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Chapter One

Introduction

1.1 PREVIOUS CONSTRUCTIONS AND KATZ'S THEORY OF p-ADIC MODULAR FORMS ON THE ORDINARY LOCUS

Let us start by giving a brief account of Katz ([34], [33] and [35]) and Bertolini-Darmon-Prasanna's ([5]) construction of p-adic L-functions over imaginary quadratic fields K in which p splits in K. The splitting assumption of Katz allows one to make use of his theory of p-adic modular forms in order to construct his and Bertolini-Darmon-Prasanna/Liu-Zhang-Zhang's p-adic L-functions, now colloquially known as the *Katz* and *BDP p-adic L-functions*, respectively. The former is also constructed for CM extensions K/L (i.e., where $L/\mathbb{Q}$ is totally real and K/L is imaginary quadratic) for which all primes of L above p split in K, and the latter was generalized by Liu-Zhang-Zhang ([46]) to the case of CM fields and weight 2 newforms. Namely, the p-adic L-functions over K which Katz, Bertolini-Darmon-Prasanna and Liu-Zhang-Zhang construct are linear functionals on the space of (p-adic) modular forms, which are obtained by evaluating p-adic differential operators applied to modular forms at ordinary CM points associated with K. This means the CM points belong to the *ordinary locus*

$$Y^{\mathrm{ord}} \subset Y,$$

which is the affinoid subdomain of (the rigid analytification of) Y obtained by removing all points which reduce to supersingular points on the special fiber (this latter locus being isomorphic to a finite union of rigid analytic open unit discs). Here, the ordinariness assumption is crucial in order to establish nice analytic properties of the p-adic L-function, namely that (p-adic) modular forms have local coordinates in neighborhoods of CM points with respect to which the differential operators alluded to above have a nice, clearly analytic expression. In Katz's setting, one views p-adic modular forms as functions on a proétale cover called the *Igusa tower*

$$Y^{\mathrm{Ig}} \to Y^{\mathrm{ord}}$$

using an explicit trivialization of the Hodge bundle (by a so-called *canonical differential*). Then on Y^{Ig}, he defines a differential operator θ called the *Atkin-Serre operator*, which sends p-adic modular forms of weight k to forms of weight $k+2$, and the nice coordinates are provided by Serre-Tate coordinates. One can express

$$\theta_{\mathrm{AS}} = (1+T)\frac{d}{dT}$$

in terms of the Serre-Tate coordinate T, and using this expression one can show easily that the family

$$\{\theta_{\mathrm{AS}}^j f\}_{j\in\mathbb{Z}_{\geq 0}}$$

for a given p-adic modular form f gives rise to a "nearly-analytic" function of j: after applying a certain Hecke operator known as *p-stabilization* to f (which corresponds to removing an Euler factor in the p-adic L-function), one can show that

$$\theta_{\mathrm{AS}}^j f^{(p)},$$

where $f^{(p)}$ denotes the p-stabilization, is an analytic function (valued in the space of p-adic modular forms) of $j \in \mathbb{Z}_p^\times$.

One could also use coordinates provided by q-expansions, if one compactifies all modular curves under our consideration; we stick to the open modular curve in this article in order to avoid boundary issues occurring at cusps, which present bigger technical issues when defining the proétale topology later.

The key property of $Y^{\mathrm{Ig}} \to Y^{\mathrm{ord}}$ which allows one to construct the differential operator θ_{AS} is the existence of the *unit root splitting* of the Hodge filtration on Y^{Ig}. Namely, one can find sections of the relative de Rham cohomology

$$\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A})|_{Y^{\mathrm{Ig}}}$$

which are *horizontal* with respect to the algebraic Gauss-Manin connection

$$\nabla : \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}) \to \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}) \otimes_{\mathcal{O}_Y} \Omega^1_Y$$

(here a section α being horizontal means that $\nabla(\alpha)=0$), and which are also eigenvectors for the canonical (Frobenius-linear) Frobenius endomorphism

$$F : Y^{\mathrm{Ig}} \to Y^{\mathrm{Ig}}$$

over W. (The reason for the terminology "unit root" is because one of the eigenvalues for F is a p-adic unit, i.e., a section of $\mathcal{O}^\times_{Y^{\mathrm{Ig}}}$, since we restrict to a covering of the ordinary locus Y^{ord}.) The unit root splitting is a functorial, F-equivariant splitting of the Hodge filtration, which allows one to then define the differential operator θ_{AS}. This uses a standard formalism of Katz which produces such a differential (weight-raising) operator, whenever a splitting of the Hodge filtration with nice properties (e.g., $\mathrm{Gal}(Y^{\mathrm{Ig}}/Y^{\mathrm{ord}})$-equivariance) exists.

Another key property of the unit root splitting is that for CM elliptic curves A, which by the theory of complex multiplication always have models over $\overline{\mathbb{Q}}$, it is induced by the splitting of $H^1_{\mathrm{dR}}(A)$ defined over $\overline{\mathbb{Q}}$ given by the eigendecomposition under the CM action. This CM splitting over $\overline{\mathbb{Q}}$ also gives rise to the real analytic Hodge decomposition over $\mathbb{C}$ from classical Hodge theory, which in that setting gives rise to the real analytic Maass-Shimura operator $\mathfrak{d}$ sending nearly holomorphic modular forms of weight k to nearly holomorphic forms of weight $k+2$. The consequence is that after normalizing by appropriate "canonical" periods

$$\Omega_p \quad \text{and} \quad \Omega_\infty,$$

one can show that given an algebraic modular form w of weight k, the values

$$\theta_{\mathrm{AS}}^j w(y)/\Omega_p^{k+2j} \quad \text{and} \quad \mathfrak{d}^j w(y)/\Omega_\infty^{k+2j}$$

at ordinary CM points $y \in Y^{\mathrm{ord}}$ belong to $\overline{\mathbb{Q}}$ and *coincide*. This observation of Katz is essential to establishing interpolation properties of the Katz and BDP/LZZ p-adic L-functions, i.e., to relate them to critical values of complex L-functions in the interpolation (Panchishkin) range. This is because such critical L-values can be expressed as period integrals over the CM torus (or finite sums over orbits of CM points) of $\mathfrak{d}^j w$, and hence by the above discussion these can be related to such p-adic period sums of $\theta_{\mathrm{AS}}^j w$ over CM points, which themselves give rise to the Katz and BDP/LZZ p-adic L-functions.

Let us elaborate on Serre-Tate coordinates and Katz's notion of p-adic modular forms, and expound on the above discussion. To fix ideas, suppose that a modular curve Y represents a fine moduli space (for example, if its topological fundamental group as an analytic space over $\mathbb{C}$ is *neat* in the sense that it has no torsion), and so it admits a universal object

$$\pi : \mathcal{A} \to Y.$$

The *Hodge bundle* is then defined as

$$\omega := \pi_* \Omega^1_{\mathcal{A}/Y}$$

and weight k modular forms can be identified with sections of $\omega^{\otimes k}$. Katz's theory of modular forms arises by constructing a nonvanishing section known as the *canonical differential*

$$\omega_{\mathrm{can}}^{\mathrm{Katz}} \in \omega(Y^{\mathrm{Ig}}),$$

and using the induced trivialization

$$\omega|_{Y^{\mathrm{Ig}}} \cong \mathcal{O}_{Y^{\mathrm{Ig}}}$$

to view modular forms on Y as functions on Y^{Ig} transforming by some weight character under the action of

$$\mathrm{Gal}(Y^{\mathrm{Ig}}/Y^{\mathrm{ord}}) \cong \mathbb{Z}_p^\times .$$

To obtain the trivialization of ω, Katz uses the simple structure of the p-divisible groups of ordinary elliptic curves, namely that they are isomorphic to

$$\mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p.$$

By the Weil pairing (or Cartier duality), such a trivialization for a given p-divisible group $A[p^\infty]$ of an ordinary elliptic curve A is determined by fixing an isomorphism

$$A[p^\infty]^{\text{ét}} \cong \mathbb{Q}_p/\mathbb{Z}_p.$$

In fact, Y^{Ig} is exactly the cover of Y^{ord} defined over $W = W(\overline{\mathbb{F}}_p)$ parametrizing such trivializations

$$\alpha : \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} A[p^\infty]^{\text{ét}},$$

or equivalently (by the previous discussion), trivializations

$$\alpha : \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} A[p^\infty]$$

of the entire p-divisible group.

Let $A_0/\overline{\mathbb{F}}_p$ be an elliptic curve corresponding to a closed geometric point y_0 on the special fiber

$$Y_0^{\mathrm{ord}} = Y^{\mathrm{ord}} \otimes_W \overline{\mathbb{F}}_p,$$

and let A/W denote any lift of A_0, i.e., with

$$A \otimes_W \overline{\mathbb{F}}_p \cong A_0,$$

corresponding to a point y on Y^{ord}. Formally completing Y^{Ig} along y_0 hence gives the formal moduli space $\hat{D}(y_0)$ of deformations of A_0 (with some level structure, which we will suppress for brevity). Since there is a canonical isomorphism

$$A[p^\infty]^{\text{ét}} \cong A_0[p^\infty](\overline{\mathbb{F}}_p),$$

then a choice of trivialization

$$\alpha_0 : \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} A_0[p^\infty](\overline{\mathbb{F}}_p)$$

fixes $A[p^\infty]^{\text{ét}}$ in the formal neighborhood $\tilde{D}(\tilde{y}_0)$ of $\tilde{y}_0 = (A_0, \alpha_0)$ in Y^{Ig}. Hence $\tilde{D}(\tilde{y}_0)$ is parametrized exactly by the connected component $A[p^\infty]^0$ of $A[p^\infty]$, and so there is an (in fact, canonical) isomorphism

$$\tilde{D}(\tilde{y}_0) \cong \hat{\mathbb{G}}_m.$$

The canonical coordinate T on the torus gives rise to the *Serre-Tate coordinate*, also denoted by T, on

$$\tilde{D}(\tilde{y}_0),$$

and on the associated residue disc

$$\tilde{D}(\tilde{y}_0) \otimes_W W[1/p]$$

(viewed as the rigid analytic generic fiber of $\tilde{D}(\tilde{Y}_0)$).

