Contents

Chapter 1

ACCUMULATION

This chapter will follow the development of the most intuitive of the big ideas of calculus, that of accumulation. We begin with the discovery of formulas for areas and volumes by the Greek philosophers Antiphon, Democritus, Euclid, Archimedes, and Pappus. This leads to the development of formulas for volumes of revolution by al-Khwarizmi, Kepler, and a host of seventeenth-century philosophers. We then move back to the fourteenth century to the application of accumulation for finding distance when the velocity is known, sketching the contributions of the Mertonian scholars and Nicole Oresme. Back in the seventeenth century, we will share in the amazement that came with the discovery of objects of infinite length yet finite volume, we will see how to turn arc lengths into areas, and we will conclude with the uses that Galileo and Newton made of accumulation to solve the greatest scientific mystery of the age: how it is possible for the earth to travel through space at incredible speeds without our experiencing the least sense of its motion.

1.1

Archimedes and the Volume of the Sphere

In 1906, Johan Ludwig Heiberg discovered a previously unknown work of Archimedes, *The Method of Mechanical Theorems*, within a thirteenth-century prayer book. The Archimedean text, which had been copied from an earlier manuscript sometime in the tenth century, had been scraped off the vellum pages so that they could be reused. Fortunately, much of the original text was still decipherable. What was readable was published in the following decade. In 1998, an anonymous collector purchased the text for two million dollars and handed it over to the Walters Art Museum in Baltimore, which has since supervised its preservation and restoration as well as its decipherment using modern scientific tools.

Archimedes wrote the *Method*, as this book has come to be known, for his contemporary and colleague Eratosthenes. In it, he explained his methods for computing areas, volumes, and moments. This text lays out the core ideas of integral calculus, including the use of infinitesimals, a technique that Archimedes hid when he wrote his formal proofs. A 2003 *NOVA* program about this manuscript claimed that

> this is a book that could have changed the history of the world. . . . If his secrets had not been hidden for so long, the world today could be a very different place. . . . We could have been on Mars today. We could have accomplished all of the things that people are predicting for a century from now. (*NOVA*, 2003)

The implication is that if the world had not lost Archimedes' *Method* for those centuries, calculus would have been developed long before. That is nonsense. As we shall see, Archimedes' other works were perfectly sufficient to lead the way toward the development of calculus. The delay was not caused by an incomplete understanding of Archimedes' methods but by the need to develop other mathematical tools. In particular, scholars needed the modern symbolic language of algebra and its application to curves before they could make substantial progress toward calculus as we know it. The development of this language and its application to analytic geometry would not be accomplished until the early seventeenth century. Even then, it took several decades to transform the "method of exhaustion" into algebraic techniques for computing areas and volumes. The work of Eudoxus, Euclid, and Archimedes was essential in the development of calculus, but not all of it was necessary, and it was far from sufficient.

Archimedes of Syracuse (circa 287–212 BCE) was the great master of areas and volumes. Although we cannot be certain of the year of his birth, the year of his death is all too sure. Sicily had allied with Carthage during the Second Punic War (218–201 BCE), the war that saw Hannibal cross the Alps with his elephants to attack Rome. The Roman general Marcellus laid a two-year siege on Syracuse, then the capital of Sicily. Archimedes was a master engineer who helped defend the city with weapons he

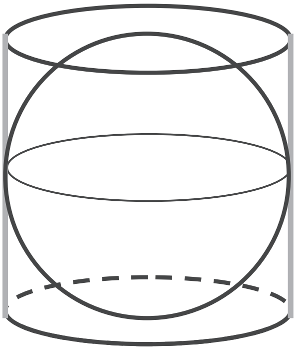

Figure 1.1. Sphere with the smallest cylinder that contains it.

invented: grappling hooks, catapults, and perhaps even mirrors to concentrate the sun's rays to burn Roman ships. Archimedes died during the sacking of the city when the Romans finally broke through the defenses. There is a story, possibly apocryphal, that General Marcellus tried to bring him to safety, but Archimedes was too engrossed in his mathematical calculations to follow.

Of his many accomplishments, Archimedes considered his greatest to be the formula for spherical volume—namely that the volume of a sphere is equal to two-thirds of the volume of the smallest cylinder that contains the sphere (see Figure 1.1). Archimedes valued this discovery so highly that he had a sphere embedded in a cylinder and the ratio 2:3 carved as his funeral monument, an object that still existed over a hundred years later when Cicero visited Syracuse.[1] To see why this gives us the usual formula for the volume of a sphere, let r be its radius. The smallest cylinder containing this sphere has a circular base of radius r and height $2r$, so its volume is

$$\text{volume of cylinder} = \pi(\text{Radius})^2(\text{Height}) = \pi r^2 \cdot 2r = 2\pi r^3.$$

Two-thirds of this is $(4/3)\pi r^3$, the volume of a sphere.

As Archimedes explained to Eratosthenes (with some elaboration on my part), he thought of the sphere as formed by rotating a circle around its diameter and imagined its volume as composed of thin slices perpendicular to the diameter. He began with a circle of diameter AB (Figure 1.2). Let X denote a point on this diameter and consider the perpendicular from X to the point C on the circle. If we rotate the area within the circle around the diameter AB, the thin slice perpendicular to the diameter at X is a disc of

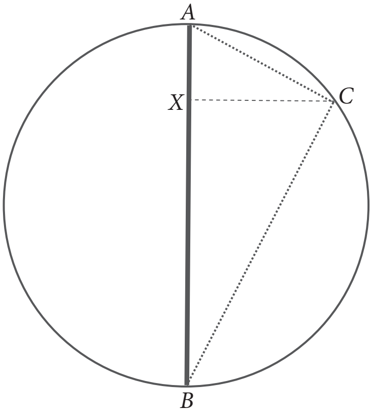

Figure 1.2. Circle with diameter *AB*.

area $\pi\overline{XC}^2$ and infinitesimal thickness ΔX. We represent the sum of the volumes of all of these discs as

$$\text{Volume of Sphere} = \sum \pi\overline{XC}^2\,\Delta X.$$

Now Archimedes relied on some simple geometry. By the Pythagorean theorem, $\overline{XC}^2 = \overline{AC}^2 - \overline{AX}^2$. Because the angle $\angle ACB$ is a right angle, triangles *AXC* and *ACB* are similar. We obtain

$$\frac{\overline{AX}}{\overline{AC}} = \frac{\overline{AC}}{\overline{AB}}, \quad \text{or} \quad \overline{AC}^2 = \overline{AX}\cdot\overline{AB}.$$

Putting these together yields

$$\begin{aligned}\text{Volume of Sphere} &= \sum \pi\overline{XC}^2\,\Delta X\\ &= \sum \pi\overline{AC}^2\,\Delta X - \sum \pi\overline{AX}^2\,\Delta X\\ &= \sum \pi\overline{AX}\cdot\overline{AB}\,\Delta X - \sum \pi\overline{AX}^2\,\Delta X.\end{aligned}$$

The second summation is the volume of a cone. If we take our same diameter *AB* and at point *X* go out to a point *D* for which $\overline{AX} = \overline{AD}$, we get an isosceles right triangle (Figure 1.3). When we rotate that triangle around the axis *AB*, we get a cone of height $\overline{AB}$ with a base of radius

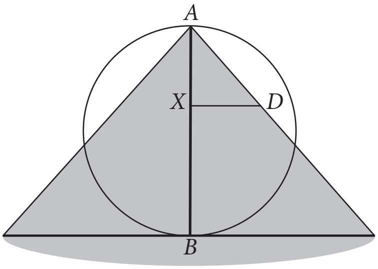

Figure 1.3. Circle with isosceles right triangle.

