© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

Contents

Prologue *xvii*
Acknowledgements *xxv*

ACT I
The Nature of Space

1 Euclidean and Non-Euclidean Geometry **3**

1.1 Euclidean and Hyperbolic Geometry 3
1.2 Spherical Geometry 6
1.3 The Angular Excess of a Spherical Triangle 8
1.4 Intrinsic and Extrinsic Geometry of Curved Surfaces 9
1.5 Constructing Geodesics via Their Straightness 11
1.6 The Nature of Space 14

2 Gaussian Curvature **17**

2.1 Introduction 17
2.2 The Circumference and Area of a Circle 19
2.3 The Local Gauss–Bonnet Theorem 22

3 Exercises for Prologue and Act I **24**

ACT II
The Metric

4 Mapping Surfaces: The Metric **31**

4.1 Introduction 31
4.2 The Projective Map of the Sphere 32
4.3 The Metric of a General Surface 34
4.4 The Metric Curvature Formula 37
4.5 Conformal Maps 38
4.6 Some Visual Complex Analysis 41
4.7 The Conformal Stereographic Map of the Sphere 44
4.8 Stereographic Formulas 47
4.9 Stereographic Preservation of Circles 49

5 The Pseudosphere and the Hyperbolic Plane **51**

5.1 Beltrami's Insight 51
5.2 The Tractrix and the Pseudosphere 52
5.3 A Conformal Map of the Pseudosphere 54
5.4 The Beltrami–Poincaré Half-Plane 56

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

5.5 Using Optics to Find the Geodesics 58
5.6 The Angle of Parallelism 60
5.7 The Beltrami–Poincaré Disc 62

6 Isometries and Complex Numbers 65

6.1 Introduction 65
6.2 Möbius Transformations 67
6.3 The Main Result 72
6.4 Einstein's Spacetime Geometry 74
6.5 Three-Dimensional Hyperbolic Geometry 79

7 Exercises for Act II 83

ACT III

Curvature

8 Curvature of Plane Curves 97

8.1 Introduction 97
8.2 The Circle of Curvature 98
8.3 Newton's Curvature Formula 100
8.4 Curvature as Rate of Turning 101
8.5 Example: Newton's *Tractrix* 104

9 Curves in 3-Space 106

10 The Principal Curvatures of a Surface 109

10.1 Euler's Curvature Formula 109
10.2 Proof of Euler's Curvature Formula 110
10.3 Surfaces of Revolution 112

11 Geodesics and Geodesic Curvature 115

11.1 Geodesic Curvature and Normal Curvature 115
11.2 Meusnier's Theorem 117
11.3 Geodesics are "Straight" 118
11.4 Intrinsic Measurement of Geodesic Curvature 119
11.5 A Simple Extrinsic Way to Measure Geodesic Curvature 120
11.6 A New Explanation of the Sticky-Tape Construction of Geodesics 120
11.7 Geodesics on Surfaces of Revolution 121
11.7.1 Clairaut's Theorem on the Sphere 121
11.7.2 Kepler's Second Law 123
11.7.3 Newton's Geometrical Demonstration of Kepler's Second Law 124
11.7.4 Dynamical Proof of Clairaut's Theorem 126
11.7.5 Application: Geodesics in the Hyperbolic Plane (Revisited) 128

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

12 The Extrinsic Curvature of a Surface 130

12.1 Introduction 130
12.2 The Spherical Map 130
12.3 Extrinsic Curvature of Surfaces 131
12.4 What Shapes Are Possible? 135

13 Gauss's *Theorema Egregium* 138

13.1 Introduction 138
13.2 Gauss's *Beautiful Theorem* (1816) 138
13.3 Gauss's *Theorema Egregium* (1827) 140

14 The Curvature of a Spike 143

14.1 Introduction 143
14.2 Curvature of a Conical Spike 143
14.3 The Intrinsic and Extrinsic Curvature of a Polyhedral Spike 145
14.4 *The Polyhedral Theorema Egregium* 147

15 The Shape Operator 149

15.1 Directional Derivatives 149
15.2 The Shape Operator S 151
15.3 The Geometric Effect of S 152
15.4 DETOUR: The *Geometry* of the Singular Value Decomposition and of the Transpose 154
15.5 The General Matrix of S 158
15.6 Geometric Interpretation of S and Simplification of [S] 159
15.7 [S] Is Completely Determined by Three Curvatures 161
15.8 Asymptotic Directions 162
15.9 Classical Terminology and Notation: The Three *Fundamental Forms* 164

16 Introduction to the Global Gauss–Bonnet Theorem 165

16.1 Some Topology and the Statement of the Result 165
16.2 Total Curvature of the Sphere and of the Torus 168
16.2.1 Total Curvature of the Sphere 168
16.2.2 Total Curvature of the Torus 169
16.3 Seeing $\mathcal{K}(\mathcal{S}_g)$ via a Thick Pancake 170
16.4 Seeing $\mathcal{K}(\mathcal{S}_g)$ via Bagels and Bridges 171
16.5 The Topological Degree of the Spherical Map 172
16.6 Historical Note 174

17 First (Heuristic) Proof of the Global Gauss–Bonnet Theorem 175

17.1 Total Curvature of a Plane Loop: Hopf's *Umlaufsatz* 175
17.2 Total Curvature of a Deformed Circle 178
17.3 Heuristic Proof of Hopf's *Umlaufsatz* 179

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

17.4 Total Curvature of a Deformed Sphere 180
17.5 Heuristic Proof of the Global Gauss–Bonnet Theorem 181

18 Second (Angular Excess) Proof of the Global Gauss–Bonnet Theorem 183

18.1 The Euler Characteristic 183
18.2 Euler's (Empirical) Polyhedral Formula 183
18.3 Cauchy's Proof of Euler's Polyhedral Formula 186
18.3.1 Flattening Polyhedra 186
18.3.2 The Euler Characteristic of a Polygonal Net 187
18.4 Legendre's Proof of Euler's Polyhedral Formula 188
18.5 Adding Handles to a Surface to Increase Its Genus 190
18.6 Angular Excess Proof of the Global Gauss–Bonnet Theorem 193

19 Third (Vector Field) Proof of the Global Gauss–Bonnet Theorem 195

19.1 Introduction 195
19.2 Vector Fields in the Plane 195
19.3 The Index of a Singular Point 196
19.4 The Archetypal Singular Points: Complex Powers 198
19.5 Vector Fields on Surfaces 201
19.5.1 The Honey-Flow Vector Field 201
19.5.2 Relation of the Honey-Flow to the Topographic Map 203
19.5.3 Defining the Index on a Surface 204
19.6 The Poincaré–Hopf Theorem 206
19.6.1 Example: The Topological Sphere 206
19.6.2 Proof of the Poincaré–Hopf Theorem 207
19.6.3 Application: Proof of the Euler–L'Huilier Formula 208
19.6.4 Poincaré's Differential Equations Versus Hopf's *Line Fields* 209
19.7 Vector Field Proof of the Global Gauss–Bonnet Theorem 214
19.8 The Road Ahead 218

20 Exercises for Act III 219

ACT IV

Parallel Transport

21 An Historical Puzzle 231

22 Extrinsic Constructions 233

22.1 Project into the Surface as You Go! 233
22.2 Geodesics and Parallel Transport 235
22.3 Potato-Peeler Transport 236

23 Intrinsic Constructions 240

23.1 Parallel Transport via Geodesics 240
23.2 The Intrinsic (aka, "Covariant") Derivative 241

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

24 Holonomy **245**

24.1 Example: The Sphere 245
24.2 Holonomy of a General Geodesic Triangle 246
24.3 Holonomy Is Additive 248
24.4 Example: The Hyperbolic Plane 248

25 An Intuitive Geometric Proof of the *Theorema Egregium* **252**

25.1 Introduction 252
25.2 Some Notation and Reminders of Definitions 253
25.3 The Story So Far 253
25.4 The Spherical Map Preserves Parallel Transport 254
25.5 The Beautiful Theorem and *Theorema Egregium* Explained 256

26 Fourth (Holonomy) Proof of the Global Gauss–Bonnet Theorem **257**

26.1 Introduction 257
26.2 Holonomy Along an *Open* Curve? 257
26.3 Hopf's Intrinsic Proof of the Global Gauss–Bonnet Theorem 258

27 Geometric Proof of the Metric Curvature Formula **261**

27.1 Introduction 261
27.2 The Circulation of a Vector Field Around a Loop 262
27.3 Dry Run: Holonomy in the Flat Plane 264
27.4 Holonomy as the Circulation of a Metric-Induced Vector Field in the Map 266
27.5 Geometric Proof of the Metric Curvature Formula 268

28 Curvature as a Force between Neighbouring Geodesics **269**

28.1 Introduction to the Jacobi Equation 269
28.1.1 Zero Curvature: The Plane 269
28.1.2 Positive Curvature: The Sphere 270
28.1.3 Negative Curvature: The Pseudosphere 272
28.2 Two Proofs of the Jacobi Equation 274
28.2.1 Geodesic Polar Coordinates 274
28.2.2 Relative Acceleration = Holonomy of Velocity 276
28.3 The Circumference and Area of a Small Geodesic Circle 278

29 Riemann's Curvature **280**

29.1 Introduction and Summary 280
29.2 Angular Excess in an n-Manifold 281
29.3 Parallel Transport: Three Constructions 282
29.3.1 Closest Vector on Constant-Angle Cone 282
29.3.2 Constant Angle within a Parallel-Transported Plane 283
29.3.3 *Schild's Ladder* 284

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

29.4 The Intrinsic (aka "Covariant") Derivative $\nabla_{\mathbf{v}}$ 284
29.5 The Riemann Curvature Tensor 286
29.5.1 Parallel Transport Around a Small "Parallelogram" 286
29.5.2 Closing the "Parallelogram" with the Vector Commutator 287
29.5.3 The General Riemann Curvature Formula 288
29.5.4 Riemann's Curvature Is a *Tensor* 291
29.5.5 Components of the Riemann Tensor 292
29.5.6 For a Given $\mathbf{w}_o$, the Vector Holonomy *Only* Depends on the *Plane* of the Loop and Its *Area* 293
29.5.7 Symmetries of the Riemann Tensor 294
29.5.8 Sectional Curvatures 296
29.5.9 Historical Notes on the Origin of the Riemann Tensor 297
29.6 The Jacobi Equation in an n-Manifold 299
29.6.1 Geometrical Proof of the Sectional Jacobi Equation 299
29.6.2 Geometrical Implications of the Sectional Jacobi Equation 300
29.6.3 Computational Proofs of the Jacobi Equation and the Sectional Jacobi Equation 301
29.7 The Ricci Tensor 302
29.7.1 Acceleration of the Area Enclosed by a Bundle of Geodesics 302
29.7.2 Definition and Geometrical Meaning of the Ricci Tensor 304
29.8 Coda 306

30 Einstein's Curved Spacetime 307

30.1 Introduction: *"The Happiest Thought of My Life."* 307
30.2 Gravitational Tidal Forces 308
30.3 Newton's Gravitational Law in Geometrical Form 312
30.4 The Spacetime Metric 314
30.5 Spacetime Diagrams 315
30.6 Einstein's Vacuum Field Equation in Geometrical Form 317
30.7 The Schwarzschild Solution and the First Tests of the Theory 319
30.8 Gravitational Waves 323
30.9 The Einstein Field Equation (with Matter) in Geometrical Form 326
30.10 Gravitational Collapse to a Black Hole 329
30.11 The Cosmological Constant: *"The Greatest Blunder of My Life."* 331
30.12 The End 333

31 Exercises for Act IV 334

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

ACT V
Forms

32 1-Forms **345**

32.1 Introduction 345
32.2 Definition of a 1-Form 346
32.3 Examples of 1-Forms 347
32.3.1 Gravitational Work 347
32.3.2 Visualizing the Gravitational Work 1-Form 348
32.3.3 Topographic Maps and the Gradient 1-Form 349
32.3.4 Row Vectors 352
32.3.5 Dirac's Bras 352
32.4 Basis 1-Forms 352
32.5 Components of a 1-Form 354
32.6 The Gradient as a 1-Form: $\mathbf{d}f$ 354
32.6.1 Review of the Gradient as a Vector: $\boldsymbol{\nabla} f$ 354
32.6.2 The Gradient as a 1-Form: $\mathbf{d}f$ 355
32.6.3 The Cartesian 1-Form Basis: $\{\mathbf{d}x^j\}$ 356
32.6.4 The 1-Form Interpretation of $df = (\partial_x f)\,dx + (\partial_y f)\,dy$ 357
32.7 Adding 1-Forms Geometrically 357

33 Tensors **360**

33.1 Definition of a Tensor: Valence 360
33.2 Example: Linear Algebra 361
33.3 New Tensors from Old 361
33.3.1 Addition 361
33.3.2 Multiplication: The Tensor Product 361
33.4 Components 362
33.5 Relation of the Metric Tensor to the Classical Line Element 363
33.6 Example: Linear Algebra (Again) 364
33.7 Contraction 365
33.8 Changing Valence with the Metric Tensor 366
33.9 Symmetry and Antisymmetry 368

34 2-Forms **370**

34.1 Definition of a 2-Form and of a p-Form 370
34.2 Example: The Area 2-Form 371
34.3 The Wedge Product of Two 1-Forms 372
34.4 The Area 2-Form in Polar Coordinates 374
34.5 Basis 2-Forms and Projections 375
34.6 Associating 2-Forms with Vectors in $\mathbb{R}^3$: Flux 376
34.7 Relation of the Vector and Wedge Products in $\mathbb{R}^3$ 379
34.8 The Faraday and Maxwell Electromagnetic 2-Forms 381

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

35 3-Forms **386**

35.1 A 3-Form Requires Three Dimensions 386
35.2 The Wedge Product of a 2-Form and 1-Form 386
35.3 The Volume 3-Form 387
35.4 The Volume 3-Form in Spherical Polar Coordinates 388
35.5 The Wedge Product of Three 1-Forms and of p 1-Forms 389
35.6 Basis 3-Forms 390
35.7 Is $\Psi \wedge \Psi \neq 0$ Possible? 391

36 Differentiation **392**

36.1 The Exterior Derivative of a 1-Form 392
36.2 The Exterior Derivative of a 2-Form and of a p-Form 394
36.3 The Leibniz Rule for Forms 394
36.4 Closed and Exact Forms 395
36.4.1 A Fundamental Result: $\mathbf{d}^2 = 0$ 395
36.4.2 Closed and Exact Forms 396
36.4.3 Complex Analysis: Cauchy–Riemann Equations 397
36.5 Vector Calculus via Forms 398
36.6 Maxwell's Equations 401

