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Author

Alexandria

The geometer and the king

Alexandria, around 300 BC.

A dinner, say: a symposium, in the palace quarter, perhaps in the Museum. Ptolemy himself in attendance: general, hero, king, deity. And the talk touches on geometry: why so hard? Why is there no easier way? The geometer – a dusty man, but pert – answers: Majesty, there is no royal road to geometry.

Ptolemy snubbed is one of the irresistible stories. The man had been a childhood friend of Alexander the Great; one of his bodyguard. Maybe his illegitimate half-brother. He was a trusted general (the name is said to mean 'warlike'): level-headed but capable of the grand gesture; a man of no nonsense.

He was one of the great survivors, too. In the twenty years of chaos that followed Alexander's death, when many abler men died, Ptolemy played and won. Of all the successors who eventually carved up Alexander's brief, continent-spanning empire, he founded the longest dynasty, the stablest country. He chose Egypt, and he never risked it for a larger realm. Fourteen Ptolemaic rulers followed him, until Cleopatra lost it all at the Battle of Actium 250 years later. The first king, then, of the last Egyptian dynasty. *Basileus* to the Greeks, pharaoh to the Egyptians. Inheritor of 3,000 years of Egyptian kingship and, yes, a god too. In 306 BC he defeated an attack on Rhodes so soundly that altars were erected to him and he acquired the title 'saviour'. By 278 BC there were Ptolemaic games in his honour: four-yearly, like the Olympics.

Ptolemy I Soter.

The geometer, by contrast – a man named Euclid; Eukleídēs – is utterly obscure, historically speaking. The story about the 'royal road' is moonshine, sadly; it is also told about another geometer (Menaechmus) and another king (Alexander), and there is little reason to suppose it really happened. Even his dates – sometime around 300 BC – are the mere surmise of authors writing centuries after him. Unlike the exceptionally well-documented Ptolemy, Euclid left no biographical traces whatever. He founded no dynasty, built no palaces. His legacy was solely intellectual.

But what a legacy. The line of his students at Alexandria outlasted his life. His book outlasted his civilisation.

What was this city, that produced such a man and such a book? Alexandria was the right setting for the *Elements of Geometry*. It was Ptolemy's greatest achievement. Alexander himself decreed the city, on the site of what had been a village and, like a dozen others, it bore his name. He never saw a stone erected, but Ptolemy took it for his capital, moving the Egyptian royal seat from Memphis. It was a Greek *polis* in a profoundly un-Greek world, a new foundation in a land where cities were 2,000 years old. Ptolemy did everything to make it splendid; it had an assembly, a council, its own coinage and its own laws. There were broad boulevards, colonnades, avenues of trees, street lamps. In 322 BC he kidnapped Alexander's corpse and put it on display in his new royal city.

It was indeed the most splendid site for a city, at the meeting place of two continents, just to the west of the mouths of the Nile. It would be a major commercial port for centuries, a strategic military site until the Second World War. Ptolemy laid the foundations of the famous lighthouse: fortress and beacon, it would be one of the seven wonders of the world when it was done, 400 feet high and topped by a statue of Zeus. It stood for 1,500 years. Blessed with such a city, people immigrated from all over the Greek world, and Alexandria became not just notoriously big and splendid but notoriously crowded and cosmopolitan, with Greeks, Macedonians, Egyptians, Jews, Syrians and more, its streets teeming like ant heaps. Within a few generations it would have more than a million inhabitants.

As well as town planning, and the sheer erection of building upon building, Ptolemy involved himself in cultural policy: and with his characteristic effectiveness. To make himself a convincing Egyptian pharaoh, he indulged in sculpture to match, and devised a new cult of 'Serapis', a blatantly invented figure with a hybrid iconography. Like all his achievements, it endured: the temple, the Serapeum at Alexandria, stood for 600 years.

And to please the Greek mind and heart there were pageants, festivals, and a palace with tapestries the gods would prize. As one contemporary put it, Alexandria had 'wealth, wrestling schools, power, tranquillity, fame, spectacles, philosophers, gold, youths, the sanctuary of the sibling gods . . . the Museum, wine, every good thing [one] could desire'.

All of which was invaluable in projecting Greek power and an idea of Greekness in a profoundly alien environment; in saying, this is what we do in the great Greek world. This is our right to rule.