$\overline{AB}$. Its volume is equal to $\frac{1}{3}\pi\overline{AB}^3$ or, as Archimedes would have understood it, as $\frac{4}{3}$rds of the volume of the smallest cylinder that contains the sphere, the cylinder of height $\overline{AB}$ and radius $\frac{1}{2}\overline{AB}$. He had now established that

$$\text{Volume of Sphere} + \frac{4}{3}\text{Volume of Cylinder} = \sum \pi\overline{AX}\cdot\overline{AB}\,\Delta X.$$

The summation on the right-hand side is problematic as it stands. Archimedes neatly finished his derivation by considering moments. One use of moments is to determine balance. The moment is the product of mass and the distance from the pivot. Two objects of different masses on a seesaw can be in balance if their moments are equal, or, equivalently, if the ratio of their masses is the reciprocal of the ratio of their distances from the pivot (Figure 1.4). Archimedes was working with volumes, not masses, but if the densities are the same, then the ratio of the volumes equals the ratio of the masses. We take our two volumes on the left side of the equality and multiply them by $\overline{AB}$, effectively placing them at distance $\overline{AB}$ to the left of our pivot (Figure 1.5).

Multiplying the right side of our equality by $\overline{AB}$ yields

$$\sum \pi\overline{AX}\cdot\overline{AB}^2\,\Delta X.$$

Now $\pi\overline{AB}^2\,\Delta X$ is the volume of a disc of radius $\overline{AB}$ and thickness ΔX. Multiplying it by $\overline{AX}$ corresponds to the moment of such a disc at distance $\overline{AX}$ from the pivot. Adding up the moments of these discs gives us the moment of a fat cylinder of radius $\overline{AB}$ that rests along the balance beam from the pivot out to distance $\overline{AB}$ (Figure 1.5). Because this is a cylinder of constant

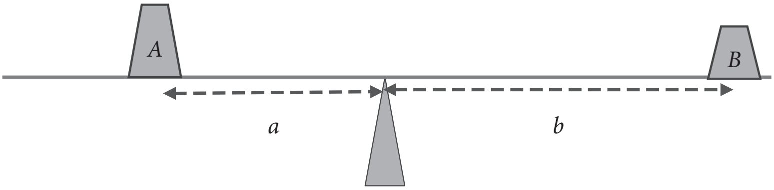

Figure 1.4. Weight A at distance a will balance weight B at distance b if $Aa = Bb$ or, equivalently, if $A/B = b/a$.

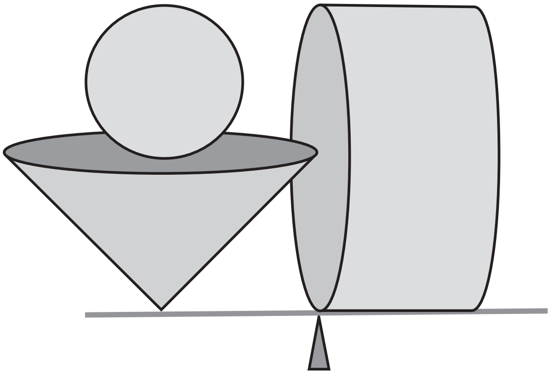

Figure 1.5. The sphere and the cone balance the fat cylinder.

radius, the total moment of all of these discs is the same as the moment were the fat cylinder to be placed at distance $\frac{1}{2}\overline{AB}$ from the pivot. The radius of the fat cylinder is $\overline{AB}$, twice the radius of the smallest cylinder that contains the sphere, so the volume of the fat cylinder is four times the volume of the cylinder that contains the sphere.

Now we can use the fact that the ratio of the volumes equals the ratio of the masses equals the reciprocal of the ratio of the distances from the pivot,

$$\frac{\text{Volume of Sphere} + \frac{4}{3}\text{Volume of Cylinder}}{4 \times \text{Volume of Cylinder}} = \frac{1}{2},$$

which gives us the result we seek,

$$\text{Volume of Sphere} = \frac{2}{3}\text{Volume of Cylinder}.$$

This argument was good enough to convince a colleague. It did not constitute a publishable proof. Archimedes would go on to supply such a proof in *On the Sphere and Cylinder*, but rather than trying to explain the intricacies of this technically challenging proof, I will illustrate the essence of the issues Archimedes faced in a much simpler example, that of demonstrating the formula for the area of a circle.

1.2

The Area of the Circle and the Archimedean Principle

Archimedes built on a technique that was much older. He credited the idea of using infinitely thin slices to find areas and volumes to Eudoxus of Cnidus who lived in the fourth century BCE on the southwest coast of what is today Turkey. Eudoxus had used this method of slicing to discover that the volume of a pyramid or cone is one-third the area of the base times the height. Even before Eudoxus, Antiphon of Athens (fifth century BCE) is credited with discovering that the area of a circle is equal to the area of a triangle with height equal to the radius of the circle and base given by the circumference of the circle.

In modern notation, we define π as the ratio of the circumference of a circle to its diameter,[2] so the circumference is π times the diameter, or $2\pi r$. The area of the triangle is half the height times the base, which is

$$\frac{1}{2}r \cdot 2\pi r = \pi r^2,$$

the familiar formula for the area of a circle. The formula emerges if we consider building a circle out of very thin triangles (see Figure 1.6). The triangles have heights that are close to the radius of the circle, and these heights approach the radius as the triangles get thinner. The sum of the bases of the triangles is close to the circumference of the circle, and again gets closer as the triangles get thinner. The total area of all of the triangles is the sum of half the base times the height, which is equal to half the sum of the bases times the height. This approaches half the circumference (the sum of the bases) times the radius.

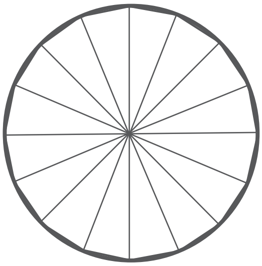

Figure 1.6. A circle approximated by thin triangles.

What I now give is a slight paraphrasing and elaboration of Archimedes proof of the formula for the area of a circle. It relies on Proposition 1 from Book X of Euclid's *Elements*.