Katz uses the above description of formal neighborhoods on Y^{Ig} around closed points of the special fiber as being canonically isomorphic to $\hat{\mathbb{G}}_m$ in order to construct the canonical differential ω_{can} mentioned before; in terms of the Serre-Tate coordinate on a residue disc D, the canonical differential is just given by

$$\omega_{\text{can}}^{\text{Katz}}|_D = dT/T.$$

Using tensorial powers of the canonical differential, modular forms, viewed as sections of powers $\omega^{\otimes k}$ of the Hodge bundle ω restricted to Y^{ord}, can be identified as functions on Y^{Ig}. Since the canonical differential transforms by

$$d^*\omega_{\text{can}}^{\text{Katz}} = d\omega_{\text{can}}^{\text{Katz}}$$

for

$$d \in \mathbb{Z}_p^\times \cong \text{Gal}(Y^{\text{Ig}}/Y^{\text{ord}}),$$

then we can even identify a modular form of weight k, i.e., a section of $w \in \omega^{\otimes k}(Y^{\text{ord}})$, as a function f of *weight* k on Y^{Ig}, via

$$w|_{Y^{\text{Ig}}} = f \cdot \omega_{\text{can}}^{\text{Katz}, \otimes k},$$

where weight $k \in \mathbb{Z}$ means that f transforms as

$$d^* f = d^{-k} f \tag{1.1}$$

for

$$d \in \mathbb{Z}_p^\times = \text{Gal}(Y^{\text{Ig}}/Y^{\text{ord}}).$$

Katz also uses this viewpoint to generalize modular forms to *p-adic modular forms of weight* $k \in \mathbb{Z}_p^\times$, which are functions on Y^{Ig} which have weight $k \in \mathbb{Z}_p^\times$ in the same way as defined above.

1.2 OUTLINE OF OUR THEORY OF p-ADIC ANALYSIS ON THE SUPERSINGULAR LOCUS AND CONSTRUCTION OF p-ADIC L-FUNCTIONS

The key question addressed by this article is that of developing a satisfactory theory of p-adic analysis of modular forms on the supersingular locus of modular curves, and subsequently to construct "supersingular" p-adic L-functions for Rankin-Selberg families $\mathbb{V}$ of twist families of automorphic representations

$$(\pi_w)_K \times \chi^{-1}$$

for anticyclotomic characters χ over an imaginary quadratic field $K/\mathbb{Q}$, where

$$\pi_w$$

is the automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$ attached to a normalized new eigenform w (i.e., a newform or Eisenstein series),

$$(\pi_w)_K$$

denotes its base change to an automorphic representation of $GL_2(\mathbb{A}_K)$, and χ varies through a family of anticyclotomic Hecke characters over K. Here, "supersingular" means that we assume that p is inert or ramified in K. This is analogous, outside the splitting assumption on p, to the "ordinary" setting in which Katz and Bertolini-Darmon-Prasanna/Liu-Zhang-Zhang construct their one-variable p-adic L-functions. In fact our theory addresses the ordinary and supersingular settings uniformly by working on an affinoid subdomain

$$\mathcal{Y}_x \subset \mathcal{Y}$$

of the p-adic universal cover

$$\mathcal{Y} \to Y$$

(defined below); in fact, $\mathcal{Y}_x$ contains a natural cover

$$\mathcal{Y}^{\mathrm{Ig}} \to Y^{\mathrm{Ig}},$$

and restricting our theory to $\mathcal{Y}^{\mathrm{Ig}}$ allows one to recover the one-variable p-adic L-functions in the ordinary case, as well as Katz's theory of p-adic modular forms on Y^{Ig}.

One motivation for the construction of supersingular Rankin-Selberg p-adic L-functions is to develop special value formulas in the same style as those of Katz and Bertolini-Darmon-Prasanna, where in the former case a special value of the Katz p-adic L-function is related to the p-adic logarithm of elliptic units attached to K, and in the latter case the special value formula is a "p-adic Waldspurger formula" (following the terminology of [46]) involving the p-adic formal logarithm of a Heegner point attached to K (when a Heegner hypothesis holds for K and level N of w). Indeed, we succeed in proving such a formula in the case $p \nmid N$ in Section 9, though in future work we expect to remove both $p \nmid N$ as well as relax the Heegner hypothesis on N, which would simply necessitate considering more general quaternionic Shimura curves than modular curves.

We seek to develop a satisfactory theory of p-adic analysis on the supersingular locus, namely a notion of p-adic modular forms on the supersingular locus

$$Y^{\mathrm{ss}} = Y \setminus Y^{\mathrm{ord}}$$

which "behaves well" with respect to some differential operator d; more precisely, this means there is some notion of "weight" which is raised by 2 under the action of d, and given a p-adic modular form f,

$$d^j f$$

or some stabilization

$$(d^j f)^\flat$$

gives rise to some p-adic analytically well-behaved family. To do this, there are several technical difficulties which must be overcome. One of which is that there is no obvious canonical differential with which to trivialize ω over a cover in order to view modular forms as functions on the cover (in the same way as $\omega_{\mathrm{can}}^{\mathrm{Katz}}$ does so for ω on $Y^{\mathrm{Ig}} \to Y^{\mathrm{ord}}$). It is also a difficulty that there is no "canonical line" in the p-divisible group of a supersingular curve as there is for

$$\mu_{p^\infty} \subset A[p^\infty]$$

when A is ordinary. Hence there is no natural splitting of the Hodge filtration with which to define a differential operator d analogous to the Atkin-Serre operator in the ordinary setting, and even if one were to construct such an operator, there is no obvious analogue of the Serre-Tate coordinate under which to locally express p-adic modular forms f and study the analytic properties of $d^j f$.

Another difficulty with defining a satisfactory p-adic Maass-Shimura operator on Y^{ss} comes from the lack of unit root splitting, whose construction comes from a horizontal basis for the Gauss-Manin connection defined as sections of the relative étale cohomology $\mathcal{H}^1_{\text{ét}}(\mathcal{A})$ over Y^{Ig} which are eigenvectors of the canonical Frobenius. This unit root splitting in the ordinary case gives a splitting of the Hodge filtration

$$0 \to \omega|_{Y^{\mathrm{ord}}} \to \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A})|_{Y^{\mathrm{ord}}} \to \omega^{-1}|_{Y^{\mathrm{ord}}} \to 0$$

as an exact sequence of $\mathcal{O}_{Y^{\mathrm{ord}}}$-modules, where $\mathcal{O}_Y$ denote the rigid analytic structure sheaf on Y. It is this functorial splitting, which is algebraically defined and coincides with the real analytic Hodge splitting at CM points, which gives rise to the ordinary p-adic Maass-Shimura operator θ_{AS} with the desired algebraicity properties. Note that unlike in the complex analytic setting, we do not have to extend the sheaf of rigid functions (the analogue of holomorphic functions) to a large sheaf (of "real analytic functions") in order to obtain the Hodge decomposition, as long as we restrict to $Y^{\mathrm{ord}} \subset Y$.

To overcome these difficulties, we generalize the strategy of Katz and in some sense emulate the construction of the complex analytic Maass-Shimura operator by working on the full p-adic universal cover

$$\mathcal{Y} \to Y$$

and by extending our coefficients from the structure sheaf to some larger sheaf of periods containing it (viewing this as analogous with extending holomorphic functions to real analytic functions).

Let us elaborate a little on the motivation of this strategy and how it works. As no unit root basis of the de Rham cohomology exists outside of Y^{ord}, we instead consider the moduli space of all horizontal bases of étale cohomology. This moduli space is representable by the *p-adic universal cover* $\mathcal{Y} \to Y$ (which we define more explicitly in the next paragraph), and with universal object being given by

$$(\mathcal{A}, \alpha_\infty) \to \mathcal{Y}$$

where α_∞ is the universal full p^∞-level structure. We then use a relative p-adic de Rham comparison theorem to view α_∞ as a universal horizontal basis for relative de Rham cohomology; unlike in the ordinary case, this comparison involves extending the structure sheaf to a certain period sheaf $\mathcal{O}\mathbb{B}^+_{\mathrm{dR},Y}$ (where this is really a sheaf on the proétale site $Y_{\mathrm{proét}}$) first constructed by Scholze in [56]. From this horizontal "framing" of the relative de Rham cohomology $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A})$, we get a "Hodge–de Rham period" measuring the position of the Hodge filtration and the "Hodge-Tate period" measuring the position of the Hodge-Tate filtration, as considered by Scholze in loc. cit., and use these periods to construct a relative Hodge decomposition which we use as a substitute for the unit root splitting. This splitting is in fact defined over an "intermediate period sheaf"

$$\mathcal{O}_\Delta := \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}}/(t),$$

equipped with natural connection

$$\nabla : \mathcal{O}_\Delta \to \mathcal{O}_\Delta \otimes_{\mathcal{O}_\mathcal{Y}} \Omega^1_\mathcal{Y}$$

which is $\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}}/(t)$-linear, induced by the natural connection

$$\nabla : \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Omega^1_{\mathcal{Y}}$$

which is $\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}}$-linear. Moreover, there is a natural map

$$\mathcal{O}_{\mathcal{Y}} \subset \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}} \overset{\mathrm{mod}\ t}{\twoheadrightarrow} \mathcal{O}_{\Delta}$$

which is in fact an inclusion compatible with connections, and such that its composition with the natural map

