Contents

CHAPTER

1

Rudiments of complex analysis

We begin by recalling some standard definitions and notations. Throughout this book the complex plane will be denoted by $\mathbb{C}$. Every $z \in \mathbb{C}$ has a unique representation of the form $z = x + iy$ in which $x, y \in \mathbb{R}$ and i is the imaginary unit, so $i^2 = -1$. We call x and y the ***real*** and ***imaginary parts*** of z and we write

$$x = \mathrm{Re}(z) \qquad \text{and} \qquad y = \mathrm{Im}(z).$$

The ***complex conjugate*** of z is the complex number defined by

$$\bar{z} = x - iy.$$

The relations

$$\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z}) \qquad \text{and} \qquad \mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$$

are easily verified. Note that $z \in \mathbb{R}$ if and only if $z = \bar{z}$.

The ***absolute value*** (or ***modulus***) of $z = x + iy$ is the non-negative number

$$|z| = \sqrt{x^2 + y^2}.$$

Evidently $|z| > 0$ if and only if $z \neq 0$, and the relation

$$|z|^2 = z\bar{z}$$

holds.

If $z \neq 0$, the complex number $z/|z|$ has absolute value 1 and can be represented by the complex exponential $e^{i\theta} = \cos\theta + i \sin\theta$ for some $\theta \in \mathbb{R}$, called an ***argument*** of z, which is unique up to addition of an integer multiple of 2π. The expression

$$z = |z|\, e^{i\theta}$$

is called the ***polar representation*** of z.

We will reserve the following notations for the open disk of radius r centered at p and the unit disk centered at the origin:

$$\mathbb{D}(p, r) = \{z \in \mathbb{C} : |z - p| < r\} \qquad \mathbb{D} = \mathbb{D}(0, 1).$$

Unless otherwise stated, when we write $z = x + iy$ we mean x, y are real, so $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. Similarly, for a complex-valued function f, when we write $f = u + iv$ we mean u, v are real-valued, so $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$.

1.1 What is a holomorphic function?

Our point of departure is the notion of complex differentiability which is fundamental to everything that follows.

DEFINITION 1.1. Suppose f is a complex-valued function defined in an open neighborhood of some $p \in \mathbb{C}$. We say f is ***(complex) differentiable*** at p if the limit

$$f'(p) = \lim_{z \to p} \frac{f(z) - f(p)}{z - p}$$

exists. The number $f'(p)$ is called the ***(complex) derivative*** of f at p.

As usual, differentiability implies continuity. In other words, if $f'(p)$ exists, then f is continuous at p:

$$\lim_{z \to p} f(z) - f(p) = \lim_{z \to p} \frac{f(z) - f(p)}{z - p} (z - p) = f'(p) \cdot 0 = 0.$$

The basic rules of differentiation that we learn in calculus hold for complex derivatives.

THEOREM 1.2.

(i) *Suppose f and g are differentiable at p. Then the sum $f + g$ and the product fg are differentiable at p and*

$$\begin{aligned}(f+g)'(p) &= f'(p) + g'(p) \\ (fg)'(p) &= f'(p)g(p) + f(p)g'(p).\end{aligned}$$

Moreover, if $g(p) \neq 0$, the quotient f/g is differentiable at p and

$$\left(\frac{f}{g}\right)'(p) = \frac{f'(p)g(p) - f(p)g'(p)}{(g(p))^2}.$$

(ii) *Suppose g is differentiable at p and f is differentiable at $g(p)$. Then the composition $f \circ g$ is differentiable at p and the "chain rule" holds:*

$$(f \circ g)'(p) = f'(g(p))\, g'(p).$$

The assumption $g(p) \neq 0$ in (i) combined with continuity of g at p implies that g is non-zero in an open neighborhood of p, so the quotient f/g is well defined in that neighborhood.

Proof. (i) The results for the sum and product are easy to prove. For the quotient rule, first consider the special case where $f = 1$ everywhere. Writing

$$\frac{\dfrac{1}{g(z)} - \dfrac{1}{g(p)}}{z-p} = -\frac{g(z)-g(p)}{z-p} \cdot \frac{1}{g(z)g(p)},$$

letting $z \to p$, and using continuity of g at p, we obtain $(1/g)'(p) = -g'(p)/(g(p))^2$. The quotient rule now follows from this and the product rule applied to $f/g = f \cdot 1/g$.

(ii) Define

$$\varepsilon(w) = \begin{cases} \dfrac{f(w) - f(g(p))}{w - g(p)} - f'(g(p)) & w \neq g(p) \\ 0 & w = g(p). \end{cases}$$

Then ε is continuous at $g(p)$ and the relation

$$f(w) - f(g(p)) = (f'(g(p)) + \varepsilon(w))\,(w - g(p))$$

holds throughout an open neighborhood of $g(p)$. Setting $w = g(z)$, it follows that

$$\frac{(f \circ g)(z) - (f \circ g)(p)}{z-p} = \big(f'(g(p)) + \varepsilon(g(z))\big)\,\frac{g(z)-g(p)}{z-p}.$$

As $z \to p$, $g(z) \to g(p)$ by continuity, so $\varepsilon(g(z)) \to 0$. Hence the right side tends to $f'(g(p))\,g'(p)$. □

EXAMPLE 1.3 (Polynomials). It is immediate from the definition that the identity function $f(z) = z$ is differentiable everywhere and $f'(z) = 1$ for all z. By repeated application of Theorem 1.2(i), it follows that every polynomial $f(z) = \sum_{n=0}^{d} a_n z^n$ is differentiable everywhere and $f'(z) = \sum_{n=1}^{d} n a_n z^{n-1}$ for all z.

EXAMPLE 1.4. The smooth function $f(z) = z\bar{z} = |z|^2$ is complex differentiable only at the origin. In fact, for $z \neq 0$,

$$\frac{f(p+z) - f(p)}{z} = \frac{(p+z)(\bar{p}+\bar{z}) - p\bar{p}}{z} = p\,\frac{\bar{z}}{z} + \bar{p} + \bar{z}.$$

But $\bar{z}/z$ does not have a limit as $z \to 0$, since $\bar{z}/z = 1$ if z is real while $\bar{z}/z = -1$ if z is purely imaginary. It follows that the right side of the above equation has a limit as $z \to 0$ if and only if $p = 0$, and that $f'(0) = 0$.

Under the canonical isomorphism $\mathbb{C} \to \mathbb{R}^2$ given by $z = x + iy \mapsto (x, y)$, every complex-valued function $f = u + iv$ can be identified with the map into the plane

$\mathbb{R}^2$ given by $f(x,y) = (u(x,y), v(x,y))$. To understand the relation between the complex derivative of f as defined above and the real derivative of f as a map into the plane, let us first introduce a few useful notations. The partial differentiation operators $\partial/\partial x$ and $\partial/\partial y$ acting on smooth real-valued functions can be naturally extended to complex-valued functions. Explicitly, if $f = u + iv$, we set

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}. \tag{1.1}$$

For complex-variable computations, it will be convenient to introduce two new differential operators defined by

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right), \tag{1.2}$$

so

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} \qquad \frac{\partial f}{\partial y} = i\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right). \tag{1.3}$$

It is important to keep in mind that the operators $\partial/\partial z$ and $\partial/\partial \bar{z}$ are not defined as partial differentiation with respect to z and $\bar{z}$. After all, z and $\bar{z}$ are not independent variables!

EXAMPLE 1.5. By the definition (1.2),

$$\frac{\partial}{\partial z}(z) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x+iy) = 1 \qquad \frac{\partial}{\partial z}(\bar{z}) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x-iy) = 0$$

$$\frac{\partial}{\partial \bar{z}}(z) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x+iy) = 0 \qquad \frac{\partial}{\partial \bar{z}}(\bar{z}) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x-iy) = 1.$$

Since it is easy to verify the product rule for $\partial/\partial z$ and $\partial/\partial \bar{z}$ (see problem 3), it follows that these linear operators act on polynomials in z and $\bar{z}$ in the following way:

$$\frac{\partial}{\partial z}\left(\sum_{j,k} a_{jk}\, z^j\, \bar{z}^k\right) = \sum_{j,k} j a_{jk}\, z^{j-1}\, \bar{z}^k \qquad \frac{\partial}{\partial \bar{z}}\left(\sum_{j,k} a_{jk}\, z^j\, \bar{z}^k\right) = \sum_{j,k} k a_{jk}\, z^j\, \bar{z}^{k-1}.$$

Observe that these are the answers we would have obtained if we had taken “partial derivatives” with respect to z and $\bar{z}$.

EXAMPLE 1.6. The operators $\partial/\partial z$ and $\partial/\partial \bar{z}$ act on $\log|z|$ as follows:

$$\frac{\partial}{\partial z}\log|z| = \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\log(x^2+y^2) = \frac{1}{2}\,\frac{x-iy}{x^2+y^2} = \frac{1}{2z},$$

$$\frac{\partial}{\partial \bar{z}}\log|z| = \frac{1}{4}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\log(x^2+y^2) = \frac{1}{2}\,\frac{x+iy}{x^2+y^2} = \frac{1}{2\bar{z}}.$$

If we write $\log|z|$ as $\frac{1}{2}\log(z\bar{z})$ and take "partial derivatives" with respect to z and $\bar{z}$, we obtain

$$\frac{\partial f}{\partial z} = \frac{1}{2}\frac{\bar{z}}{z\bar{z}} = \frac{1}{2z} \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{z}{z\bar{z}} = \frac{1}{2\bar{z}},$$

which agree with the previous computations.

In both of the above examples, we could formally consider z and $\bar{z}$ as independent variables and compute $\partial f/\partial z$ and $\partial f/\partial \bar{z}$ by "partial differentiation" with respect to the corresponding variable, pretending the other is fixed. Such formal computations are not totally meaningless and in fact there are practical situations where their legitimacy can be justified. We will provide such a justification at the end of this section.