37 Integration **404**

37.1 The Line Integral of a 1-Form 404
37.1.1 Circulation and Work 404
37.1.2 Path-Independence $\Longleftrightarrow$ Vanishing Loop Integrals 405
37.1.3 The Integral of an Exact Form: $\varphi = \mathbf{d}f$ 406
37.2 The Exterior Derivative as an Integral 406
37.2.1 $\mathbf{d}$(1-Form) 406
37.2.2 $\mathbf{d}$(2-Form) 409
37.3 Fundamental Theorem of Exterior Calculus (Generalized Stokes's Theorem) 411
37.3.1 Fundamental Theorem of Exterior Calculus 411
37.3.2 Historical Aside 411
37.3.3 Example: Area 412
37.4 The Boundary of a Boundary Is Zero! 412
37.5 The Classical Integral Theorems of Vector Calculus 413
37.5.1 $\Phi = 0$-Form 413
37.5.2 $\Phi = 1$-Form 414
37.5.3 $\Phi = 2$-Form 415
37.6 Proof of the Fundamental Theorem of Exterior Calculus 415
37.7 Cauchy's Theorem 417
37.8 The Poincaré Lemma for 1-Forms 418
37.9 A Primer on de Rham Cohomology 419
37.9.1 Introduction 419

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

37.9.2 A Special 2-Dimensional Vortex Vector Field 419
37.9.3 The Vortex 1-Form Is Closed 420
37.9.4 Geometrical Meaning of the Vortex 1-Form 420
37.9.5 The Topological Stability of the Circulation of a Closed 1-Form 421
37.9.6 The First de Rham Cohomology Group 423
37.9.7 The Inverse-Square Point Source in $\mathbb{R}^3$ 424
37.9.8 The Second de Rham Cohomology Group 426
37.9.9 The First de Rham Cohomology Group of the Torus 428

38 Differential Geometry via Forms 430

38.1 Introduction: Cartan's Method of Moving Frames 430
38.2 Connection 1-Forms 432
38.2.1 Notational Conventions and Two Definitions 432
38.2.2 Connection 1-Forms 432
38.2.3 WARNING: Notational Hazing Rituals Ahead! 434
38.3 The Attitude Matrix 435
38.3.1 The Connection Forms via the Attitude Matrix 435
38.3.2 Example: The Cylindrical Frame Field 436
38.4 Cartan's Two Structural Equations 438
38.4.1 The Duals $\boldsymbol{\theta}^i$ of $\mathbf{m}_i$ in Terms of the Duals $\mathbf{d}x^j$ of $\mathbf{e}_j$ 438
38.4.2 Cartan's First Structural Equation 439
38.4.3 Cartan's Second Structural Equation 440
38.4.4 Example: The Spherical Frame Field 441
38.5 The Six Fundamental Form Equations of a Surface 446
38.5.1 Adapting Cartan's Moving Frame to a Surface: The Shape Operator and the Extrinsic Curvature 446
38.5.2 Example: The Sphere 447
38.5.3 Uniqueness of Basis Decompositions 447
38.5.4 The Six Fundamental Form Equations of a Surface 448
38.6 Geometrical Meanings of the Symmetry Equation and the Peterson–Mainardi–Codazzi Equations 449
38.7 Geometrical Form of the Gauss Equation 450
38.8 Proof of the Metric Curvature Formula and the *Theorema Egregium* 451
38.8.1 Lemma: Uniqueness of $\boldsymbol{\omega}_{12}$ 451
38.8.2 Proof of the Metric Curvature Formula 451
38.9 A New Curvature Formula 452
38.10 Hilbert's Lemma 453
38.11 Liebmann's Rigid Sphere Theorem 454
38.12 The Curvature 2-Forms of an n-Manifold 455
38.12.1 Introduction and Summary 455
38.12.2 The Generalized Exterior Derivative 457

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

38.12.3 Extracting the Riemann Tensor from the Curvature 2-Forms 459
38.12.4 The Bianchi Identities Revisited 459
38.13 The Curvature of the Schwarzschild Black Hole 460

39 Exercises for Act V 465

Further Reading 475
Bibliography 485
Index 491

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

Chapter 1

Euclidean and Non-Euclidean Geometry

1.1 Euclidean and Hyperbolic Geometry

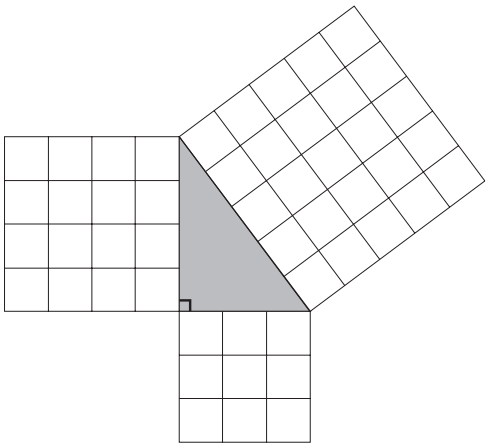
Differential Geometry is the application of calculus to the geometry of space that is *curved*. But to understand space that is curved we shall first try to understand space that is *flat*.

We inhabit a natural world pervaded by curved objects, and if a child asks us the meaning of the word "flat," we are most likely to answer in terms of the *absence* of curvature: a smooth surface *without* any bumps or hollows. Nevertheless, the very earliest mathematicians seem to have been drawn to the singular simplicity and uniformity of the flat plane, and they were rewarded with the discovery of startlingly beautiful facts about geometric figures constructed within it. With the benefit of enormous hindsight, some of these facts can be seen to *characterize* the plane's flatness.

One of the earliest and most profound such facts to be discovered was Pythagoras's Theorem. Surely the ancients must have been awed, as any sensitive person must remain today, that a seemingly unalloyed fact about *numbers*,

$$3^2 + 4^2 = 5^2,$$

in fact has *geometrical* meaning, as seen in [1.1].[1]



[1.1] *Pythagoras's Theorem: the geometry of* $3^2 + 4^2 = 5^2$.

While Pythagoras himself lived in Greece around 500 BCE, the theorem bearing his name was discovered much earlier, in various places around the world. The earliest known example of such knowledge is recorded in the Babylonian clay tablet (catalogued as "Plimpton 322") shown in [1.2], which was unearthed in what is now Iraq, and which dates from about 1800 BCE.

The tablet lists ***Pythagorean triples***:[2] integers (a, b, h) such that h is the hypotenuse of a right triangle with sides a and b, and therefore $a^2 + b^2 = h^2$. Some of these ancient examples are impressively large, and it seems clear that they did not stumble upon them, but rather possessed a mathematical process for generating solutions. For example, the fourth row of the tablet records the fact that $13500^2 + 12709^2 = 18541^2$.

The deeper knowledge that underlay these ancient results is not known,[3] but to find the first evidence of the "modern," logical, deductive approach to mathematics we must jump 1200 years into the future of the clay tablet. Scholars believe that it was Thales of Miletus (around 600 BCE)

[1] We repeat what was said in the Prologue: *equations* are labelled with parentheses (*round* brackets)—(...), while *figures* are labelled with *square* brackets—[...].

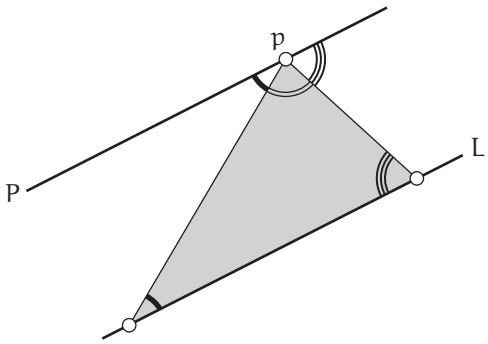
[2] In fact the tablet only records *two* members (a, h) of the triples (a, b, h).

[3] In the seventeenth century, Fermat and Newton reconstructed and generalized a *geometrical* method of generating the general solution, due to Diophantus. See Exercise 5.

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

[1.2] *Plimpton 322: A clay tablet of Pythagorean triples from 1800 BCE.*



[1.3] *Euclid's Parallel Axiom:* P *is the* unique *parallel to* L *through* p*, and the angle sum of a triangle is* π.

who first pioneered the idea of deducing new results from previously established ones, the logical chain beginning at a handful of clearly articulated assumptions, or ***axioms***.

Leaping forward again, 300 years beyond Thales, we find one of the most perfect exemplars of this new approach in Euclid's *Elements*, dating from 300 BCE. This work sought to bring order, clarity, and rigour to geometry by deducing everything from just five simple axioms, the fifth and last of which dealt with parallel lines.

Defining two lines to be ***parallel*** if they do not meet, Euclid's Fifth Axiom[4] is illustrated in [1.3]:

> ***Parallel Axiom.*** *Through any point* p *not on the line* L *there exists precisely one line* P *that is parallel to* L.

But the character of this axiom was more complex and less immediate than that of the first four, and mathematicians began a long struggle to dispense with it as an assumption, instead seeking to show that it must be a logical *consequence* of the first four axioms.

This tension went unresolved for the next *2000 years*. As the centuries passed, many attempts were made to prove the Parallel Axiom, and the number and intensity of these efforts reached a crescendo in the 1700s, but all met with failure.

Yet along the way useful *equivalents* of the axiom emerged. For example: *There exist similar triangles of different sizes* (Wallis in 1663; see Stillwell (2010)). But the very first equivalent was already present in Euclid, and it is the one still taught to every school child: *the angles in a triangle add up to two right angles.* See [1.3].

The explanation of these failures only emerged around 1830. Completing a journey that had begun 4000 years earlier, Nikolai Lobachevsky and János Bolyai independently announced the

[4]Euclid did not state his axiom in this form, but it is logically equivalent.

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

discovery of an entirely new form of geometry (now called ***Hyperbolic Geometry***) taking place in a new kind of plane (now called the ***hyperbolic plane***). In this Geometry the first four Euclidean axioms hold, but the parallel axiom does *not*. Instead, the following is true:

> ***Hyperbolic Axiom.*** *There are at least two parallel lines through* p *that do not meet* L. (1.1)

These pioneers explored the logical consequences of this axiom, and by purely abstract reasoning were led to a host of fascinating results within a rich new geometry that was bizarrely different from that of Euclid.

[1.4] *Johann Heinrich Lambert (1728–1777).*

Many others before them, perhaps most notably Saccheri (in 1733; see Stillwell 2010) and Lambert (in 1766; see Stillwell 2010), had discovered some of these consequences of (1.1), but their aim in exploring these consequences had been to find a *contradiction*, which they believed would finally prove that Euclidean Geometry to be the One True Geometry.

Certainly Saccheri believed he had found a clear contradiction when he published "Euclid Freed of Every Flaw." But Lambert is a much more perplexing case, and he is perhaps an unsung hero in this story. His results penetrated so deeply into this new geometry that it seems impossible that he did not at times believe in the reality of what he was doing. Regardless of his motivation and beliefs[5], Lambert (shown in [1.4]) was certainly the first to discover a remarkable fact[6] about the angle sum of a triangle under axiom (1.1), and his result will be central to much that follows in Act II.

Nevertheless, Lobachevsky and Bolyai richly deserve their fame for having been the first to recognize and fully embrace the idea that they had discovered an entirely new, consistent, non-Euclidean Geometry. But what this new geometry really *meant*, and what it might be useful for, even they could not say.[7]

Remarkably and surprisingly, it was the *Differential Geometry of curved surfaces* that ultimately resolved these questions. As we shall explain, in 1868 the Italian mathematician Eugenio Beltrami finally succeeded in giving Hyperbolic Geometry a concrete interpretation, setting it upon a firm and intuitive foundation from which it has since grown and flourished. Sadly, neither Lobachevsky nor Bolyai lived to see this: they died in 1856 and 1860, respectively.

This non-Euclidean Geometry had in fact already manifested itself in various branches of mathematics throughout history, but always in disguise. Henri Poincaré (beginning around 1882) was the first not only to strip it of its camouflage, but also to recognize and exploit its power

[5]I thank Roger Penrose for making me see that Lambert deserves greater credit than he is usually granted. Penrose did so by means of the following analogy: "Should we not give credit to Einstein for the cosmological constant, even if he introduced it for the wrong reasons? And should we blame him for later retracting it, calling it the "greatest blunder of my life"? Or what about General Relativity itself, which Einstein seemed to become less and less convinced was the right theory (needing to be replaced by some kind of non-singular unified field theory) as time went on?" [Private communication.]

[6]If you cannot wait, it's (1.8).

[7]Lobachevsky did in fact put this geometry to use to evaluate previously unknown integrals, but (at least in hindsight) this particular application must be viewed as relatively minor.

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

in such diverse areas as Complex Analysis, Differential Equations, Number Theory, and Topology. Its continued vitality and centrality in the mathematics of the 20th and twenty-first centuries is demonstrated by Thurston's work on 3-manifolds, Wiles's proof of ***Fermat's Last Theorem***, and Perelman's proof of the ***Poincaré Conjecture*** (as a special case of Thurston's ***Geometrization Conjecture***), to name but three examples.

In Act II we shall describe Beltrami's breakthrough, as well as the nature of Hyperbolic Geometry, but for now we wish to explore a different, simpler kind of non-Euclidean Geometry, one that was already known to the Ancients.

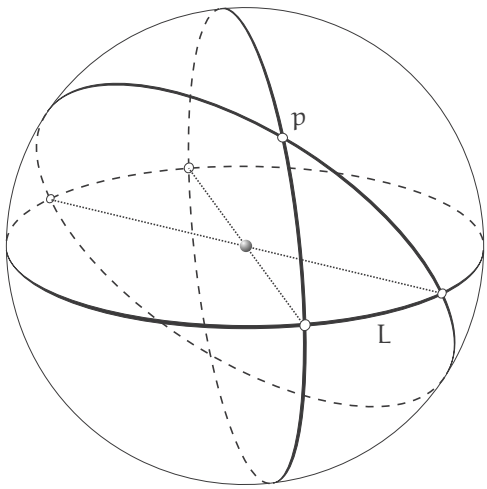
1.2 Spherical Geometry

To construct a non-Euclidean Geometry we must deny the existence of a unique parallel. The Hyperbolic Axiom assumes two or more parallels, but there is one other logical possibility—*no* parallels:

> ***Spherical Axiom.*** *There are* no *lines through* p *that are parallel to* L*:* every *line meets* L. (1.2)

Thus there are actually *two* non-Euclidean[8] geometries: spherical and hyperbolic.

As the name suggests, Spherical Geometry can be realized on the surface of a sphere—denoted $\mathbb{S}^2$ in the case of the *unit* sphere—which we may picture as the surface of the Earth. On this sphere, what should be the analogue of a "straight line" connecting two points on the surface? Answer: the shortest route between them! But if you wish to sail or fly from London to New York, for example, what *is* the shortest route?



[1.5] *The great circles of* $\mathbb{S}^2$ *intersect in pairs of antipodal points.*

The answer, already known to the ancient mariners, is that the shortest route is an arc of a ***great circle***, such as the equator, obtained by cutting the sphere with a plane passing through its centre. In [1.5] we have chosen L to be the equator, and it is clear that (1.2) is satisfied: every line through p meets L in a pair of ***antipodal*** (i.e., diametrically opposite) points.

In the plane, the shortest route is also the *straightest* route, and in fact the same is true on the sphere: in a precise sense to be discussed later, the great circle trajectory bends neither to the right nor to the left as it traverses the spherical surface.