And so, the Museum. The Mouseion: the shrine of the Muses. It was royally subsidised, with scholars in discipline after discipline. Their head was a priest of the Muses, and they included poets, grammarians, historians, philosophers, doctors, natural philosophers, geographers, machine-builders, astronomers, and of course geometers. It was partly Ptolemy's own impulse, partly the creation of Demetrios of Phaleron, a famous pupil of Aristotle brought over from Athens to oversee the creation of the new institution. It had courts, covered walks and gardens, a dining hall and an observatory. The staff numbered perhaps forty, and they spent their time researching, writing and sometimes teaching. They held learned symposia, some attended by the king. They were a remarkable collection of people, sometimes compared a little sourly with the collection of animals that Ptolemy also founded: 'well-fed bookworms, arguing endlessly in the Muses' birdcage'. If there were bookworms, there were certainly books, too; the library at Alexandria would become the most famous in the world, though it seems to have been organised a little later, under Ptolemy's son.

All of which is how the famous Greek mathematician ended up working in Egypt. Was Euclid another part of Ptolemy's collection, someone brought in to swell the ranks of the Museum? It is not certain whether he was native to Alexandria or an immigrant, although at this early date in the city's life the latter is much the more likely. An immigrant from where? His austere prose gives no hint of an accent: unlike Archimedes, in the next generation, who wrote in the Doric dialect of Syracuse.

What came to Alexandria in Euclid's person (and perhaps in the persons of other mathematicians; it is not clear whether or not he was the only one) was a well-established tradition of Greek geometry. Greeks liked to have things to think about. They liked hobbies. Some Greeks raced chariots, some talked about philosophy, some occupied themselves with politics. From perhaps the late fifth century BC onwards, some did geometry.

What was it like? Perhaps it is clearest to think of Greek geometry as an outgrowth of the Greek love of talk, of disputation. For its geometry was nothing if it was not a performance.

Draw a line, a square, a circle. Reason out loud as you draw; play to the inevitable audience. From such beginnings the long-lived game of geometrical reasoning took shape. The figure of the geometer drawing in the sand remains part of the image of ancient Greek mathematics to this day: raking in the 'learned dust', as the Roman orator Cicero put it. He memorably evoked Archimedes as a man of 'dust and drawing-stick'. (Though have you ever tried to draw a detailed diagram in dry sand? Pieces of clay, wax tablets or, for showing to a larger audience, wooden boards seem more probable.)

The number of Greek mathematicians was never very large, and they had to write their ideas down to preserve what they had found out about their lines and circles; there weren't enough of them, it seems, for a purely oral transmission to stand a chance. So a genre was born: a special style of mathematical writing. That style would come to define mathematics in the West for more than two millennia, as constraining as any poetic metre, and as long-lived. Its components were the statement (of something to be proved); the diagram, with its points labelled with letters; and a chain of reasoning from things already known to things newly proved. That chain ended with the proposed, intended result, and the section – the 'proposition' – was rounded off with the note that 'this is what was to be proved': *hóper édei deîxai*; *quod erat demonstrandum*; QED. In some cases it was 'what was to be drawn'. Here is an example:

How to draw an equilateral triangle

Start with any straight line; call its ends A and B.