We continue identifying $f = u + iv$ with the map into the plane given by $f(x, y) = (u(x, y), v(x, y))$. By definition, this map is real differentiable at p if there is a necessarily unique linear map $Df(p) : \mathbb{R}^2 \to \mathbb{R}^2$, called the ***real derivative*** of f at p, such that

$$\lim_{(x,y)\to(0,0)} \frac{\| f(p + (x, y)) - f(p) - Df(p)(x, y)\|}{\|(x, y)\|} = 0.$$

Here $\|(x, y)\| = \sqrt{x^2 + y^2}$ is the Euclidean norm which agrees with the absolute value of $x + iy$ as a complex number. Equivalently, we can express this condition as the first-order Taylor approximation formula: For all $(x, y) \in \mathbb{R}^2$ sufficiently close to the origin $(0, 0)$,

$$f(p + (x, y)) = f(p) + Df(p)(x, y) + \varepsilon(x, y), \tag{1.4}$$

where the "error term" $\varepsilon(x, y)$ satisfies $\|\varepsilon(x, y)\|/\|(x, y)\| \to 0$ as $(x, y) \to (0, 0)$. It is easy to see that in the standard basis of $\mathbb{R}^2$, the linear map $Df(p)$ is represented by the 2×2 matrix of partial derivatives

$$Df(p) = \begin{bmatrix} \dfrac{\partial u}{\partial x}(p) & \dfrac{\partial u}{\partial y}(p) \\ \dfrac{\partial v}{\partial x}(p) & \dfrac{\partial v}{\partial y}(p) \end{bmatrix}. \tag{1.5}$$

For convenience, let us use the subscript notation for our differential operators. Thus,

$$f_x = \frac{\partial f}{\partial x}, \qquad f_y = \frac{\partial f}{\partial y}, \qquad f_z = \frac{\partial f}{\partial z}, \qquad f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}}.$$

Suppose f has a real derivative at p so (1.4) holds. Using the matrix (1.5) for $Df(p)$, we see that

$$Df(p)(x, y) = \begin{bmatrix} u_x(p) & u_y(p) \\ v_x(p) & v_y(p) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xu_x(p) + yu_y(p) \\ xv_x(p) + yv_y(p) \end{bmatrix},$$

which by (1.1) and (1.3) can be identified with the complex number

$$\begin{aligned}(xu_x(p)+yu_y(p))+i(xv_x(p)+yv_y(p)) &= xf_x(p)+yf_y(p)\\ &= \frac{1}{2}(z+\bar{z})(f_z(p)+f_{\bar{z}}(p))+\frac{1}{2}(z-\bar{z})(f_z(p)-f_{\bar{z}}(p))\\ &= zf_z(p)+\bar{z}f_{\bar{z}}(p).\end{aligned}$$

Hence, in our complex-variable notation the Taylor formula (1.4) reads

$$f(p+z)=f(p)+zf_z(p)+\bar{z}f_{\bar{z}}(p)+\varepsilon(z), \tag{1.6}$$

where $\varepsilon(z)/z \to 0$ as $z \to 0$. If $f_{\bar{z}}(p)=0$, we obtain

$$\frac{f(p+z)-f(p)}{z}=f_z(p)+\frac{\varepsilon(z)}{z}.$$

Letting $z \to 0$, it follows that the complex derivative $f'(p)$ exists and is equal to $f_z(p)$.

Conversely, suppose $f'(p)$ exists, so

$$f(p+z)=f(p)+f'(p)z+\varepsilon(z),$$

where $\varepsilon(z)/z \to 0$ as $z \to 0$. Then the complex multiplication $z \mapsto f'(p)z$, viewed as a linear map $\mathbb{R}^2 \to \mathbb{R}^2$, satisfies the condition (1.4). Hence the real derivative $Df(p)$ exists and $Df(p)(x,y)$ can be identified with $f'(p)\cdot(x+iy)$. If $f'(p)=\alpha+i\beta$, we have

$$f'(p)\cdot(x+iy)=(\alpha x-\beta y)+i(\beta x+\alpha y),$$

which shows $Df(p)$ is represented by the matrix

$$Df(p)=\begin{bmatrix}\alpha & -\beta\\ \beta & \alpha\end{bmatrix}. \tag{1.7}$$

Comparing with (1.5), we see that $\alpha=u_x(p)=v_y(p)$ and $\beta=-u_y(p)=v_x(p)$. In particular,

$$f_{\bar{z}}(p)=\frac{1}{2}(f_x(p)+if_y(p))=\frac{1}{2}((\alpha+i\beta)+i(-\beta+i\alpha))=0.$$

Let us summarize our findings in the following

THEOREM 1.7. *For a given complex-valued function $f=u+iv$ defined in an open neighborhood of $p\in\mathbb{C}$, the following conditions are equivalent:*

(i) The complex derivative $f'(p)$ exists.
(ii) The real derivative $Df(p)$ exists and

$$f_{\bar{z}}(p)=0.$$

(iii) The real derivative $Df(p)$ exists and

$$u_x(p)=v_y(p) \qquad u_y(p)=-v_x(p).$$

Under any of these conditions, we have

$$f'(p) = f_z(p) = f_x(p) = -i f_y(p).$$

EXAMPLE 1.8. The polynomial $f(z) = z^2 = (x^2 - y^2) + i(2xy)$ has the complex derivative $f'(z) = 2z$ for all z. Furthermore,

$$f_x = 2x + i2y = 2z \qquad f_y = -2y + i2x = i2z$$

$$f_z = 2z \qquad f_{\bar{z}} = 0,$$

which are consistent with Theorem 1.7.

EXAMPLE 1.9. Consider the continuous function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^5/|z|^4$ for $z \neq 0$ and $f(0) = 0$. Write $f = u + iv$, where for $(x, y) \neq (0, 0)$

$$u(x, y) = \frac{x^5 - 10x^3y^2 + 5xy^4}{(x^2 + y^2)^2} \qquad \text{and} \qquad v(x, y) = \frac{y^5 - 10x^2y^3 + 5x^4y}{(x^2 + y^2)^2}$$

and $u(0, 0) = v(0, 0) = 0$. Thus

$$u_x(0, 0) = \lim_{x \to 0} \frac{u(x, 0)}{x} = 1 \qquad \text{and} \qquad v_y(0, 0) = \lim_{y \to 0} \frac{v(0, y)}{y} = 1$$

and similarly

$$u_y(0, 0) = \lim_{y \to 0} \frac{u(0, y)}{y} = 0 \qquad \text{and} \qquad v_x(0, 0) = \lim_{x \to 0} \frac{v(x, 0)}{x} = 0,$$

so the pair of conditions in Theorem 1.7(iii) holds. However, the complex derivative $f'(0)$ does not exist since

$$\frac{f(z)}{z} = \left(\frac{z}{|z|} \right)^4$$

does not have a limit as $z \to 0$. For example, this quotient tends to 1 when z tends to 0 along the real line, while it tends to -1 when z tends to 0 along the line $\mathrm{Re}(z) = \mathrm{Im}(z)$. Note that this example does not contradict Theorem 1.7 since $Df(0)$ does not exist.

Here is another important observation: Suppose $f'(p) = \alpha + i\beta$ so $Df(p)$ has the matrix form (1.7). If $f'(p) \neq 0$, then $\det(Df(p)) = \alpha^2 + \beta^2 > 0$, which means $Df(p)$ is orientation-preserving. Moreover, the matrix (1.7) can be decomposed as

$$Df(p) = \begin{bmatrix} \sqrt{\alpha^2 + \beta^2} & 0 \\ 0 & \sqrt{\alpha^2 + \beta^2} \end{bmatrix} \begin{bmatrix} \dfrac{\alpha}{\sqrt{\alpha^2 + \beta^2}} & \dfrac{-\beta}{\sqrt{\alpha^2 + \beta^2}} \\ \dfrac{\beta}{\sqrt{\alpha^2 + \beta^2}} & \dfrac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \end{bmatrix}.$$

Geometrically, this is a rotation by the angle $\arccos(\alpha/\sqrt{\alpha^2 + \beta^2})$ about the origin, followed by a dilation by the factor $\sqrt{\alpha^2 + \beta^2}$. Alternatively, the action of $Df(p)$ can be identified with the complex multiplication by $f'(p)$, which amounts to a rotation by the argument of $f'(p)$ followed by a dilation by the factor $|f'(p)|$. This geometric description shows that $Df(p)$ is an angle-preserving linear transformation in the sense that the angle between any two non-zero vectors v_1, v_2 is the same as the angle between their images $Df(p)v_1, Df(p)v_2$. Such linear maps are often called ***conformal*** because they preserve shapes (but not necessarily scales).

COROLLARY 1.10. *Suppose f has a non-zero complex derivative at $p \in \mathbb{C}$. Then the real derivative $Df(p) : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation-preserving conformal linear transformation.*

For several alternative characterizations of conformal linear transformations in dimension 2, see problem 9. We will return to the issue of angle preservation in chapters 4 and 6.

Coined by Cauchy's students C. A. Briot and J. C. Bouquet, "holomorphic" comes from the Greek *ὅλος* (whole) and *μορφή* (form). According to R. Remmert, the widespread adoption of the notation $\mathscr{O}$ for holomorphic appears to have been purely accidental.

DEFINITION 1.11. Let $U \subset \mathbb{C}$ be non-empty and open. A function $f : U \to \mathbb{C}$ is called ***holomorphic*** in U if $f'(p)$ exists for every $p \in U$. The set of all holomorphic functions in U is denoted by $\mathscr{O}(U)$. Elements of $\mathscr{O}(\mathbb{C})$ are called ***entire functions***.

It follows from Theorem 1.2 that sums, products, and compositions of holomorphic functions are holomorphic. In particular, pointwise addition and multiplication of functions make $\mathscr{O}(U)$ into a commutative ring with identity.

EXAMPLE 1.12 (Ratios). By Theorem 1.2(i), if $f, g \in \mathscr{O}(U)$ and $g \neq 0$ in U, then $f/g \in \mathscr{O}(U)$. An important example is provided by ***rational functions***: If f and g are polynomials in z, g not identically zero, and if $p_1, \ldots, p_n$ are all the roots of the polynomial equation $g(z) = 0$, then the rational function f/g is holomorphic in $\mathbb{C} \smallsetminus \{p_1, \ldots, p_n\}$.

The following is an immediate corollary of Theorem 1.7:

THEOREM 1.13. *Suppose $f = u + iv$ is real differentiable as a map $U \to \mathbb{R}^2$. Then $f \in \mathscr{O}(U)$ if and only if*

$$f_{\bar{z}} = 0, \tag{1.8}$$

or equivalently,

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x \tag{1.9}$$

throughout U. In this case,

$$f' = f_z = f_x = -i f_y.$$

The Cauchy-Riemann equations had been studied earlier in the 18th century by d'Alembert and Euler.

The pair of equations (1.9) are classically known as the ***Cauchy-Riemann equations***. The equivalent form (1.8) is called the ***complex Cauchy-Riemann equation***.

