

$$\hat{a}_n = \frac{e^{i\omega_n t}}{2\, N_n} \int \frac{dx}{L}\, e^{-ik_n x} \left[\phi(t, x) + i\, v^2 \frac{\pi(t, x)}{\omega_n}\right], \tag{2.33}$$

so that

$$\begin{aligned}
[\hat{a}_n, \hat{a}^\dagger_{n'}] &= \frac{e^{i(\omega - \omega_{n'})t}}{4\, N_n\, N_{n'}} \int \frac{dx\, dx'}{L^2}\, e^{-ik_n x + ik_{n'} x'} \\
&\quad \left[\phi(t, x) + iv^2 \frac{\pi(t, x)}{\omega_n},\ \phi(t, x') - iv^2 \frac{\pi(t, x')}{\omega_{n'}}\right] \\
&= \frac{e^{i(\omega - \omega_{n'})t}}{4\, N_n\, N_{n'}} \int \frac{dx\, dx'}{L^2}\, e^{-ik_n x + ik_{n'} x'}\, \hbar\, v^2 \left[\frac{\delta(x - x')}{\omega_{n'}} + \frac{\delta(x - x')}{\omega_n}\right] \\
&= \frac{\hbar\, v^2}{2\, N_n^2\, L\, \omega_n} \delta_{nn'}
\end{aligned} \tag{2.34}$$

and, in similar fashion, $[\hat{a}_n, \hat{a}_{n'}] = [\hat{a}_n^\dagger, \hat{a}^\dagger_{n'}] = 0$. By choosing the normalization in equation (2.30), we find that each of the $\hat{a}_n$, $\hat{a}_n^\dagger$ pair of operators satisfy the same algebra as the creation and annihilation operators of the simple harmonic oscillator. Indeed we will find that this identification is accurate and will henceforth call $\hat{a}_n^\dagger$ a creation operator and $\hat{a}_n$ an annihilation operator.

We are now in position to evaluate the Hamiltonian. Because ϕ has two terms, $\hat{a}_n$ and $\hat{a}_n^\dagger$, the Hamiltonian

$$\begin{aligned}
H &= \int dx \left[\frac{1}{2\, v^2} \left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x}\right)^2\right] \\
&= \int dx \sum_{n,n'} N_n\, N_{n'} \\
&\quad \times \left[\frac{-\omega_n\, \omega_{n'}}{2\, v^2} \left(\hat{a}_n e^{-i\psi_n} - \hat{a}_n^\dagger e^{+i\psi_n}\right) \left(\hat{a}_{n'} e^{-i\psi_{n'}} - \hat{a}^\dagger_{n'} e^{+i\psi_{n'}}\right)\right. \\
&\quad \left. - \frac{k_n\, k_{n'}}{2} \left(\hat{a}_n e^{-i\psi_n} - \hat{a}_n^\dagger e^{+i\psi_n}\right) \left(\hat{a}_{n'} e^{-i\psi_{n'}} - \hat{a}_n^\dagger e^{+i\psi_{n'}}\right)\right]
\end{aligned} \tag{2.35}$$

will have four pieces, although they come in pairs because H is Hermitian. In the exponents, we have used the shorthand $\psi_n = (\omega_n t - k_n x)$ to save space.

The integral over x will imply that the momenta are either equal or opposite. By using

$$\int dx\, e^{ik_n x}\, e^{-ik_{n'} x} = L\, \delta_{n,n'}\,,$$
$$\int dx\, e^{ik_n x}\, e^{ik_{n'} x} = L\, \delta_{n,-n'}\,, \tag{2.36}$$

we can write equation (2.35) as a single sum over the momentum variable. Both of these cases have $\omega_n = \omega_{n'}$, and by inserting the normalization factor, we have

$$H = \sum_n \frac{\hbar\, v^2}{2\,\omega_n} \left[\frac{1}{2} \left(-\frac{\omega_n^2}{v^2} + k_n^2 \right) \left(e^{-2i\omega_n t} \hat{a}_n\, \hat{a}_{-n} + e^{+2i\omega_n t} \hat{a}_n^\dagger\, \hat{a}_{-n}^\dagger \right) + \frac{1}{2} \left(+\frac{\omega_n^2}{v^2} + k_n^2 \right) \left(\hat{a}_n\, \hat{a}_n^\dagger + \hat{a}_n^\dagger\, \hat{a}_n \right) \right]. \tag{2.37}$$