$$\theta : \mathcal{O}_{\Delta} \twoheadrightarrow \hat{\mathcal{O}}_{\mathcal{Y}}$$

where $\hat{\mathcal{O}}_{\mathcal{Y}}$ is the p-adically completed structure sheaf on $\mathcal{Y}$ is the natural map

$$\mathcal{O}_{\mathcal{Y}} \to \hat{\mathcal{O}}_{\mathcal{Y}}.$$

Here θ is induced by the natural relative analogue

$$\theta : \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}} \twoheadrightarrow \hat{\mathcal{O}}_{\mathcal{Y}}$$

of Fontaine's map $\theta : B^+_{\mathrm{dR}} \twoheadrightarrow \mathbb{C}_p$. Here,

$$t \in \mathbb{B}^+_{\mathrm{dR},Y}(\mathcal{Y})$$

is a global analogue of Fontaine's "$2\pi i$" and is a global section of a period sheaf $\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}}$ on $\mathcal{Y}$, which is itself a relative version of Fontaine's ring of periods B^+_{dR}. We call $\mathcal{O}_{\Delta}$ "intermediate" because it, in the sense above, lies in between $\mathcal{O}_{\mathcal{Y}}$ and $\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}}$. In analogy with having to extend from holomorphic to real analytic functions on the complex universal cover $\mathcal{H}$ in order to define the complex analytic Hodge decomposition, we view $\mathcal{O}_{\Delta}$ as a sheaf of "p-adic nearly holomorphic (or rigid) functions on the p-adic universal cover $\mathcal{Y}$."

Let us go into more detail on the construction of $\mathcal{Y}$. On geometric points, it has a moduli-theoretic interpretation moduli space parametrizing elliptic curves with full p^∞-level structure represented by a $GL_2(\mathbb{Z}_p)$-profinite-étale cover $\mathcal{Y}$ of Y (viewing the latter as an adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$) called the *p-adic universal cover* (or *infinite-level modular curve*)

$$\mathcal{Y} = \varprojlim_i Y(p^i),$$

as considered by Scholze in [56] and Scholze-Weinstein in [62]. Here $Y(p^i)$ is the modular curve obtained by adding full p^i-level structure to the moduli space represented by Y, and $\mathcal{Y}$ is an adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ which is an object in the proétale site $Y_{\text{proét}}$. Here, the full universal p^∞-level structure α_∞ is just a trivialization of the Tate module of $\mathcal{A}$

$$\alpha_\infty : \hat{\mathbb{Z}}_{p,\mathcal{Y}}^{\oplus 2} \xrightarrow{\sim} T_p\mathcal{A}|_{\mathcal{Y}},$$

here $\hat{\mathbb{Z}}_{p,\mathcal{Y}}$ is the "constant sheaf" on $\mathcal{Y}$ associated with $\mathbb{Z}_p$, except that sections are continuous functions into $\mathbb{Z}_p$ where the latter has the p-adic (and not discrete) topology. Now let $\mathcal{O}_Y$ denote the proétale structure sheaf on $Y_{\text{proét}}$. Using the Hodge–de Rham comparison theorem of Scholze ([57]), we then have a natural inclusion

$$\mathcal{H}^1_{\text{dR}}(\mathcal{A}) \otimes_{\mathcal{O}_Y} \mathcal{O}\mathbb{B}^+_{\text{dR},Y} \overset{\iota_{\text{dR}}}{\subset} \mathcal{H}^1_{\text{ét}}(\mathcal{A}) \otimes_{\hat{\mathbb{Z}}_{p,Y}} \mathcal{O}\mathbb{B}^+_{\text{dR},Y}$$

on $Y_{\text{proét}}$ compatible with filtrations (on the left, the convolution of the Hodge filtration on $\mathcal{H}^1_{\text{dR}}(\mathcal{A})$ and the natural filtration on $\mathcal{O}\mathbb{B}^+_{\text{dR},Y}$, and on the right is just the filtration on $\mathcal{O}\mathbb{B}^+_{\text{dR},Y}$) and connections (on the left, the convolution of the Gauss-Manin connection on $\mathcal{H}^1_{\text{dR}}(\mathcal{A})$ and the natural connection on $\mathcal{O}\mathbb{B}^+_{\text{dR},Y}$ via the Leibniz rule, and on the right is just the connection on $\mathcal{O}\mathbb{B}^+_{\text{dR},Y}$). Pulling back to $\mathcal{Y}$, we then have

$$\mathcal{H}^1_{\text{dR}}(\mathcal{A}) \otimes_{\mathcal{O}_Y} \mathcal{O}\mathbb{B}^+_{\text{dR},Y}|_{\mathcal{Y}} \overset{\iota_{\text{dR}}}{\hookrightarrow} \mathcal{H}^1_{\text{ét}}(\mathcal{A}) \otimes_{\hat{\mathbb{Z}}_{p,Y}} \mathcal{O}\mathbb{B}^+_{\text{dR},Y}|_{\mathcal{Y}} \xrightarrow[\sim]{\alpha_\infty^{-1}} (\mathcal{O}\mathbb{B}^+_{\text{dR},\mathcal{Y}} \cdot t^{-1})^{\oplus 2} \tag{1.2}$$

where the last isomorphism uses the universal p^∞-level structure α_∞ and the isomorphism

$$\mathcal{H}^1_{\text{ét}}(\mathcal{A}) \cong T_pA(-1)$$

given by the Weil pairing. We also use the fact that there is a natural isomorphism

$$\hat{\mathbb{Z}}_{p,\mathcal{Y}}(-1) = \hat{\mathbb{Z}}_{p,\mathcal{Y}} \cdot t^{-1},$$

as t is a period for the cyclotomic character.

We note that there is a natural sublocus

$$\mathcal{Y}^{\text{Ig}} = \{\hat{\mathbf{z}} = \infty\} \subset \mathcal{Y}$$

which parametrizes ordinary elliptic curves A together with a trivialization $\alpha : \mathbb{Z}_p^{\oplus 2} \xrightarrow{\sim} T_pA$ of their Tate modules T_pA and with

$$\alpha_{\mathbb{Z}_p \oplus \{0\}} : \mathbb{Z}_p \cong T_pA^0 \subset T_pA$$

trivializing the canonical line in T_pA, viewed as arithmetic p^∞-level structures

$$\alpha : \mathbb{Z}_p(1) \oplus \mathbb{Z}_p \xrightarrow{\sim} T_pA,$$

together with a trivialization

$$\mathbb{Z}_p(1) \cong \mathbb{Z}_p,$$

and it is clear that these two data are equivalent to a full p^∞-level structure

$$\alpha : \mathbb{Z}_p \oplus \mathbb{Z}_p \xrightarrow{\sim} T_p A.$$

Thus,

$$\mathcal{Y}^{\mathrm{Ig}} \to Y^{\mathrm{Ig}}$$

is a natural $\mathbb{Z}_p \rtimes \mathbb{Z}_p^\times$-cover of the Igusa tower Y^{Ig}, and a B-cover of Y^{ord}, where $B \subset GL_2(\mathbb{Z}_p)$ denotes the subgroup of upper triangular matrices.

Using (1.2), one sees that

$$\alpha_{\infty,1} := \alpha_\infty|_{\hat{\mathbb{Z}}_{p,\mathcal{Y}} \oplus \{0\}} \quad \text{and} \quad \alpha_{\infty,2} := \alpha_\infty|_{\{0\} \oplus \hat{\mathbb{Z}}_{p,\mathcal{Y}}}$$

form a horizontal basis for the connection ∇. Moreover, upon making the identification (via the Weil pairing)

$$T_p\mathcal{A} \cong \mathrm{Hom}(\mathcal{A}[p^\infty], \mu_{p^\infty}),$$

we get a natural map

$$HT_{\mathcal{A}} : T_p\mathcal{A} \otimes_{\hat{\mathbb{Z}}_{p,Y}} \mathcal{O}_Y \to \omega, \qquad \alpha \mapsto \alpha^* \frac{dT}{T}$$

where dT/T is the canonical invariant differential on μ_{p^∞}. (It is also sometimes customary to denote $HT_{\mathcal{A}} = d\log$, as $d\log T = dT/T$.) We then define the *fake Hasse invariant* as

$$\mathfrak{s} := HT_{\mathcal{A}}(\alpha_{\infty,2}),$$

and in fact we have that on the restriction to $\mathcal{Y}^{\mathrm{Ig}}$,

$$\mathfrak{s}|_{\mathcal{Y}^{\mathrm{Ig}}} = \omega_{\mathrm{can}}^{\mathrm{Katz}}|_{\mathcal{Y}^{\mathrm{Ig}}}. \tag{1.3}$$

Consider the affinoid subdomain

$$\mathcal{Y}_x = \{\mathfrak{s} \neq 0\} \subset \mathcal{Y}.$$

We note that $\mathfrak{s} \in \omega(\mathcal{Y}_x)$ is a nonvanishing global section, i.e., a generator. Then let $\mathfrak{s}^{-1} \in \omega^{-1}(\mathcal{Y}_x)$ the generator corresponding to $\mathfrak{s}$ under Poincaré duality. The trivialization

$$\omega|_{\mathcal{Y}^{\mathrm{Ig}}} \cong \mathcal{O}_{\mathcal{Y}^{\mathrm{Ig}}}$$

induced by (1.3), along with the universal level structure on $\mathcal{Y}^{\mathrm{Ig}}$ given by $\alpha_\infty|_{\mathcal{Y}^{\mathrm{Ig}}}$, gives rise to a p-adic differential operator (the Atkin-Serre operator)

$$\theta_{\mathrm{AS}} : \mathcal{O}_{\mathcal{Y}^{\mathrm{Ig}}} \to \mathcal{O}_{\mathcal{Y}^{\mathrm{Ig}}}$$

with nice p-adic analytic properties, as seen using Serre-Tate coordinates. The key to these nice p-adic properties is the identity (see [34, Main Theorem 3.7.2])

$$\sigma(\omega_{\mathrm{can}}^{\mathrm{Katz},\otimes 2}) = d\log T$$

where T is the Serre-Tate coordinate, and

$$\sigma : \omega^{\otimes 2} \xrightarrow{\sim} \Omega_Y^1 \tag{1.4}$$

is the Kodaira-Spencer isomorphism.