> Two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out. (Euclid, 1956, vol. 3, p. 14)

What this tells us is that if we have two positive quantities, leave one fixed and keep removing half from the other, then eventually (in a finite number of steps) the amount that remains of the quantity that has been successively halved will be less than the amount left unchanged. Today this is known as the *Archimedean Principle*, even though it goes back at least to Euclid. It may seem so obvious as not to be worth mentioning, but it should be noted that it explicitly rules out the possibility of an infinitesimal, a quantity that is larger than zero but smaller than any positive real number. If we allowed the fixed quantity to be an infinitesimal and the other to be a positive real number, then no matter how many times we take half of the real number, it will always be larger than the infinitesimal.

Theorem 1.1 (Archimedes, from *Measurement of a Circle*). *The area of a circle is equal to the area of a right triangle whose height is the radius of the circle and whose length is the circumference.*

Proof. Following Archimedes' proof, we will demonstrate that the area of the circle is exactly equal to the area of the triangle by showing that it is neither smaller than the area of the triangle nor larger than the area of the

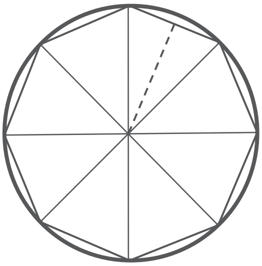

Figure 1.7. A circle with an inscribed octagon. The dashed line shows the height of one of the triangles.

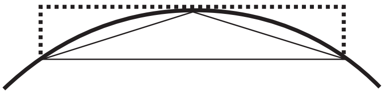

Figure 1.8. Comparing the area between the circle and the first polygon to the area between the circle and the polygon with twice as many sides.

triangle. We first assume that A, the area of the circle, is strictly larger than T, the area of the triangle, i.e., that $A - T > 0$.

We consider an inscribed polygon, such as the octagon shown in Figure 1.7. We let P denote the area of the polygon. Because this polygon is inscribed in the circle, its area is less than that of the circle, $A - P > 0$. The area of the polygon is the sum of the areas of the triangles. Because each triangle has height less than the radius of the circle and the sum of the lengths of the bases of the triangles is less than the circumference of the circle, the area of the polygon is also less than the area of the triangle, $P < T$.

We now form a new polygon with twice as many sides by inserting a vertex on the circle exactly halfway between each pair of existing vertices. We label its area P'. I claim that $A - P'$ is less than half of $A - P$. To see why this is so, consider Figure 1.8. It is visually evident that the area that is filled by adding extra sides accounts for more than half of the area between the circle and the original polygon. We continue to double the number of sides until we get an inscribed polygon of area P^* for which $A - P^* < A - T$. The Archimedean principle promises us that this will happen eventually. When it does, then $P^* > T$.

But the polygon of area P^* is still an inscribed polygon, so $P^* < T$. Our assumption that the area of the circle is larger than T cannot be correct.

What if the area of the circle is strictly less than T? In that case, $T - A > 0$, and we let P be the area of a circumscribed polygon (see Figure 1.9). The height of each triangle that makes up our polygon is now equal to the radius,

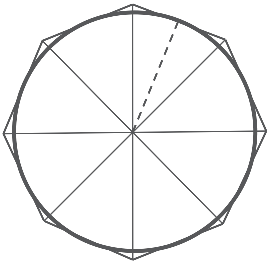

Figure 1.9. A circle with a circumscribed octagon. The dashed line shows the height of one of the triangles.

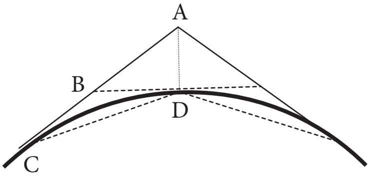

Figure 1.10. Comparing the area between the circle and a circumscribed polygon to the area between the circle and a circumscribed polygon with twice as many sides.

but the perimeter of the polygon is strictly greater than the circumference of the circle, so $P > T$.

Once again we double the number of sides of the polygon by inserting a new vertex exactly halfway between each existing pair of vertices, and we let P' denote the area of the new polygon. Figure 1.10 shows how much of the area $P - A$ is removed when we double the number of sides. Because $BC = BD$, it follows that AB is more than half of AC. Comparing triangle ACD and BCD, they both have the same height (perpendicular distance from D to the line through AC) and the base of ACD is more than twice the base of BCD, it follows that doubling the number of sides takes away more than half of the area between the polygon and the circle, $P' - A < \frac{1}{2}(P - A)$.

We repeat this until $P^* - A < T - A$. This implies that $P^* < T$, contradicting the fact that every circumscribed polygon has an area greater than T. Because A can be neither strictly greater than T nor less than T, it must be exactly equal to T. □

The proof we have just seen may seem cumbersome and pedantic. Most people would be convinced by Figure 1.6. The problem is that such an argument relies on accepting "infinitely many" and "infinitely small" as meaningful quantities. Hellenistic philosophers *were* willing to use these as useful fictions that could help them discover mathematical formulas.

They were not willing to embrace them as sufficient to establish the validity of a mathematical result.

In the seventeenth century, philosophers engaged in heated debates over whether it was legitimate to derive results from nothing more than an analysis of infinitely thin slices. One sees in the work of both Newton and Leibniz a recognition of the power of arguments that rest on the use of infinitesimals, combined with a reluctance to abandon the rigor that Archimedes insisted upon. This reluctance would dissipate under the influence of the Bernoullis and Euler in the eighteenth century, but the problems this engendered would come roaring back in the early nineteenth in the form of apparent contradictions and paradoxes. In chapter 4, we will see how Cauchy recast the arguments of Archimedes and his Hellenistic successors into the precise language of limits in order to establish the modern foundations of calculus.

1.3
Islamic Contributions

In the centuries following Archimedes, mathematics declined as the Roman Empire grew. There never were many people who could read and understand the works of Euclid or Archimedes, much less build upon them. The continuation of their work required an unbroken chain of teachers and students steeped in these methods. For several centuries, Alexandria remained the one bright center of learning in the Eastern Mediterranean, but even there the number of teachers gradually declined.

One of the final flashes of mathematical brilliance occurred in the early fourth century CE with Pappus of Alexandria (circa 290–350 CE), the last great geometer of the Hellenistic world. His *Synagoge* or *Collection* was written as a commentary on and companion to the great Greek geometric texts that still existed in his time. In many cases, the original texts have since disappeared. Our knowledge of what they contained, even the fact of their existence, rests solely on what Pappus wrote about them. One of these lost books is *Plane Loci* by Apollonius of Perga (circa 262–190 BCE). Pappus preserved the statements of Apollonius's theorems, but not the proofs. As we shall see, these tantalizing hints of Hellenistic accomplishments would provide direct inspiration for Fermat, Descartes, and their contemporaries in the seventeenth century.

In the Greco-Roman world, virtually all mathematical work ceased in the late fifth century when the Musaeum of Alexandria—the Temple of the Muses—and its associated library and schools were suppressed because of their pagan associations.[3] All was not lost, however. The rise of the Abbasid empire in the eighth century would see renewed interest and significant new developments in mathematics.