There are other ways of constructing the great circles on the Earth that do not require thinking about planes passing through the completely inaccessible centre of the Earth. For example, on a globe you may map out your great circle journey by holding down one end of a piece of string on London and pulling the string tightly over the surface so that the other end is on New York. The taut string has

[8]Nevertheless, the reader should be aware that in modern usage "non-Euclidean Geometry" is usually synonymous with "Hyperbolic Geometry."

For general queries, contact webmaster@press.princeton.edu

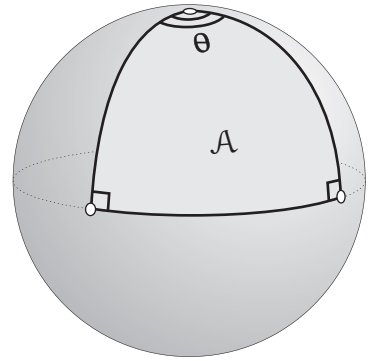
© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

automatically found the shortest, straightest route—the shorter[9] of the two arcs into which the great circle through the two cities is divided by those cities.

With the analogue of straight lines now found, we can "do geometry" within this spherical surface. For example, given three points on the surface of the Earth, we can connect them together with arcs of great circles to obtain a "triangle." Figure [1.6] illustrates this in the case where one vertex is located at the north pole, and the other two are on the equator.

But if this non-Euclidean Spherical Geometry was already used by ancient mariners to navigate the oceans, and by astronomers to map the spherical night sky, what then was so shocking and new about the non-Euclidean geometry of Lobachevsky and Bolyai?

The answer is that this Spherical Geometry was merely considered to be inherited from the *Euclidean* Geometry of the 3-dimensional space in which the sphere resides. No thought was given in those times to the sphere's internal 2-dimensional geometry as representing an alternative to Euclid's plane. Not only did it violate Euclid's fifth axiom, it also violated a much more basic one (Euclid's first axiom) that we can always draw a unique straight line connecting two points, for this fails when the points are antipodal.



[1.6] *A particularly simple "triangle" on the sphere.*

On the other hand, the Hyperbolic Geometry of Lobachevsky and Bolyai was a much more serious affront to Euclidean Geometry, containing familiar lines of infinite length, yet flaunting multiple parallels, ludicrous angle sums, and many other seemingly nonsensical results. Yet the 21-year-old Bolyai was confident and exuberant in his discoveries, writing to his father, *"From nothing I have created another entirely new world."*

We end with a tale of tragedy. Bolyai's father was a friend of Gauss, and sent him what János had achieved. By this time Gauss had himself made some important discoveries in this area, but had kept them secret. In any case, János had seen further than Gauss. A kind word in public from Gauss, the most famous mathematician in the world, would have assured the young mathematician a bright future. But Nature and nurture sometimes conspire to pour extraordinary mathematical gifts into a vessel marred by very ordinary human flaws, and Gauss's reaction to Bolyai's marvellous discoveries was mean-spirited and self-serving in the extreme.

First, Gauss kept Bolyai in suspense for six months, then he replied as follows:

> Now something about the work of your son. You will probably be shocked for a moment when I begin by saying that *I cannot praise it*, but I cannot do anything else, since to praise it would be to praise myself. The whole content of the paper, the path that your son has taken, and the results to which he has been led, agree almost everywhere with my own meditations, which have occupied me in part for 30–35 years.

Gauss did however "thank" Bolyai's son for having "saved him the trouble"[10] of having to write down theorems he had known for decades.

János Bolyai never recovered from the surgical blow delivered by Gauss, and he abandoned mathematics for the rest of his life.[11]

---

[9]If the two points are antipodal, such as the north and south poles, then the two arcs are the *same* length. Furthermore, the great circle itself is no longer unique: *every* meridian is a great circle connecting the poles.

[10]Gauss had previously denigrated Abel's discovery of elliptic functions in precisely the same manner; see Stillwell (2010, p. 236).

[11]If this depresses you, turn your thoughts to the uplifting counterweight of Leonhard Euler. An intellectual volcano erupting with wildly original thoughts (some of which we shall meet later) he was also a kind and generous spirit. We cite one, parallel

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

1.3 The Angular Excess of a Spherical Triangle

As we have said, the parallel axiom is equivalent to the fact that the angles in a triangle sum to π. It follows that both the spherical axiom and the hyperbolic axiom must lead to geometries in which the angles do *not* sum to π. To quantify this departure from Euclidean Geometry, we introduce the ***angular excess***, defined to be the amount $\mathcal{E}$ by which the angle sum exceeds π:

$$\mathcal{E} \equiv (\text{angle sum of triangle}) - \pi.$$

For example, for the triangle shown in [1.6], $\mathcal{E} = (\theta + \frac{\pi}{2} + \frac{\pi}{2}) - \pi = \theta$.

A crucial insight now arises if we compare the triangle's angular excess with its area $\mathcal{A}$. Let the radius of the sphere be R. Since the triangle occupies a fraction $(\theta/2\pi)$ of the northern hemisphere, $\mathcal{A} = (\theta/2\pi)\, 2\pi R^2 = \theta\, R^2$. Thus,

$$\mathcal{E} = \frac{1}{R^2}\,\mathcal{A}. \tag{1.3}$$

[1.7] *Thomas Harriot (1560–1621).*

In 1603 the English mathematician Thomas Harriot (see [1.7]) made the remarkable discovery[12] that this relationship holds for *any* triangle Δ on the sphere; see [1.8a]. Harriot's elementary but ingenious argument[13] goes as follows.

Prolonging the great-circle sides of Δ divides the surface of the sphere into eight triangles, the four triangles labelled Δ, Δ_α, Δ_β, Δ_γ each being paired with a congruent antipodal triangle. This is clearer in [1.8b]. Since the area of the sphere is $4\pi R^2$, we deduce that

$$\mathcal{A}(\Delta) + \mathcal{A}(\Delta_\alpha) + \mathcal{A}(\Delta_\beta) + \mathcal{A}(\Delta_\gamma) = 2\pi R^2. \tag{1.4}$$

On the other hand, it is clear in [1.8b] that Δ and Δ_α together form a wedge whose area is a fraction $(\alpha/2\pi)$ of the area of the sphere:

$$\mathcal{A}(\Delta) + \mathcal{A}(\Delta_\alpha) = 2\alpha R^2.$$

Similarly,

$$\mathcal{A}(\Delta) + \mathcal{A}(\Delta_\beta) = 2\beta R^2,$$
$$\mathcal{A}(\Delta) + \mathcal{A}(\Delta_\gamma) = 2\gamma R^2.$$

---

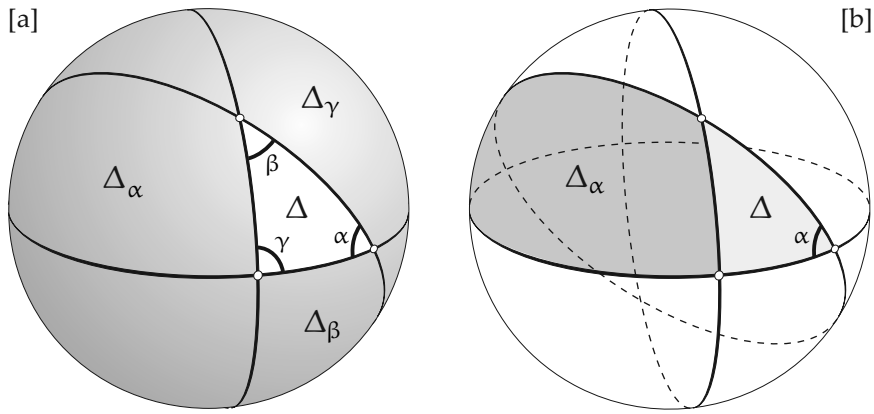
example. When the then-unknown 19-year-old Lagrange wrote to him with overlapping discoveries in the calculus of variations, Euler wrote back: "... I deduced this myself. However, I decided to conceal this until you publish your results, since in no way do I want to take away from you any of the glory that you deserve." See Gindikin (2007, p. 216). Incidentally, Euler also personally intervened to rescue Lambert's career!

[12]This discovery is most often attributed to Girard, who rediscovered it about 25 years later.

[13]This argument was later rediscovered by Euler in 1781.

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.



[1.8] *Harriot's Theorem (1603):* $\mathcal{E}(\Delta) = \mathcal{A}(\Delta)/R^2$.

Adding these last three equations, we find that

$$3\mathcal{A}(\Delta) + \mathcal{A}(\Delta_\alpha) + \mathcal{A}(\Delta_\beta) + \mathcal{A}(\Delta_\gamma) = 2(\alpha+\beta+\gamma)R^2. \tag{1.5}$$

Finally, subtracting (1.4) from (1.5), we find that

$$\mathcal{A}(\Delta) = R^2(\alpha+\beta+\gamma-\pi) = R^2\,\mathcal{E}(\Delta),$$

thereby proving (1.3).

1.4 Intrinsic and Extrinsic Geometry of Curved Surfaces

The mathematics associated with this stretched-string construction of a "straight line" will be explored in depth later in the book. For now we merely observe that the construction can be applied equally well to a *non*spherical surface, such as the crookneck squash shown in [1.9].

Just as on the sphere, we stretch a string over the surface, thereby finding the shortest, straightest route between two points, such as a and b. Provided that the string can slide around on the surface easily, the tension in the string ensures that the resulting path is as short as possible. Note that in the case of cd, we must imagine that the string runs over the *inside* of the surface.

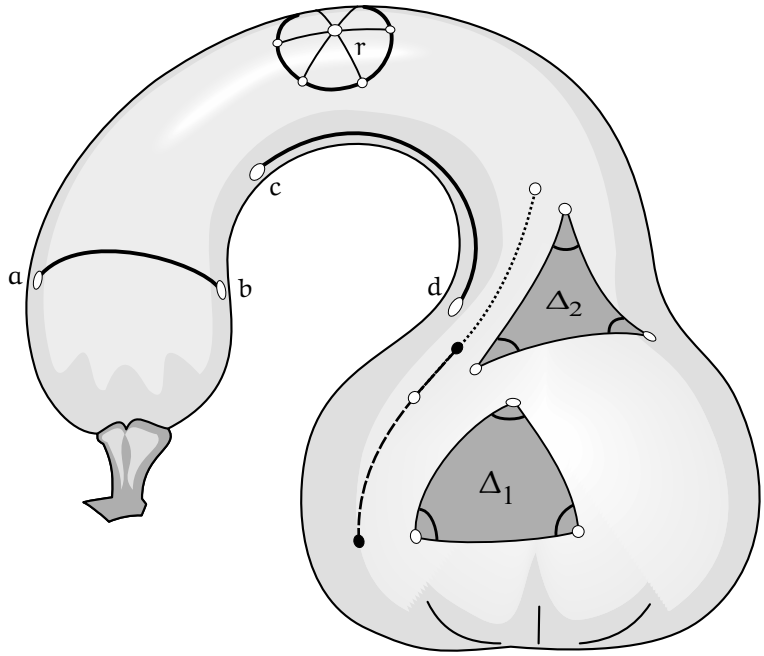
In order to deal with all possible pairs of points in a uniform way, it is therefore best to imagine the surface as made up of two thinly separated layers, with the string trapped between them. On the other hand, this is only useful for thought experiments, not actual experiments. We shall overcome this obstacle shortly by providing a *practical* method of constructing these straightest curves on the surface of a physical object, even if the patch of surface bends the wrong way for a string to be stretched tightly over the outside of the object.

These shortest paths on a curved surface are the equivalent of straight lines in the plane, and they will play a crucial role throughout this book—they are called ***geodesics***. Thus, to use this new word, we may say that geodesics in the plane are straight lines, and geodesics on the sphere are great circles.

But even on the sphere the *length-minimizing* definition of "geodesic" is provisional, because we see that nonantipodal points are connected by *two* arcs of the great circle passing through them: the short one (which *is* the shortest route) and the long one. Yet the long arc is every bit as much a geodesic as the short one. There is the additional complication on the sphere that antipodal points

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.



[1.9] *The* **intrinsic geometry** *of the surface of a crookneck squash:* **geodesics** *are the equivalents of straight lines, and triangles formed out of them may possess an angular excess of either sign, depending on how the surface bends:* $\mathcal{E}(\Delta_1) > 0$ *and* $\mathcal{E}(\Delta_2) < 0$.

are connected by *multiple* geodesics, and this *nonuniqueness* occurs on more general surfaces, too. What *is* true is that any two points that are *sufficiently close together* can be joined by a unique geodesic segment that is the shortest route between them.

Just as a line segment in the plane can be extended indefinitely in either direction by laying down overlapping segments, so too can a geodesic segment be extended on a curved surface, and this extension is unique. For example, in [1.9] we have extended the dashed geodesic segment connecting the black dots, by laying down the overlapping dotted segment between the white points.

Because of the subtleties associated with the length-minimizing characterization of geodesics, before long we will provide an alternative, purely *local* characterization of geodesics, based on their straightness.

With these caveats in place, it is now clear how we should define distance within a surface such as [1.9]: the distance between two sufficiently close points a and b is the length of the geodesic segment connecting them.

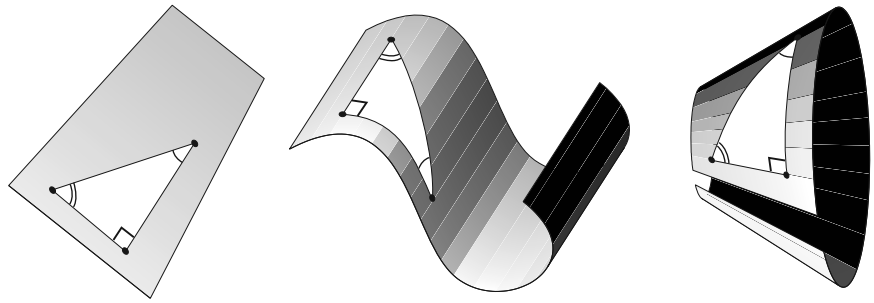
Figure [1.9] shows how we may then define, for example, a "circle of radius r and centre c" as the locus of points at distance r from c. To construct this ***geodesic circle*** we may take a piece of string of length r, hold one end fixed at c, then (keeping the string taut) drag the other end round on the surface. But just as the angles in a triangle no longer sum to π, so now the circumference of a circle no longer is equal to $2\pi r$. In fact you should be able to convince yourself that for the illustrated circle the circumference is *less* than $2\pi r$.

Given three points on the surface, we may join them with geodesics to form a ***geodesic triangle***; [1.9] shows two such triangles, Δ_1 and Δ_2:

- Looking at the angles in Δ_1, it seems clear that they sum to *more* than π, so $\mathcal{E}(\Delta_1) > 0$, like a triangle in Spherical Geometry.

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.



[1.10] *Bending a piece of paper changes the extrinsic geometry, but not the intrinsic geometry.*

- On the other hand, it is equally clear that the angles of Δ_2 sum to *less* than π: $\mathcal{E}(\Delta_2) < 0$, and (as we shall explain) this opposite behaviour is in fact exhibited by triangles in *Hyperbolic Geometry*. Note also that if we construct a circle in this saddle-shaped part of the surface, the circumference is now *greater* than $2\pi r$.