EXAMPLE 1.14. The ***exponential function*** $\exp : \mathbb{C} \to \mathbb{C}$ defined by

$$\exp(z) = e^z = e^x\, e^{iy} = e^x(\cos y + i \sin y)$$

is entire. In fact,

$$\frac{\partial}{\partial x} \exp = e^x\, e^{iy} \quad \text{and} \quad \frac{\partial}{\partial y} \exp = ie^x\, e^{iy}, \quad \text{so} \quad \frac{\partial}{\partial \bar{z}} \exp = 0.$$

It follows that

$$\exp' = \frac{\partial}{\partial z} \exp = \exp\,.$$

The basic identity

$$\exp(z+w) = \exp(z)\ \exp(w) \qquad \text{for } z, w \in \mathbb{C}$$

can be proved as follows: Fix w and set $f(z) = \exp(z+w)$. By the chain rule, $f'(z) = \exp(z+w) = f(z)$. Since $\exp \neq 0$, the ratio $g = f/\exp$ is entire and $g' = 0$ everywhere by the quotient rule. It follows that g is a constant function (this can be seen, for example, by noting that $g' = 0$ implies that the real and imaginary parts of g have vanishing partial derivatives in the plane, hence are constant). Since $g(0) = \exp(w)$, we conclude that $g(z) = \exp(z+w)/\exp(z) = \exp(w)$ for all z, as required.

EXAMPLE 1.15. Let $\varphi : [0,1] \to [0,1]$ be a ***Cantor function***, i.e., a continuous non-decreasing function which satisfies $\varphi(0) = 0$, $\varphi(1) = 1$, and $\varphi' = 0$ a.e. (the graph of such φ is often called a "devil's staircase"). Extend φ to a map $\mathbb{R} \to \mathbb{R}$ by setting $\varphi(x+n) = \varphi(x) + n$ for $0 \leq x \leq 1$ and $n \in \mathbb{Z}$. Define $f : \mathbb{C} \to \mathbb{C}$ by

As is customary, "a.e." is short for "almost everywhere," that is, outside a set of Lebesgue measure zero.

$$f(x+iy) = x + i(y + \varphi(x)).$$

Then f is a homeomorphism, with $f_x = 1$ and $f_y = i$; hence $f_z = 1$ and $f_{\bar{z}} = 0$ a.e. on $\mathbb{C}$. However, f is not holomorphic since φ', and hence f_z, fails to exist everywhere. This does not contradict Theorem 1.13 because f is not real differentiable.

REMARK 1.16. The implication $f_{\bar{z}} = 0 \Longrightarrow f \in \mathscr{O}(U)$ holds under much weaker conditions than real differentiability in Theorem 1.13. For example, a generalization of a classical theorem of Looman and Menshov asserts that if $f : U \to \mathbb{C}$ is continuous, f_z and $f_{\bar{z}}$ exist outside a countable set in U, and $f_{\bar{z}} = 0$ a.e. in U, then $f \in \mathscr{O}(U)$ [**GM**]. Another well-known result of the same flavor is "Weyl's lemma" which is important in the theory of quasiconformal maps (see [**A3**]).

We end this section with a brief justification for computing f_z and $f_{\bar{z}}$ as partial derivatives. Suppose $F(z,w)$ is holomorphic in each variable near a point $(p, \bar{p}) \in \mathbb{C} \times \mathbb{C}$. This means there is an $r > 0$ such that $z \mapsto F(z,w)$ is holomorphic in $\mathbb{D}(p,r)$ for each fixed $w \in \mathbb{D}(\bar{p}, r)$ and $w \mapsto F(z,w)$ is holomorphic in $\mathbb{D}(\bar{p}, r)$ for each fixed $z \in \mathbb{D}(p,r)$. Set $z = x+iy$ and $w = s+it$. Then F, viewed as a function of the four real variables (x,y,s,t), can be shown to be real differentiable. Consider the function $f(z) = F(z, \bar{z}) = F(x,y,x,-y)$ which is defined in some neighborhood of p. Using the symbol D_j for partial differentiation with respect to the j-th variable, we apply the chain rule to obtain

$$f_x = D_1F + D_3F \qquad \text{and} \qquad f_y = D_2F - D_4F,$$

where the left sides of these equations are evaluated at z and the right sides are evaluated at $(z, \bar{z}) = (x, y, x, -y)$. This gives

$$f_z = \tfrac{1}{2}(f_x - if_y) = \tfrac{1}{2}(D_1F - iD_2F) + \tfrac{1}{2}(D_3F + iD_4F)$$
$$f_{\bar{z}} = \tfrac{1}{2}(f_x + if_y) = \tfrac{1}{2}(D_1F + iD_2F) + \tfrac{1}{2}(D_3F - iD_4F).$$

Denote by D_zF and D_wF the complex derivatives of F with respect to each variable when the other is kept fixed. Then $(D_1F - iD_2F)/2 = D_zF$ and $D_3F + iD_4F = 0$ since F is holomorphic in w. Similarly, $(D_3F - iD_4F)/2 = D_wF$ and $D_1F + iD_2F = 0$ since F is holomorphic in z. It follows that

$$f_z = D_zF \qquad \text{and} \qquad f_{\bar{z}} = D_wF.$$

This means f_z and $f_{\bar{z}}$ are obtained by taking the partial derivatives of $F(z, w)$ with respect to z and w, respectively, and then substituting $w = \bar{z}$.

In Example 1.5, this result can be applied to the polynomial function $F(z, w) = \sum_{j,k} a_{jk}\, z^j\, w^k$ to justify the given formulas. In Example 1.6, it can be applied to $F(z, w) = \log(zw)$ which, as we will see in chapter 2, has well-defined holomorphic branches in each variable in a small neighborhood of $(p, \bar{p})$ provided that $p \neq 0$.

1.2 Complex analytic functions

For every $p \in \mathbb{C}$ and every sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers, we can form the ***power series***

$$\sum_{n=0}^{\infty} a_n\, (z-p)^n \tag{1.10}$$

in the complex variable z. Such series provide an abundance of examples of holomorphic functions and play a central role in complex analysis, especially the classical function theory according to Weierstrass. For now, the basic fact that we need to know (or remember) is that each power series (1.10) has a ***disk of convergence*** about p characterized by the property that it converges within this disk and diverges outside of it. Moreover, we can effectively compute the radius of this disk once we know the coefficients a_n or merely their asymptotic behavior as $n \to \infty$. This fact is stated more precisely in the following

THEOREM 1.17 (Cauchy, 1821). *Consider the power series* (1.10) *and define*

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \in [0, +\infty]. \tag{1.11}$$

Then (1.10) *converges absolutely and uniformly in the disk* $\mathbb{D}(p, r)$ *for every* $r < R$ *and diverges at every point* z *with* $|z - p| > R$.

This result is also attributed to Hadamard who rediscovered it in 1888.

The number R is called the ***radius of convergence*** of the power series (1.10). Observe that the possibilities $R = 0$ or $R = +\infty$ have not been excluded.

PROOF. First consider the power series inside the disk of radius R. If $R = 0$ there is nothing to prove, so assume $R > 0$ and let $0 < r < s < R$. The definition of R shows that there is an integer $N \geq 1$ such that $|a_n|\, s^n < 1$ for all $n \geq N$. If $|z - p| < r$, then

$$\sum_{n=N}^{\infty} |a_n|\, |z-p|^n \leq \sum_{n=N}^{\infty} |a_n|\, r^n = \sum_{n=N}^{\infty} |a_n|\, s^n \left(\frac{r}{s}\right)^n \leq \sum_{n=N}^{\infty} \left(\frac{r}{s}\right)^n .$$

Since $r/s < 1$, the far right geometric series converges, which proves that the power series converges absolutely and uniformly in $\mathbb{D}(p, r)$.

Now consider the power series outside the disk of radius R. If $R = +\infty$ there is nothing to prove, so assume $R < +\infty$ and let $r > s > R$. The definition of R shows that there is an increasing sequence S of positive integers such that $|a_n|\, s^n > 1$ for all $n \in S$. If $|z - p| = r$, then

$$|a_n|\, |z-p|^n = |a_n|\, s^n \left(\frac{r}{s}\right)^n > \left(\frac{r}{s}\right)^n$$

whenever $n \in S$. Since $r/s > 1$, it follows that the power series diverges since its general term fails to converge to zero. □

The behavior of power series on the circle of convergence $|z - p| = R$ is much more subtle. In fact, no general statement similar to Theorem 1.17 can be made for what should be happening on this circle.

EXAMPLE 1.18. The power series

$$\sum_{n=0}^{\infty} z^n \qquad \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad \sum_{n=1}^{\infty} \frac{z^n}{n}$$

all have a radius of convergence of 1. The first diverges everywhere on the unit circle $\partial\mathbb{D}$ since its general term z^n does not tend to zero when $|z| = 1$. The second converges uniformly on $\partial\mathbb{D}$ since it is dominated by the convergent series $\sum 1/n^2$. The third converges at every point of $\partial\mathbb{D}$ other than 1 (see problem 13).

More on the behavior of power series on the circle of convergence will be discussed in chapter 10.

DEFINITION 1.19. Let $U \subset \mathbb{C}$ be non-empty and open. We call a function $f : U \to \mathbb{C}$ ***complex analytic*** if for every disk $\mathbb{D}(p, r) \subset U$ there exists a power series $\sum_{n=0}^{\infty} a_n\, (z - p)^n$ which converges to $f(z)$ whenever $z \in \mathbb{D}(p, r)$.

It is a fundamental fact that a function is complex analytic in U if and only if it is holomorphic in U. The following theorem proves the "only if" part of this statement. The "if" part, which is more difficult, will be proved in Theorem 1.37.

THEOREM 1.20. *Let $f : U \to \mathbb{C}$ be complex analytic. Then*

(i) $f \in \mathscr{O}(U)$ and f' is also complex analytic in U.

(ii) All higher derivatives $f^{(k)}$ for $k \geq 1$ exist and are complex analytic in U. Moreover, the power series representation of the higher derivatives are obtained by term-by-term differentiation of that of f, that is, if $\mathbb{D}(p,r) \subset U$ and

$$f(z) = \sum_{n=0}^{\infty} a_n\,(z-p)^n \qquad \text{for } z \in \mathbb{D}(p,r), \tag{1.12}$$

then the representation

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\,a_n\,(z-p)^{n-k}$$

holds in $\mathbb{D}(p,r)$.

(iii) The coefficients $\{a_n\}$ of the power series (1.12) *are given by*

$$a_n = \frac{f^{(n)}(p)}{n!} \qquad (n \geq 0). \tag{1.13}$$

In particular, $\{a_n\}$ is uniquely determined by f, so any power series in $z - p$ which converges to f in some disk in U centered at p must coincide with (1.12).