By the above discussion, $\mathfrak{s}$ seems like a natural candidate to extend Katz's idea of viewing p-adic modular forms (sections of ω) as functions to the (non-Galois) covering $\mathcal{Y}_x \to Y$. However, unlike in Katz's situation on $\mathcal{Y}^{\mathrm{Ig}}$, in our situation the splitting of (a lift of) the Hodge filtration which we define and use will require extending coefficients from $\mathcal{O}_{\mathcal{Y}_x}$ to a larger sheaf $\mathcal{O}_{\Delta,\mathcal{Y}_x}$ (which can be viewed as "the sheaf of nearly rigid functions," in analogy to extending coefficients from the sheaf of real analytic functions to the sheaf of nearly holomorphic functions in order to define the Hodge decomposition in the complex analytic situation), and with respect to this splitting $\mathfrak{s}$ will *not* be the most convenient choice for trivializing $\omega \otimes_{\mathcal{O}_Y} \mathcal{O}_{\Delta,\mathcal{Y}_x}$. Instead, we will trivialize using the generator

$$\omega_{\mathrm{can}} := \frac{\mathfrak{s}}{y_{\mathrm{dR}}} \in (\omega \otimes_{\mathcal{O}_Y} \mathcal{O}_{\Delta,\mathcal{Y}_x})(\mathcal{Y}_x)$$

where $y_{\mathrm{dR}} \in \mathcal{O}_{\Delta,\mathcal{Y}_x}(\mathcal{Y}_x)^\times$ is a certain p-adic period associated with $\mathfrak{s}$. Hence this induces a trivialization

$$\omega \otimes_{\mathcal{O}_Y} \mathcal{O}_{\Delta,\mathcal{Y}_x} \cong \mathcal{O}_{\Delta,\mathcal{Y}_x}.$$

One can show that $y_{\mathrm{dR}} = 1$ on the sublocus $\mathcal{Y}^{\mathrm{Ig}} \subset \mathcal{Y}$, and so we have

$$\omega_{\mathrm{can}}|_{\mathcal{Y}^{\mathrm{Ig}}} = \omega_{\mathrm{can}}^{\mathrm{Katz}}|_{\mathcal{Y}^{\mathrm{Ig}}}. \tag{1.5}$$

In analogy with (1.4), we also have

$$\sigma(\omega_{\mathrm{can}}^{\otimes,2}) = dz_{\mathrm{dR}} \tag{1.6}$$

where $z_{\mathrm{dR}} = \mathbf{z}_{\mathrm{dR}} \pmod{t}$ for the fundamental de Rham period $\mathbf{z}_{\mathrm{dR}}$, which we describe in more detail below. We note that the analogy between (1.4) and (1.6), along with (1.5), suggests that z_{dR} provides the correct analogue of $\log T$. It is this observation which later leads to our notion of the $q_{\mathrm{dR}} = \exp(z_{\mathrm{dR}} - \bar{z}_{\mathrm{dR}})$-coordinate as an analogue (and extension) for the Serre-Tate coordinate T, and q_{dR}-expansions as analogues (and extensions) of Serre-Tate T-expansions.

We can also use ω_{can} to generalize Katz's notion of p-adic modular forms. Let $\mathcal{U} \subset \mathcal{Y}_x$ be a subadic space, let $\lambda : \mathcal{Y} \to Y$ denote the natural projection, and let $\lambda(\mathcal{U}) = U$. Then letting

$$\Gamma = \mathrm{Gal}(\mathcal{U}/U) \subset \mathrm{Gal}(\mathcal{Y}/Y) = GL_2(\mathbb{Z}_p),$$

we have a natural map

$$\omega^{\otimes k}|_U(U) \overset{\lambda^*}{\hookrightarrow} \omega^{\otimes k}|_{\mathcal{U}}(\mathcal{U}) \hookrightarrow (\omega \otimes_{\mathcal{O}_Y} \mathcal{O}_{\Delta,\mathcal{Y}_x}|_{\mathcal{U}})^{\otimes k}(\mathcal{U}) \xrightarrow[\sim]{\omega_{\mathrm{can}}^{\otimes k}} \mathcal{O}_{\Delta,\mathcal{Y}_x}|_{\mathcal{U}}(\mathcal{U}).$$

In fact, the image under this map consists of sections $f \in \mathcal{O}_{\Delta,\mathcal{Y}_x}|_{\mathcal{U}}(\mathcal{U})$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* f = (ad - bc)^{-k}(cz_{\mathrm{dR}} + a)^k f \tag{1.7}$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. In this situation, we say that f has *weight k for Γ on $\mathcal{U}$*. We note that when $\mathcal{U} = \mathcal{Y}^{\mathrm{Ig}}$ and so $U = Y^{\mathrm{ord}}$, and

$$\Gamma = B \subset GL_2(\mathbb{Z}_p)$$

the subgroup of upper triangular matrices, and then (1.7) becomes

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^* f = d^{-k} f. \tag{1.8}$$

In particular, f descends to a section in $\mathcal{O}_Y(Y^{\mathrm{Ig}})$ and we recover Katz's notion (1.1) of a p-adic modular form of weight k. Our main interest, which is defining a satisfactory notion of p-adic modular form on the supersingular locus, will involve the case $\mathcal{U} = \mathcal{Y}^{\mathrm{ss}}, U = Y^{\mathrm{ss}}$ and $\Gamma = GL_2(\mathbb{Z}_p)$.

Let us now elaborate on the construction of our splitting of (a lift of) the Hodge filtration alluded to above, which is crucial to the construction of the p-adic Maass-Shimura operator and its algebraicity properties. Unlike in Katz's theory, outside of $\mathcal{Y}^{\mathrm{Ig}}$, $\alpha_{\infty,1}$ and $\alpha_{\infty,2}$ do not generate either the Hodge or Hodge-Tate filtrations, and instead we are led to consider certain relative periods

$$\mathbf{z}_{\mathrm{dR}} \in \mathcal{O}\mathbb{B}^+_{\mathrm{dR},Y}(\mathcal{Y}_x), \hat{\mathbf{z}} \in \hat{\mathcal{O}}_Y(\mathcal{Y}_x),$$

where $\hat{\mathcal{O}}_Y = \hat{\mathcal{O}}_Y^+[1/p]$ and where

$$\hat{\mathcal{O}}_Y^+ = \varprojlim_n \mathcal{O}_Y^+/p^n \subset \hat{\mathcal{O}}_Y = (\varprojlim_n \mathcal{O}_Y^+/p^n)[1/p]$$

denotes the p-adic completion integral structure sheaf

$$\mathcal{O}_Y^+ \subset \mathcal{O}_Y = \mathcal{O}_Y^+[1/p].$$

Both of the above periods can be viewed as sections of an ambient period presheaf

$$\hat{\mathcal{O}}\mathbb{B}_{\mathrm{dR},Y}^+ := \mathcal{O}\mathbb{B}_{\mathrm{dR},Y}^+ \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_Y,$$

where $\mathcal{O}\mathbb{B}_{\mathrm{dR},Y}^+$ is the usual de Rham period sheaf as considered in [57]. The *Hodge–de Rham period* $\mathbf{z}_{\mathrm{dR}} \in \mathcal{O}\mathbb{B}_{\mathrm{dR},Y}^+(\mathcal{Y}_x)$ measures the position of the Hodge filtration

$$\omega|_{\mathcal{Y}_x} = \mathfrak{s} \cdot \hat{\mathcal{O}}_{\mathcal{Y}_x} \subset \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}) \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}_x} \overset{\iota_{\mathrm{dR}}}{\hookrightarrow} \mathcal{H}^1_{\text{ét}}(\mathcal{A}) \otimes_{\hat{\mathbb{Z}}_{p,Y}} \hat{\mathcal{O}}\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}_x}$$
$$\xrightarrow[\sim]{\alpha_\infty^{-1}} (\hat{\mathcal{O}}\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}_x} \cdot t^{-1})^{\oplus 2}.$$

The *Hodge-Tate period* $\hat{\mathbf{z}} \in \hat{\mathcal{O}}_Y(\mathcal{Y}_x) \subset \mathcal{O}\mathbb{B}_{\mathrm{dR},Y}^+(\mathcal{Y}_x)$ measures the position of the Hodge-Tate filtration

$$\omega^{-1}|_{\mathcal{Y}_x} = \mathfrak{s}^{-1} \cdot \hat{\mathcal{O}}_{\mathcal{Y}_x} \subset \mathcal{H}^1_{\text{ét}}(\mathcal{A}) \otimes_{\hat{\mathbb{Z}}_{p,Y}} \hat{\mathcal{O}}\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}_x} \xrightarrow[\sim]{\alpha_\infty^{-1}} (\hat{\mathcal{O}}\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}_x} \cdot t^{-1})^{\oplus 2}.$$

Using these periods, and recalling our notation $\mathcal{O}_{\Delta,\mathcal{Y}_x} = \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}_x}/(t)$ and defining $\hat{\mathcal{O}}_{\Delta,\mathcal{Y}_x} = \mathcal{O}_{\Delta,\mathcal{Y}_x} \otimes_{\mathcal{O}_{\mathcal{Y}_x}} \hat{\mathcal{O}}_{\mathcal{Y}_x}$, one can construct a Hodge decomposition

$$T_p\mathcal{A} \otimes_{\hat{\mathbb{Z}}_{p,Y}} \hat{\mathcal{O}}_{\Delta,\mathcal{Y}_x} \xrightarrow{\sim} (\omega \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_{\Delta,\mathcal{Y}_x}) \oplus (\omega^{-1} \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_{\Delta,\mathcal{Y}_x} \cdot t) \tag{1.9}$$

where the projection onto the first factor is given by $HT_{\mathcal{A}}$ (i.e., the inclusion of the first factor is a section of $HT_{\mathcal{A}}$), and so this gives a splitting of the Hodge-Tate filtration.