The concept of a geodesic belongs to the so-called ***intrinsic geometry*** of the surface—a fundamentally new view of geometry, introduced by Gauss (1827). It means the geometry that is knowable to tiny, ant-like, intelligent (but 2-dimensional!) creatures living *within* the surface. As we have discussed, these creatures can, for example, define a geodesic "straight line" connecting two nearby points as the shortest route within their world (the surface) connecting the two points. From there they can go on to define triangles, and so on. Defined in this way, it is clear that the intrinsic geometry is unaltered when the surface is bent (as a piece of paper can be) into quite different shapes in space, as long as distances *within* the surface are not stretched or distorted in any way. To the ant-like creatures within the surface, such changes are utterly undetectable.

Under such a bending, the so-called ***extrinsic geometry*** (how the surface sits in space) most certainly does change. See [1.10]. On the left is a flat piece of paper on which we have drawn a triangle Δ with angles $(\pi/2)$, $(\pi/6)$, and $(\pi/3)$. Of course $\mathcal{E}(\Delta) = 0$. Clearly we can bend such a flat piece of paper into either of the two (extrinsically) curved surfaces on the right.[14] However, *intrinsically* these surfaces have undergone no change at all—they are both as flat as a pancake! The illustrated triangles on these surfaces (into which Δ is carried by our stretch-free bending of the paper) are identical to the ones that intelligent ants would construct using geodesics, and in both cases $\mathcal{E} = 0$: geometry on these surfaces is Euclidean.

Even if we take a patch of a surface that is intrinsically *curved*, so that a triangle within it has $\mathcal{E} \neq 0$, it too can generally be bent somewhat without stretching or tearing it, thereby altering its extrinsic geometry while leaving its intrinsic geometry unaltered. For example, cut a ping pong ball in half and gently squeeze the rim of one of the hemispheres, distorting that circular rim into an oval (but not an oval lying in a single plane).

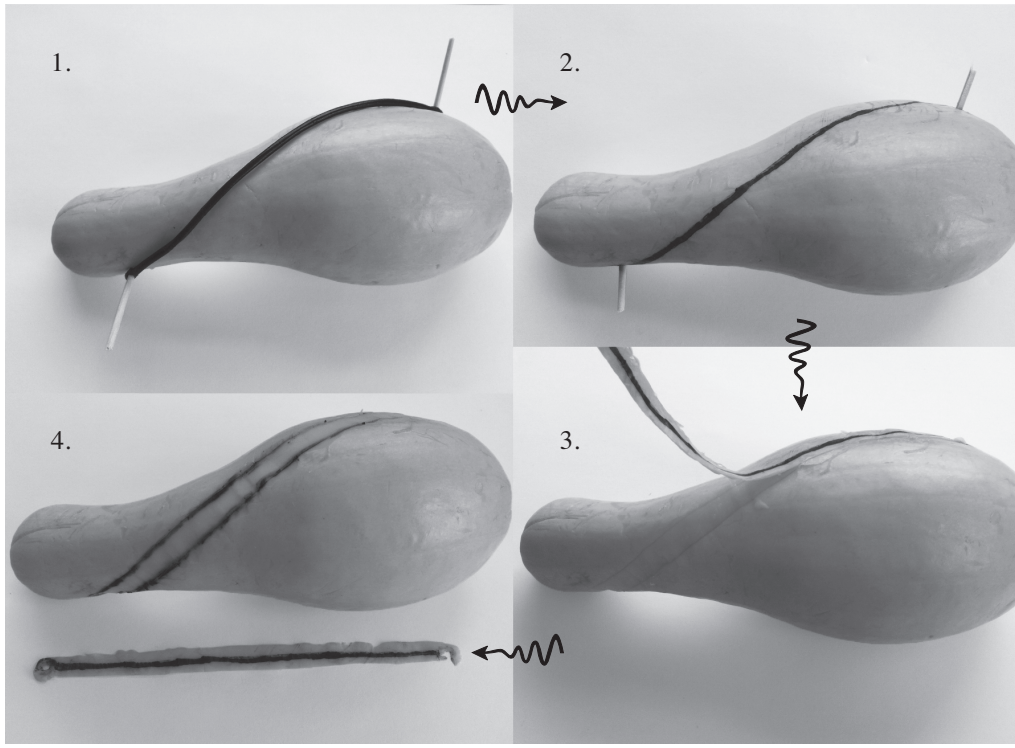
1.5 Constructing Geodesics via Their Straightness

We have already alluded to the fact that geodesics on a surface have at least *two* characteristics in common with lines in the plane: (1) they provide the *shortest* route between two points that are not too far apart *and* (2) they provide the "straightest" route between these points. In this section we seek to clarify what we mean by "straightness," leading to a very simple and *practical* method of constructing geodesics on a physical surface.

[14] But note that we must first trim the edges of the rectangle to bend it into the shape on the far right.

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.



[1.11] *On the curved surface of a fruit or vegetable, peel a narrow strip surrounding a geodesic, then lay it flat on the table. You will obtain a* straight line *in the plane!*

Most texts on Differential Geometry pay scant attention to such practical matters, and it is perhaps for this reason that the construction we shall describe is surprisingly little known in the literature.[15] In sharp contrast, in this book we *urge* you to explore the ideas by all means possible: theoretical contemplation, drawing, computer experiments, and (especially!) physical experiments with actual surfaces. Your local fruit and vegetable shop can supply your laboratory with many interesting shapes, such as the yellow summer squash shown in [1.11].

We can now use this vegetable to reveal the hidden straightness of geodesics via an experiment that we hope you will repeat for yourself:

1. On a fruit or vegetable, construct a geodesic by stretching a string over its curved surface.
2. Use a pen to trace the path of the string, then remove the string.
3. Make shallow incisions on either side of (and close to) the inked path, then use a vegetable peeler or small knife to remove the narrow strip of peel between the two cuts.
4. Lay the strip of peel flat on the table, and witness the marvellous fact that the geodesic within the peeled strip has become a *straight line* in the plane!

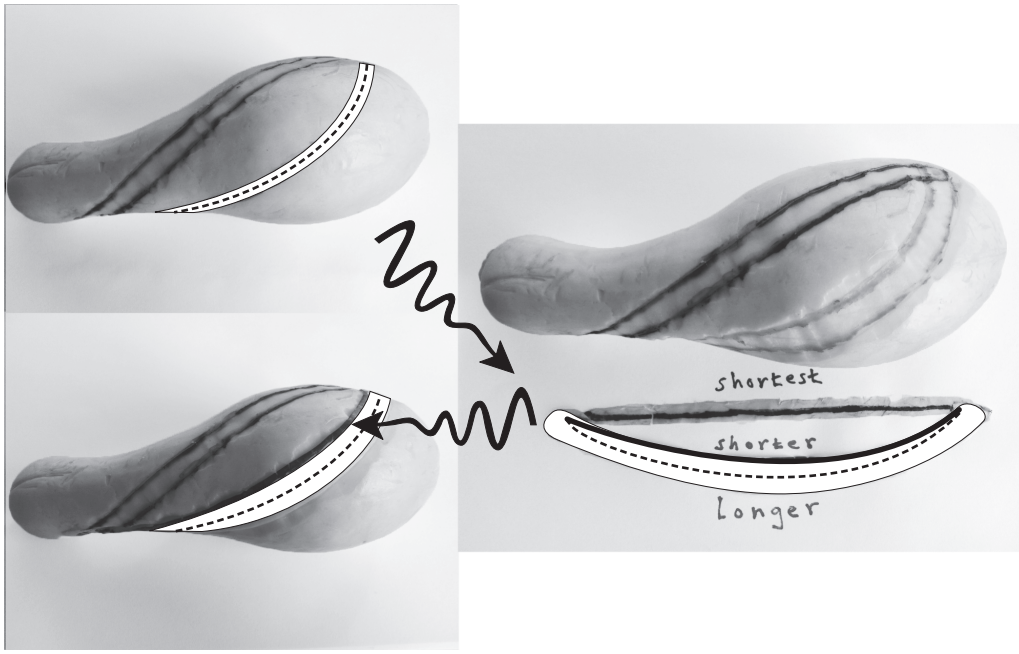
But why?!

To understand this, first let us be clear that although the strip is free to bend in the direction perpendicular to the surface (i.e., perpendicular to itself), it is *rigid* if we try to bend it sideways, tangent to the surface. Now let us employ proof by contradiction, and imagine what would happen if such a peeled geodesic did *not* yield a straight line when laid flat on the table. It is both a

[15]One of the rare exceptions is Henderson (1998), which we strongly recommend to you; for more details, see the *Further Reading* section at the end of this book.

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.



[1.12] *Suppose that the illustrated dotted path is a geodesic such that a narrow (white) strip surrounding it does* not *become a straight line when laid flat in the plane. But in that case we can shrink the dotted path in the plane (towards the shortest, straight-line route in the plane) thereby producing the solid path. But if we then reattach the strip to the surface, this solid path is still shorter than the original dotted path, which was supposed to be the shortest path within the surface—a contradiction!*

drawback and an advantage of conducting such physical experiments that they will simply not *permit* us to construct something that is impossible, as is required in our desired mathematical proof by contradiction. Nevertheless, let us *suppose* that there exists a geodesic path, such as the dotted one shown on top left of [1.12], that when peeled and laid flat on the table (on the right) does *not* become a straight line.

The shortest route between the ends of this dotted (nonstraight) plane curve is the straight line connecting them. (As illustrated, this is the path of the *true* geodesic we already found using the string—but pretend you don't know that for now!) Thus we may shorten the dotted curve by deforming it slightly towards this straight, shortest route, yielding the solid path along the edge of the peeled strip. Therefore, after reattaching the strip to the surface (bottom left) the solid curve provides a shorter route over the surface than the dotted one, which we had supposed to be the shortest: a contradiction! Thus we have proved our previous assertion:

> *If a narrow strip surrounding a segment* G *of a geodesic is cut out of a surface and laid flat in the plane, then* G *becomes a segment of a* straight line. (1.6)

We are now very close to the promised simple and practical construction of geodesics. Look again at step 3 of [1.11], where we peeled off the strip of surface. But imagine now that we are *reattaching* the strip to the surface, instead. Ignore the history of how we got to this point: what are we actually doing right now in this reattachment process? We have picked up a narrow straight strip (of three-dimensional peel—but mathematically idealized as a two-dimensional strip) and we have unrolled it back onto the surface into the shallow channel from which we cut it. But here

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

is the crucial observation: this shallow channel need not exist—the *surface* decides where the strip must go as we unroll it!

Thus, as a kind of time-reversed converse of (1.6), we obtain a remarkably simple and practical method[16] of constructing geodesics on a physical surface:

> *To construct a geodesic on a surface, emanating from a point* p *in direction* **v**, *stick one end of a length of narrow sticky tape down at* p *and unroll it onto the surface, starting in the direction* **v**. (1.7)

(Note, however, that this does *not* provide a construction of the geodesic connecting p to a specified *target* point q.)

If this construction seems too simple to be true, please try it on any curved surface you have to hand. You can check that the sticky tape[17] is indeed tracing out a geodesic by stretching a string over the surface between two points on the tape: the string will follow the same path as the tape. But note that, as a promised bonus, this new tape construction works on *any* part of a surface, even where the surface is concave towards you, so that the stretched-string construction breaks down.

Of course all of this is a concrete manifestation of a mathematical idealization. A totally flat narrow strip of tape of nonzero width *cannot*[18] be made to fit perfectly on a genuinely curved surface, but its centre line *can* be made to rest on the surface, while the rest of the tape is tangent to the surface.

1.6 The Nature of Space

Let us return to the history of the discovery of non-Euclidean Geometry, and take our first look at how these two new geometries differ from Euclid's.

As we have said, Euclidean Geometry, is characterized by the vanishing of $\mathcal{E}(\Delta)$. Note that, unlike the original formulation of the parallel axiom, *this statement can be checked against experiment*: construct a triangle, measure its angles, and see if they add up to π. Gauss may have been the first person to ever conceive of the possibility that physical space might not be Euclidean, and he even attempted the above experiment, using three mountain tops as the vertices of his triangle, and using light rays for its edges.

Within the accuracy permitted by his equipment, he found $\mathcal{E} = 0$. Quite correctly, Gauss did not conclude that physical space is definitely Euclidean in structure, but rather that if it is *not* Euclidean then its deviation from Euclidean Geometry is extremely small. But he did go so far as to say (see Rosenfeld 1988, p. 215) that he wished that this non-Euclidean Geometry might apply to the real world. In Act IV we shall see that this was a prophetic statement.

---

[16] This important fact is surprisingly hard to find in the literature. After we (re)discovered it, more than 30 years ago, we began searching, and the earliest mention of the underlying idea we could find at that time was in Aleksandrov (1969, p. 99), albeit in a less practical form: he imagined pressing a flexible metal ruler down onto the surface. Later, the basic idea also appeared in Koenderink (1990), Casey (1996), and Henderson (1998). *However*, we have since learned that the essential idea (though not in our current, *practical* form) goes all the way back to Levi-Civita, more than a century ago! See the footnote on page 236.

[17] We recommend using masking tape (aka painter's tape) because it comes in bright colours, and once a strip has been created, it can be detached and reattached repeatedly, with ease. A simple way to create narrow strips (from the usually wide roll of tape) is to stick a length of tape down onto a kitchen cutting board, then use a sharp knife to cut down its length, creating strips as narrow as you please.

[18] This is a consequence of a fundamental theorem we shall meet later, called the *Theorema Egregium*.

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

Although Gauss had bragged to friends that he had anticipated the Hyperbolic Geometry of Lobachevsky and Bolyai by decades, even he had unknowingly been scooped on some of its central results.

In 1766 (eleven years before Gauss was born) Lambert rediscovered Harriot's result on the sphere and then broke totally new ground in pursuing the analogous consequences of the Hyperbolic Axiom (1.1). First, he found that a triangle in Hyperbolic Geometry (if such a thing even existed) would behave *oppositely* to one in Spherical Geometry:

- In Spherical Geometry the angle sum of a triangle is greater than π: $\mathcal{E} > 0$.
- In Hyperbolic Geometry the angle sum of a triangle is less than π: $\mathcal{E} < 0$.

Thus a hyperbolic triangle behaves like a triangle drawn on a saddle-shaped piece of surface, like Δ_2 in [1.9]. Later we shall see that this is no accident.

Furthermore, Lambert discovered the crucial fact that $\mathcal{E}(\Delta)$ is again simply proportional to $\mathcal{A}(\Delta)$:

In both Spherical and Hyperbolic Geometry,

$$\mathcal{E}(\Delta) = \mathcal{K}\mathcal{A}(\Delta), \tag{1.8}$$

where $\mathcal{K}$ is a constant that is positive in Spherical Geometry, and negative in Hyperbolic Geometry.