PROOF. Define $g(z) = \sum_{n=1}^{\infty} na_n\,(z-p)^{n-1}$. Note that by $\lim_{n\to\infty} \sqrt[n]{n} = 1$ and the formula (1.11), the power series with coefficients $\{na_n\}$ has the same radius of convergence as the power series with coefficients $\{a_n\}$, so g converges in $\mathbb{D}(p,r)$. We will show that for every $z_0 \in \mathbb{D}(p,r)$, $f'(z_0)$ exists and is equal to $g(z_0)$. This will prove (i). Evidently, (ii) follows by induction from (i), and (iii) follows from (ii).

After replacing $z - p$ by z, we may assume $p = 0$. Fix $z_0 \in \mathbb{D}(0,r)$ and take any $\varepsilon > 0$. Choose s such that $|z_0| < s < r$. Since the power series of g converges absolutely in $\mathbb{D}(0,r)$, we can find an integer $N \geq 2$ such that

$$\sum_{n=N+1}^{\infty} n|a_n|\,s^{n-1} < \varepsilon.$$

For $z \neq z_0$ in $\mathbb{D}(0,r)$, write

$$\begin{aligned}\frac{f(z)-f(z_0)}{z-z_0} - g(z_0) &= \sum_{n=2}^{\infty} a_n \left(\frac{z^n - z_0^n}{z - z_0} - nz_0^{n-1} \right) \\ &= \left(\sum_{n=2}^{N} + \sum_{n=N+1}^{\infty} \right) a_n\,(z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1} - nz_0^{n-1}).\end{aligned}$$

The first (finite) sum tends to 0 as $z \to z_0$. The second sum has its absolute value bounded above by $\sum_{n=N+1}^{\infty} 2n|a_n|\, s^{n-1} < 2\varepsilon$ if $|z| < s$. Hence,

$$\limsup_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \right| \leq 2\varepsilon.$$

Since ε was arbitrary, it follows that $\lim_{z\to z_0} (f(z) - f(z_0))/(z - z_0)$ exists and is equal to $g(z_0)$. □

EXAMPLE 1.21. The radius of convergence of the power series $f(z) = \sum_{n=0}^{\infty} z^n$ is 1, hence $f \in \mathscr{O}(\mathbb{D})$. The formula for the sum of a geometric series shows that in fact $f(z) = 1/(1-z)$. Term-by-term differentiation of this power series, which is legitimate by Theorem 1.20, yields other useful formulas. For example, it follows that

$$\sum_{n=1}^{\infty} n\, z^{n-1} = \left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2},$$

so

$$\sum_{n=1}^{\infty} n\, z^n = \frac{z}{(1-z)^2} \qquad \text{for } |z| < 1.$$

Differentiating once more, we obtain

$$\sum_{n=1}^{\infty} n^2\, z^{n-1} = \left(\frac{z}{(1-z)^2}\right)' = \frac{1+z}{(1-z)^3},$$

so

$$\sum_{n=1}^{\infty} n^2\, z^n = \frac{z(1+z)}{(1-z)^3} \qquad \text{for } |z| < 1.$$

Continuing inductively, we can find closed expressions (as rational functions in z) for the power series $\sum_{n=1}^{\infty} n^p\, z^n$ in the unit disk for every positive integer p.

EXAMPLE 1.22. Since $\lim_{n\to\infty} 1/\sqrt[n]{n!}$ is easily seen to be 0, the radius of convergence of the power series $f(z) = \sum_{n=0}^{\infty} z^n/n!$ is $+\infty$. Hence, by Theorem 1.20, f is an entire function with $f(0) = 1$, and term-by-term differentiation gives $f' = f$. The exponential function exp defined in example 1.14 also satisfies $\exp(0) = 1$ and $\exp' = \exp$. It follows that the ratio $g = f/\exp$ is entire, $g(0) = 1$, and $g' = 0$ everywhere by the quotient rule, which shows g is the constant function 1. Thus, we arrive at the following alternative formula for the exponential function:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \text{for all } z \in \mathbb{C}.$$

1.3 Complex integration

We now turn to integration of complex-valued functions along curves. Our standing assumption in this section is that all curves are piecewise smooth. This regularity assumption greatly simplifies the arguments but it is not essential, as one can fashion a definition to allow more general curves such as those that are merely rectifiable. However, such generalizations are not worth the extra effort: The integration theory we are about to develop will be applied almost exclusively to holomorphic functions and such integrals, as we will see in chapter 2, depend only on the "homology class" of the curve. This means the integral along an arbitrary curve (rectifiable or not) can be defined as the integral along any piecewise smooth curve in the same homology class. Thus, one can ultimately arrive at the most general definition using only the special case treated here.

Let $U \subset \mathbb{C}$ be a non-empty open set. A ***curve*** in U is a continuous map $\gamma : [a,b] \to U$, where $[a,b] = \{t \in \mathbb{R} : a \leq t \leq b\}$. We call $\gamma(a)$ the ***initial point*** and $\gamma(b)$ the ***end point*** of γ. For simplicity we often say that γ is a curve ***from*** $\gamma(a)$ ***to*** $\gamma(b)$. γ is a ***closed*** curve if $\gamma(a) = \gamma(b)$. We denote by $|\gamma|$ the ***image*** of γ as a subset of $\mathbb{C}$, that is, $|\gamma| = \{\gamma(t) : t \in [a,b]\}$. We say γ is ***piecewise*** C^1 if there is a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that for each $1 \leq k \leq n$, γ is continuously differentiable in the open interval (t_{k-1}, t_k), and the one-sided limits $\lim_{t \to t_{k-1}^+} \gamma'(t)$ and $\lim_{t \to t_k^-} \gamma'(t)$ exist.

Throughout this section we will assume that all curves are piecewise C^1*, even if it is not explicitly mentioned.*

DEFINITION 1.23. Let $\gamma : [a,b] \to \mathbb{C}$ be a curve and $f : |\gamma| \to \mathbb{C}$ be a continuous function. The ***integral of f along γ*** is the complex number defined by

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt. \tag{1.14}$$

Here $\gamma' = d\gamma/dt$ is defined at all but finitely many points of $[a,b]$.

By writing $f = u + iv$ and $\gamma(t) = x(t) + iy(t)$, and separating the real and imaginary parts, we see that the integral in (1.14) can be written in terms of a pair of classical "line integrals" along γ:

$$\int_\gamma f(z)\, dz = \int_\gamma (u\, dx - v\, dy) + i \int_\gamma (v\, dx + u\, dy). \tag{1.15}$$

It is easy to see that the right side of (1.14) remains unchanged if we reparametrize γ. In fact, if $\varphi : [c,d] \to [a,b]$ is a C^1 orientation-preserving homeomorphism and $\eta = \gamma \circ \varphi$, then by the change of variable formula in calculus,

$$\begin{aligned} \int_c^d f(\eta(t))\, \eta'(t)\, dt &= \int_c^d f(\gamma(\varphi(t)))\, \gamma'(\varphi(t))\, \varphi'(t)\, dt \\ &= \int_a^b f(\gamma(s))\, \gamma'(s)\, ds. \end{aligned}$$

In particular, the domain of γ can always be chosen to be the unit interval $[0,1]$ by precomposing it with an affine map, namely by considering the reparametrized curve $t \mapsto \gamma((1-t)a+tb)$ instead.

Given $\gamma:[0,1]\to\mathbb{C}$, the ***reverse curve*** $\gamma^-:[0,1]\to\mathbb{C}$ is defined by $\gamma^-(t)=\gamma(1-t)$. Since

$$\begin{aligned}\int_0^1 f(\gamma^-(t))\,(\gamma^-)'(t)\,dt &= -\int_0^1 f(\gamma(1-t))\,\gamma'(1-t)\,dt\\ &= -\int_0^1 f(\gamma(t))\,\gamma'(t)\,dt,\end{aligned}$$

we see that

$$\int_{\gamma^-} f(z)\,dz = -\int_\gamma f(z)\,dz.$$

There is an obvious way of combining two curves whenever the end point of one is the initial point of the other: If $\gamma,\eta:[0,1]\to U$ are curves such that $\gamma(1)=\eta(0)$, we can define the ***product*** $\gamma\centerdot\eta:[0,1]\to U$ by

$$(\gamma\centerdot\eta)(t)=\begin{cases}\gamma(2t) & t\in[0,1/2]\\ \eta(2t-1) & t\in[1/2,1].\end{cases}$$

This amounts to first traveling along γ and then along η, both with twice the usual speed so as to finish the journey in unit time. The additivity property of the integral shows that the relation

$$\int_{\gamma\centerdot\eta} f(z)\,dz = \int_\gamma f(z)\,dz + \int_\eta f(z)\,dz \tag{1.16}$$

holds for every continuous function $f:|\gamma|\cup|\eta|\to\mathbb{C}$.

Closely related to the complex integral is the notion of the line integral of $f:U\to\mathbb{C}$, viewed as a scalar function, as one learns in calculus:

$$\int_\gamma f(z)\,|dz| = \int_0^1 f(\gamma(t))\,|\gamma'(t)|\,dt. \tag{1.17}$$

For example, the case $f=1$ gives

$$\int_\gamma |dz| = \int_0^1 |\gamma'(t)|\,dt.$$

This is by definition the ***length*** of γ for which we use the notation $\operatorname{length}(\gamma)$. It is evident that

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz|.$$

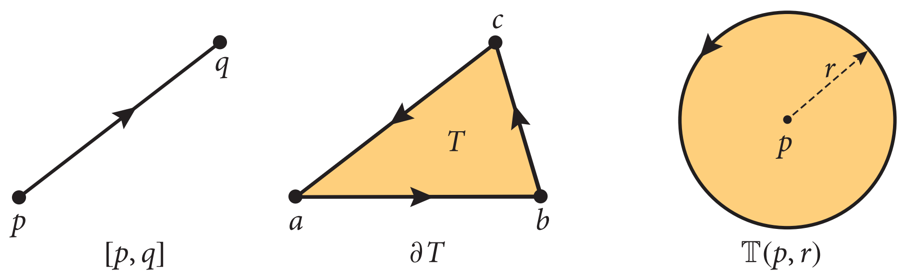

Figure 1.1. Three basic curves that frequently arise in complex integration. From left to right: an oriented segment, the oriented boundary of a triangle, and an oriented circle.

This proves the following useful inequality which we will frequently invoke when estimating integrals:

$$\left|\int_\gamma f(z)\,dz\right| \le \sup_{z\in|\gamma|} |f(z)| \cdot \text{length}(\gamma). \tag{1.18}$$

If we denote the supremum of $|f|$ on $|\gamma|$ by M and the length of γ by L, then (1.18) reads $|\int_\gamma f(z)\,dz| \le ML$. This is why (1.18) is informally referred to as ***the ML-inequality***.

