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Chapter One

Introduction

This text has several parts:

In the first part of the text we develop a small-cancellation theory over cube complexes. When the cube complex is 1-dimensional, we obtain the classical small-cancellation theory, as well as the closely related Gromov graphical small-cancellation theory.

It is hard to say what the main result is in the first part, since it seems the definitions are more important than the theorems. For this and the second part, the reader might wish to scan the table of contents to get a feel for what is going on. We give the following sample result to give an idea of the scope here. In ordinary small-cancellation theory, when $W_1, \ldots, W_r$ represent distinct conjugacy classes, the presentation $\langle a, b, \ldots \mid W_1^{n_1}, \ldots, W_r^{n_r} \rangle$ is "small-cancellation" for sufficiently large n_i. In analogy with this we have the following:

C6-sample. *Let X be a nonpositively curved cube complex. Let $Y_i \to X$ be a local-isometry with Y_i compact for $1 \leq i \leq r$ such that each $\pi_1 Y_i$ is malnormal, and $\pi_1 Y_i, \pi_1 Y_j$ do not share any nontrivial conjugacy classes. Then $\langle X \mid \widehat{Y}_1, \ldots, \widehat{Y}_r \rangle$ is a "small-cancellation" cubical presentation for sufficiently large "girth" finite covers $\widehat{Y}_i \to Y_i$.*

Many other general small-cancellation theories have been propounded. For instance two such graded theories directed especially towards Burnside groups were produced by Olshanskii and McCammond. Stimulated by Gromov's ideas of small-cancellation over word-hyperbolic groups, there have been later important works of Olshanskii, followed by more recent theories "over relatively hyperbolic groups" by Osin [Osi06] and Groves-Manning [GM08]. The theory we propose is decidedly more geometric, and arguably favors explicitness over scope. However, although it may be more limited by presupposing a nonpositively curved cube complex as a starting point, it has the advantage of not presupposing (relative) hyperbolicity—yet some form of hyperbolicity must lurk inside for there to be any available small-cancellation.

In the second part of the text we impose additional conditions that lead to the existence of a wallspace structure on the resulting small-cancellation presentation. We can illustrate the nature of the results with the following sample:

B6-sample. *Let G be an infinite word-hyperbolic group acting properly and cocompactly on a CAT(0) cube complex. Let $H_1, \ldots, H_k$ be quasiconvex subgroups that are not commensurable with G. And suppose that each H_i has separable hyperplane stabilizers. There exist finite index subgroups $H'_1, \ldots, H'_k$ such that the quotient $G/\langle\langle H'_1, \ldots, H'_k \rangle\rangle$ has a codimension-1 subgroup.*

Here $\langle\langle A, B, \ldots \rangle\rangle$ denotes the normal closure of $\{A \cup B \cup \cdots\}$ in the group.

In the third part of the text, we probe further and seek a virtually special cubulation.

We then prove the following:

Theorem A (Special Quotient Theorem)**.** *Let G be a word-hyperbolic group that is virtually the fundamental group of a compact special cube complex. Let $H_1, \ldots, H_r$ be quasiconvex subgroups of G. Then there are finite index subgroups $H'_i \subset H_i$ such that: $G/\langle\langle H'_1, H'_2, \ldots, H'_r \rangle\rangle$ is virtually special.*

We then prove the following:

Theorem B (Quasiconvex Hierarchy $\Rightarrow$ Virtually Special)**.** *Let G be a word-hyperbolic group with a quasiconvex hierarchy, in the sense that it can be decomposed into trivial groups by finitely many HNN extensions and amalgamated free products along quasiconvex subgroups. Then G is virtually special.*

There are two important applications of the virtual specialness of groups with a quasiconvex hierarchy: It is applied to hyperbolic 3-manifolds with a geometrically finite incompressible surface to reveal their virtually special structure. This resolves the subgroup separability problem for fundamental groups of such manifolds. It also completes a proof that Haken hyperbolic 3-manifolds are virtually fibered. It is also applied to resolve Baumslag's conjecture on the residual finiteness of one-relator groups with torsion.

The fourth part of the text deals with groups that are hyperbolic relative to virtually abelian subgroups, and provides similar structural results for many such groups when they also have quasiconvex hierarchies.

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