Several interesting observations can be made in connection with this result:

- Although there are no qualitative differences between them, there are nevertheless *infinitely many* different Spherical Geometries, depending on the value of the positive constant $\mathcal{K}$. Likewise, each negative value of $\mathcal{K}$ yields a different Hyperbolic Geometry.
- Since the angle sum of a triangle cannot be negative, $\mathcal{E} \geqslant -\pi$. Thus in Hyperbolic Geometry ($\mathcal{K} < 0$) we have the strange and surprising result that *no triangle can have an area greater than* $|\pi/\mathcal{K}|$.
- From (1.8) we deduce that two triangles of different size cannot have the same angles. In other words, in non-Euclidean Geometry, *similar triangles do not exist!* (This accords with Wallis's 1663 discovery that the existence of similar triangles is equivalent to the Parallel Axiom.)
- Closely related to the previous point is the fact that in non-Euclidean Geometry there exists an *absolute unit of length*. (Gauss himself found it to be an exciting possibility that this purely mathematical fact might be realized in the physical world.) For example, in Spherical Geometry we could define this absolute unit of length to be the side of *the* equilateral triangle having, for instance, angle sum 1.01π. Similarly, in Hyperbolic Geometry we could define it to be the side of *the* equilateral triangle having angle sum 0.99π.
- A somewhat more natural way of defining the absolute unit of length is in terms of the constant $\mathcal{K}$. Since the radian measure of angle is defined as a ratio of lengths, $\mathcal{E}$ is a pure number. On the other hand, the area $\mathcal{A}$ has units of (length)^2. It follows that $\mathcal{K}$ must have units of $1/(\text{length})^2$, and so there exists a length R such that $\mathcal{K}$ can be written as follows: $\mathcal{K} = +(1/\mathsf{R}^2)$ in Spherical Geometry; $\mathcal{K} = -(1/\mathsf{R}^2)$ in Hyperbolic Geometry. Of course in Spherical Geometry we already know that the length R occurring in the formula $\mathcal{K} = +(1/\mathsf{R}^2)$ is simply the

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

radius of the sphere. Later we will see that this length R occurring in the formula $\mathcal{K} = -(1/R^2)$ can be given an equally intuitive and *concrete* interpretation in Hyperbolic Geometry.

- The smaller the triangle, the harder it is to distinguish it from a Euclidean triangle: only when the linear dimensions are a significant fraction of R will the differences become discernable. For example, we humans are small compared to the radius of the Earth, so if we find ourselves in a boat in the middle of a lake, its surface appears to be a Euclidean plane, whereas in reality it is part of a sphere. This Euclidean illusion for small figures is the reason that Gauss chose the largest possible triangle to conduct his light-ray experiment, thereby increasing his chances of detecting any small curvature that might be present in the space through which the light rays travelled.

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

Index

aberration formula, 93
Absolute Differential Calculus, 231
amplification, 43, 156
amplitwist, 73, 85, 156, 417
  amplification of, 43
  Cauchy's Theorem, 417
  Cauchy–Riemann equations, 397
  definition of, 43
  of z^2, 43
  of z^m, 44, 90
  twist of, 43
analytic mapping, 42
angle of parallelism, 60
angular excess
  definition of, 8
  equivalence to holonomy, 247
  in a 3-manifold, 282
  is additive, 23
  of polygon, 19, 26, 83
  of quadrilateral, 26
  of spherical polygon, 147
  proportional to area, 15
angular momentum, 128
  and geodesics in $\mathbb{H}^2$, 129
anti-self-dual 2-form, 470
anticonformal
  definition of, 39
antipodal points, 49, 83
antipodal triangle, 8
Apollo 15, 308
Apollonius, xviii
Archimedes, 86
Archimedes–Lambert projection, 85
area of surface element, 38
Arnol'd
  and Newton's *Principia*, xviii
  as champion of Newton's geometrical method, xx
  Newtonian problem, 24
asymptotic, 60
asymptotic direction, 162
  at parabolic point, 164
  effect of Shape Operator on, 163
  generalized definition, 164
Atiyah
  The Faustian Offer, xvii
  Atiyah–Singer Index Theorem, 165
Attitude Matrix, 435–438

Bach, BWV 101, xxv
Baker, Michael, xxvii
banana skin
  spherical image under deformation, 139
  Theorema Egregium illustrated with, 140
banana, fried
  Honey-Flow on, 202
  topographic map of, 203
Banchoff, Thomas
  and *Theorema Egregium*, 252
  and Polyhedral *Theorema Egregium*, 148
  author's debt to, xxv
  Global Gauss–Bonnet Theorem via pancakes, 171
Barish, Barry C., 231
Beautiful Theorem, 139
  geometric proof of, 255
  predates *Theorema Egregium* by eleven years, 253
Beethoven
  death of, 138
  Grosse Fuge, xxv
Beltrami
  conformal metric of $\mathbb{H}^3$ and $\mathbb{H}^n$, 79
  curvature at hyperbolic point, 164
  discovered Beltrami–Poincaré disc model, 62
  discovered Beltrami–Poincaré half-plane, 57
  discovered SVD, 154
  discovery of concrete interpretation of Hyperbolic Geometry, 5, 51
  explanation of horosphere, 82
  portrait of, 51
  pseudospherical surfaces, 22
Beltrami–Laplace equation, 84
Beltrami–Poincaré disc, 63
  apparent size of objects within, 63
  conformal mapping to half-plane, 70
  curvature of, 90
  geodesics of, 62
  mapping to half-plane, 90
  metric of, 62, 90
Beltrami–Poincaré half-plane, 56
  apparent size of objects within, 57
  conformal mapping to disc, 70
  connection Forms of, 472
  discovery by Beltrami, 57
  geodesics of, 60
  mapping to disc, 90
  semi-circular geodesics within, 57
Berger, Marcel, 131
  on known proofs of *Theorema Egregium*, 252
  on new proof of *Theorema Egregium*, 252
Bernoulli, Johann
  Brachistochrone Problem, 58
  geodesics, 119
Berry phase, 246
Berry, Sir Michael, 246, 478
Bianchi, 298
Bianchi Identities, 295
  Algebraic (First), 295, 337
  Differential (Second), 295, 298, 327, 332, 340, 413
  via Forms, 459–460
binormal, 107
  only spins, 219
black hole, 329
  as solution of Einstein's Vacuum Field Equation, 460–464
  Birkhoff's Theorem, 329
  Cosmic Censorship Hypothesis, 464
  curvature of, 460–464
  event horizon, 329, 330, 464
  formation of, 329–331
  Kerr solution, 322
  maximum survival time within, 331
  Sagittarius A*, 331, 464
  Schwarzschild radius, 321, 329
  Schwarzschild Solution, 320
  spacetime depiction of birth, 330
  spacetime singularity, 330
  spaghettification, 330
  supermassive, 331
  Tolman–Oppenheimer–Volkoff limit, 329
  versus naked singularity, 464
Blaschke
  metric notation, 37
Bolyai
  co-discovered horosphere, 82
  discovery of Hyperbolic Geometry, 4
  initally sought 3-dimensional non-Euclidean Geometry, 79
  treatment of by Gauss, 7
Bolyai–Lobachevsky Formula, 61, 62
Bonnet
  Local Gauss–Bonnet Theorem, 174
  unaware of Global Gauss–Bonnet Theorem, 174
Bowler, Michael G., xxvii
Boy, Werner, 174
Brachistochrone Problem, 58
Brahe, Tycho, 123
Bramson, Brian D., xxvii
branch points, 170
bridge (between bagels), 171
Brouwer degree, 173
Brouwer, L.E.J., 173
Burnett-Stuart, George, xxiii, xxvii

Carathéodory, 81
Cartan
  "debauch of indices", xvii
  Differential Geometry via Forms, 430–474
  discovery of Forms, xxii
  portrait of, 345

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

Cartan's , 456
Cartan's Razor, 462
Cartan's Structural Equations, 438–445
  First Equation, 439
  Second Equation, 440
catenoid, 480
Catmull, Edwin, xxvi
Cauchy
  polyhedra as hollow surfaces, 186
Cauchy's Theorem, 398, 417
Cauchy–Riemann equations, 85, 397
Celestial Mechanics, 210
celestial sphere, 44, 77
Central Cylindrical Projection, 86
central force field, 123
Central Projection, 32
centre of curvature, 98
Chandrasekhar
  and $\asymp$–notation, xviii
  and Newton's *Principia*, xviii
Chern, 221, 252, 435, 481
  Characteristic Classes, 481
Cicero, 86
circle of curvature, 98
circulation, 404
circulation of vector field, 263
Clairaut's Theorem
  angular momentum explanation of, 128
  dynamical proof of, 126–128
  experimental investigation of, 27
  on the sphere, 121
Clairaut's theorem, 122
Clairaut, Alexis Claude, 123
CMC surfaces, 480
Codazzi–Mainardi Equations, *see* Peterson–Mainardi–Codazzi Equations
commutator, 287–288, 407, 409
complex inversion, 67
  is rotation of Riemann sphere, 69
complex mapping, 41
  amplitwist of, 43
  analytic, 42
  anticonformality, 398
  Cauchy's Theorem, 398, 417
  Cauchy–Riemann equations, 398
  conformality, 41
  Pólya vector field of, 201
cone
  curvature of, 134, 142
  geodesics on, 27
  spike of, 143
conformal
  amplitwist, 43
  Cauchy–Riemann Equations, 397
  coordinates, 40, 84
  curvature formula, 41, 89
  definition of, 39
  map, 39
  map of hyperbolic plane, 56
  map of pseudosphere, 54–55
  mapping between hyperbolic models, 63
  mapping of $\mathbb{C}$, 41
  Mercator projection of sphere, 86
  metric, 39
  property of stereographic projection, 45
  stereographic metric of sphere, 47
conformal disc model, 63
conjugate point, 269, 272
connection 1-forms, 432–435
  definition of, 433
  differing conventions, 433
conservation of energy, 327, 340, 406
convex, 186
coordinates
  breakdown of, 37
  conformal, 40, 84
  homogeneous, 70
  isothermal, 40, 84
  orthogonal, 37
  projective, 70
Cosmic Censorship Hypothesis, 464
cosmological constant, Λ, 332
Costa, Celso, 480
Coxeter, 63
crosspoint, 196
curl (of vector field), 264, 398
Curvatura Integra, 165
curvature
  and exponential operator, 339
  as angular excess per unit area, 18
  as attractive force, 272
  as rate of turning, 101
  as repulsive force, 272, 274
  as spreading of normals, 131, 132
  as stiffness of spring, 272
  Cartesian formula, 222
  centre of, 98
  circle of, 98
  conformal formula, 41, 89, 472
  controls area of small circle, 20, 278–279
  controls circumference of small circle, 20, 28, 278–279
  definition for plane curve, 98
  dependence of sign on shape, 19
  Euclidean Geometry *characterized* by vanishing of, 472
  Euler's Formula for surfaces, 110
  extrinsic equals intrinsic, 142
  extrinsic formula for, 134
  Gaussian, 17
  geometric formula for, 99
  Hilbert's Theorem on surfaces of constant negative, 22, 52
  holonomy as measure of, 246
  intrinsic, 17
  mean, 110, 130, 222
  metric formula, 38
  metric formula (forms proof of), 451–452
  metric formula (geometric proof of), 261–268
  Minding's Theorem on surfaces of constant, 21
  Newton's formula for, 100
  nonisometric surfaces with equal curvature, 224
  normal, 115
  of n-saddle, 224
  of black hole, 460–464
  of cone, 134, 142
  of conical spike, 143–145
  of cosine graph, 99
  of cylinder, 132, 134, 142
  of hyperbolic point, 164
  of monkey saddle, 224
  of parabola, 99
  of polyhedral spike, 145–147
  of polyhedral spike, defined, 147
  of pseudosphere, 53, 142
  of saddle, 224
  of sphere, 132, 134, 142
  of spike, defined, 144
  of surface of revolution, 114, 142
  of the Beltrami–Poincaré disc, 90
  of the Beltrami–Poincaré half-plane, 251
  of the Universe, 474
  of torus, 21, 90, 142
  of tractrix, 105
  parametric formula, 103
  polar formula, 221, 224
  principal, 109
  radius of, 98
  related to acceleration, 102
  related to force, 98
  Rodrigues's extrinsic formula for, 134
  sign defined by orientation, 135
  sign for plane curves, 99
  simplified formula for unit speed orbit, 104
  surface of revolution, 220, 221
  surfaces of constant, 89
  surfaces of constant negative, 22, 52
  surfaces of constant positive, 21
  Theorema Egregium, *see Theorema Egregium*
  total, 165
  umbilic point, 110
  via Shape Operator, 222
curvature vector, 107
  geodesic curvature component, 117
  normal curvature component, 117
cylinder
  curvature of, 134, 142
  extrinsic curvature, 132

Darboux vector, 219
Darboux, Jean-Gaston, 219, 430
dark energy, 333
de Rham cohomology, 397, 419–429
  Closed 1-Form Deformation Theorem, 423
  Closed 2-Form Deformation Theorem, 427
  cohomologous 1-forms, 423
  cohomology class, 423
  first de Rham cohomology group, 423
  first de Rham cohomology group of torus, 428–429
  inverse-square point source, 424–426
  period of 1-form, 423

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

period of 2-form, 427
second de Rham cohomology group, 426
topological circulation, 421–423
vortex 1-form, 419–421
Dedekind, 298
degree (of mapping), 173, 177–180, 182
Descartes
anticipation of Euler's Polyhedral Formula, 184
determinant, 153
coordinate-independence of, 153
of Shape Operator, 153
Differential Equations
integral curves, 211
Poincaré's Qualitative Theory of, 211
Differential Forms, *see* Forms
Differential Geometry via Forms
1-forms in terms of basis, 447
2-forms in terms of basis, 448
adapted frame field, 446
adapted frame field of sphere, 447
area 2-form, 447
attitude matrix, 435–438
Bianchi Identities, 459–460
black hole curvature, 460–464
Cartan's First Structural Equation, 439, 458
Cartan's Method of Moving Frames, 430–432
Cartan's Razor, 462
Cartan's Second Structural Equation, 440
Cartan's Second Structural Equation (generalized), 456
Cartan's Two Structural Equations, 438–445
Cartan, Élie, 430
Codazzi Equations in geometric form, 450
connection 1-forms, 432–435
differing conventions, 433
connection equations, 434
curvature 2-forms, 455–460
curvature 2-forms (defined), 456
curvature 2-forms (notation), 456
curvature 2-forms (of 2-surface), 473
curvature matrix (of curvature 2-forms), 456
cylindrical frame field, 436–438
Darboux, Jean-Gaston, 430
dual 1-form basis, 438, 439, 430–474
extra structure of orthonormal frame, 431
extrinsic curvature 2-form, 446, 448
First Structural Equations of 2-surface, 448
Frenet Frame, 430
Frenet–Serret Equations, 430
Gauss Equation in geometric form, 450
Gauss Equation of 2-surface, 448
generalized exterior derivative, 457
Hilbert's Lemma, 453–454
Hilbert's Lemma curvature formula, 452
Liebmann's Rigid Sphere Theorem, 454–455
moving frame field, 432
O'Neill championed Forms approach in 1966, 431
Peterson–Mainardi–Codazzi Equations of 2-surface, 448
principal frame field, 449
Riemann tensor via curvature 2-forms, 457, 459
Shape Operator, 446
Shape Operator symmetry, 449
six Fundamental Form equations of a 2-surface, 448
spherical frame field, 441–445
Symmetry Equation (meaning of), 449
Symmetry Equation of 2-surface, 448
dipole, 196, 199
streamlines, 227
Dirac bra-ket, 352
directional derivative, 149
meaning of, 151
discontinuity (of vector field), 195
dodecahedron, 185
tesselation of sphere, 26
Dombrowski, Peter, 138, 252
dual (of vectors), 347
dual polyhedra, 227
Dupin indicatrix, 111, 162, 204
elliptic, 111
hyperbolic, 111
parabolic, 111
Dupin, Charles, 110, 111
Durian, 143
Dyck, Walther, 174, 215