EXAMPLE 1.24 (Oriented segments). For $p, q \in \mathbb{C}$, let $[p, q]$ denote the oriented line segment traversed once from p to q (see Fig. 1.1 left). We take $\gamma : [0, 1] \to \mathbb{C}$ defined by $\gamma(t) = (1-t)p + tq$ as the standard parametrization of $[p, q]$. In the interest of simplifying our notations, we often denote the image $|\gamma|$ by $[p, q]$ as well. Thus, for any continuous function $f : [p, q] \to \mathbb{C}$,

$$\int_{[p,q]} f(z)\,dz = (q-p)\int_0^1 f((1-t)p + tq)\,dt.$$

Note that $\int_{[q,p]} f(z)\,dz = -\int_{[p,q]} f(z)\,dz$ since $[q, p]$ is the reverse of $[p, q]$.

EXAMPLE 1.25 (Oriented triangle boundaries). Let T be the closed triangle with vertices a, b, c labeled counterclockwise. We use the notation $T = \triangle abc$, so $\triangle abc = \triangle bca = \triangle cab$. By definition, the oriented boundary ∂T is the product $([a, b] \centerdot [b, c]) \centerdot [c, a]$, that is, the closed curve obtained from the segment $[a, b]$, followed by $[b, c]$, followed by $[c, a]$ (see Fig. 1.1 middle). Again, to simplify notations, we denote the corresponding subset of the plane by ∂T as well. By (1.16), for any continuous function $f : \partial T \to \mathbb{C}$,

$$\int_{\partial T} f(z)\,dz = \int_{[a,b]} f(z)\,dz + \int_{[b,c]} f(z)\,dz + \int_{[c,a]} f(z)\,dz.$$

EXAMPLE 1.26 (Oriented circles). For $p \in \mathbb{C}$ and $r > 0$, let $\mathbb{T}(p, r)$ denote the oriented circle $|z - p| = r$ traversed once in the counterclockwise direction (see Fig. 1.1 right). We take $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(t) = p + re^{it}$ as the standard parametrization of $\mathbb{T}(p, r)$, and often denote the image $|\gamma| = \{z \in \mathbb{C} : |z - p| = r\}$ by $\mathbb{T}(p, r)$ as well (the distinction is easily understood from the context).

Evidently, for any continuous function $f : \mathbb{T}(p,r) \to \mathbb{C}$,

$$\int_{\mathbb{T}(p,r)} f(z)\, dz = ir \int_0^{2\pi} f(p + re^{it})e^{it}\, dt.$$

As a special case, consider a continuous complex-valued function f defined on the ***unit circle*** $\mathbb{T} = \mathbb{T}(0,1)$. The integral of f as a scalar function can be expressed as a complex integral:

$$\int_0^{2\pi} f(e^{it})\, dt = \int_{\mathbb{T}} f(z)\, \frac{dz}{iz}.$$

More generally, the ***Fourier coefficients*** of f, defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\, e^{-int}\, dt \qquad (n \in \mathbb{Z}),$$

can be expressed as the complex integrals

$$\hat{f}(n) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(z)}{z^{n+1}}\, dz.$$

The following elementary observation will be useful:

LEMMA 1.27 (Continuous dependence on vertices)**.** *Let $f : U \to \mathbb{C}$ be continuous and $T = \triangle abc$ be a closed triangle in U. Then, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|a - a'|$, $|b - b'|$, and $|c - c'|$ are all less than δ, then $T' = \triangle a'b'c' \subset U$ and*

$$\left| \int_{\partial T} f(z)\, dz - \int_{\partial T'} f(z)\, dz \right| < \varepsilon.$$

PROOF. Since the integral along the oriented boundary of a triangle is the sum of three integrals along oriented segments, it suffices to prove continuous dependence for oriented segments. Fix $[a,b] \subset U$ and let V be any open neighborhood of $[a,b]$ whose closure $\overline{V}$ is a compact subset of U. Given $\varepsilon > 0$, use uniform continuity of f on V to find $0 < \delta < \varepsilon$ such that $|f(z) - f(w)| < \varepsilon$ whenever $z, w \in V$ and $|z - w| < \delta$. We can also arrange that $[a', b'] \subset V$ whenever $|a - a'| < \delta$ and $|b - b'| < \delta$. Let $[a', b']$ be any such segment and note that if $\gamma(t) = (1-t)a + tb$ and $\eta(t) = (1-t)a' + tb'$, then

$$\begin{aligned} |\gamma(t) - \eta(t)| &\leq (1-t)|a - a'| + t|b - b'| < \delta, \\ |\gamma'(t) - \eta'(t)| &= |(b-a) - (b' - a')| \leq |b - b'| + |a - a'| < 2\delta \end{aligned}$$

for all $0 \leq t \leq 1$. Hence,

$$\begin{aligned} \left| \int_{[a,b]} f(z)\, dz - \int_{[a',b']} f(z)\, dz \right| &= \left| \int_0^1 \left[f(\gamma(t))\, \gamma'(t) - f(\eta(t))\, \eta'(t) \right] dt \right| \\ &= \left| \int_0^1 \left[\big(f(\gamma(t)) - f(\eta(t)) \big) \gamma'(t) + f(\eta(t)) \big(\gamma'(t) - \eta'(t) \big) \right] dt \right| \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 |f(\gamma(t)) - f(\eta(t))|\,|\gamma'(t)|\,dt + \int_0^1 |f(\eta(t))|\,|\gamma'(t) - \eta'(t)|\,dt \\
&\leq |b-a| \int_0^1 |f(\gamma(t)) - f(\eta(t))|\,dt + 2\delta \int_0^1 |f(\eta(t))|\,dt \\
&\leq |b-a|\,\varepsilon + 2\delta \sup_{z\in V} |f(z)| \leq \big(|b-a| + 2\sup_{z\in V}|f(z)|\big)\,\varepsilon,
\end{aligned}
$$

which proves the asserted continuity. □

A primitive is what students of calculus call "antidetrivative."

DEFINITION 1.28. A function $F \in \mathscr{O}(U)$ is called a ***primitive*** of a continuous function $f : U \to \mathbb{C}$ if $F'(z) = f(z)$ for all $z \in U$.

Suppose F is a primitive of f and $\gamma : [0,1] \to U$ is a piecewise C^1 curve. By the chain rule, the relation $(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t)$ holds for all but finitely many $t \in [0,1]$ (see problem 6). Since $F'(\gamma(t))\gamma'(t)$ is piecewise continuous on $[0,1]$ with at worst jump discontinuities, the fundamental theorem of calculus shows that

$$
\begin{aligned}
\int_\gamma f(z)\,dz &= \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt = \int_0^1 F'(\gamma(t))\,\gamma'(t)\,dt \\
&= \int_0^1 (F\circ\gamma)'(t)\,dt = F(\gamma(1)) - F(\gamma(0)).
\end{aligned}
$$

THEOREM 1.29. *A continuous function $f : U \to \mathbb{C}$ has a primitive in U if and only if $\int_\gamma f(z)\,dz = 0$ for every closed curve γ in U.*

PROOF. First suppose f has a primitive F. If $\gamma : [0,1] \to U$ is a closed curve, then $\gamma(0) = \gamma(1)$, so

$$
\int_\gamma f(z)\,dz = F(\gamma(1)) - F(\gamma(0)) = 0.
$$

Conversely, suppose f integrates to zero along every closed curve in U. To show f has a primitive, it suffices to consider the case when U is connected (and therefore path-connected); the general case follows by applying this case to each connected component of U. If γ, η are two curves in U with the same initial and end points, then the product $\gamma \centerdot \eta^-$ is a closed curve. Hence, by additivity (1.16) and our assumption,

$$
\int_\gamma f(\zeta)\,d\zeta - \int_\eta f(\zeta)\,d\zeta = \int_\gamma f(\zeta)\,d\zeta + \int_{\eta^-} f(\zeta)\,d\zeta = \int_{\gamma\centerdot\eta^-} f(\zeta)\,d\zeta = 0.
$$

Now fix a point $p \in U$. For any $z \in U$ use path-connectivity of U to find a curve γ in U from p to z and define

$$
F(z) = \int_\gamma f(\zeta)\,d\zeta.
$$

By the above remark, the right-hand side is independent of the choice of γ and yields a well-defined function $F : U \to \mathbb{C}$. Let us show that F is a primitive of f. Fix $z_0 \in U$ and choose $r > 0$ small enough so that $\mathbb{D}(z_0, r) \subset U$. Let $z \in \mathbb{D}(z_0, r)$ and let γ be any curve in U from p to z_0. The product $\gamma \centerdot [z_0, z]$ is then a curve in U from p to z. By additivity,

$$F(z) - F(z_0) = \int_{\gamma \centerdot [z_0,z]} f(\zeta)\, d\zeta - \int_\gamma f(\zeta)\, d\zeta = \int_{[z_0,z]} f(\zeta)\, d\zeta,$$

so if $z \neq z_0$,

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0,z]} (f(\zeta) - f(z_0))\, d\zeta. \tag{1.19}$$

Since f is continuous at z_0, for each $\varepsilon > 0$ we can find a $0 < \delta < r$ such that $|f(\zeta) - f(z_0)| < \varepsilon$ whenever $|\zeta - z_0| < \delta$. Since $|z - z_0| < \delta$ implies $|\zeta - z_0| < \delta$ for every $\zeta \in [z_0, z]$, the ML-inequality (1.18) applied to the right side of (1.19) gives

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \frac{1}{|z - z_0|} \cdot \varepsilon \cdot \text{length}([z_0, z]) = \varepsilon$$

whenever $0 < |z - z_0| < \delta$. Thus, $F'(z_0)$ exists and is equal to $f(z_0)$. Since $z_0 \in U$ was arbitrary, we conclude that F is a primitive of f in U. $\square$

EXAMPLE 1.30. For every integer $n \neq -1$, the power function $f(z) = z^n$ has a primitive $F(z) = z^{n+1}/(n+1)$. It follows from Theorem 1.29 that $\int_\gamma z^n\, dz = 0$ if γ is any closed curve in the punctured plane $\mathbb{C} \smallsetminus \{0\}$ and $n \neq -1$, or if γ is any closed curve in $\mathbb{C}$ and $n \geq 0$.

The case $n = -1$ is completely different: For any $r > 0$,

$$\int_{\mathbb{T}(0,r)} \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{re^{it}}\, rie^{it}\, dt = 2\pi i \neq 0.$$

Note that the result is independent of the radius r. It follows from Theorem 1.29 that *the function* $z \mapsto 1/z$ *does not have a primitive in any punctured neighborhood of* 0.