Harun al-Rashid (763 or 766–809 CE) was the fifth Abbasid caliph or ruler. Stories of his exploits figure prominently in the classic tales of the *One Thousand and One Nights*. The Abbasids were descendants of the Prophet Muhammad's youngest uncle, and they took control of most of the Islamic world in 750. In 762 they moved their capital from Damascus to Baghdad. Among al-Rashid's supreme accomplishments was the founding of the Bayt al-Hikma or House of Wisdom. It was a center for the study of mathematics, astronomy, medicine, and chemistry. Its library collected and translated important scientific texts gathered from the Hellenistic Mediterranean, Persia, and India, and it ushered in a great flowering of Islamic[4] science that would last until the Mongol invasions of the thirteenth century.

Thabit ibn Qurra (836–901) was one of the scholars of the House of Wisdom who built on the work of both Greek and Islamic scholars. One of his accomplishments was the rediscovery of the formula for the volume of a paraboloid, the solid formed when a parabola is rotated about its main axis. Although this result had been known to Archimedes, there is every indication that ibn Qurra discovered it anew.

Cast into modern language, the derivation of this formula begins with recognition that a parabola is characterized as a curve for which the distance from the major axis is proportional to the square root of the distance along the major axis from the vertex. In modern algebraic notation, if the vertex is located at $(0,0)$ and x is the distance from the vertex, then y, the distance from the axis, can be represented by $y = a\sqrt{x}$ (Figure 1.11). The cross-sectional area of the paraboloid at distance x is $\pi\left(a\sqrt{x}\right)^2 = \pi a^2 x$. To approximate the volume over $0 \le x \le b$, we slice the paraboloid into n discs of thickness b/n. At $x = ib/n$, for each $0 \le i < n$, the volume of the disc is

$$\pi a^2 \frac{ib}{n} \times \frac{b}{n} = \frac{\pi a^2 b^2}{n^2} i.$$

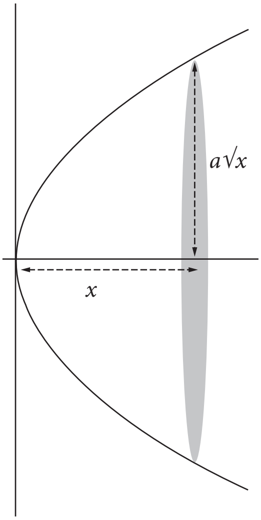

Figure 1.11. Cross-section of a paraboloid.

We now add the volumes of the individual discs,[5]

$$\frac{\pi a^2 b^2}{n^2}\,(0+1+2+\cdots+(n-1)) = \frac{\pi a^2 b^2}{n^2} \times \frac{n^2-n}{2} = \frac{\pi a^2 b^2}{2} - \frac{\pi a^2 b^2}{2n}.$$

As we take larger values of n (and thinner discs), the second term can be made as small as we wish, guaranteeing that the actual value can be neither smaller nor larger than $\pi a^2 b^2/2$.

Ibn al-Haytham (965–1039) demonstrated the power of this approach when he showed how to calculate the volume of the solid obtained by rotating this area about a line perpendicular to the axis of the parabola (Figure 1.12). If the parabolic curve is represented by $y = b\sqrt{x/a}$, where $0 \le y \le b$, then the radius of the disc at height ib/n is given by

$$a - \frac{ay^2}{b^2} = a - \frac{a(ib/n)^2}{b^2},$$

and the volume of the disc at height $y = ib/n$ is

$$\pi\left(a - \frac{a(ib/n)^2}{b^2}\right)^2 \times \frac{b}{n} = \pi a^2 b\left(\frac{1}{n} - \frac{2i^2}{n^3} + \frac{i^4}{n^5}\right). \tag{1.1}$$

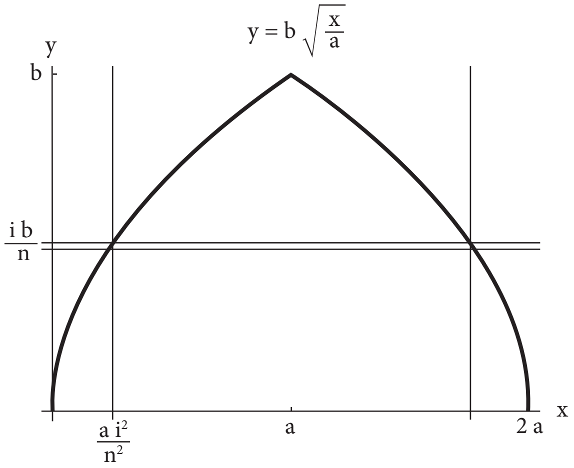

Figure 1.12. A vertical cross-section of al-Haytham's solid of revolution showing the horizontal slice.

It only remains to sum this expression over i from 1 to $n-1$. We need closed formulas for $1^2+2^2+3^2+\cdots+(n-1)^2$ and $1^4+2^4+3^4+\cdots+(n-1)^4$.

In his text *On Spirals*, Archimedes derived the formula for the sum of squares by showing that if

$$S(n) = (n+1)n^2 + (1+2+\cdots+n) = (n+1)n^2 + \frac{n(n+1)}{2},$$

then

$$S(n+1) - S(n) = 3(n+1)^2.$$

Since $S(1) = 3$, it follows that

$$S(n) = 3\left(1^2+2^2+\cdots+n^2\right),$$

or, equivalently,

$$1^2+2^2+\cdots+n^2 = \frac{(n+1)n^2}{3} + \frac{n(n+1)}{6}.$$

Abu Bakr al-Karaji (953–c. 1029) had discovered the formula for the sum of cubes,

$$1^3+2^3+\cdots+n^3=(1+2+\cdots+n)^2=\frac{n^2(n+1)^2}{4}.$$

Once he had guessed the formula, it was easy to verify by observing that the right side is 1 when $n=1$, and the right side increases by $(n+1)^3$ when n is replaced by $n+1$.

Beyond the cubes, the problem gets harder because the formulas are not easy to guess. The genius of al-Haytham was to show how to use a known formula for the sum of the first n kth powers to find the formula for the sum of the first n $k+1$st powers. He did this using specific sums, but his approach translates easily into a general statement. Seeking a formula for the sum of the first n $k+1$st powers, we begin with

$$(n+1)\left(1^k+2^k+\cdots+n^k\right).$$

We distribute $n+1$ through the sum, breaking it into two pieces so that $(n+1)i^k$ becomes

$$(i+(n+1-i))\,i^k=i^{k+1}+(n+1-i)i^k.$$

It follows that

$$\begin{aligned}(1.2)\quad (n+1)\left(1^k+2^k+\cdots+n^k\right)&=\left(1^{k+1}+2^{k+1}+\cdots+n^{k+1}\right)\\&\quad+n\cdot 1^k+(n-1)2^k+\cdots+1\cdot n^k\\&=\left(1^{k+1}+2^{k+1}+\cdots+n^{k+1}\right)\\&\quad+\left(1^k+2^k+\cdots+n^k\right)\\&\quad+\left(1^k+2^k+\cdots+(n-1)^k\right)+\\&\quad+\cdots+\left(1^k+2^k\right)+1^k.\end{aligned}$$