At this stage "a miracle occurs" and the $\hat{a}_n\, \hat{a}_{-n}$ and $\hat{a}_n{}^\dagger \hat{a}_{-n}^\dagger$ terms disappear because $\omega_n^2 = k_n^2 v^2$. If we use the creation operator commutation rule, we obtain

$$H = H_0 + E_0\,, \tag{2.38}$$

with

$$\boxed{H_0 = \sum_n \hbar\, \omega_n\, \hat{a}_n^\dagger\, \hat{a}_n} \tag{2.39}$$

and

$$E_0 = \sum_n \frac{1}{2} \hbar\, \omega_n\,. \tag{2.40}$$

Here E_0 is the *zero-point energy*, which we will discuss in section 3.6. It provides a constant shift in energy that we will ignore for now. The other part of the Hamiltonian H_0 is quite promising. We see the *number operator* for each mode $\hat{a}_n^\dagger\, \hat{a}_n$ emerging, with an associated energy $E_n = \hbar\, \omega_n$. Note that it was the field commutation relation that fixed the normalization and, therefore, required the energy of the n-th mode to be $\hbar\, \omega_n$.

2.4 States filled with quanta

Once we identify the energy eigenstates of the theory and confirm that $\hat{a}_n^\dagger\, \hat{a}_n$ acts like a number operator, we will have reached our goal. This is easily fulfilled by using our experience with the simple harmonic oscillator. The states can be constructed by defining an "empty" state—the *vacuum*, which by definition is annihilated by all the annihilation operators

$$\hat{a}_n|0\rangle = 0 \quad \text{for all } n\,. \tag{2.41}$$

After this, we can construct new states by acting with the $\hat{a}_n$ and $\hat{a}_n^\dagger$ operators on the vacuum $|0\rangle$. Let us start with the action of a single $\hat{a}_n^\dagger$ operator, and let us define

$$|n\rangle = \hat{a}_n^\dagger\,|0\rangle\,. \tag{2.42}$$

This operation produces energy eigenstates, as can be readily verified

$$H|n\rangle = \sum_{n'} \hbar\,\omega_{n'}\,\hat{a}_{n'}^\dagger\,\hat{a}_{n'}\,(\hat{a}_n^\dagger|0\rangle) \tag{2.43}$$

$$= \sum_{n'} \hbar\,\omega_{n'}\,\hat{a}_{n'}^\dagger\,([\hat{a}_{n'},\,\hat{a}_n^\dagger] + \hat{a}_n^\dagger\,\hat{a}_{n'})|0\rangle \tag{2.44}$$

$$= \sum_{n'} \hbar\,\omega_{n'}\,\hat{a}_{n'}^\dagger\,\delta_{n,n'}|0\rangle = \hbar\,\omega_n|n\rangle\,. \tag{2.45}$$

This construction gives a state with energy $\hbar\,\omega_n$. We are thus led to interpret $|n\rangle$ as a *single particle state* that contains one quantum of the state with energy $\hbar\,\omega_n$. Here the word *particle* is used to mean a quantum carrying energy $\hbar\,\omega$. We will see later that we can define additional operators associated to observables such as momentum, charge, etc., and that the action of a creation operator on the vacuum gives eigenstates of all these operators, which is exactly as expected for a single particle state with those quantum numbers. For this reason, from now on, we will use interchangeably the terms "state $|n\rangle$" and "particle in n-th state."