Eddington, Sir Arthur, 322, 325
Einstein
1905 discovery of Special Relativity, 382
1915 discovery of General Relativity, 327
1916 prediction of gravitational waves, 323
1917 introduction of cosmological constant, Λ, 332
aberration formula, 93
and precession of orbit of Mercury, 322
curved spacetime, 82
discovered General Relativity *without* parallel transport, 232
electromagnetic motivation of relativity, 382
final Gravitational Field Equation, 327
first attempt at Gravitational Field Equation, 327
first three predictions of General Relativity, 322
geometric form of field equation, 328
Greatest Blunder, 332
Happiest Thought, 307
ignorance of Bianchi Identity in 1915, 327
need for new theory of gravity, 231
portrait of, 320
prediction of bending of light, 322
Special Theory of Relativity, 74, 75
tensor, 328
Vacuum Field Equation, 319
Einstein summation convention, 292
Einstein tensor, 328
conservation of, 340
Electromagnetism
1-form potential, 402
conservation of electric charge, 470
electric 1-form, 383
electric 2-form, 383
electric field, 383
Faraday 2-form, 383, 402, 469
Faraday's Law of Electromagnetic Induction, 415
Heaviside's simplification of Maxwell's Equations, 382
Hodge duality, 385, 402
magnetic 1-form, 383
magnetic 2-form, 383
magnetic field, 383
Maxwell 2-form, 384, 402, 469
Maxwell's Equations (1873 publication), 382
Maxwell's Equations (source-free), 384
Maxwell's Equations (via Forms), 401–403
Maxwell's Equations (with sources), 384
photons (spinning), 470
self-duality, 470
spacetime source of, 402, 469
vector potential, 402
elliptic point, 111, 134, 136
energy–momentum tensor, 326
Enneper, 164
Equation of Geodesic Deviation, 302
Escher, M. C., 63
Euclid's *Elements*, 4, 185
Euler characteristic
adding handle reduces by two, 191
definition of, 167, 184
determines degree of spherical map, 174
Hopf's calculation of, 227
of polygonal net, 187
related to degree, 217
vanishes for hollow handle, 192
Euler's Polyhedral Formula, 184
Cauchy's proof, 186–188
Euler's own proof, 184
Legendre's proof, 188–190, 226
Platonic solids for the, 185
Euler's Relation (for homogeneous functions), 471

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

Euler, Leonhard
  breakthrough on curvature of surfaces, 109
  curvature formula, 110
  curvature formula (original form), 110
  description of polyhedra, 183
  generosity to young Lagrange, 7
  geodesics, 119
  impossibility of perfect map, 31
  portrait of, 183
  rescued Lambert's career, 8
  rigid motions of the sphere, 73
Euler–L'Huilier Formula, 190
  proof of, 209
event (in spacetime), 74
event horizon, 329, 330, 464
extended complex plane, 44
exterior derivative, *see* Forms
exterior product, 372
extrinsic geometry
  definition of, 11

faithfullness of map, 33
Faltings, Gerd, 24
Faraday
  2-form, 383
  admiration of Maxwell, 382
  first electric generator, 381
  introduced flux concept, 377
  laboratory, 381
  portrait of, 381
Fermat, 58
Fermat's Last Theorem, 6, 58
Fermat's Principle, 58
Feynman
  Newtonian derivation of Snell's Law, 58
  on Newton, 126
  quantum-mechanical explanation of Fermat's Principle, 58
fiducial vector field, 205
field of line elements, 211
fixed points, 77
flux
  as 2-form, 378, 408
  associated vector, 377
  definition of, 378
  introduced by Faraday, 377
  refined by Maxwell, 377
Focus, 196, 200
folding paper, 221
Forms
  1-forms
    "contravariant" explained, 465
    "covariant" explained, 465
    as duals of vectors, 347
    basis, 352–354
    Cartesian basis, 356–357
    components of, 354
    connection 1-forms, 432–435
    contraction of, 347
    definition of, 346–347
    Dirac bras as, 352
    Dirac delta function, 465
    direction (sense, orientation), 349
    dual basis, 352
    electromagnetic potential, 402
    examples of, 347–352
    field of, 347
    geometrical addition of, 357–359
    gradient, 350, 351
    gradient (defined), 355
    gravitational work as, 347–349
    interpretation of $\mathbf{d}f$, 357
    kernel of, 349
    notation, 346
    row vectors as, 352
    stack, 349
    topographic map, 350
    visualized, 349
  2-forms
    anti-self-dual, 470
    area 2-form, 371, 373
    area 2-form (polar), 374–375
    area formula for surface, $f = const.$, 467
    as flux, 378, 398
    associated vector (definition of), 377
    basis, 375–376
    basis (as area projection), 376
    basis (geometric meaning), 377
    definition of, 370
    factorizes in $\mathbb{R}^3$, 466
    Faraday, 383, 402
    flux, 408
    flux as vector, 376–378
    Maxwell, 402
    need not factorize in $\mathbb{R}^4$, 466
    self-dual, 470
    symplectic manifolds, 370
    vector product as wedge product, 379–381
    via wedge product, 372–374
  3-forms
    basis, 390
    factorizes in $\mathbb{R}^4$, 467
    need at least three dimensions, 386
    volume (Cartesian), 387
    volume (spherical polar), 389
  4-forms
    volume, 391
  Bianchi Identities, 459–460
  Cauchy's Theorem, 398, 417
  Cauchy–Riemann Equations, 397
  closed, 419–429, 467, 471
  closed (definition), 396
  de Rham cohomology, 397, 419–429
  discovery by Cartan, 345
  exact, 419–429, 467, 471
  exact (definition), 396
  exterior derivative
    generalized to vectors, 457–460
  exterior derivative ($\mathbf{d}$)
    as integral, 406–411
    closed Forms (definition), 396
    exact Forms (definition), 396
    explanation of $\mathbf{d}^2 = 0$, 413
    Fundamental result: $\mathbf{d}^2 = 0$, 395–396
    Leibniz Rule (Product Rule) for 0-forms, 356
    Leibniz Rule (Product Rule) for p-forms, 394–395
    of 0-form, 355
    of 1-form, 392–394
    of 2-form, 394
    of p-form, 394
    Poincaré Lemma, 396, 418
  exterior product, 372
  gauge freedom, 396
  gauge transformation, 396
  Maxwell's Equations, 401–403
  p-forms
    definition of, 370
    need at least p dimensions, 386
  Poincaré Lemma, 396, 418
  potential, 396
  Vector Calculus
    curl, 398
    divergence, 399
    flux 2-form, 398
    Gibbs and Heaviside, 274
    identities via Forms, 400, 468
    integral theorems, 413–415
    irrotational, 399
    via Forms, 398–400
  vector potential, 402
  wedge product, 372–374
    as vector product, 379–381
    definition of, 372
    geometry of, 374
    of 2-form and 1-form, 387
    of three 1-forms, 390
    vector product formula, 380
    volume formula, 381
Frenet Approximation, 220
Frenet frame, 106
Frenet–Serret Equations, 108, 220, 430, 434
  variable speed, 219
Friedmann, Alexander, 474
Friedmann–Lemaître–Robertson–Walker Universe, 474
Fundamental Forms
  are *not* Differential Forms, 164
  first, 164
  second, 164
  third, 164
fundamental group, 207
Fundamental Theorem of Exterior Calculus, 411–412
  boundary of a boundary is zero: $\partial^2 = 0$, 413
  Cauchy's Theorem, 417
  circulation, 404
  Divergence Theorem, 415
  explanation of $\mathbf{d}^2 = 0$, 413
  exterior derivative as integral, 406–411
  Faraday's Law of Electromagnetic Induction, 415
  flux 2-form, 408
  Gauss's Theorem, 415
  Green's Theorem, 414
  history of, 411–412

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

integral theorems of vector calculus, 413–415
line integral of 1-form, 404–406
line integral of 1-form (path-independence), 405–406
line integral of exact 1-form, 406
named by N.M.J. Woodhouse, 412
Penrose precedent for FTEC terminology, 412
proof of, 415–417
role of Stokes, 411
statement of, 411
Stokes's Theorem, 414
work, 404

Galileo, 308
Gamow, George, 332
gauge freedom, 396
gauge transformation, 396
Gauss
and possibility of absolute unit of length, 15
and the spherical map, 131
Beautiful Theorem, 138
curvature, *see* curvature
discovery of intrinsic geometry, 11, 31
Dombrowski's analysis of, 138
experimental test of the curvature of physical space, 14
first curvature formula, 40
General Investigations of Curved Surfaces, 3, 17
introduction of metric, 31
isometries versus bendings, 141
metric notation, 36, 37
motto, 138
portrait of, 17
reaction to Riemann's ideas, 298
rotations of sphere as Möbius transformations, 73
Theorema Egregium, 138, 140, 142
treatment of Abel, 7
treatment of Bolyai, 7
treatment of Riemann, 297
unaware of Global Gauss–Bonnet Theorem, 174
Gauss map, 131
Gauss's Integral Theorem, *see* Fundamental Theorem of Exterior Calculus
Gauss's Lemma, 274
via computation, 337
visualized, 275
Gauss–Bonnet Theorem
General Local, 336
Global, *see* Global Gauss–Bonnet Theorem
Local, 22, 174, 336
Gaussian curvature, *see* curvature
General Relativity, *see* Gravity
Generalized Stokes's Theorem (GST), *see* Fundamental Theorem of Exterior Calculus (FTEC)
genus, 166
increasing by adding handles, 191
of sphere, 166
of torus, 166
of two-holed doughnut, 166
Genzel, Reinhard, 464
geodesic
as equivalent of straight line, 9
as shortest route, 9
as straightest route, 11, 13, 118
as taut string, 9, 118
equation of, 244
of $\mathbb{H}^3$, 79
of Beltrami–Poincaré disc model, 62
of Beltrami–Poincaré half-plane, 60, 128–129
of cone, 27
of pseudosphere, 28, 89
parallel transport via, 240–241
possible nonuniqueness of, 10
relative acceleration, 270
sticky-tape construction of, 14, 239
vanishing geodesic curvature, 119
via parallel transport, 235–236, 238, 239
geodesic circle, 10
geodesic curvature, 115
extrinsic construction, 120
intrinsic formula for, 244
intrinsic measurement of, 119
on a cone, 334
on a sphere, 334
on touching surfaces, 334
used to construct geodesics, 120
vanishes for geodesics, 119
vector, 117, 243
via intrinsic differentiation, 335
geodesic equation, 244, 286
geodesic polar coordinates, 274–276
geodesic triangle, 10
geometric inversion, 68, 81, 82
geometrization conjecture, 6
geometrized units, 328
GGB, *see* Global Gauss–Bonnet Theorem
Ghez, Andrea, 464
Gibbs, Josiah Willard, 274
gimel (from the Hebrew alphabet), 75
Global Gauss–Bonnet Theorem, 167
discovered by Kronecker and Dyck, 174
for sphere, 168–169
for torus, 90, 169–170
Hopf's intrinsic proof, 258–260
intuitive interpretation of, 167
paradoxes?, 225
some predictions of, 224
via angular excess, 194
via Bagels and Bridges, 171–172
via folded membrane, 182
via thick pancake, 171
via topological degree, 173
via vector fields, 217
Goldbach's Conjecture, 183
Goldbach, Christian, 183
gradient 1-form, 351
gradient vector, 354–355
graph (topological), 186
gravitation, *see* Gravity
gravitational lensing, 325
Gravitational Spinor, *see* Weyl Curvature
gravitational wave astronomy, 325
gravitational waves, 323–326
curvature of, 323
depiction of oscillating tidal forces, 324
details of first detection (14th of September, 2015), 325
Einstein's 1916 prediction of, 231
enormous energy of, 325
field lines of tidal forces, 323
first detection of, 231
harnessed for gravitational wave astronomy, 325
naming convention for, 325
gravity
bending of light, 322
Big Bang, 474
Birkhoff's Theorem, 329
birth of a black hole, 330
black hole, 329
Cosmological Einstein Field Equation, 332, 474
cosmological constant, Λ, 332, 474
curvature of Friedmann–Lemaître–Robertson–Walker Universe, 474
dark energy, 333
eclipses and the tides, 339
Einstein (matter) Field Equation, 327
Einstein (matter) Field Equation (*geometrical* form), 328
Einstein (vacuum) Field Equation, 319
Einstein tensor, 328, 340
Einstein's initial Gravitational Field Equation, 327
eliminated in free fall, 307
energy-momentum tensor, 326
event horizon, 329
geometrical signature of inverse-square attraction, 314
geometrical signature of inverse-square law, 313
geometrized units, 328
gravitational waves, 323–326
gravitational work as 1-form, 347–349
neutron star, 329
neutron star collision, 331
Newton's apple, 307
Newton's explanation of the ocean tides, 311
Newton's Inverse-Square Law, 98, 307–309
precession of orbit of Mercury, 322
redshifting of light, 322
repulsion of negative energy, 333
spacetime singularity, 464
spacetime tidal forces, 317
spherical Schwarzschild field, 320
standard model (cosmological), 474
static Universe, 332
stellar evolution, 329
stress–energy tensor, 326

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

gravity (*continued*)
tidal force field around spherical mass, 309
tidal forces, 308–312
Tolman–Oppenheimer–Volkoff limit, 329
Universe's expansion is accelerating, 332
Weyl curvature, 340, *see* Weyl curvature
great circle, 6
Green's theorem, *see* Fundamental Theorem of Exterior Calculus
Green, George, 411
group, 65
GST, *see* Fundamental Theorem of Exterior Calculus