The key to simplifying this relationship is the fact that the formula for the sum of the first n kth powers is of the form $n^{k+1}/(k+1)+p_k(n)$ where p_k is a polynomial of degree at most k. As al-Haytham knew, this is true for $k=1$, 2, and 3. The remainder of this derivation establishes that if it is true for the exponent k, then it holds for the exponent $k+1$. We make this substitution on both sides of equation (1.2).

$$(n+1)\left(\frac{n^{k+1}}{k+1}+p_k(n)\right)=\left(1^{k+1}+2^{k+1}+\cdots+n^{k+1}\right)+\frac{1}{k+1}$$

$$\left(n^{k+1}+(n-1)^{k+1}+\cdots+1^{k+1}\right)+p_k(n)$$

$$+p_k(n-1)+p_k(n-2)+\cdots+p_k(1)$$

$$\frac{n^{k+2}}{k+1}+\frac{n^{k+1}}{k+1}+np_k(n)+p_k(n)=\frac{k+2}{k+1}\left(1^{k+1}+2^{k+1}+\cdots+n^{k+1}\right)+p_k(n)$$

$$+p_k(n-1)+p_k(n-2)+\cdots+p_k(1).$$

Multiplying through by $(k+1)/(k+2)$ and solving for the sum of the $k+1$st powers, we get the desired relationship

$$1^{k+1}+2^{k+1}+\cdots+n^{k+1}=\frac{n^{k+2}}{k+2}+p_{k+1}(n), \tag{1.3}$$

where $p_{k+1}(n)$ is a polynomial in n of degree at most $k+1$.[6]

Now returning to the expression for the volume of each disc, equation (1.1), we can add these volumes:

$$\text{total volume}=\sum_{i=1}^{n}\pi a^2 b\left(\frac{1}{n}-\frac{2i^2}{n^3}+\frac{i^4}{n^5}\right)$$

$$=\pi a^2 b\left(1-\frac{2}{n^3}\left(\frac{n^3}{3}+p_2(n)\right)+\frac{1}{n^5}\left(\frac{n^5}{5}+p_4(n)\right)\right)$$

$$=\pi a^2 b\left(\frac{8}{15}+\frac{2p_2(n)}{n^3}+\frac{p_4(n)}{n^5}\right).$$

Since p_k is a polynomial of degree at most k, we can make the last two terms as small as we wish by taking n sufficiently large. This tells us that the volume of our solid can be neither larger nor smaller than $\frac{8}{15}$ths of the volume of the cylinder in which it sits, or $8\pi a^2 b/15$.

1.4

The Binomial Theorem

Fourth powers had never occurred to the Hellenistic philosophers whose mathematics was rooted in geometry, for they would suggest a fourth dimension. But by the end of the first millennium in the Middle East, in India, and in China astronomers and philosophers were using polynomials of arbitrary degree. Sometime around the year 1000, almost simultaneously within these three mathematical traditions, the binomial theorem appeared,

$$(a+b)^n = \sum_{k=0}^{n} C_k^n a^k b^{n-k},$$

where C_k^n is the $k+1$st entry of the $n+1$st row in the triangular arrangement

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \vdots & & & & & &
\end{array}$$

Each entry is recognized as the sum of the two diagonally above, what today we call *Pascal's triangle*.[7] The initial purpose of this expansion was to find roots of polynomials,[8] but they would come to play many important roles in mathematics. In particular, the binomial theorem provides a means of finding sums of arbitrary positive integer powers.

The starting point for deriving a formula for the sum of kth powers is an observation of Pascal's triangle that was made many times by many different philosophers. In Figure 1.13, we see that if we start at any point along the right-hand edge and add up the terms along a southwest diagonal, then

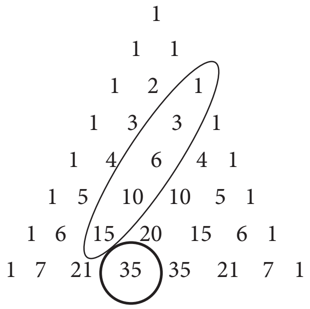

Figure 1.13. The sum of terms down a diagonal, starting from the edge, is always equal to the next term down the opposite diagonal.

wherever we choose to stop, the sum of those numbers is equal to the next number southeast of the number at which we stopped. It is not particularly difficult to see why this is so. For instance, if we take the example in the figure,

$$1 + 3 + 6 + 10 + 15 = 35,$$

$1 + 3$ is the same as summing 3 and the 1 that lies immediately to its right. From the way this triangle is constructed, $3 + 1$ equals the number directly below them and to the right of the 6. The sum of the first three terms down the diagonal is equal to the sum of the last term and the number immediately to its right. The sum of the 6 and the 4 is equal to the number immediately below them, which is the number immediately to the right of the 10 that lies along the diagonal. Wherever we choose to stop, the sum of the terms along the diagonal is equal to the last term plus the term to its right, which is the number directly below.

The earliest documented appearance of this observation occurs in an astrological text by the Spanish-Sephardic philosopher Rabbi Abraham ben Meir ibn Ezra (1090–1167). It also appears in the Chinese manuscript *Siyuan Yujian* (Jade mirror of the four origins) by Zhu Shijie, from 1303, and also in 1356 in the Indian text *Ganita Kaumudi* (Moonlight of mathematics) by Narayana Pandit (circa 1340–1400). It can be expressed as

$$C_k^k + C_k^{k+1} + C_k^{k+2} + \cdots + C_k^{k+n-1} = C_{k+1}^{k+n}. \tag{1.4}$$

As we will see in section 1.7, Pierre de Fermat would use this insight to discover the area beneath the graph of $y = x^k$ from 0 to a for arbitrary positive integer k, the formula that today we would write as

$$\int_0^a x^k\,dx = \frac{1}{k+1}a^{k+1}, \tag{1.5}$$

for any positive integer k.

1.5
Western Europe

The works of Euclid and Archimedes that were known to the European scientists of the sixteenth and seventeenth centuries had survived the Early Middle Ages in Constantinople, copied over the succeeding centuries by scribes who often had no understanding of what they were writing. By the eighth century, Euclid's *Elements* and Archimedes' *Measurement of a Circle* and *On the Sphere and Cylinder* had found their way from the Byzantine Empire to the courts of the Islamic caliphs who had them translated into Arabic. By the twelfth century, Latin translations of the Arabic had begun to appear in Europe. In the following centuries, Euclid was introduced into the university curriculum, but even the master's degree required attending lectures on at most the first six books, and students were seldom held responsible for anything beyond Book I.