However, for the purposes of using this splitting to construct a differential operator, there is a technical issue that there is no natural way to define a connection on $\hat{\mathcal{O}}\mathbb{B}^+_{\mathrm{dR},Y}$, precisely because it contains a copy of the p-adically completed structure sheaf $\hat{\mathcal{O}}_Y$: the pullback $\hat{\mathcal{O}}_{\mathcal{Y}}$ is (essentially) the structure sheaf of the perfectoid space $\hat{\mathcal{Y}} \sim \mathcal{Y}$ associated with $\mathcal{Y}$, and there are no nontrivial differentials on perfectoid spaces since differentials are infinitely divisible, and hence 0 in the p-adic completion.

There are two ways to remedy this. One is to instead replace (1.9) with another splitting

$$T_p\mathcal{A}\otimes_{\hat{\mathbb{Z}}_{p,Y}}\mathcal{O}_{\Delta,\mathcal{Y}_x}\xrightarrow{\sim}(\omega\otimes_{\mathcal{O}_Y}\mathcal{O}_{\Delta,\mathcal{Y}_x})\oplus\mathcal{L}, \tag{1.10}$$

where $\mathcal{L}$ is a free $\mathcal{O}_{\Delta,\mathcal{Y}_x}$-module of rank 1. This splitting is constructed by using the natural "horizontal" embedding $\hat{\mathbf{z}}\in\hat{\mathcal{O}}_{\mathcal{Y}_x}\subset\mathcal{O}_{\Delta,\mathcal{Y}_x}$. Now the projection onto the first factor is *not* given by $HT_{\mathcal{A}}$, but instead its kernel is *horizontal* in the sense that

$$\nabla_w(\mathcal{L})\subset\mathcal{L}$$

for any section w of $\Omega^1_Y\otimes_{\mathcal{O}_Y}\mathcal{O}_{\Delta,\mathcal{Y}_x}$. Moreover, (1.10) recovers the usual Hodge-Tate decomposition upon applying the natural map

$$\theta:\mathcal{O}_{\Delta,\mathcal{Y}_x}\twoheadrightarrow\hat{\mathcal{O}}_{\mathcal{Y}_x},$$

where $\hat{\mathcal{O}}_{\mathcal{Y}_x}$ denotes the p-adically completed structure sheaf (and θ is analogous to Fontaine's universal cover $\theta:B^+_{\mathrm{dR}}\twoheadrightarrow\mathbb{C}_p$)

$$T_p\mathcal{A}\otimes_{\hat{\mathbb{Z}}_{p,Y}}\hat{\mathcal{O}}_{\mathcal{Y}_x}\xrightarrow{\sim}(\omega\otimes_{\mathcal{O}_Y}\hat{\mathcal{O}}_{\mathcal{Y}_x})\oplus(\hat{\omega}_{\mathcal{Y}_x}(1)). \tag{1.11}$$

The other approach is to instead construct intermediate period sheaves of "nearly holomorphic coefficients"

$$\mathcal{O}_{\Delta,\mathcal{Y}_x}\subset\mathcal{O}^{\diamond}_{\Delta,\mathcal{Y}_x}\subset\mathcal{O}^{\dagger}_{\Delta,\mathcal{Y}_x}\subset\hat{\mathcal{O}}_{\Delta,\mathcal{Y}_x},$$

where $\mathcal{O}^{\dagger}_{\Delta}$ contains the Hodge-Tate period $\hat{\mathbf{z}}$ and so is large enough to construct a Hodge splitting (essentially one just adjoins a few sections in $\hat{\mathcal{O}}_{\Delta,\mathcal{Y}_x}$ to $\mathcal{O}_{\Delta,\mathcal{Y}_x}$), and on which one can also extend the natural connection on $\mathcal{O}_{\Delta}$ (namely by declaring that $\nabla(\hat{\mathbf{z}})=0$, and showing that this gives rise to a well-defined connection on $\mathcal{O}^{\dagger}_{\Delta,\mathcal{Y}_x}$ since $\hat{\mathbf{z}}$ is transcendental over $\mathcal{O}_{\Delta}$). One can then define a splitting like (1.10) using $\mathcal{O}^{\dagger}_{\Delta,\mathcal{Y}_x}$, and consequently construct a p-adic Maass-Shimura operator with coefficients in $\mathcal{O}^{\dagger}_{\Delta,\mathcal{Y}_x}$; one can even show that this differential operator is defined over the smaller sheaf of coefficients $\mathcal{O}^{\diamond}_{\Delta,\mathcal{Y}_x}$.

Both approaches have their virtues: while the first approach stays within the smaller period sheaf $\mathcal{O}_{\Delta,\mathcal{Y}_x}$, it requires the use of Fontaine's map θ in order to recover a true Hodge-Tate decomposition, whereas the second requires enlarging (slightly) to $\mathcal{O}^{\diamond}_{\Delta,\mathcal{Y}_x}$ but then does not require the use of θ. While the p-adic Maass-Shimura operators arising from each approach are different, they satisfy the same algebraicity properties at CM points and are both equal in value (after normalizing by a p-adic period) to the value of the complex Maass-Shimura operator (normalized by a complex period) at CM points. Hence, either Maass-Shimura operator can be used in order to construct our p-adic L-function.

We will follow the first approach in this outline in the introduction. Now we can define a p-adic Maass-Shimura operator d with respect to the splitting (1.10). Since (1.11) recovers the relative Hodge-Tate decomposition, it is induced at CM points by the algebraic CM splitting, and so as in Katz's theory one can show (*using* the horizontalness of (1.10)) that for an algebraic modular form $w \in \omega^{\otimes k}(Y)$, writing

$$w|_{\mathcal{Y}_x} = f \cdot \omega_{\mathrm{can}}^{\otimes k},\ f \in \mathcal{O}_{\Delta,\mathcal{Y}_x}(\mathcal{Y}_x), \qquad F \cdot (2\pi i dz)^{\otimes k},\ F \in \mathcal{O}^{\mathrm{hol}}(\mathcal{H}^+),$$

where $\mathcal{O}^{\mathrm{hol}}$ denotes the sheaf of (complex) holomorphic function and $\mathcal{H}^+ \to Y$ the complex universal cover (i.e., the complex upper half-plane), we have that the value

$$(\theta \circ d^j) f(y)/\Omega_p(y)^{k+2j}$$

at a CM point $y \in \mathcal{Y}_x$ is an algebraic number for an appropriate p-adic period $\Omega_p(y)$ (depending on y), and in fact is equal (in $\overline{\mathbb{Q}}$) to the algebraic number

$$\mathfrak{d}^j F(y)/\Omega_\infty(y)^{k+2j}$$

at the same CM point $y \in \mathcal{Y}_x$ for an appropriate complex period $\Omega_\infty(y)$ (only depending on the image of y under the natural projection $\mathcal{Y} \to Y$):

$$(\theta \circ d^j) f(y)/\Omega_p(y)^{k+2j} = \mathfrak{d}^j F(y)/\Omega_\infty(y)^{k+2j}. \tag{1.12}$$

The key fact for proving this algebraicity is that the fiber $\mathcal{O}_{\Delta,\mathcal{Y}_x}(y)$ contains a unique copy of $\overline{\mathbb{Q}}_p$ by Hensel's lemma, and so composition

$$\overline{\mathbb{Q}}_p \subset \mathcal{O}_{\Delta,\mathcal{Y}_x}(y) \xrightarrow{\theta} \mathbb{C}_p$$

is the natural inclusion. Then since the specialization $\hat{\mathbf{z}}(y) \in \overline{\mathbb{Q}}_p$, we have

$$\theta(\hat{\mathbf{z}}(y)) = \hat{\mathbf{z}}(y),$$

and so

$$(\theta \circ d^j) f(y) = \theta(d^j f(y)) = d^j f(y),$$

and this latter value is equal (after normalizing by periods) to

$$\mathfrak{d}^j f(y)$$

since both (1.10) and the complex analytic Hodge decomposition are both induced by the CM splitting at y.