$\mathbb{H}^2$, *see* hyperbolic plane
Hairy-Ball Theorem, 472
handle (topological), 192
harmonic oscillator, 271
Harriot
angular excess theorem on sphere, 8
discovered conformality of stereographic projection, 46
discovered Snell's Law, 58
portrait of, 8
Heaviside, Oliver, 274, 382
helicoid, 224, 480
helix, 220
Henderson, David W., 252
Hilbert
and Polyhedral *Theorema Egregium*, 148
impossibility of hyperbolic plane within Euclidean space, 82
spherical map, 131
theorem on surfaces of constant negative curvature, 22, 52
worked with Schwarzschild, 319
Hipparchus, 44
Hodge duality, 377, 379, 385, 402, 468
Hodge, Sir W.V.D., 377
Hofstadter, Douglas, xxvi
holonomy
along open curve, 257–258
definition of, 245
equivalence to angular excess, 247
invariance under spherical map, 255
is additive, 248
measures relative acceleration of geodesics, 278
measures total curvature, 246
of geodesic polygon, 247
of geodesic triangle, 247
of the hyperbolic plane, 248–251
on cone and sphere, 335
on the sphere, 245–246
Riemann curvature and, 290
vector, 293
homeomorphic, 166
homeomorphism, 165
homogeneous coordinates, 70
homogeneous function (defined), 471
homogeneous functions, 471
honey-flow, 202
as othogonal trajectories of topographic map, 204
on surface of genus g, 209, 227
precise definition of, 215
reversing flow preserves indices, 217
sign of index determines orientation of covering by spherical map, 216, 217
singular points related to spherical map, 215, 216
Hooke's Law, 123, 271
Hopf
calculation of Euler characteristic, 227
indices (fractional), 213
intrinsic proof of Global Gauss–Bonnet Theorem, 258–260
line field, 211
metric notation, 37
Poincaré–Hopf Theorem, 206, 213
portrait of, 176
spherical map, 131
Umlaufsatz, 176, 225
horosphere, 82
curvature of, 92
metric of, 92
Hubble Constant, 325
Hubble's Law (1929), 332
constant of, 325
Hubble, Edwin
discovered expansion of Universe (1929), 332
discovered galaxies beyond our own (1924), 332
Huygens
finite area of pseudosphere, 88
influence on Newton, xviii
investigated pseudosphere, 22, 53
Hyperbolic Axiom, 5, 52, 60, 63
Hyperbolic Geometry
$\mathcal{K} = -1$, by convention, 56
3-dimensional, 74, 79–82
absolute unit of length, 16
angle of parallelism, 60
angular excess proportional to area, 15
as geometry of surface of constant negative curvature, 51
aymptotics, 60
Beltrami's critical discovery, 51
Bolyai–Lobachevsky Formula, 61, 62
conformal Beltrami–Poincaré disc model of, 63
conformal Beltrami–Poincaré half-plane model of, 56
conformal mapping between models, 63
curvature of $\mathbb{H}^3$, 473
definition of, 5
discovery of, 4
Euclidean and Spherical Geometries subordinate to it, 82
Euclidean illusion for small figures, 16
Euclidean plane within $\mathbb{H}^3$, 82
Euclidean sphere within $\mathbb{H}^3$, 82
horizon, 57, 79
horosphere, 82, 92
isometries of $\mathbb{H}^3$, 81
Lambert as pioneer of, 5, 15
limit rotation, 74
maximum area of a triangle, 15
metric of $\mathbb{H}^3$, 79
Poincaré as prophet of, 5
points at infinity, 57
some properties of, 15–16
sphere in $\mathbb{H}^3$, 92
standardized half-plane metric, 56
ultraparallels, 60
visualizing length within, 57
hyperbolic plane, 5, 52
$\mathcal{K} = -1$, by convention, 56
conformal Beltrami–Poincaré disc model of, 63
conformal Beltrami–Poincaré half-plane model of, 56
definition of, 57
generalized to $\mathbb{H}^3$, 79
geodesics via Clairaut's Theorem, 128–129
hemispherical $\mathbb{H}^2$ in $\mathbb{H}^3$, 80
isometries as Möbius transformations, 73
metric within $\mathbb{H}^3$, 91
pseudosphere as flawed model of, 52
within $\mathbb{H}^3$, 80
hyperbolic point, 111, 134, 136

Ibn Sahl, 58
icosahedron, 185
tesselation of sphere, 26
ideal points, 57
index, 196
as winding number, 198
determined by infinitesimal neighbourhood, 199
formal definition, 197, 205
fractional, 213
illustrated for complex powers, 200
invariance of, 197, 198
of complex powers, 198–201
of dipole, 197, 199
of honey-flow on fried banana, 204
of saddle point, 197
of vortex, 197
on a surface, 204
on a surface (illustrated), 205
index of refraction, 59
integral curve, 195
integration, *see* Fundamental Theorem of Exterior Calculus
intrinsic derivative, 241–244
definition of, 242
formal properties of, 243
geodesic curvature via, 244
of n-manifold, 284–286
Shape Operator formula for, 243
visualized, 242

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

intrinsic geometry
  definition of, 11
inversion, 68
  effect on Riemann sphere, 68
  is anticonformal, 69
  preserves circles, 69
isometries
  definition of, 61
  direct, 65
  group structure of, 65–66
  of $\mathbb{H}^3$, 81
  of flat spacetime, 74–79
  of hyperbolic plane as Möbius transformations, 73
  of Riemann sphere, 73, 90
  opposite, 65
  symmetry groups of surfaces of constant curvature, 72–74
  versus bendings, 141
isothermal coordinates, 40, 84

Jacobi Equation
  computational proof of, 301–302
  defined, 271
  geodesic polar coordinates proof of, 274–276
  holonomy proof of, 276–278
  in n-manifold, 302
  introduced, 269
  negative curvature, 272–274
  on general surface, 336
  positive curvature, 270–272
  Sectional Jacobi Equation, 299–302
  zero curvature, 269–270
Jacobi, Carl Gustav Jacob, 269
Jobs, Steve, xxvi

Kepler, 58, 123
Kepler's Laws, 124
kernel (of 1-form), 349
Kerr solution, 322
Kerr, Roy, 322
Klein
  classification of closed surfaces, 191
  reaction to Riemann's ideas, 298
  worked with Schwarzschild, 319
Kronecker delta, 353
Kronecker, Leopold, 174

Lambert
  and first theorems of Hyperbolic Geometry, 5
  career rescued by Euler, 8
  discovery of hyperbolic equivalent of Harriot's Theorem, 15
  portrait of, 5
  projection of sphere, 85
Laplace Equation, 85
Laplacian, 40
  in polar coordinates, 47
Le Verrier, Urbain, 322
Legendre, 189
Leibniz, xviii
Leibniz Rule, *see* Forms, exterior derivative
Lemaître, Georges, 474
level curves, 203
Levi-Civita connection, 242
Levi-Civita, Tullio, 231
  discoverer of parallel transport, 232
  portrait of, 232
Levy, Anthony, xxvii
L'Huilier, Simon-Antoine-Jean, 190, 226
Lie Bracket, 287
Lie, Sophus, 287
Liebmann, Heinrich
  equivalence of Möbius and Lorentz transformations, 74
  sphere rigidity theorem, 21, 454–455
light cone
  picture of, 316
LIGO detector, 231
limit rotation, 74
line field, 211
  applications to physics, 214
  as flow around barriers, 213
  fractional indices, 212
  Poincaré–Hopf Theorem applies to, 213
  singular points of, 212
Linear Algebra (Visual), 221
Listing, 166
Lobachevsky, 4
  initally sought 3-dimensional non-Euclidean Geometry, 79
  named horosphere, 82
loop, 175
Lorentz
  contraction, 75
  Special Theory of Relativity, 75
  symmetries of spacetime as Möbius transformations, 74–79
Lorentz transformations, 74–79
  as spin-matrices, 77
  as spin-transformations, 77
  boost, 78, 93
  classification of, 78
  fixed null rays of, 77
  four archetypes, 77
  four-screw, 78
  as Möbius transformations, 77
  null rotation, 79
  preservation of spacetime interval, 77
loxodrome, 87
lune, 168

Mainardi–Codazzi Equations, *see* Peterson–Mainardi–Codazzi Equations
manifold, 280
  n-dimensional, 280
  3-manifold as exemplar of n-manifold, 281
  angular excess in a 3-manifold, 282
  pseudo-Riemannian, 280
  Riemannian, 280
map
  conformal, 39
  definition of, 31
  element of area, 38
  faithfulness of, 33
  of hyperbolic plane, *see* hyperbolic plane
  of pseudosphere, 55
  of sphere, *see* sphere
Maxwell
  admiration of Faraday, 382
  and Polyhedral *Theorema Egregium*, 148
  portrait of, 382
  refined flux concept, 377
  Smith's Prize examination, 411
Maxwell's Equations, *see* electromagnetism
  via Forms, 401–403
mean curvature, 110
Mercator projection, 86
  loxodrome, 87
  rhumb line, 87
meridian, 121
Merton College, Oxford, xviii, xxvii
metric
  conformal, 39
  conformal stereographic, 47
  curvature formula, 38, 261
  definition of, 31
  formula in orthogonal coordinates, 37
  Gauss's notation differs from ours, 37
  general formula for 2-surface, 36
  in orthogonal coordinates, 262
  of $\mathbb{H}^2$ in $\mathbb{H}^3$, 91
  of $\mathbb{H}^3$, 79
  of Beltrami–Poincaré disc model, 62, 90
  of Beltrami–Poincaré half-plane model, 56
  of conformal map of pseudosphere, 55
  of hemispherical $\mathbb{H}^2$ in $\mathbb{H}^3$, 81
  of horosphere, 92
  of pseudosphere, 53
  of pseudosphere via parameterized tractrix, 88
  of torus, 90
  spacetime interval, 75
  tensor, 363–364
  used to change tensor valence, 366–368
Meusnier's Theorem, 117
Meusnier, Jean-Baptiste, 110
Minding's Theorem, 21, 276, 336
Minding, Ferdinand, 53
minimal surface, 130
Minkowski, 382
  spacetime geometry, 74
  spacetime interval, 75
  worked with Schwarzschild, 319
miracles
  false, xxiii
  true, xxiii
Misner, Charles, 475

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

Möbius
band, 166
classification of closed surfaces, 166
strip, 166
Möbius transformations, 67–74
as isometries of $\mathbb{H}^3$, 81
as Lorentz transformations, 77
as rotations of the Riemann sphere, 73, 90
as symmetry group of flat spacetime, 74–79
characterized by circle-preserving property, 81
classification of, 78
decomposition into simpler transformations, 67
elliptic, 78
explanation of matrix correspondence, 71
fixed points of, 77
form a group, 71
four achetypal Lorentz transformations, 77
hyperbolic, 78
inversion in a circle, 67
isometries of Riemann sphere, 73
loxodromic, 78
matrix representation of, 70–71
nonsingular, 67
normalized, 71
normalized matrix defined uniquely up to sign, 71
parabolic, 79
rotations of sphere via antipodal points, 73
singular, 67
symmetry groups of surfaces of constant curvature, 66, 72–74
moment of force, 128
monkey saddle, 136
complex equation, 137
curvature of, 224
generalized, 137
Morgan, Frank, xxvi
Riemannian Geometry, 481

naked singularity, 464
Needham, Faith, xxviii
Needham, Guy, xxvi
Needham, Hope, xxviii
Needham, Mary, xxvii
Nel, Stanley, xxvi
nematic field, 478
network (topological), 186
neutron star, 329, 331
Newcomb, Simon, 322
Newton
and Celestial Mechanics, 210
and the falling apple, 307
elliptical orbits in linear field, 123
embraced geometrical methods in 1680s, xviii
explained Kepler's Laws, 124
explained the ocean tides, 311
general curvature formula, 100
geometric curvature formula, 99
geometric definition of tractrix, 52
investigated tractrix, 22
Law of Gravitation, 308
parametric curvature formula, 103
Principia, xviii–xx
Principia, area is the clock, 124
Principia, Lemma II, 99
Principia, Proposition 1, 124
Principia, Proposition 2, 126
Principia, Proposition 10, 123
Principia Proposition 31, 309
priority battle with Leibniz, xviii
proof of Kepler's Second Law, 124–126
Second Law of Motion, 97
shunned his 1665 calculus in the *Principia*, xviii
synthetic method of fluxions, xviii
Westfall's biography of, xviii
Ultimate Equality, *see* Ultimate Equality
Newtonmas, 98
Nobel Prize (2011), 332
Nobel Prize (2017), 231
Nobel Prize (2020), 464
non-Euclidean Geometry, *see* Hyperbolic Geometry
nonorientable, 166
normal curvature, 115
normal curvature vector, 243
normal map, 131
null cone
picture of, 316
Riemann sphere representation of, 76
null vector, 76
Nye, J. F., 478

O'Neill, Barrett
championed Forms approach to Differential Geometry, 431, 475
championed Shape Operator, 164
octahedron, 185, 227
optics, 58
orientable surface, 165
orthogonal coordinates, 37
orthogonal linear transformation, 222
osculating plane, 106

Pappus, xviii
centroid theorem, 90
parabolic point, 111, 134, 136
Parallel Axiom
definition of, 4
via angle sum of a triangle, 4
via similar triangles, 4, 15
parallel transport
discovery of, 232
geodesics and, 235–236
in n-manifold via constant-angle cone, 282–283
in n-manifold via parallel-transported plane, 283–284
in n-manifold via Schild's Ladder, 284
preserved by spherical map, 255
used to define intrinsic derivative, 241–244
via geodesics, 240–241
via potato-peeler, 236–239
via projection into surface, 233–235
parallelepiped volume, 381
Pauli Exclusion Principle, 329
Penrose
and *Theorema Egregium*, 252
author's debt to, xxv
Cosmic Censorship Hypothesis, 464
Escher diagram, 63
labelling of light rays with complex numbers, 92
Nobel Prize for Physics, 464
on Lambert, 5
pioneering use of 2-spinors, 70
studied under Hodge, 377
Perelman, 6
Perlmutter, Saul, 332
Peterson–Mainardi–Codazzi Equations, 448
geometric meaning of, 450
use in Hilbert's Lemma, 453
phase portrait, 195
photons (spinning), 470
Pixar, xxvi
planar point, 136
surrounded by negative curvature, 137
surrounded by positive curvature, 137
Plato, 185
Platonic Forms, xxiii
Platonic solids, 185, 227
topologically determined, 186, 226
uniqueness of, 226
Poincaré, *see* Beltrami–Poincaré
half-plane
and celestial mechanics, 210
and Euler characteristic, 184
as father of topology, 165
as prophet of Hyperbolic Geometry, 5
discovery of Möbius isometries of $\mathbb{H}^3$, 81
rediscovery of hyperbolic disc model, 62
rediscovery of hyperbolic half-plane model, 57
Special Theory of Relativity, 75
Poincaré Conjecture, 6
Poincaré disc, *see* Beltrami–Poincaré disc
Poincaré Lemma, 396, 418
Poincaré–Hopf Theorem, 206
also applies to line fields, 213
Hairy Ball Theorem, 472
on sphere, 206, 213
on torus, 207
physical applications of, 477
proof of, 207–208
point at infinity, 44
polar decomposition, 222