Euclid's *Elements*, in Campanus's Latin translation of an Arabic text, was the first mathematics book of any significance to be printed. This was in Venice in 1482. It was followed in 1505 by a translation from a Greek manuscript based on a commentary on the *Elements* by Theon of Alexandria (circa 355–405 CE). Until 1808 when François Peyrard discovered an earlier version of the *Elements* in the Vatican library, the standard edition of Euclid's *Elements* was the 1572 translation by Commandino of Theon's commentary.[9]

The survival of Archimedes' work was even more tenuous. In addition to the Arabic texts, there were two Greek manuscripts, probably copied around the tenth century in Constantinople, that each contained several of his works. These are believed to have been taken to Sicily by the Normans when they conquered that kingdom in the eleventh century. At the defeat of Manfred of Sicily at the Battle of Benevento in 1266, the Archimedean manuscripts were sent to the Vatican in Rome where three years later they were translated into Latin. In 1543, Niccolò Tartaglia published Latin translations of *Measurement of a Circle*, *Quadrature of the Parabola*, *On the Equilibrium of Planes*, and Book I of *On Floating Bodies*. The following year, all of the known works of Archimedes were published in the original Greek together with a Latin translation.[10]

Federico Commandino (1509–1575) translated into Latin and then published works of many of the Greek masters: Euclid, Archimedes, Aristarchus of Samos, Hero of Alexandria, and Pappus of Alexandria. The translation into Latin and publication of Pappus's *Collection*, which would inspire both Fermat and Descartes, was completed in 1588 by his student Guidobaldo del Monte (1545–1607). Commandino and others, including Francesco Maurolico (1494–1575), expanded on Archimedes' results, especially the problem of finding centers of gravity. Maurolico determined the center of gravity of a paraboloid using inscribed and circumscribed discs of constant thickness, calculating the respective centers of gravity of these stacks of discs and showing that the distance from the apex to the center of gravity can be neither larger nor smaller than two-thirds the distance from the apex to the base.[11]

Over the following decades, the Dutch engineer Simon Stevin (1548–1620) and the Roman philosopher—and frequent correspondent of Galileo—Luca Valerio (1552–1618) applied the Archimedean techniques to determine areas, volumes, and centers of mass. As Baron[12] has pointed out, the work of Maurolico, Commandino, Stevin, and Valerio is entirely within the framework of the formal proofs received from Archimedes. In the next century, scholars searching for "quick results and simplified techniques" would begin to loosen these strictures and adopt the use of infinitesimals. By the mid-seventeenth century, these tools were sufficiently well established that Cavalieri, Torricelli, Gregory of Saint-Vincent, Fermat, Descartes, Roberval, and their successors were able to apply them to the production of many of the common formulas for solids of revolution.

The first systematic treatment of volumes of solids of revolution was the *Nova steriometria doliorum vinariorum* (New solid geometry of wine barrels) published by Johannes Kepler (1571–1630) in 1615. It included formulas for the volumes of 96 different solids formed by rotating part of a conic section about some axis. An example is the volume of an

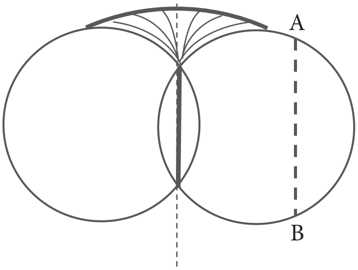

Figure 1.14. An apple formed by rotating a circle about one of its chords.

apple, formed by rotating a circle around a vertical chord of that circle (see Figure 1.14). Abandoning Archimedean rigor, Kepler established this result by considering the apple as composed of infinitely many thin cylindrical shells. We take one of the vertical chords such as *AB*, rotate it around the central axis, and find the surface area of this cylinder. The volume of the solid is obtained by adding up these surface areas. In practical terms, what he did was to take these cylinders, unroll each into a rectangle, and then assemble the rectangles into a solid whose volume he could compute. It is what today we refer to as the *shell method*.

There is a simpler way of computing volumes of solids of revolution that had been known to Pappus of Alexandria in the fourth century CE. In his *Collection* of the known geometric results of his time, he stated that the volume of a solid of revolution is proportional to the product of the area of the region that is rotated to form the solid and the distance from the center of gravity to the axis. Unfortunately, all that has survived is the statement of this theorem with no indication of how Pappus justified it. In 1640, Paul Guldin (1577–1643), a Swiss Jesuit trained in Rome and a regular correspondent of Kepler, published a statement and proof of this theorem in his book *De centro gravitatis*.[13]

1.6

Cavalieri and the Integral Formula

Bonaventura Cavalieri (1598–1647) was strongly influenced by Kepler. A student of Benedetto Castelli (1578–1643) who had studied with Galileo, Cavalieri began an extensive correspondence with Galileo in 1619 and discovered Kepler's *Stereometrica* around 1626. He obtained a professorship

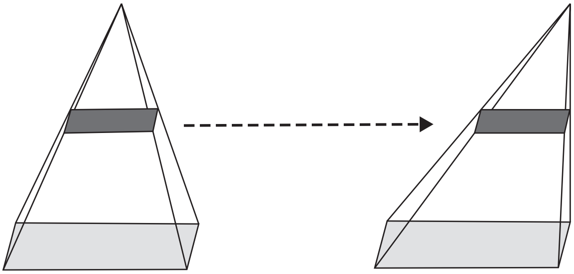

Figure 1.15. Solids with the same cross-sections have identical volumes.

in mathematics at the University of Bologna in 1629, two years after he had finished much of the work on his *Geometria indivisibilibus*. It would not be published until 1635. Galileo had been working along similar lines, and it has been suggested[14] that Cavalieri may have been waiting for Galileo to publish these results.

Cavalieri proceeded from the assumption that areas can be built up from one-dimensional lines and solids are composed of two-dimensional *indivisibles*. These were not just infinitely thin sheets. Cavalieri explicitly rejected the idea that solids could be thought of as built from three-dimensional but infinitesimally thin sheets. His starting point for computing volumes was the observation, going back to Democritus (circa 460–370 BCE), that if two solids have the same height and congruent cross-sections at each intermediate height, then they must have the same volume (Figure 1.15). Democritus had used this argument to prove that the area of any pyramid is one-third the area of the base times the height, but making the step to the assumption that the solid actually *is* a stack of these two-dimensional cross-sections went too far for many. Guldin was one of many vociferous critics.

Cavalieri's *Geometria* contains the first derivation of a formula equivalent to the integral formula for x^k. Though Cavalieri only carried this up to the integral of x^9, that was far enough that anyone could see what the general formula had to be. In explaining Cavalieri's work, it is important to recognize that this was written before the development of analytic geometry, the ability to represent a relationship such as $y=x^k$ as a graph with an area beneath it. What we today interpret as an integral Cavalieri understood as simply a sum, a sum involving lines used to build up an area.

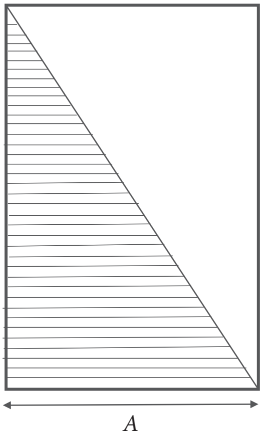

Figure 1.16. The triangular region is composed of lines of variable length $0 \leq \ell \leq A$.

We begin with the triangular region in Figure 1.16 which shows some of the lines that make up this triangle. Cavalieri thought of the area of this region as the sum of the lengths of all of these lines, $\sum \ell$. The area of the entire rectangle[15] is the sum of lines of equal length A, $\sum A$. The first step for Cavalieri was the fact that

$$\frac{\sum \ell}{\sum A} = \frac{1}{2};$$

the area of the triangle is half the area of the rectangle.