It is the algebraicity of $\theta \circ d^j$ at CM points, and moreover the fact that it is equal in value to complex Maass-Shimura derivatives, which makes it applicable to questions regarding interpolation of critical L-values and hence construction of p-adic L-functions. Ultimately, for the construction of the latter, it is necessary to understand the analytic behavior of

$$(\theta \circ d^j)f$$

around CM points y, and here the framework for understanding such analytic properties is provided by q_{dR}-expansions of modular forms, given by a q_{dR}-expansion map

$$\omega^{\otimes k}|_{\mathcal{Y}_x} \xrightarrow[\sim]{\omega_{\mathrm{can}}^{\otimes k}} y_{\mathrm{dR}}^k \mathcal{O}_{\mathcal{Y}_x} \overset{q_{\mathrm{dR}}-\exp}{\hookrightarrow} \hat{\mathcal{O}}_{\mathcal{Y}_x}[\![q_{\mathrm{dR}}-1]\!] \subset \mathcal{O}_{\Delta,\mathcal{Y}_x} \xrightarrow[\sim]{\omega_{\mathrm{can}}^{\otimes k}} \omega^{\otimes k} \otimes_{\mathcal{O}_Y} \mathcal{O}_{\Delta,\mathcal{Y}_x}. \tag{1.13}$$

A key fact is that on the supersingular locus $\mathcal{Y}^{\mathrm{ss}} \subset \mathcal{Y}_x$, (1.13) *coincides* with the natural inclusion

$$\omega|_{\mathcal{Y}^{\mathrm{ss}}} \hookrightarrow \omega \otimes_{\mathcal{O}_Y} \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}^{\mathrm{ss}}} \xrightarrow{\bmod t} \omega \otimes_{\mathcal{O}_Y} \mathcal{O}_{\Delta,\mathcal{Y}^{\mathrm{ss}}}, \tag{1.14}$$

which is induced by the composition

$$\mathcal{O}_{\mathcal{Y}} \subset \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\mathcal{Y}} \xrightarrow{\bmod t} \mathcal{O}_\Delta$$

which turns out to be an inclusion. In fact, recalling that $\hat{\mathcal{O}}_Y$ denotes the p-adic completion of the structure sheaf $\mathcal{O}_Y$, we have a natural inclusion

$$\hat{\mathcal{O}}_{\mathcal{Y}}[\![q_{\mathrm{dR}}-1]\!] \subset \mathcal{O}_\Delta$$

which is compatible with the natural connections on each sheaf, and which is in fact an *equality* on $\mathcal{Y}^{\mathrm{ss}}$:

$$\hat{\mathcal{O}}_{\mathcal{Y}^{\mathrm{ss}}}[\![q_{\mathrm{dR}}-1]\!] = \mathcal{O}_{\Delta,\mathcal{Y}^{\mathrm{ss}}}.$$

Hence we see that, at least on the supersingular locus $\mathcal{Y}^{\mathrm{ss}}$, q_{dR} provides the correct coordinate when viewing a rigid modular form

$$w \in \omega^{\otimes k}(\mathcal{Y}^{\mathrm{ss}}) \subset (\omega \otimes_{\mathcal{O}_Y} \mathcal{O}_{\Delta,\mathcal{Y}^{\mathrm{ss}}})^{\otimes k}(\mathcal{Y}^{\mathrm{ss}}) \xrightarrow[\sim]{\omega_{\mathrm{can}}^{\otimes k}} \mathcal{O}_{\Delta,\mathcal{Y}^{\mathrm{ss}}}(\mathcal{Y}^{\mathrm{ss}})$$

as a "nearly rigid function." The coordinate

$$q_{\mathrm{dR}} \in \mathcal{O}_{\Delta,\mathcal{Y}_x}(\mathcal{Y}_x)$$

plays the role analogous to that of the Serre-Tate coordinate, and in fact the q_{dR}-expansion of a modular form recovers the Serre-Tate expansion upon restricting to $\mathcal{Y}^{\mathrm{Ig}}$ (due to the fact that $\omega_{\mathrm{can}}|_{\mathcal{Y}^{\mathrm{Ig}}} = \omega_{\mathrm{can}}^{\mathrm{Katz}}$).

In fact, one can write down an explicit formula for $\theta \circ d^j$ in terms of q_{dR}-coordinates

$$\theta \circ d^j = \sum_{i=0}^{j} \binom{j}{i} \binom{j+k-1}{i} i! \left(-\frac{\theta(y_{\mathrm{dR}})}{\hat{\mathbf{z}}}\right)^i \theta \circ \left(\frac{q_{\mathrm{dR}} d}{dq_{\mathrm{dR}}}\right)^{j-i}. \qquad (1.15)$$

On $\mathcal{Y}^{\mathrm{Ig}}$, as we noted before, $\hat{\mathbf{z}} = \infty$ and so we have

$$(\theta \circ d^j)|_{\mathcal{Y}^{\mathrm{Ig}}} = \theta \circ \left(\frac{q_{\mathrm{dR}} d}{dq_{\mathrm{dR}}}\right)^j |_{\mathcal{Y}^{\mathrm{Ig}}} = \theta \circ \theta_{\mathrm{AS}}^j = \theta_{\mathrm{AS}}^j$$

where the last equality follows from the fact that

$$\frac{dq_{\mathrm{dR}}}{q_{\mathrm{dR}}}|_{\mathcal{Y}^{\mathrm{Ig}}} = dz_{\mathrm{dR}}|_{\mathcal{Y}^{\mathrm{Ig}}} = dT|_{\mathcal{Y}^{\mathrm{Ig}}}$$

and that

$$\mathcal{O}_{\mathcal{Y}_x} \subset \mathcal{O}_{\Delta,\mathcal{Y}_x} \overset{\theta}{\twoheadrightarrow} \hat{\mathcal{O}}_{\mathcal{Y}_x}$$

is the natural completion map. Hence, again restricting to $\mathcal{Y}^{\mathrm{Ig}} \subset \mathcal{Y}_x$, we recover Katz's theory.

In order to construct the p-adic L-function, we consider the image of a modular form

$$w \in \omega^{\otimes k}(Y)$$

under the q_{dR}-expansion map (1.13), and study the growth of the coefficients of its q_{dR}-expansion around supersingular CM points y:

$$\omega^{\otimes k}|_{\mathcal{Y}^{\mathrm{ss}}} \xrightarrow{q_{\mathrm{dR}}-\mathrm{exp}} \hat{\mathcal{O}}_{\mathcal{Y}^{\mathrm{ss}}}[\![q_{\mathrm{dR}}-1]\!] \xrightarrow{\text{stalk at } y} \hat{\mathcal{O}}_{\mathcal{Y}^{\mathrm{ss}},y}[\![q_{\mathrm{dR}}-1]\!] = \hat{\mathcal{O}}_{\mathcal{Y}^{\mathrm{ss}},y}[\![q_{\mathrm{dR}}^{1/p^b}-1]\!]$$

for any $b \in \mathbb{Q}$, where the last equality is just a formal change of variables. By the remarks above involving (1.13) and (1.14), since $y \in \mathcal{Y}^{\mathrm{ss}}$, we see that the above map coincides with the natural map

$$\omega^{\otimes k}|_{\mathcal{Y}^{\mathrm{ss}}} \hookrightarrow \omega^{\otimes k} \otimes_{\mathcal{O}_Y} \mathcal{O}_{\Delta,\mathcal{Y}^{\mathrm{ss}}} \xrightarrow{\text{stalk at } y} \omega^{\otimes k} \otimes_{\mathcal{O}_Y} \mathcal{O}_{\Delta,\mathcal{Y}^{\mathrm{ss}},y}$$

and hence is compatible with the natural connections on all sheaves; in particular, this compatibility shows that the formula (1.15) gives the action of the p-adic Maass-Shimura operator $\theta^j \circ d^j$. One of the main results of Section 6 is that for appropriate

$$b \in \mathbb{Q}$$

(a priori depending on y, but we later show that b is the same for all y in a certain CM orbit), in fact we have that the above map factors through

$$\omega^{\otimes k}|_{\mathcal{Y}^{\mathrm{ss}}} \to \hat{\mathcal{O}}^{+}_{\mathcal{Y}^{\mathrm{ss}},y}[\![q_{\mathrm{dR}}^{1/p^b} - 1]\!][1/p]. \tag{1.16}$$

Proving the integrality of the q_{dR}^{1/p^b}-expansion involves the consideration of another *Lubin-Tate period* $\mathbf{z}_{\mathrm{LT}}$ coming from the Rapoport-Zink uniformization of the infinite-level supersingular locus $LT_\infty \to \mathcal{Y}^{\mathrm{ss}}$ by the Lubin-Tate space LT_∞ at infinite level. This aforementioned period comes from the Grothendieck-Messing crystalline period map $LT_\infty \to \mathbb{P}^1$ associated to this Rapoport-Zink space. Viewing LT_∞ as an object in the proétale site of Y, one can in fact show that there is a canonical isomorphism

$$\mathcal{O}_{\Delta,LT_\infty} \cong \hat{\mathcal{O}}_{LT_\infty}[\![\mathbf{z}_{\mathrm{LT}} - \bar{\mathbf{z}}_{\mathrm{LT}}]\!], \tag{1.17}$$

where $\bar{\mathbf{z}}_{\mathrm{LT}}$ denotes $\mathbf{z}_{\mathrm{LT}}$ viewed as a section in $\hat{\mathcal{O}}_{LT_\infty}$, which in turn has a natural (horizontal) embedding into $\mathcal{O}_{\Delta,LT_\infty}$. From the above isomorphism, one can show another natural isomorphism

$$\mathcal{O}_{\Delta,\mathcal{Y}^{\mathrm{ss}}} \cong \hat{\mathcal{O}}_{\mathcal{Y}^{\mathrm{ss}}}[\![q_{\mathrm{dR}} - 1]\!] = \hat{\mathcal{O}}_{\mathcal{Y}^{\mathrm{ss}}}[\![q_{\mathrm{dR}}^{1/p^b} - 1]\!]. \tag{1.18}$$

Using a variant of the Dieudonné-Dwork lemma for integrality of power series, one can show that integrality of coefficients (at certain geometric stalks) of the power series expansion (1.17) transfer to integrality in the q_{dR}^{1/p^b}-expansion (1.18).