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

Pólya, 201
  mechanical proof of Snell's Law, 87
Pólya vector field, 201
  divergence-free and curl-free, 324
  physical examples of, 201
polygonal net, 186
polyhedral spike, 145
  curvature of, 146, 147
  spherical image of, 147
Polyhedral *Theorema Egregium*, 147
  attributed to Hilbert, 148
  discovered by Maxwell, 148
  visualization of, 146
polyhedron
  and Theorema Egregium, 147
  curvature of, 145–147
  dual, 227
positive definite, 222
positive semidefinite, 222
potential, 396
Pound, Robert, 322
Prime Meridian, 274
principal curvatures, 109
  as eigenvalues of Shape Operator, 152
  extrinsic curvature in terms of, 134
  Unit Speed Formula, 113, 114
principal directions, 109
  as eigenvectors of Shape Operator, 152
  as symmetry directions, 111, 133
  rate of rotation of normal along, 134
principal normal, 107
principal radii of curvature
  of pseudosphere, 105
  of surface of revolution, 112
  of torus, 112
Principia, *see* Newton
projective coordinates, 70
projective geometry, 70
projective map, 32
projective model, 32
pseudosphere, 22
  as flawed model of Hyperbolic Plane, 52
  building your own, 54, 88
  conformal map of, 54–55
  conformal metric of, 55
  constant negative curvature of, 53
  curvature of, 142
  definition of, 53
  element of area, 55
  finite area of, 88
  geodesics of, 28, 89
  geodesics via Clairaut's Theorem, 128–129
  metric of, 53
  principal radii of curvature, 105, 113
  rim, 56
  tractrix, as generator of, 22
  via parameterized tractrix, 88
Ptolemy, 44, 58
punctured plane, 419
Pythagoras, 3
Pythagoras's Theorem
  characterizes flatness, 3
Pythagorean triples
  Babylonian examples, 3
  definition of, 3
  general formula for, 25

radius of curvature, 98
Rebka, Glen, 322
redshift
  gravitational, 322
  of galaxies, 332
refraction, 58
rhumb line, 87
Ricci Calculus, 231
Ricci tensor, 302–306
  definition of, 305
  effect on bundle of geodesics, 305
  geometrical meaning of, 304–306
  notation, 304
  sign conventions, 305
  symmetry of, 305
Ricci, Gregorio, 231, 298
  portrait of, 303
Riemann
  as father of topology, 165
  Darrigol's analysis of, 298
  definition of genus, 166
  discovered Differential Bianchi Identity, 298
  Foundations of Geometry lecture (1854), 297
  French Academy Prize Essay (1861), 298
  Gauss's reaction to lecture by, 298
  Klein's reaction to, 298
  portrait of, 281
  Spivak's analysis of, 298
Riemann curvature operator, 287, 290
  antisymmetry of, 290
Riemann sphere, 44
  as representation of null cone in spacetime, 76
  fixed points under Möbius transformation, 77
  rotated by complex inversion, 69
  rotations as Möbius transformations, 73, 90
  used to label light rays, 92
Riemann tensor, 281
  and exponential operator, 339
  and vector commutator, 287–288
  antisymmetry of, 290
  changing valence of, 367
  components of, 292–293
  defined, 290
  defined via parallel transport, 286–287
  different notational conventions, 290
  history of, 297–298
  is a tensor, 291–292
  number of components, 281, 338
  Riemann curvature operator, 287
  standard definition of, 360
  symmetries of, 294–295, 337
  vector holonomy, 293
  visualization of, 289
  Weyl curvature, *see* Weyl curvature
Riess, Adam G., 332
Robertson, 474
Rodrigues, Olinde, 17
  discovered extrinsic curvature formula, 134
  introduced spherical map, 131
Rodrigues–Gauss map, 131
rotation matrix, 152
Royal Institution of London, 381
Royal Observatory at Greenwich, 274

Saccheri, 5
Saddle Point, 196
Sagittarius A*, 331, 464
Schild's Ladder, 284
Schild, Alfred, 284
Schmidt, Brian P., 332
Schutz, Bernard, 325
Schwarzschild radius, 321, 329
Schwarzschild Solution, 320
  interior, 321
Schwarzschild, Karl, 319, 321
  portrait of, 321
Second Fundamental Form, 151
sectional curvature, 282, 296–297
Segerman, Henry, 191
self-adjoint matrix, 153
self-dual 2-form, 470
Shape Operator, 151
  also called the Second Fundamental Form, 164
  Cartesian formula, 222
  curvature formula, 222
  curvature interpretation of, 159
  curvature interpretation visualized, 160
  determinant of, 153
  determined by three normal curvatures, 161
  diagonalized matrix of, 153
  effect on asymptotic directions, 163
  eigenvectors and eigenvalues of, 152, 223
  general matrix of, 158, 161
  geometric meaning of components, 160
  is linear, 152
  is symmetric, 153
  matrix representation of, 152
  of saddle, 223
  visualization of, 150
Singer
  Atiyah–Singer Index Theorem, 165
  spherical map, 131
singular point, 195
singular value decomposition
  discovered by Beltrami, 154
  geometric derivation of, 154–156
  in $\mathbb{R}^3$, 222
  matrix form of, 155
  singular values of, 154
  statement of, 154
  twist of, 154
  visualization of, 155
Sink, 200
skew symmetric, 222

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

Snell, 58
Snell's Law, 58
  Fermat's two proofs of, 58
  Generalized, 59
  mechanical proof of, 87
  Newtonian proof of, 58
  Ptolemy's experiments, 58
soap films, 130
Source, 200
  physics of, 201
spacetime
  4-velocity, 316
  4-vector, 74
  absolute, observer-independent structure of, 75
  causal structure of, 316
  diagrams, 315–316
  event, 74, 316
  event horizon, 330
  Gravitational Field Equation, 319
  interval, 75
  light cone, 316
  Lorentz transformation, 76
  metric, 314–315
  metric tensor, 314
  Minkowski, 74
  null cone, 316
  null vector, 315
  singularity of, 330, 464
  spacelike vector, 315
  spherical Schwarzschild geometry, 320
  symmetries as Möbius Transformations, 74–79
  tetrad, 315
  tidal forces, 317
  timelike vector, 315
  Weyl curvature, 319
  Weyl curvature formula, 340
  world-line, 315
spacetime interval, 75
  preservation by Lorentz transformation, 77
  vivid interpretation of, 75
spaghettification, 330
Special Theory of Relativity, 74, 382
  aberration formula, 93
  discrepancy between clocks in relative motion, 75
  Einstein, 75
  Einstein's 1905 paper, 382
  Lorentz, 75
  Lorentz transformation, 76
  Minkowski, 74
  Poincaré, 75
  spacetime interval, 75
Spectral Theorem, 157
sphere
  Archimedes–Lambert metric, 86
  Archimedes–Lambert projection, 85
  area in cylindrical polar coordinates, 88
  area of small circle on, 20
  central cylindrical projection, 86
  central projection, 32, 83
  central-cylindrical metric, 86
  circumference of small circle on, 20, 28
  Clairaut's theorem, 121
  curvature of, 134, 142
  extrinsic curvature, 132
  genus of, 166
  Global Gauss–Bonnet Theorem for, 168–169
  holonomy on, 245–246
  in $\mathbb{H}^3$, 92
  isometries as Möbius Transformations, 73
  Liebmann's rigidity theorem, 21
  longitude-latitude metric formula, 34
  lune of, 168
  Mercator projection, 86
  other surfaces with same intrinsic geometry, 21
  preservation of antipodal points used to derive rotations of, 73
  projective map, 32
  projective metric formula, 33
  projective model, 32
  Riemann, 44
  rigidity of, 454–455
  stereographic metric, 47, 83
  stereographic projection of, 44
  tessellation of, 26
  vector fields on, 206
Spherical Geometry
  absolute unit of length, 15
  Euclidean illusion for small figures, 16
  geodesics are great circles, 6
  Harriot's angular excess theorem, 8
  perpendicular bisector of, 25
  Spherical Axiom, 6
spherical map, 131
  and orientation, 135
  covers closed surface, 214
  curvature as local expansion of area by, 132
  degree determined by Euler characteristic, 174, 217
  degree of, 180
  folds of image, 182
  "Gauss map", 131
  generalized, 144
  index of, 178
  negative covering by, 173
  of bridged bagels, 174
  of thick pancake, 173, 174
  positive covering by, 173
  precedence for terminology, 131
  preserves holonomy, 255
  preserves parallel transport, 255
  related to honey-flow, 215
  "Rodrigues–Gauss map", 131
  topological degree of, 173
spin-matrix, 77
spin-transformation, 77
2-spinor, 70
spinorial objects, 71
stable node, 200
Star Trek
  Captain Kirk, 191, 482
  Dr. McCoy, 38
  forms proof of ("*Star Trek* phaser") metric curvature formula, 452
  geometric proof of ("*Star Trek* phaser") metric curvature formula, 266
  Mr. Spock, 191, 476, 482
  NCC-1701, 435
  ("*Star Trek* phaser") metric curvature formula, 38
  The City on the Edge of Forever, 38
  The Doomsday Machine, 191
stereographic projection, 44
  conformality, 46
  formulas, 47–49
  image of line, 45
  image of point, 44
  preservation of metric under Möbius transformation, 73
  preserves circles, 49
  used to identify light rays with complex numbers, 77
Stiefel vector field, 208, 227
Stiefel, Eduard, 208
stiffness, 271
Stillwell, xxvii
  Mathematics and Its History, xxi, xxvii
  transformation of Escher diagram, xxvii, 63, 64
Stillwell, Elaine, xxvii
Stokes's Theorem, *see* Fundamental Theorem of Exterior Calculus
streamline, 195
stress-energy tensor, 326
surface of revolution
  Clairaut's Theorem, *see* Clairaut's Theorem
  curvature of, 114, 142, 220, 221
  meridian of, 121
  normal of, 168
  of constant curvature, 89
  of constant positive curvature, 21
  of tractrix is pseudosphere, 22
  parallel of, 123
  principal radii of curvature, 112
  total curvature, 169
surfaces of constant mean curvature, 480
SVD, *see* singular value decomposition
symmetric matrix, 153
  geometric meaning of, 156

Tabachnikov, Sergei, 58
Tensor Calculus, 231
tensors, 360–369
  addition of, 361
  and Linear Algebra, 361–362, 364–365
  antisymmetric, 369, 370
  basis tensors, 362
  changing valence, 366–368
  components of, 362–363
  contraction, 365–366
  contravariant indices, 363
  covariant indices, 363
  definition of, 360

For general queries, contact webmaster@press.princeton.edu

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

Einstein tensor, 328, 340
index lowering, 367
index raising, 367
matrix multiplication, 466
metric, 363–364
preliminary definition, 291
product of, 361
Riemann, *see* Riemann tensor
skew symmetric, 369
symmetric, 369
valence of, 360
tessellation
of the plane, 25
of the sphere, 26
tetrad, 315
Thales, 3
Theaetetus, 185
Theorema Egregium, 138, 140
Beautiful Theorem as precursor of, 138
bending vs isometry, 141
confirmed for specific surfaces, 142
equality of extrinsic and intrinsic curvature, 142
experimental test of, 141
geometric proof of, 255
illustrated with banana skin, 140
polyhedral version of, 147
proved using Forms, 451–452
simple geometric proof of, 252–256
statement of, 140
Thomson, William (Lord Kelvin), 148, 411
Thorne, Kip S., 231, 475
Thurston, 6, 477
tidal forces, 308–312
around spherical mass, 309
field lines of a gravitational wave, 323
spacetime depiction of, 317
Weyl curvature, 319
Weyl curvature formula, 340
Tolman–Oppenheimer–Volkoff limit, 329
topographic map, 203, 349, 350
preservation of orthogonality, 204
topological defect, 478
topological degree, 173, 215
topological joke, 191
topological mapping, 165
topological transformation, 165
topologically equivalent, 166
topology, 165
Toponogov, V. A.
spherical map, 131
torsion, 106, 107, 164
curvature in terms of, 164
torus, 18, 89
curvature of, 21, 90, 142
genus of, 166
Global Gauss–Bonnet Theorem for, 169–170
metric of, 90
principal radii of curvature, 112
total curvature of, 90
vector fields on, 207
with g handles, 191
total curvature, 165
TOV, *see* Tolman–Oppenheimer–Volkoff limit
trace (of energy-momentum), 327
trace of matrix, 161
invariance of, 162
tractrix
as generator of pseudosphere, 22
as geodesic of pseudosphere, 28
curvature of, 105
equation of, 52
mechanical definition of, 52
Newton's geometric definition of, 52
parametric description of, 88
tractroid, 22
transpose of matrix, 153
composed with original matrix, 158
geometric meaning of, 156
geometric visualization of, 157
has same determinant, 157
triangulation, 187
twist, 43, 154

Ultimate Equality
$\asymp$–notation, xviii
adoption of $\asymp$–notation by Chandrasekhar, xviii
basic properties of, xix
definition, xviii
Newton's use of in the *Principia*, xviii–xx
of vectors, 151
practice exercises using, 24
used to calculate $(\sin\theta)'$ and $(\cos\theta)'$, 24
used to calculate $(\tan\theta)'$, xix
used to calculate $(x^3)'$, 24
ultra-parallel, 60
umbilic, 110
Umlaufsatz, 176
proof of, 225
unitary, 72
universal cover, 56
Unstable Node, 200

valence (of tensor), 360
VCA, *see* Visual Complex Analysis, in bibliography
Vector Calculus, *see* Forms: Vector Calculus
vector field, 195
and differential equations, 211
archetypal singular points, 198–201
centre, 200
circulation of, 263
commutator, 287–288, 407, 409
conservative, 399
crosspoint, 196
curl of, 264, 398
dipole, 196, 199, 200
divergence, 399
divergence-free, 324
double saddle point, 200
fiducial, 205
focus, 196, 200
generalization to line field, 211
Honey-Flow, 202
index, 196–198
index on surface, 205
integral curve, 195
irrotational, 399
on S_g with $-\chi$ saddle points, 227
on S_g with one singular point, 227
on a surface, 202
on sphere, 206
on torus, 207
phase portrait, 195
quadrupole, 200
reference, 205
saddle point, 196, 200
singular point, 195
singularity, 419
sink, 200
source, 200
stable node, 200
Stiefel, 208, 227
streamline, 195
unstable node, 200
vortex, 196, 200
vorticity, 399
with barriers, 213
vector potential, 402
vector product (as wedge product), 380
vector space (defined), 347
vortex, 196, 200
Voss, Aurel, 298

Wachter, 82
Walker, 474
Wallis, 4, 15
Ward, Richard S., xxvii
Weber, Wilhelm, 298
Weingarten Map, 151
Weiss, Rainer, 231
Weyl curvature, 319
conformal invariance of, 341
explicit formula for, 340
Penrose's "fingerprint" of, 478
requires $n \geqslant 4$, 281
Wheeler
General Relativity in one sentence, 307
impact on the author's life, xviii, 482
Wiles, 6
winding number, 198, 422
Wolf, Robert, 221
Woodhouse, N.M.J., 412
work, 404
wormhole, 172, 225
puzzle, 172, 225

Zeitz, Paul, xxvi
"Zeno's Revenge", 57

For general queries, contact webmaster@press.princeton.edu