Instead of simply summing the lengths of the lines that constitute the triangle, he now summed their squares. If we place a square of base ℓ^2 on each line, we get a pyramid, which we have seen was long known to have volume equal to one-third of the rectangular solid formed by stacking squares of equal size $A \times A$,

$$\frac{\sum \ell^2}{\sum A^2} = \frac{1}{3}.$$

Cavalieri now stepped into the unknown by considering the ratio of the sum of cubes of the lines in the triangle to the sum of cubes of A. He accomplished this using the equality

$$(x+y)^3 + (x-y)^3 = 2x^3 + 6xy^2. \tag{1.6}$$

Instead of summing ℓ^3 as ℓ decreases from A to 0, he added $(A/2+\ell)^3$ as ℓ decreases from $A/2$ to 0 and $A/2-\ell$ as ℓ increases from 0 to $A/2$,[16]

Figure 1.17. Pierre de Fermat.

$$\sum_{0\le\ell\le A} \ell^3 = \sum_{0\le\ell\le A/2} \left(\left(\frac{A}{2}+\ell\right)^3 + \left(\frac{A}{2}-\ell\right)^3\right).$$

He could now use equation 1.6 and the formula he knew for $\sum \ell^2$,

$$\begin{aligned}\sum_{0\le\ell\le A} \ell^3 &= \sum_{0\le\ell\le A/2} \left(2\left(\frac{A}{2}\right)^3 + 6\left(\frac{A}{2}\right)\ell^2\right)\\ &= \frac{1}{4}\sum_{0\le\ell\le A/2} A^3 + 3A \sum_{0\le\ell\le A/2} \ell^2\\ &= \frac{1}{4}\sum_{0\le\ell\le A/2} A^3 + A \sum_{0\le\ell\le A/2} \left(\frac{A}{2}\right)^2\\ &= \frac{1}{4}\sum_{0\le\ell\le A/2} A^3 + \frac{1}{4}\sum_{A/2\le\ell\le A} A^3\\ &= \frac{1}{4}\sum_{0\le\ell\le A} A^3.\end{aligned}$$

He proceeded up to $\sum \ell^9$, in each case using the identity

$$(x+y)^k + (x-y)^k = 2x^k + 2C_2^k x^{k-2} y^2 + 2C_4^k x^{k-4} y^4 + \cdots$$

and the formulas he had already found to show that, for $1 \le k \le 9$,

$$\frac{\sum \ell^k}{\sum A^k} = \frac{1}{k+1}.$$

If you rotate the rectangle by 90° counter-clockwise, you see that he has demonstrated that the area under the curve $y = x^k$, $0 \le x \le A$, is equal to

$$\sum_{0 \le \ell \le A} \ell^k = \frac{1}{k+1} \sum_{0 \le \ell \le A} A^k = \frac{1}{k+1} A^{k+1}.$$

Unfortunately, few people in 1635 realized what he had accomplished. Cavalieri's great work was almost unreadable.[17] What people would come to know of Cavalieri's mathematics was due to Torricelli's 1644 explanation in *Opera geometrica*. By this time, Fermat and Descartes had established algebraic geometry for graphing algebraic relationships, and they and others had found simpler routes to the integral formula.

1.7

Fermat's Integral and Torricelli's Impossible Solid

In 1636, Pierre de Fermat (1601–1665) wrote to two of his colleagues in Paris, Marin Mersenne (1588–1648) and Gilles de Roberval (1602–1675), announcing that he had discovered a general method for finding the area beneath the graph of the curve $y = x^k$ for positive integer k. Within a month, Roberval responded, stating that this result had to rely on the fact that (in modern notation)

$$\sum_{j=1}^{n} j^k > \frac{n^{k+1}}{k+1} > \sum_{j=1}^{n-1} j^k, \tag{1.7}$$

for all positive integers k and n. Fermat was clearly disappointed that Roberval caught on so quickly, but expressed his doubts that Roberval was able to justify this pair of inequalities.

Reconstructing Fermat's proof as best we can[18] and casting it in modern notation, the proof begins with the fact that the binomial coefficients can be written as

$$C_k^{k+j-1} = \frac{j(j+1)(j+2)\cdots(j+k-1)}{k!}.$$

We expand the numerator as a polynomial in j,

$$C_k^{k+j-1} = \frac{1}{k!}\left(j^k + a_1 j^{k-1} + a_2 j^{k-2} + \cdots + a_k\right), \tag{1.8}$$

where the coefficients a_i are integers. Combining equation (1.4) with equation (1.8), we obtain,

$$\frac{1}{k!}\sum_{j=1}^{n}\left(j^k + a_1 j^{k-1} + a_2 j^{k-2} + \cdots + a_k\right) = \frac{n(n+1)(n+2)\cdots(n+k)}{(k+1)!}. \tag{1.9}$$

We can express the sum of kth powers in terms of sums of lower powers,

$$\begin{aligned}\sum_{j=1}^{n} j^k = {} & \frac{k!}{(k+1)!}n(n+1)(n+2)\cdots(n+k) \\ & -\sum_{j=1}^{n}\left(a_1 j^{k-1} + a_2 j^{k-2} + \cdots + a_k\right).\end{aligned} \tag{1.10}$$

We use the inductive assumption[19] that the sum of mth powers from 1^m up to n^m is a polynomial in n of degree $m+1$. We have seen this to be true for $m =$ 1, 2, and 3 and can assume it to be true up to $m = k-1$. Equation (1.10) is then expressed as

$$\sum_{j=1}^{n} j^k = \frac{1}{k+1}\, n^{k+1} + \text{a polynomial in } n \text{ of degree at most } k. \tag{1.11}$$

To find the area under the curve $y = x^k$, we subdivide the interval from 0 to a into n subintervals of equal width, a/n (Figure 1.18). The combined area of the inscribed rectangles is

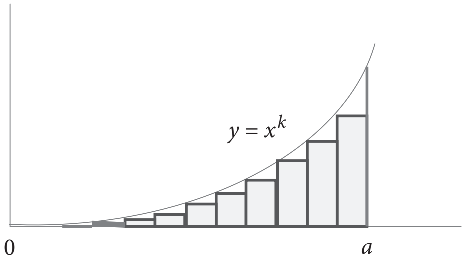

Figure 1.18. Inscribed rectangles of width a/n below the graph of $y = x^k$.

$$\begin{aligned}\sum_{j=0}^{n-1}\left(\frac{aj}{n}\right)^k \frac{a}{n} &= \frac{a^{k+1}}{n^{k+1}}\sum_{j=0}^{n-1} j^k \\ &= \frac{a^{k+1}(n-1)^{k+1}}{(k+1)n^{k+1}} + \text{a sum of terms involving negative powers of } n.\end{aligned}$$

This can be brought as close as we wish to $a^{k+1}/(k+1)$ by taking n sufficiently large.