Given

$$w \in \omega^{\otimes k}(Y),$$

we can construct the p-adic L-function associated with w by considering sums of the images

$$w(q_{\mathrm{dR}}^{1/p^b})_y$$

of w under (1.16), where the subscript y denotes the stalks at various orbits of CM points y on $\mathcal{Y}^{\mathrm{ss}}$, then applying the normalized Maass-Shimura operators

$$p^{bj}\theta \circ d^j$$

using the formula (1.15). By the formula, we see that as long as p-adic valuations

$$|y_{\mathrm{dR}}(y)|, \quad |\hat{\mathbf{z}}(y)|$$

of the specializations

$$y_{\mathrm{dR}}(y), \quad \hat{\mathbf{z}}(y)$$

of the p-adic periods

$$y_{\mathrm{dR}}, \quad \hat{\mathbf{z}}$$

satisfy certain bounds, then images in the stalks

$$p^{bj}(\theta \circ d^j)w(q_{\mathrm{dR}}^{1/p^b})_y$$

"converge" to some p-adic continuous function in $j \in \mathbb{Z}/(p-1) \times \mathbb{Z}_p$, in a sense we now make more precise. Define the *stabilization* by

$$w^\flat(q_{\mathrm{dR}}^{1/p^b})_y = w(q_{\mathrm{dR}}^{1/p^b})_y - \frac{1}{p}\sum_{j=0}^{p-1} w(\zeta_p^j q_{\mathrm{dR}}^{1/p^b})_y. \tag{1.19}$$

We also denote

$$w^\flat(q_{\mathrm{dR}}^{1/p^b})_y = f^\flat(q_{\mathrm{dR}}^{1/p^b})_y \cdot w_{\mathrm{can},y}^{\otimes k}. \tag{1.20}$$

One can show directly from (1.15) that

$$p^{bj}(\theta \circ d^j)w^\flat(q_{\mathrm{dR}}^{1/p^b})_y$$

is a p-adic continuous function of $j \in \mathbb{Z}/(p-1) \times \mathbb{Z}_p$. Then for any $j_0 \in \mathbb{Z}_{\geq 0}$ we have

$$\lim_{m\to\infty} p^{b(j_0+p^m(p-1))}(\theta \circ d^{j_0+p^m(p-1)})w(q_{\mathrm{dR}}^{1/p^b})_y = p^{bj_0}(\theta \circ d^{j_0})w^\flat(q_{\mathrm{dR}}^{1/p^b})_y. \tag{1.21}$$

Roughly, summing

$$p^{bj}(\theta \circ d^j)w^\flat(q_{\mathrm{dR}}^{1/p^b})_y$$

against anticyclotomic Hecke characters χ evaluated at ideals corresponding to y for y over an appropriate CM orbit (associated with an order $\mathcal{O}$ of an imaginary quadratic field K) gives the construction of our p-adic continuous L-function. In reality, we will be able to bound the p-adic periods at CM points y' which are related to the natural CM points y associated with $\mathcal{O}$ via

$$y' \cdot \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} = y,$$

where $q = p$ if $p > 2$ and $q = 8$ if $p = 2$. As a consequence, the $q_{\mathrm{dR}}^{1/q}$-expansions at the $\mathcal{O}$-orbit of y' are well-behaved, and we then construct our p-adic L-function by summing

$$w_j'(q_{\mathrm{dR}}^{1/q})_{y'}$$

against anticyclotomic Hecke characters evaluated at an $\mathcal{O}$ orbit of y', where

$$w'_j = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^* (q^j(\theta \circ d^j)w^\flat)$$

is a certain Hecke translate of the Maass-Shimura derivatives, which allows us to relate our p-adic L-function to period sums over CM orbits of y, and hence obtain our interpolation property using the algebraicity theorem (1.12). Namely, values of the p-adic L-function in a certain range are equal to certain algebraic normalizations of central critical L-values associated with the Rankin-Selberg family (w, χ).

We end this outline with a few remarks on how we obtain the p-adic Waldspurger formula in Section 9, focusing on the case when $k=2$. A key property of the p-adic Maass-Shimura operator d^j is that it sends p-adic modular forms of weight k in the sense of (1.7) to modular forms of weight $k+2j$. Hence the limit

$$\lim_{m\to\infty} p^{bp^m(p-1)}(\theta \circ d^{p^m(p-1)})w^\flat(q_{\mathrm{dR}})_y$$

converges to a p-adic modular form of weight 0 on some small affinoid neighborhood of y, for some subgroup

$$\Gamma \subset GL_2(\mathbb{Z}_p).$$

Let K_p denote the p-adic completion of K with respect to a fixed embedding

$$\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p.$$

In fact, one can show that on some affinoid

$$\mathcal{U} \supset \mathcal{Y}^{\mathrm{Ig}} \sqcup \mathcal{C},$$

where $\mathcal{C} \subset \mathcal{Y}$ is a locus of CM points associated with K such that

$$\mathrm{Gal}(\mathcal{C}/C) \cong \mathcal{O}_{K_p}^\times \subset GL_2(\mathbb{Z}_p)$$

(induced by some embedding

$$K_p \hookrightarrow M_2(\mathbb{Z}_p);$$

the subadic space $\mathcal{C}$ itself does not depend on this choice of embedding), the limit

$$G := \lim_{m\to\infty} p^{bp^m(p-1)}(\theta \circ d^j)w^\flat(q_{\mathrm{dR}})|_{\mathcal{U}} \tag{1.22}$$

converges to a p-adic modular form of weight 0 on $\mathcal{U}$ for some Γ with

$$B \subset \Gamma \subset GL_2(\mathbb{Z}_p).$$

In particular, by restriction it induces a rigid function on $\mathcal{Y}^{\mathrm{Ig}} \sqcup \mathcal{U}$, which is of weight 0 for B on $\mathcal{Y}^{\mathrm{Ig}}$ and of weight 0 for Γ on $\mathcal{U}$. This means that G descends to a section G on an affinoid open

$$U \subset Y'$$

for some finite étale cover

$$Y' \to Y;$$

here we use the fact that while

$$\mathcal{Y}^{\mathrm{Ig}} \subset \mathcal{Y}$$

is not affinoid open, its image on any finite cover is isomorphic to a copy of the ordinary locus

$$Y^{\mathrm{ord}} \subset Y,$$

which, being the complement of a finite union of residue discs (the supersingular locus), is an (admissible) affinoid open. In particular, G is rigid on U, and one can show using Coleman's theory of integration that on

$$U \cap Y'^{\mathrm{ord}},$$

G is in fact equal to the formal logarithm

$$\log_{w^\flat} |_{U \cap Y'^{\mathrm{ord}}}$$

for some p-stabilization $w^\flat$ of the newform w. (Here p-stabilization denotes the image of w under some explicit Hecke operator at p.) Then the rigidity of G on U implies that dG is a rigid 1-form on U, and so by the theory of Coleman integration the rigid primitive G on $\mathcal{U}$ is unique up to constant, which implies

$$G = \log_{w^\flat} |_U. \tag{1.23}$$

Since the relevant special value of our p-adic L-function corresponds to evaluating (1.22) on an orbit of the CM point y, one sees that we arrive at our p-adic Waldspurger formula by evaluating (1.23) at an appropriate Heegner point.

1.3 MAIN RESULTS

We now finally state our main results. We adopt the notation of Chapter 8, and the reader should refer to there for precise definitions and assumptions.

Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and view all number fields as embedded in $\overline{\mathbb{Q}}$. Let p denote a prime number. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and fix embeddings

$$i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \quad i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p.$$

Fix an imaginary quadratic extension $K/\mathbb{Q}$. Now suppose the prime p is inert or ramified in K, and let w be a new eigenform (i.e., a newform or Eisenstein series) of weight $k \geq 2$ for $\Gamma_1(N)$ and nebentype ϵ_w, where $p \nmid N$, $N \geq 4$. Let $a_p(w)$ denote the Hecke eigenvalue of T_p. Suppose that

1. k is even *or* $p > 2$, and
2. N satisfies the Heegner hypothesis, i.e., that each prime $\ell | N$ splits or ramifies in K, and if $\ell^2 | N$ then ℓ splits in K.

Let A be a fixed elliptic curve with CM by an order $\mathcal{O}_c \subset \mathcal{O}_K$ of conductor $p \nmid c$ also with $(c, Nd_K) = 1$, let

$$\alpha : \mathcal{O}_{K_p} \xrightarrow{\sim} T_p A$$

be a choice of full p^∞-level structure as in Choice 8.6, let

$$y = (A, \alpha) \in \mathcal{C}(\overline{K}_p, \mathcal{O}_{\overline{K}_p})$$

and let $\Omega_p(y)$ and $\Omega_\infty(y)$ be the associated periods as in Definition 5.45, and also let

$$w|_{\mathcal{H}^+} = F \cdot (2\pi i dz)^{\otimes k}.$$

Then for Hecke characters $\chi \in \Sigma$, in the notation of Chapter 8, Section 8.2, we have that the values $L(F, \chi^{-1}, 0)$ are central critical. On a certain subset $\Sigma_+ \subset \Sigma$, characters satisfy root number conditions so that these central critical L-values are nonvanishing, and so present as candidates for interpolation. Given an algebraic Hecke character χ, let $\check{\chi}$ denote its p-adic avatar, and let $\mathbb{N}_K : \mathbb{A}_K^\times \to \mathbb{C}^\times$ denote the norm character, which has infinity type $(1,1)$. We let $\overline{\check{\Sigma}}$ denote the p-adic closure of the p-adic avatar $\check{\Sigma}_+$ (in the space of functions on $\mathbb{A}_K^{(p\infty)}$, equipped with the uniform convergence topology) of Σ_+: we note that one can naturally view $\check{\Sigma} \subset \overline{\check{\Sigma}}$. Finally, suppose (A, t, α) is a suitable CM point on infinite level (see Choice 8.6), let

$$\theta(\Omega_p)(A, t, \alpha) \in \mathbb{C}_p^\times$$

be the period as in Definition 5.45 (see also Propositions 7.3 and 7.7), so that denoting by $\mathfrak{p}$ the prime of K above p, $\#\kappa = \#\mathcal{O}/\mathfrak{p}$ the order of the residue field at $\mathfrak{p}$, and e the ramification index of $K_{\mathfrak{p}}/\mathbb{Q}_p$ ($=1$ if p is inert in K, $=2$ if p is ramified in K), we have

$$|\theta(\Omega_p)(A,t,\alpha)| = |2|p^{\frac{1}{p-1}-\frac{1}{e(\#\kappa-1)}} = \begin{cases} |2|p^{\frac{p}{p^2-1}} & p \text{ inert in } K \\ |2|p^{\frac{1}{2(p-1)}} & p \text{ ramified in } K. \end{cases}$$

Let

$$\Omega(A,t) \in \overline{\mathbb{Q}}^\times$$

be the period as in Definition 8.10. We collect our results into one Main Theorem.