1.4 Cauchy's theory in a disk

Our primary goal in this section is to prove that every holomorphic function in a disk has a primitive. Somewhat surprisingly, all the local properties of holomorphic functions are consequences of this central fact of Cauchy's theory. The special case of a disk will be enough for our purposes here; general domains and global issues will be dealt with in chapter 2.

According to Theorem 1.29, the existence of a primitive is equivalent to having vanishing integrals along all closed curves. Convexity of the disk allows us to replace the latter with something far simpler in terms of triangles.

THEOREM 1.31. *Let $D \subset \mathbb{C}$ be an open disk and $f : D \to \mathbb{C}$ be continuous. Suppose $\int_{\partial T} f(z)\, dz = 0$ for every closed triangle $T \subset D$. Then f has a primitive in D.*

Proof. Let p be the center of D and define

$$F(z) = \int_{[p,z]} f(\zeta)\, d\zeta \qquad \text{for } z \in D.$$

We show that F is a primitive of f. Take distinct points $z_0, z \in D$ and apply the condition $\int_{\partial T} f(\zeta)\, d\zeta = 0$ to the closed triangle T with vertices p, z, z_0 to obtain

$$F(z) - F(z_0) = \int_{[p,z]} f(\zeta)\, d\zeta - \int_{[p,z_0]} f(\zeta)\, d\zeta = \int_{[z_0,z]} f(\zeta)\, d\zeta.$$

The rest of the argument, that is, dividing by $z - z_0$ and letting $z \to z_0$ to show that $F'(z_0) = f(z_0)$, is identical to the proof of Theorem 1.29. □

The problem of constructing primitives in D is thus reduced to showing that every $f \in \mathcal{O}(D)$ satisfies the triangle condition of Theorem 1.31. If we knew that the derivative f' is continuous (which is true but we have not yet proved it), this would be an easy consequence of Green's theorem. To see this, suppose $f \in \mathcal{O}(D)$ and *assume* f' is continuous in D. Then the partial derivatives of $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ are continuous in D and Green's theorem together with the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ shows that for every closed triangle $T \subset D$,

$$\int_{\partial T} (u\, dx - v\, dy) = \iint_T (-v_x - u_y)\, dx\, dy = 0$$

and

$$\int_{\partial T} (v\, dx + u\, dy) = \iint_T (u_x - v_y)\, dx\, dy = 0.$$

Hence, by (1.15), $\int_{\partial T} f(z)\, dz = 0$.

It was Goursat's key observation that the triangle condition for a holomorphic function can be proved directly without any reference to Green's theorem and continuity of the derivative.

Goursat's formulation of Theorem 1.32 was in fact more complicated. It was A. Pringsheim who in 1901 realized it suffices to consider triangles.

THEOREM 1.32 (Goursat, 1900). *If $f \in \mathcal{O}(U)$, then $\int_{\partial T} f(z)\, dz = 0$ for every closed triangle $T \subset U$.*

Proof. Fix a closed triangle $T \subset U$ and set $I = \int_{\partial T} f(z)\, dz$. Connect the midpoints of the edges of T to form four congruent triangles, each having half the diameter of T. It is easy to see that I is the sum of the integrals of f along the oriented boundaries of these four triangles (see Fig. 1.2). Hence, one of these triangles, which we call T_1, satisfies

$$\left| \int_{\partial T_1} f(z)\, dz \right| \geq \frac{1}{4} |I|.$$

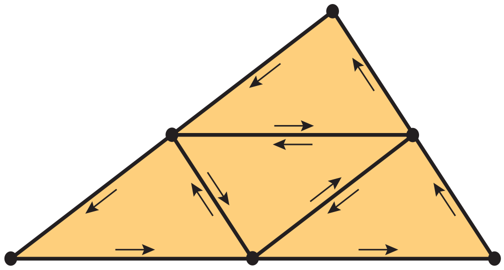

Figure 1.2. The integral along the oriented boundary of the large triangle is equal to the sum of the integrals along the oriented boundaries of the four smaller ones because each internal edge is traversed twice in opposite directions, so its net contribution to the integral is zero.

Replacing T by T_1 in the above construction and continuing inductively, we obtain a nested sequence $T \supset T_1 \supset T_2 \supset T_3 \supset \cdots$ of closed triangles with the properties

$$\operatorname{diam}(T_n) = 2^{-n} \operatorname{diam}(T) \qquad \text{and} \qquad \left| \int_{\partial T_n} f(z)\, dz \right| \geq 4^{-n} |I|.$$

Here "diam" denotes the Euclidean diameter.

The nested intersection $\bigcap_{n=1}^{\infty} T_n$ is a single point $p \in U$. By the assumption, $f'(p)$ exists, so given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z) - f(p) - f'(p)(z-p)| \leq \varepsilon |z-p| \quad \text{whenever} \quad |z-p| < \delta.$$

Choose n large enough that $\operatorname{diam}(T_n) < \delta$. If $z \in \partial T_n$, then $|z-p| \leq \operatorname{diam}(T_n)$, so

$$|f(z) - f(p) - f'(p)(z-p)| \leq \varepsilon \operatorname{diam}(T_n).$$

Observe that by Theorem 1.29,

$$\int_{\partial T_n} (f(p) + f'(p)(z-p))\, dz = 0$$

since the integrand has a primitive $f(p)z + (1/2)f'(p)(z-p)^2$. Hence, by the *ML*-inequality (1.18),

$$\begin{aligned} 4^{-n}|I| \leq \left| \int_{\partial T_n} f(z)\, dz \right| &= \left| \int_{\partial T_n} (f(z) - f(p) - f'(p)(z-p))\, dz \right| \\ &\leq \varepsilon \ \operatorname{diam}(T_n) \ \operatorname{length}(\partial T_n) \\ &= \varepsilon\, 2^{-n} \operatorname{diam}(T) \cdot 2^{-n} \operatorname{length}(\partial T), \end{aligned}$$

which implies

$$|I| \leq \varepsilon \ \operatorname{diam}(T) \ \operatorname{length}(\partial T).$$

Since this is true for every $\varepsilon > 0$, we must have $I = 0$. □

Edouard Jean-Baptiste Goursat (1858–1936)

Theorems 1.31 and 1.32 put together now imply the following

THEOREM 1.33. *Let $D \subset \mathbb{C}$ be an open disk and $f \in \mathscr{O}(D)$. Then f has a primitive in D.*

Combining Theorem 1.29 and Theorem 1.33, we arrive at

THEOREM 1.34 (Cauchy's theorem in a disk, 1825)**.** *Let $D \subset \mathbb{C}$ be an open disk and $f \in \mathscr{O}(D)$. Then for every closed curve γ in D,*

$$\int_\gamma f(z)\, dz = 0.$$

REMARK 1.35. Here is a minor technical point that will be exploited in the next result: Cauchy's Theorem 1.34 remains true under the apparently weaker assumption that f is continuous in D and holomorphic in $D \setminus \{p\}$ for some $p \in D$. To see this, it suffices to show that $\int_{\partial T} f(z)\, dz = 0$ for every closed triangle $T \subset D$. If $T \subset D \setminus \{p\}$, this follows from Theorem 1.32, so assume $p \in T$. First consider the case where p is on the boundary of T. By slightly moving a vertex of T, we can find a triangle T', arbitrarily close to T, for which $p \notin T'$. Since $\int_{\partial T'} f(z)\, dz = 0$ and since by Lemma 1.27 the integral along the boundary of a triangle depends continuously on vertices, we conclude that $\int_{\partial T} f(z)\, dz = 0$. If p belongs to the interior of $T = \triangle abc$, write $\int_{\partial T} f(z)\, dz$ as the sum of the integrals along the boundaries of $\triangle abp$, $\triangle bcp$, and $\triangle cap$, and reduce to the previous case.

Later we will see that such a point p is not really exceptional, so under the above assumptions $f \in \mathscr{O}(D)$ (compare Example 1.40 or Theorem 3.5).

THEOREM 1.36 (Cauchy's integral formula in a disk)**.** *Let $D \subset \mathbb{C}$ be an open disk and $f \in \mathscr{O}(D)$. If $\overline{\mathbb{D}}(p, r) \subset D$, then*

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \text{for } z \in \mathbb{D}(p, r).$$

In particular, the values of f on the circle $\mathbb{T}(p, r)$ uniquely determine the values of f inside the disk $\mathbb{D}(p, r)$.

PROOF. Fix $z \in \mathbb{D}(p, r)$ and define $g : D \to \mathbb{C}$ by

$$g(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z. \end{cases}$$

Evidently g is continuous in D and holomorphic in $D \setminus \{z\}$. Hence by Remark 1.35, $\int_{\mathbb{T}(p,r)} g(\zeta)\, d\zeta = 0$. This gives

$$\frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z) \cdot \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{1}{\zeta - z}\, d\zeta.$$

To finish the proof, we need to show that the integral on the right is $2\pi i$. Take the parametrization of $\mathbb{T}(p, r)$ defined by $\gamma(t) = z + \rho(t) e^{it}$ for $t \in [0, 2\pi]$, where $\rho(t)$ is

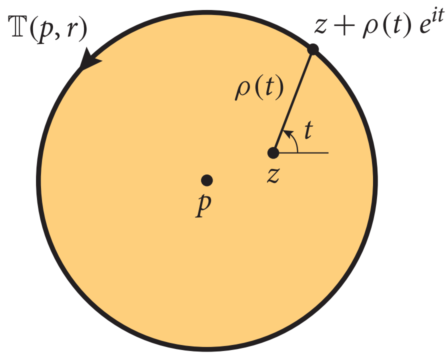

Figure 1.3. Parametrizing the oriented circle $\mathbb{T}(p,r)$ as seen from an off-center point z, used in the proof of Theorem 1.36.

the unique positive number which satisfies $|z+\rho(t)e^{it}-p|=r$ (see Fig. 1.3). It is easy to check that $t\mapsto\rho(t)$ is continuously differentiable. Hence

$$\begin{aligned}\int_{\mathbb{T}(p,r)}\frac{1}{\zeta-z}\,d\zeta &= \int_0^{2\pi}\frac{\gamma'(t)}{\gamma(t)-z}\,dt=\int_0^{2\pi}\frac{(\rho'(t)+i\rho(t))\,e^{it}}{\rho(t)\,e^{it}}\,dt\\ &=\int_0^{2\pi}\frac{\rho'(t)}{\rho(t)}\,dt+2\pi i\\ &=\log(\rho(2\pi))-\log(\rho(0))+2\pi i=2\pi i,\end{aligned}$$

where the last equality holds since $\rho(2\pi)=\rho(0)$. □

More general versions of Theorems 1.34 and 1.36 will be proved in chapter 2. For now, let us collect some corollaries of these basic results. The first one is the converse of Theorem 1.20:

THEOREM 1.37 (Holomorphic implies complex analytic). *Every $f\in\mathcal{O}(U)$ is complex analytic in U: In every disk $\mathbb{D}(p,r)\subset U$ there is a power series representation*

$$f(z)=\sum_{n=0}^{\infty}a_n\,(z-p)^n$$

where the coefficients $\{a_n\}$ are given by

$$a_n=\frac{f^{(n)}(p)}{n!}=\frac{1}{2\pi i}\int_{\mathbb{T}(p,s)}\frac{f(\zeta)}{(\zeta-p)^{n+1}}\,d\zeta \tag{1.20}$$

for any $0<s<r$.