On a technical point, we have explicitly worked out the action of the commutator when using the Hamiltonian, which is how the calculation proceeds. However, the way to think about this calculation is to mentally say "the annihilation operator $\hat{a}_{n'}$ annihilates the particle n'." This can be represented pictorially with a *contraction*

$$\sum_{n'} \hat{a}_{n'}^\dagger \underbrace{\hat{a}_{n'}|n}\rangle = \sum_{n'} \hat{a}_{n'}^\dagger \underbrace{\hat{a}_{n'}\hat{a}_n^\dagger}|0\rangle = \sum_{n'} \hat{a}_{n'}^\dagger[\hat{a}_{n'},\hat{a}_n^\dagger]|0\rangle = \sum_{n'} \hat{a}_{n'}^\dagger\delta_{n,n'}|0\rangle = |n\rangle\,, \tag{2.46}$$

indicating that the given annihilation operator removes the creation operator. The calculation is given by the commutator but the result is indicated by the contraction. To acquire familiarity with contraction, you should work out the slightly more difficult case with two quanta

$$H|n_1,n_2\rangle = \sum_{n'} \hbar\omega'_n\hat{a}_{n'}^\dagger \underbrace{\hat{a}_{n'}|n_1},n_2\rangle + \sum_{n'} \hbar\omega'_n\hat{a}_{n'}^\dagger \underbrace{\hat{a}_{n'}|n_1,n_2}\rangle$$

$$= (\hbar\omega_1 + \hbar\omega_2)|n_1,n_2\rangle\,. \tag{2.47}$$

From now on we will just show the contractions and the calculation behind it will be implied.

This construction allows us to construct all the states with all possible values of the energy. Because these are bosons, there can be more than one particle in a given energy state. For example, the normalized state

$$|n_1, 3\,n_2, n_3\rangle = \frac{1}{\sqrt{3!}} \hat{a}^\dagger_{n_3} \hat{a}^\dagger_{n_2} \hat{a}^\dagger_{n_2} \hat{a}^\dagger_{n_2} \hat{a}^\dagger_{n_1} |0\rangle \tag{2.48}$$

has energy

$$H|n_1, 3\,n_2, n_3\rangle = (\hbar\,\omega_1 + 3\,\hbar\,\omega_2 + \hbar\,\omega_3)\,|n_1, 3\,n_2, n_3\rangle\,. \tag{2.49}$$

We have finally arrived back to 1905 with quanta with the correct energy-frequency relation.

Our pathway to this point has been somewhat formal, in the sense that we have used the standard formalism for both classical mechanics and quantum mechanics. We just "turned the crank" and watched what emerged. The end product is quite intuitive. States are filled with quanta with the correct energy, and each quantum corresponds to a field that solves the wave equation. The operator character of fields, which seems nonintuitive at the start, is not so scary because it just turns into a number operator that counts the fields.[5] A different pedagogic approach would be to start with the existence of quanta and work from there to the idea of quantized fields. This is now easy to do—you are invited to read this chapter backward![6] Nevertheless, proceeding the way that we have done reinforces the point that the idea of "quanta of a field" is not a separate hypothesis from the basic postulates of quantum mechanics. It follows uniquely from the standard procedures of classical and quantum physics.

Let us recap what we have done here, because, perhaps without noticing, we have accomplished something extremely deep. We started from a discrete set of many particles and took the continuum limit. When quantizing the system in its continuum limit, we came up with a discrete set of states (such as the state $|n_1, 3\,n_2, n_3\rangle$ discussed in equation (2.48)). Each of these states is associated with what we call a set of "elementary particles"! Two main points here need to be stressed. The first point is that these "particles" have nothing to do with the original particles that make up the string. Actually, a lot of information has been lost by taking the continuum limit. (We started from a model with three parameters m, a, and k and ended up with a model with a single parameter v, so information has clearly been lost in the process.) The "elementary particles" found by quantization have thus very little to do with the "actual" particles that make up, at a microscopic level, our system. The second point is that Quantum Field Theory teaches us that we should not think as much in terms of individual particles as we should think in terms of excitations

[5]Later we will see that the operator also gives the correct counting factors for transitions due to interactions.