Theorem 1.1 (Theorems 8.9, 8.14, 9.10). *There is a p-adic continuous function*

$$\mathcal{L}_{p,\alpha}(w,\cdot) : \overline{\check{\Sigma}}_+ \to \mathbb{C}_p$$

that satisfies the following interpolation property. Let $q=p$ if $p>2$ and $q=8$ if $p=2$. Then for all $\chi \in \Sigma_+$, when w is a newform we have

$$\mathcal{L}_{p,\alpha}(w,\check{\chi}) = \frac{(q\theta(\Omega_p)(A,t,\alpha))^{k+2j}}{\Omega(A,t)^j} \cdot \Xi_p(w,\chi) \cdot i_p\left(L^{\mathrm{alg}}(F,\chi^{-1},0)\right) \qquad (1.24)$$

and when w is an Eisenstein series ($F = E_k^{\psi_1,\psi_2}$, see Definition 8.3) we have

$$\mathcal{L}_{p,\alpha}(w,\check{\chi}) = \frac{(q\theta(\Omega_p)(A,t,\alpha))^{k+2j}}{\Omega(A,t)^j} \cdot \Xi_p(w,\chi) \cdot i_p\left(L^{\mathrm{alg}}(E_k^{\psi_1,\psi_2},\chi^{-1},0)\right), \qquad (1.25)$$

where $L^{\mathrm{alg}}(F,\chi^{-1},0)$ and $L^{\mathrm{alg}}(E_k^{\psi_1,\psi_2},\chi^{-1},0)$ are certain algebraic normalizations of square roots of the Rankin-Selberg central L-value $L((\pi_w)_K \times \chi^{-1}, 1/2)$ and Hecke central L-value $L^{\mathrm{alg}}(E_k^{\psi_1,\psi_2},\chi^{-1},0)$ as defined in Definition 8.5, and

$$\Xi_p(w,\chi) = \begin{cases} 1 - a_p(w)^2\chi^{-1}(p)\frac{p-1}{p+1} - \frac{1}{p^2} & p \text{ inert in } K \\ 1 - a_p(w)\chi^{-1}(\mathfrak{p})\frac{p-1}{p} - \frac{1}{p^2} & p \text{ ramified in } K. \end{cases}$$

We have the following "p-adic Waldspurger formula": For any $\chi \in \check{\Sigma}_- \subset \overline{\check{\Sigma}}_+$ of infinity type $(k-(j+1), j+1)$ where $0 \le j \le r := k-2$, that

$$\mathcal{L}_{p,\alpha}(w,\check{\chi}) = \frac{(q\theta(\Omega_p)(A,t,\alpha))^{k-2(j+1)}}{\Omega(A,t)^{-(j+1)}} \Xi_p(w,\check{\chi}) \cdot \frac{c^{-j}}{j!} \mathrm{AJ}_F(\Delta_{\chi\mathbb{N}_K^{-1}})(w \wedge \omega_{A_0}^j \eta_{A_0}^{r-j}),$$

where

$$\mathrm{AJ}_F(\Delta_{\chi\mathbb{N}_K^{-1}})(w \wedge \omega_{A_0}^j \eta_{A_0}^{r-j})$$

is the p-adic Abel-Jacobi image of a specific generalized Heegner cycle as defined in Chapter 10, Section 9.2 (which depends on some fixed elliptic curve A_0 *with CM by* $\mathcal{O}_K$*). In particular, we have the following application toward the Beilinson-Bloch conjecture. Let* χ *take values in* E*. In particular, if*

$$\mathcal{L}_{p,\alpha}(w,\check{\chi}) \neq 0, \tag{1.26}$$

then

$$\epsilon_w \epsilon_\chi \mathbb{N}_K^{-1} \Delta_{\chi\mathbb{N}_K^{-1}} \in \varepsilon_w \varepsilon_{\chi\mathbb{N}_K^{-1}} \mathrm{CH}_0^{r+1}(X_r)(F)_E^{\chi\mathbb{N}_K^{-1}}$$

is nontrivial, where the right-hand side denotes the $\varepsilon_w\varepsilon_{\chi\mathbb{N}_K^{-1}}$*-isotypic component of an appropriate Chow group for the underlying (Chow) motive attached to* (w,χ^{-1})*.*

When $k=2$*, we have a simpler statement. Let* H_c *denote the ring class field associated with the order* $\mathcal{O}_c$*. For any character* $\chi : \mathcal{C}\ell(\mathcal{O}_c) \to \overline{\mathbb{Q}}_p^\times$ *with* $\check{\mathbb{N}}_K\chi \in \check{\Sigma}_- \subset \overline{\check{\Sigma}}_+$*, we have*

$$\mathcal{L}_{p,\alpha}(w,\check{\mathbb{N}}_K\chi) = \Omega(A,t)\Xi_p(w,\check{\mathbb{N}}_k\chi)\cdot \log_w P_K(\chi)$$

where $P_K(\chi) \in \mathrm{Jac}(Y_1(N))(H_c)$ *is the Heegner point as defined in Section 9.*

In particular, if

$$\mathcal{L}_{p,\alpha}(w,\check{\mathbb{N}}_K\chi) \neq 0,$$

then $P_K(\chi)$ *projects via a modular parametrization to a non-torsion point in* $A_w(H_c)^\chi$*, where* $A_w/\mathbb{Q}$ *is the* GL_2*-type abelian variety associated uniquely up to isogeny with* w*. Then by the Gross-Zagier formula and Kolyvagin, we have*

$$\mathrm{rank}_{\mathbb{Z}} A_w(H_c)^\chi = \dim_{\mathbb{Q}} A_w = \mathrm{ord}_{s=1} L(A_w,\chi,s).$$

1.4 SOME REMARKS ON OTHER WORKS IN SUPERSINGULAR IWASAWA THEORY

Finally, we point out that there has been much groundbreaking work done in supersingular Iwasawa theory by many other authors. We give a brief summary of some of these results. In our setting of Iwasawa theory for imaginary quadratic fields in which p is inert or ramified, Rubin ([54]), following methods of Katz and invoking the machinery of Coleman power series, succeeded in constructing 1-variable continuous p-adic L-functions in the Lubin-Tate direction (i.e., for characters of type $(k,0)$ and varying $k \in \mathbb{Z}/(p-1)\times\mathbb{Z}_p$). In [53], he also formulated analogues of supersingular main conjectures assuming certain conjectures on the structure of the Iwasawa module of universal norms of local units. Agboola-Howard ([1]) also developed anticyclotomic main conjectures for imaginary quadratic fields, assuming the aforementioned conjecture on the structure of local units (though even with this assumption, they did not construct an analytic anticyclotomic p-adic L-function, as we do in this book). Schneider-Teitelbaum ([63]), using their p-adic Fourier theory and Coleman power series, also constructed distributions interpolating Hecke L-values over imaginary quadratic fields in which p is inert or ramified.

In the GL_2-setting, particularly for newforms attached to elliptic curves, there has also been great progress, although not directly related to our situation. Previous works have mainly addressed the Iwasawa theory of families of twists $V \otimes \chi$ where the weight of V is greater than the weight of the characters χ. In this case the Galois representations in consideration are supersingular at p exactly when V itself is supersingular at p, since the Hodge-Tate weights of V dominate those of χ. In contrast, we address the case where the weight of the χ's is at least the weight of V, and hence the twists are supersingular precisely when the character χ is supersingular at p (i.e., p is inert or ramified in K), since the Hodge-Tate weights of χ dominate those of V. The former situation, however, already has potent applications to the Birch and Swinnerton-Dyer conjecture. For V attached to elliptic curves over $\mathbb{Q}$, see the fundamental work of Pollack ([50]) who introduced "$+/-$" constructions in order to produce 1-variable (cyclotomic) measures from the classical distributions attached to elliptic curves with good supersingular reduction at p (the construction of which is due to Višik [66], Amice-Vélu [2], and Mazur-Tate-Teitelbaum [47]). Kobayashi, soon after Pollack, gave an algebraic construction of these "$+/-$" p-adic L-functions (see [39]) by defining a suitable "$+/-$" Coleman map and evaluating on Kato's Euler system (see [32]). Later, Sprung ([65]) extended this to the $a_p \neq 0$ case by constructing an appropriate generalization of the "$+/-$" Coleman map. From their algebraic constructions of p-adic L-functions, Kobayashi and Sprung were also able to formulate appropriate "$+/-$" cyclotomic main conjectures in the non-CM case, and use Kato's Euler system to prove one divisibility of these main conjectures. Pollack-Rubin soon afterward formulated and proved the CM analogue of this main conjecture ([51]), building on Kobayashi's construction and Rubin's previous work on the Euler system of elliptic units ([55]). Kim was also able to generalize Kobayashi's constructions to 2 variables in certain height 1 settings ([38]). In more general settings, for elliptic curves there is also the work of Wan ([67]), who proved the supersingular analogue of Skinner-Urban's GL_2 main conjecture ([64]), and of Castella-Wan ([11]), who formulated and proved $+/-$ analogues of Perrin-Riou's main conjecture on Heegner points ([49]). See also the works addressing more general settings such as that of Lei-Loeffler-Zerbes ([45]), Büyükboduk-Lei ([7]), and Castella-Çiperiani-Skinner-Sprung ([10]), who also addressed the setting of a general elliptic modular form.

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