PROOF. Fix $0 < s < r$ and a point $z \in \mathbb{D}(p,s)$. For any $\zeta \in \mathbb{T}(p,s)$,

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - p)\left[1 - \left(\dfrac{z-p}{\zeta - p}\right)\right]} = \frac{1}{\zeta - p} \sum_{n=0}^{\infty} \left(\frac{z-p}{\zeta - p}\right)^n.$$

Here the geometric series converges uniformly in ζ since its general term has absolute value $|z-p|/s < 1$ independent of ζ. Thus, we can integrate this series term-by-term on the circle $\mathbb{T}(p,s)$. By Theorem 1.36, we obtain

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}(p,s)} \sum_{n=0}^{\infty} \frac{f(\zeta)(z-p)^n}{(\zeta - p)^{n+1}}\, d\zeta = \sum_{n=0}^{\infty} a_n\, (z-p)^n,$$

where the a_n are given by (1.20). This proves that f can be represented by the power series $\sum_{n=0}^{\infty} a_n\, (z-p)^n$ in $\mathbb{D}(p,s)$. Since this holds for every $s < r$, Theorem 1.20(iii) shows that the power series with the same coefficients must converge to $f(z)$ for all $z \in \mathbb{D}(p,r)$. □

It follows from Theorem 1.20 that

COROLLARY 1.38. *If $f \in \mathscr{O}(U)$, then $f' \in \mathscr{O}(U)$. Therefore, the k-th derivative $f^{(k)}$ exists and belongs to $\mathscr{O}(U)$ for every $k \geq 1$.*

In particular, by Theorem 1.7, a differentiable map $f : U \to \mathbb{R}^2$ which satisfies the Cauchy-Riemann equation $f_{\bar{z}} = 0$ throughout U is automatically C^∞-smooth.

The following converse of Theorem 1.32 is a useful criterion for deciding when a continuous function is holomorphic:

Giacinto Morera (1856–1909)

THEOREM 1.39 (Morera, 1886). *Suppose $f : U \to \mathbb{C}$ is continuous and $\int_{\partial T} f(z)\, dz = 0$ for every closed triangle $T \subset U$. Then $f \in \mathscr{O}(U)$.*

PROOF. Let $D \subset U$ be a disk. By Theorem 1.31, f has a primitive F in D. Since $F \in \mathscr{O}(D)$ and since the derivative of a holomorphic function is holomorphic by Corollary 1.38, it follows that $f = F' \in \mathscr{O}(D)$. As this holds for every disk $D \subset U$, we conclude that $f \in \mathscr{O}(U)$. □

EXAMPLE 1.40 (Lines are removable). Let $U \subset \mathbb{C}$ be open and L be a straight line which intersects U. Suppose $f : U \to \mathbb{C}$ is a continuous function which is holomorphic in $U \smallsetminus L$. We prove that f is holomorphic in U by showing that $\int_{\partial T} f(z)\, dz = 0$ for every triangle $T \subset U$. First assume that the interior of T is disjoint from L. Then, by moving the vertices of T slightly, we can find a triangle $T' \subset U \smallsetminus L$, arbitrarily close to T. By Goursat's Theorem 1.32, $\int_{\partial T'} f(z)\, dz = 0$. Since the integral along the boundary of a triangle depends continuously on vertices by Lemma 1.27, we must have $\int_{\partial T} f(z)\, dz = 0$. If the interior of T meets L, write T as the union of at most three triangles with pairwise disjoint interiors, each meeting L along a vertex or an edge, and reduce to the previous case.

This shows in particular that points are removable: If f is continuous in U and holomorphic in $U \smallsetminus \{p\}$, then $f \in \mathscr{O}(U)$. More general removability results are discussed in Theorem 3.5 and in chapter 10.

REMARK 1.41. Morera's theorem holds if we replace triangles with other special families of closed sets with nice boundaries. A typical example, which turns out to be more convenient in some situations, is the family of closed rectangles, or even squares. See problem 25.

THEOREM 1.42 (Cauchy's estimates, 1835). *Suppose f is continuous on $\overline{\mathbb{D}}(p,r)$ and holomorphic in $\mathbb{D}(p,r)$. Then,*

$$|f^{(n)}(p)| \leq \frac{n!}{r^n} \sup_{|z-p|=r} |f(z)| \qquad (n \geq 0). \tag{1.21}$$

The example $f(z)=z^n$ in the unit disk $\mathbb{D}$ shows that the bound in (1.21) is optimal for each n.

PROOF. Take $0<s<r$ and represent f by a power series $\sum_{n=0}^{\infty} a_n\,(z-p)^n$ in $\mathbb{D}(p,s)$. By (1.20),

$$|f^{(n)}(p)| = n!\,|a_n| = \frac{n!}{2\pi}\left|\int_{\mathbb{T}(p,s)} \frac{f(z)}{(z-p)^{n+1}}\,dz\right|,$$

which by the ML-inequality implies

$$|f^{(n)}(p)| \leq \frac{n!}{2\pi} \cdot \sup_{|z-p|=s} \frac{|f(z)|}{|z-p|^{n+1}} \cdot 2\pi s = \frac{n!}{s^n} \sup_{|z-p|=s} |f(z)|.$$

Letting $s \to r$, we obtain (1.21). □

Cauchy's estimates lead to various quantitative results on holomorphic functions which have no counterpart in the smooth category. Here we prove two basic but important statements of this type.

THEOREM 1.43. *If a holomorphic function f maps the disk $\mathbb{D}(p,r)$ into the disk $\mathbb{D}(q,R)$, then $|f'(p)| \leq R/r$.*

Note that we have *not* assumed $q=f(p)$. In particular, if $f:\mathbb{D}\to\mathbb{D}$ is holomorphic, then $|f'(0)| \leq 1$. This is a basic version of the so-called "Schwarz lemma" which has deep applications and will be discussed at length in chapters 4, 11, and 13.

PROOF. Take $0<s<r$ and apply (1.21) to the function $g=f-q$:

$$|f'(p)| = |g'(p)| \leq \frac{1}{s} \sup_{|z-p|=s} |g(z)| \leq \frac{R}{s}.$$

Letting $s \to r$ proves the result. □

Liouville's theorem was known to Cauchy in 1844.

THEOREM 1.44 (Liouville, 1847). *Every bounded entire function is constant.*

PROOF. Let $f \in \mathscr{O}(\mathbb{C})$ and $|f(z)| < M$ for all $z \in \mathbb{C}$. Then f maps any disk $\mathbb{D}(p,r)$ into $\mathbb{D}(0,M)$, so by Theorem 1.43, $|f'(p)| \leq M/r$. Letting $r \to +\infty$, we obtain $f'(p) = 0$. Since this holds for every $p \in \mathbb{C}$, f must be constant. □

Joseph Liouville (1809–1882)

EXAMPLE 1.45 (The fundamental theorem of algebra). Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 1$, so $\lim_{z\to\infty} P(z) = \infty$. If $P(z) \neq 0$ for all z, then $f(z) = 1/P(z)$ is entire and $\lim_{z\to\infty} f(z) = 0$. Hence there is an $R > 0$ such that $|f(z)| \leq 1$ whenever $|z| \geq R$. Since by continuity f is bounded on the closed disk $\overline{\mathbb{D}}(0,R)$, it follows that f is bounded on the plane. Liouville's theorem then implies that f is constant, which is a contradiction. Thus, P has at least one root z_1 and we can write $P(z) = (z - z_1)P_1(z)$ for some polynomial P_1 of degree $d-1$. If $d - 1 = 0$ so P_1 is constant, stop. Otherwise repeat the argument with P_1 in place of P to find a root z_2 of P_1, and so on. This process stops after d steps and shows that P factors as $P(z) = a(z - z_1)(z - z_2)\cdots(z - z_d)$ for some $a, z_1, \ldots, z_d \in \mathbb{C}$. Thus, *every complex polynomial of degree $d \geq 1$ has precisely d roots counting multiplicities.*

We end this section with a useful theorem which, roughly speaking, says that the integral of a function which depends holomorphically on a parameter is a holomorphic function of that parameter, and differentiation under the integral sign is legitimate. We formulate a simple version of the theorem which will be sufficient for our purposes. One should note, however, that the result holds in much more general settings (see problem 27).

THEOREM 1.46. *Let $U \subset \mathbb{C}$ be open and $\varphi : U \times [a,b] \to \mathbb{C}$ be a continuous function such that for each $t \in [a,b]$, $z \mapsto \varphi(z,t)$ is holomorphic in U with derivative $\varphi'(z,t)$. Then, the function $f : U \to \mathbb{C}$ defined by*

$$f(z) = \int_a^b \varphi(z,t)\, dt$$

is holomorphic and we can differentiate under the integral sign:

$$f'(z) = \int_a^b \varphi'(z,t)\, dt \qquad \text{for all } z \in U.$$

PROOF. Fix $p \in U$ and take $r > 0$ such that $\overline{\mathbb{D}}(p,r) \subset U$. Let $0 < |z - p| < r/2$. By Theorem 1.36,

$$\varphi(z,t) - \varphi(p,t) = \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \varphi(\zeta,t) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - p} \right) d\zeta,$$

so

$$\frac{\varphi(z,t) - \varphi(p,t)}{z - p} = \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{\varphi(\zeta,t)}{(\zeta - z)(\zeta - p)}\, d\zeta.$$

Since $|\zeta - z| > r/2$ whenever $|\zeta - p| = r$, we obtain the following estimate using the ML-inequality:

$$\left| \frac{\varphi(z,t) - \varphi(p,t)}{z-p} \right| \leq \frac{1}{2\pi} \cdot M \cdot \frac{2}{r^2} \cdot 2\pi r = \frac{2M}{r}.$$

Here M is the supremum of $|\varphi|$ on the compact set $\overline{\mathbb{D}}(p,r) \times [a,b]$. If $\{z_n\}$ is any sequence in $U \smallsetminus \{p\}$ which tends to p, then

$$g_n(t) = \frac{\varphi(z_n,t) - \varphi(p,t)}{z_n - p}$$

is a sequence of continuous functions on $[a,b]$ which converges pointwise to $\varphi'(p,t)$ and is bounded by $2M/r$ for all large n. Hence, by Lebesgue's dominated convergence theorem, the function $t \mapsto \varphi'(p,t)$ is integrable on $[a,b]$ and

$$\lim_{n\to\infty} \frac{f(z_n) - f(p)}{z_n - p} = \lim_{n\to\infty} \int_a^b g_n(t)\, dt = \int_a^b \varphi'(p,t)\, dt.$$

Since this holds for every sequence $z_n \to p$, we conclude that $f'(p)$ exists and equals $\int_a^b \varphi'(p,t)\, dt$. □

REMARK 1.47. Under the assumptions of the above theorem, the derivative $(z,t) \mapsto \varphi'(z,t)$ is in fact continuous on $U \times [a,b]$ (see problem 26). Thus, the result holds when $\varphi(z,t)$ is replaced with $\varphi'(z,t)$, and a simple induction proves the formula

$$f^{(n)}(z) = \int_a^b \varphi^{(n)}(z,t)\, dt \qquad \text{for all } z \in U,$$

where $\varphi^{(n)}(z,t)$ is the n-th derivative of $\varphi(z,t)$ with respect to z.