The combined area of the circumscribed rectangles is

$$\begin{aligned}\sum_{j=1}^{n}\left(\frac{aj}{n}\right)^k \frac{a}{n} &= \frac{a^{k+1}}{n^{k+1}}\sum_{j=1}^{n} j^k \\ &= \frac{a^{k+1}n^{k+1}}{(k+1)n^{k+1}} + \text{a sum of terms involving negative powers of } n,\end{aligned}$$

which also can be brought as close as we wish to $a^{k+1}/(k+1)$ by taking n sufficiently large. The area is $a^{k+1}/(k+1)$.[20]

Evangelista Torricelli (1608–1647) was another student of Castelli, earning his tuition by serving as Castelli's secretary. He began his correspondence with Galileo in 1632 and spent the last few months of Galileo's life with him, from October 1641 until January 1642. In his *Opera geometrica*, published in 1644, Torricelli embraced the language of indivisibles that Cavalieri had espoused, but he explicitly stated that his indivisibles do have "a thickness which is always equal and uniform,"[21] even though it is infinitesimal.

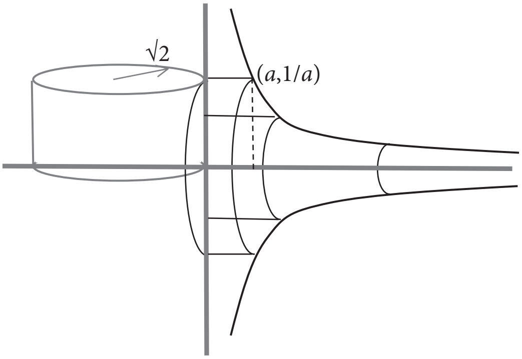

Figure 1.19. Torricelli's acute hyperbolic solid.

Torricelli is best known today—and at the time made his reputation—for the discovery of an infinitely long solid of revolution of finite volume, what he called an *acute hyperbolic solid*. This is the solid obtained by rotating about the horizontal axis the region bounded above by $y = 1/a$ for $0 \leq x \leq a$ and by $y = 1/x$, for all $x \geq a$, where a is strictly positive. Specifically, what he proved is that the volume of this solid is equal to the volume of the cylinder of radius $\sqrt{2}$ and height $1/a$ (see Figure 1.19). In other words, the volume of this infinitely long solid is the finite value $2\pi/a$.

The proof proceeds by decomposing the acute hyperbolic solid into hollow cylinders of infinitesimal thickness. The hollow cylinder at height y has radius y and circumference $2\pi y$, while the distance from the base to the hyperbolic curve is $1/y$. Every cylinder, irrespective of the value of y, has the same surface area: 2π, which is the area of a circle of radius $\sqrt{2}$. We therefore can match the volume of the acute hyperbolic solid to that of the cylinder formed by discs of radius $\sqrt{2}$ stacked from $y = 0$ to $y = 1/a$.

Torricelli shared this discovery with Cavalieri in 1641, who wrote back,

> I received your letter while in bed with fever and gout . . . but in spite of my illness I enjoyed the savory fruits of your mind, since I found infinitely admirable that infinitely long hyperbolic solid which is equal to a body finite in all the three dimensions. And having spoken about it to some of my philosophy students, they agreed that it seemed truly marvelous and extraordinary that that could be.[22]

In 1643, Cavalieri communicated this result, though not the proof, to Jean-François Niceron in Paris. He passed it on to Mersenne, and soon the entire mathematical world knew about it. Torricelli published two proofs the following year as part of his *Opera geometrica*, one using the method of indivisibles as described in the previous paragraph, the other employing the classical Archimedean approach in which he demonstrated that the volume of his solid could be neither larger nor smaller than that of the cylinder of radius $\sqrt{2}$ and height $1/a$.

Torricelli's result truly shocked the mathematical establishment. He later recorded that Roberval had not believed the result when he first learned of it and had attempted to disprove it.[23] The fact that the initial proof used Cavalieri's indivisibles cast considerable doubt on their reliability, which is why Torricelli realized that he also needed to provide a justification with full Archimedean rigor.

1.8

Velocity and Distance

If accumulation were no more than a way of calculating areas, volumes, and moments, it would have provided us with an interesting set of results, but hardly the historical foundation for a major branch of mathematics. What made accumulation the powerful tool it is today was the discovery of the connection to instantaneous velocity. If we know the velocity at each point in time, then we can accumulate small changes in distance to find the total distance that has been traveled. This is not a simple or obvious idea. More than one calculus student has been mystified by the fact that we can find distances by calculating areas under curves.

Today, we take the concept of velocity of an object at a particular moment in time for granted. It confronts us every time we look at a speedometer. Yet explaining what it means requires some subtlety. The fifth century BCE philosopher Zeno of Elea described the paradox of instantaneous velocity: An arrow is always either in motion or at rest. At a single instant, it cannot be in motion, for to be in motion is to change position, and if it did change position in an instant, then that instant would have a duration and could be subdivided. Therefore, at each instant, the arrow is at rest. But if the arrow is at rest at every instant, then it is *always* at rest, and so it never moves.[24]

Aristotle answered this paradox by denying the existence of instants in time, consequently denying the existence of an instantaneous velocity. To Aristotle and his successors, this was not a great loss. The motion they studied was uniform motion, either linear or circular. There was no general treatment of velocity as the ratio of distance traveled to the time required or even as a magnitude in its own right.[25] But in the fourteenth century scholars in Oxford and Paris began to study velocity as something that has a magnitude at each instant of time and to explore what could be said when velocity is not uniform.

The first of the great European universities was established in Bologna in 1088. Others soon followed. The Greek classics, which were now being translated from Arabic, provided grist for the scholars who gathered there. They sought to understand these works. Soon they would transcend them.

Merton College in Oxford was established in 1264. Starting around 1328, a remarkable group of Mertonian scholars—Thomas Bradwardine, William Heytesbury, Richard Swineshead, and John Dumbleton—began their explorations of velocity. The first of their accomplishments was to separate kinematics, the quantitative study of motion, from dynamics, the study of the causes of motion. The idea of describing a moving object with no reference to what set that object in motion or maintained its motion was new. For the first time, scholars began to speak of velocity as a magnitude.[26]

The earliest description of instantaneous velocity can be found in William Heytesbury's 1335 manuscript, *Rules for Solving Sophisms*. He made it clear that instantaneous velocity, the velocity at a single instant of time, is not affected in any way by how far the object has moved, but is "measured by the path which *would* be described by the most rapidly moving point if, in a period of time, it were moved uniformly at the same degree of velocity."[27] This was an adequate definition that would serve for close to 500 years. It is not how we define instantaneous velocity today. Our modern definition was not fully articulated until the early nineteenth century. It is based on limits and the algebra of inequalities and can be found in section 4.2.

Heytesbury went on to consider the motion of an object that is uniformly accelerated, whose velocity increases at a constant rate. He argued that the distance traveled by an object that starts with an initial velocity and accelerates or decelerates uniformly to some final velocity is the same as the distance traveled by an object moving at the mean velocity, the velocity

(continued...)

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