[6]Seriously, it is a good exercise to map out how you would explain to a novice the idea of a quantum field starting from the experimental evidence of quanta with $E = \hbar\,\omega$.

of a single field. This explains why our Universe contains so many identical electrons, for instance: those electrons are not many different particles, but they are many different excitations of a single field. This is nearly the same as going from a description of the sea as a set of many waves to a description where it is a single body of water carrying a number of waves.

2.5 Connection with normal modes

The example we started with (the Lagrangian in equation (2.1)), was a specific system, but our analysis is actually valid for all many body systems near equilibrium. It is sometimes said that "a quantum field is an infinite number of harmonic oscillators." This can be seen from the expression for the Hamiltonian. It is also perhaps useful to go back to the discrete case and carry out the quantization procedure before taking the continuum limit. This is a solution via the normal mode technique.

If we start from the most general Lagrangian describing a system on N degrees of freedom near equilibrium,

$$L = \sum_i \left[\frac{m}{2} \dot{y}_i^2 - V(y_i) \right] \tag{2.50}$$

with a potential

$$V = \frac{1}{2} \sum_{ij} v_{ij}\, y_i\, y_j \,, \tag{2.51}$$

where v_{ij} is a real, symmetric $N \times N$ matrix, then this system can be solved by using normal mode techniques. The normal frequencies ω_n are the entries of the diagonal, positive, $N \times N$ matrix Ω found by solving

$$\det(m\,\Omega^2 - v) = 0\,. \tag{2.52}$$

The normal coordinates are then found via the modal matrix A_{in}

$$y_j = \sum_n A_{jn}\, \xi_n \quad \text{or} \quad \xi_n = \sum_j (A^T)_{nj}\, y_j\,. \tag{2.53}$$

This procedure decouples the harmonic oscillators

$$L = \sum_n \left[\frac{1}{2} \dot{\xi}_n^2 - \frac{1}{2} \omega_n^2\, \xi_n^2 \right]. \tag{2.54}$$

Each normal mode would then have an independent solution

$$\xi_n(t) = N\,(a_n e^{-i\omega_n t} + a_n^* e^{i\omega_n t}) \tag{2.55}$$

and the general solution would be a mixture of normal modes

$$y_j(t) = \sum_n A_{jn}\, \xi_n(t)\,. \tag{2.56}$$

Quantization then takes place independently for each normal mode

$$p_n = \frac{\partial L}{\partial \dot{\xi}_n} = \dot{\xi}_n \quad \text{with} \quad [\xi_n, p_{n'}] = i\hbar\, \delta_{n,n'}\,. \tag{2.57}$$

The coefficients in the normal mode expansion now need to become operators. Choosing

$$\xi_n = \sqrt{\frac{\hbar}{2\,\omega_n}}\,(\hat{a}_n\, e^{-i\omega_n t} + \hat{a}_n^\dagger\, e^{i\omega_n t})$$

and

$$p_n = -i\sqrt{\frac{\hbar\,\omega_n}{2}}\,(\hat{a}_n\, e^{-i\omega_n t} - \hat{a}_n^\dagger\, e^{i\omega_n t}) \tag{2.58}$$

and imposing the commutation rules $[\hat{a}_n, \hat{a}_{n'}^\dagger] = \delta_{n,n'}$, this leads to the Hamiltonian

$$H = \sum_n \hbar\omega_n \left(\hat{a}_n^\dagger \hat{a}_n + \frac{1}{2}\right) \tag{2.59}$$

and states as described in section 2.4.

The lesson of this exercise is that the states are the quanta of the normal modes. In field theory, the normal modes are wave solutions $A_{jn} \to e^{ik_n x}$ with the continuum identification $x = ja$. In the continuum limit the number of normal modes becomes infinite, hence the identification of the field quantization with an infinite number of normal modes.

Chapter summary: You have done it! You now understand how the usual rules of quantum mechanics lead to quanta of a field. We have found the commutation rules for fields and have seen how they can be expressed in terms of creation/annihilation operators and that can lead to an intuitive construction of the states of the system. We have defined the theory starting from the Lagrangian and, by following the rules, have expressed the related Hamiltonian in terms of the number operator. These are some of the most important lessons of Quantum Field Theory. There is much more to explore.

Index