The following corollary of the above theorem will be used repeatedly:

COROLLARY 1.48. *Let $\gamma : [0,1] \to \mathbb{C}$ be a piecewise C^1 curve and $g : |\gamma| \to \mathbb{C}$ be a continuous function. Then, for each integer $n \geq 1$, the function*

$$f(z) = \int_\gamma \frac{g(\zeta)}{(\zeta - z)^n}\, d\zeta$$

is holomorphic in $\mathbb{C} \smallsetminus |\gamma|$, and

$$f'(z) = n \int_\gamma \frac{g(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \qquad \text{for all } z \in \mathbb{C} \smallsetminus |\gamma|.$$

PROOF. This follows from Theorem 1.46 applied to $\varphi : (\mathbb{C} \smallsetminus |\gamma|) \times [0,1] \to \mathbb{C}$ defined by

$$\varphi(z,t) = \frac{g(\gamma(t))\, \gamma'(t)}{(\gamma(t) - z)^n}.$$

(Technically, we need to break up $[0,1]$ into finitely many intervals in which γ' is continuous and add up the corresponding integrals, but that is a trivial matter.) $\square$

EXAMPLE 1.49 (Cauchy's integral formula for higher derivatives). A special case of the above corollary is Cauchy's integral formula. If $f \in \mathscr{O}(U)$ and $\overline{\mathbb{D}}(p,r) \subset U$, then

$$\frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

defines a holomorphic function in $\mathbb{C} \smallsetminus \mathbb{T}(p,r)$. By Theorem 1.36, this function coincides with f inside the disk $\mathbb{D}(p,r)$. Differentiation under the integral sign then gives

$$f'(z) = \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \qquad \text{for } z \in \mathbb{D}(p,r).$$

It follows by induction that for every $n \geq 0$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \qquad \text{for } z \in \mathbb{D}(p,r).$$

Observe that for $z = p$ this is the formula (1.20) that we derived earlier.

1.5 Mapping properties of holomorphic functions

DEFINITION 1.50. Suppose $f \in \mathscr{O}(U)$ and f is not identically zero in the disk $\mathbb{D}(p,r) \subset U$. Let $f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n$ be the power series representation of f in $\mathbb{D}(p,r)$. The smallest integer m with the property $a_m \neq 0$ is called the ***order*** of p and is denoted by $\operatorname{ord}(f,p)$. Thus, $\operatorname{ord}(f,p) \geq 1$ if and only if $f(p) = 0$. We call p a ***simple zero*** of f if $\operatorname{ord}(f,p) = 1$.

Alternatively, $\operatorname{ord}(f,p)$ can be described as the unique integer $m \geq 0$ for which f can be factored as

$$f(z) = (z-p)^m f_1(z)$$

with $f_1 \in \mathscr{O}(U)$ and $f_1(p) \neq 0$. The function f_1 is given by $(z-p)^{-m} f(z)$ in $U \smallsetminus \{p\}$. It is holomorphic in U since it is represented by the power series $\sum_{n=m}^{\infty} a_n (z-p)^{n-m}$ in $\mathbb{D}(p,r)$.

EXAMPLE 1.51 (Holomorphic L'Hôpital's rule). Suppose f and g are holomorphic in some neighborhood of p, with $\operatorname{ord}(f,p) = \operatorname{ord}(g,p) = m \geq 1$. Write $f(z) = (z-p)^m f_1(z)$ and $g(z) = (z-p)^m g_1(z)$, where f_1 and g_1 are non-zero and holomorphic near p. Since

$$f_1(p) = \frac{f^{(m)}(p)}{m!} \qquad \text{and} \qquad g_1(p) = \frac{g^{(m)}(p)}{m!},$$

it follows that

$$\lim_{z \to p} \frac{f(z)}{g(z)} = \frac{f_1(p)}{g_1(p)} = \frac{f^{(m)}(p)}{g^{(m)}(p)}.$$

Let us call $U \subset \mathbb{C}$ a ***domain*** if U is non-empty, open, and connected.

LEMMA 1.52. *Suppose $U \subset \mathbb{C}$ is a domain and $f \in \mathscr{O}(U)$. If the zero-set $f^{-1}(0) = \{z \in U : f(z) = 0\}$ has an accumulation point in U, then $f = 0$ everywhere in U.*

Connectivity of U is essential here: If U is the disjoint union of non-empty open sets U_1 and U_2, and if $f = 0$ in U_1 and $f = 1$ in U_2, then $f \in \mathscr{O}(U)$ and $f^{-1}(0) = U_1$ has accumulation points in U, but f is not identically zero in U.

PROOF. Let E be the non-empty set of accumulation points of $f^{-1}(0)$ in U. Then E is closed in U, and $E \subset f^{-1}(0)$ by continuity of f. Suppose $p \in E$ and there is a disk $\mathbb{D}(p, r) \subset U$ in which f is not identically zero. Then we can write $f(z) = (z - p)^m f_1(z)$, where $m = \operatorname{ord}(f, p) \geq 1$, $f_1 \in \mathscr{O}(U)$, and $f_1(p) \neq 0$. By continuity, f_1 does not vanish in some neighborhood of p. It follows that p is the only zero of f in this neighborhood, contradicting the fact that $p \in E$. Thus, if $\mathbb{D}(p, r) \subset U$, then f is identically zero in $\mathbb{D}(p, r)$ and therefore $\mathbb{D}(p, r) \subset E$. This shows that E is an open set. Since U is connected, we must have $E = f^{-1}(0) = U$. $\square$

Since every domain $U \subset \mathbb{C}$ is a countable union of open disks, it is clear that every uncountable subset of U must have an accumulation point in U. It follows from the above lemma that *a non-constant holomorphic function in a domain has at most countably many zeros, all of which are isolated.* Another immediate corollary is

THEOREM 1.53 (The identity theorem). *Suppose $U \subset \mathbb{C}$ is a domain, $f, g \in \mathscr{O}(U)$, and the set $\{z \in U : f(z) = g(z)\}$ has an accumulation point in U. Then $f = g$ everywhere in U.*

EXAMPLE 1.54. The complex cosine and sine are the entire functions defined by

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

They extend the usual cosine and sine functions defined on the real line. It follows from Theorem 1.53 that any trigonometric identity between cosine and sine that holds on $\mathbb{R}$ must continue to hold in $\mathbb{C}$. For example, the identities $\cos^2 z + \sin^2 z = 1$, $\sin(2z) = 2\sin z \cos z$, and $\cos(2z) = \cos^2 z - \sin^2 z$ remain valid for all $z \in \mathbb{C}$.

EXAMPLE 1.55. Suppose $f \in \mathscr{O}(\mathbb{C})$ has the power series representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If every coefficient a_n is real, then clearly $f(\mathbb{R}) \subset \mathbb{R}$. Conversely, suppose $f(\mathbb{R}) \subset \mathbb{R}$ and consider the entire function

$$g(z) = \overline{f(\overline{z})} = \sum_{n=0}^{\infty} \overline{a_n}\, z^n.$$

Since $f(z)$ is real when z is real, we have $g = f$ on the real line. By Theorem 1.53, $g = f$ everywhere in $\mathbb{C}$. Uniqueness of power series then shows that every a_n is real.

Our next goal is to prove the fundamental fact that the image of a domain under a non-constant holomorphic function is open (Theorem 1.62). This will follow from a much stronger result on the local behavior of holomorphic functions (Theorem 1.59).

LEMMA 1.56. *If $f \in \mathscr{O}(U)$, the function $g : U \times U \to \mathbb{C}$ defined by*

$$g(\zeta, z) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$

is continuous.

Proof. Clearly g is continuous off the diagonal $\{(z,z) : z \in U\}$, so it is enough to check continuity of g at a diagonal point (p,p). Let $\varepsilon > 0$ be given. Since f' is continuous at p, there is an $r > 0$ such that

$$|f'(z) - f'(p)| < \varepsilon \qquad \text{whenever } z \in \mathbb{D}(p,r). \tag{1.22}$$

Let $\zeta, z \in \mathbb{D}(p,r)$. If $\zeta = z$, then $|g(\zeta,z) - g(p,p)| = |f'(z) - f'(p)| < \varepsilon$. If $\zeta \neq z$, then

$$\frac{f(\zeta) - f(z)}{\zeta - z} = \frac{1}{\zeta - z} \int_{[z,\zeta]} f'(w)\, dw = \int_0^1 f'(\gamma(t))\, dt,$$

where $\gamma(t) = (1-t)z + t\zeta$. Hence,

$$\begin{aligned} |g(\zeta,z) - g(p,p)| &= \left| \frac{f(\zeta) - f(z)}{\zeta - z} - f'(p) \right| = \left| \int_0^1 [\, f'(\gamma(t)) - f'(p)\,]\, dt \right| \\ &\leq \int_0^1 |f'(\gamma(t)) - f'(p)|\, dt \leq \varepsilon, \end{aligned}$$

where the last inequality holds since by (1.22), $|f'(\gamma(t)) - f'(p)| < \varepsilon$ for every $t \in [0,1]$. □

THEOREM 1.57 (Holomorphic inverse function theorem). *Suppose $f \in \mathscr{O}(U)$, $p \in U$, and $f'(p) \neq 0$. Then, there exist open neighborhoods $V \subset U$ of p and $W \subset \mathbb{C}$ of $f(p)$ such*

(continued...)

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