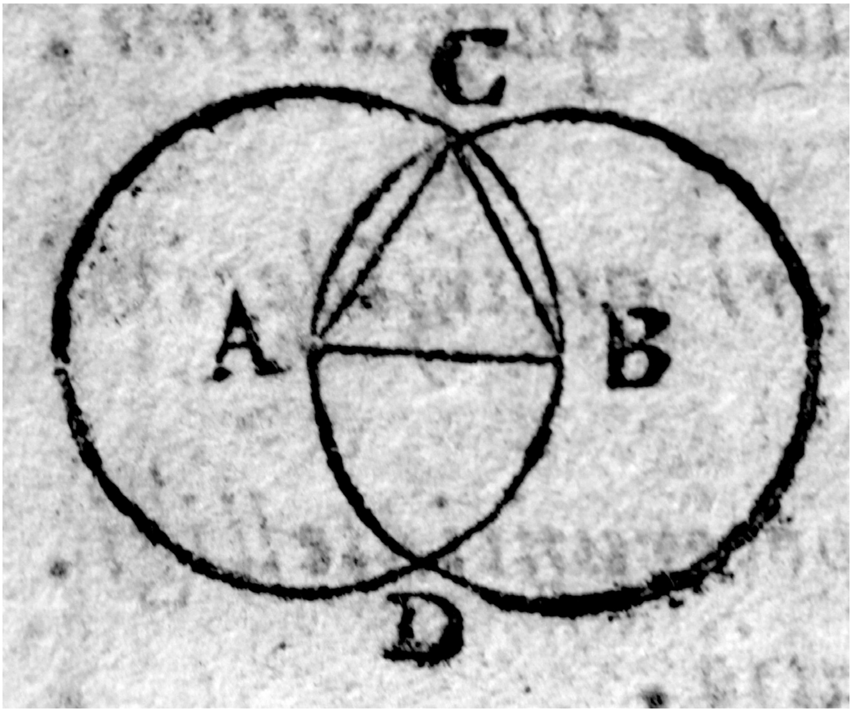
Now draw two circles, each with its radius equal to that line: one centred at A, the other centred at B.

The two circles will cross at two points. Pick one of them and call it C. Now join up A, B and C. They make an equilateral triangle.

Why?

Because of the way the distance from A to B was used to find C. C is that same distance from A, and C is also that same distance from B. In other words, all three of the triangle's sides – AB, BC and CA – are the same length. So it's an equilateral triangle. Which is what was to be drawn.

The same ancient sources that report when Euclid lived also report that there were written collections of geometrical learning in Greek by perhaps 400 BC: a century before him. They are quite informative about the subjects involved, and even some of the specific results and techniques. The written matter itself has not survived, though, throwing it all into doubt. There is a strong temptation to make up genealogies for mathematical ideas when real evidence is lacking. So yes, maybe the study of the circle was



Constructing an equilateral triangle.

done by the Pythagoreans; likewise the work on numbers and their properties. Very probably the work on ratios was done by a geometer called Eudoxus early in the fourth century BC. Some work on the regular solids originated with another, named Theaetetus. The claim that there were, before Euclid, well-developed books called *Elements of Geometry* seems distinctly more doubtful.

What was Euclid's role, then? He took all the easier material known to the Greek geometers of his day and put it all together in a single book. He organised it, both on the large scale and the small. Certainly he added some new matter of his own, though no one can now confidently identify what was new and what was not. Historians continue to argue – and they always will – about how much was compilation and how much composition in Euclid's book. His was a work of collection, like Ptolemy's; Euclid the

Museum artefact became himself a curator, the *Elements* a smaller museum.

Yet this museum contained a world. It displayed the geometrical style of prose, in a relentless, ritualistic tramp of proposition after proposition: 400 of them, arranged in thirteen 'books' or chapters. Every verb was perfect, imperative and passive: 'let a circle have been drawn'. There was something hypnotic about it; something infinitely calm. The book began with definitions: what was meant by a line? A point? A circle? It continued with the simplest manipulations of lines and shapes in two dimensions: how to draw different kinds of triangle; how to divide a line or an angle in two. The fact that, in a triangle, any two sides add up to more than the third. The Epicurean philosophers thought that last piece of geometrical information was 'evident even to an ass' since 'if straw is placed at one extremity of the sides, an ass in quest of provender will make his way along the one side and not by way of the two others'.

Euclid did not care how obvious any of it was. He arranged and exemplified a toolbox of the basic techniques and results that he had inherited: ways of arguing, ways of proving; facts that geometers commonly assumed or used but seldom proved in full. At the end of the first book he placed 'Pythagoras' theorem'. Draw a triangle, with one of its angles a right angle. Using its shortest side as a base, draw a square, whose side is the same length as that side of the triangle. Repeat the procedure with the triangle's two longer sides, so that you end up with three squares of different sizes, resting flush against the three sides of the triangle. Now, it turns out, the areas of the two smaller squares will add up to that of the larger one: a startling fact, not evident to any ass, which Euclid proved in his characteristically meticulous manner.

And so the ideas and the diagrams became harder and more complicated throughout the book. There were purely geometrical sections: a description of how to draw a regular pentagon or hexagon inside a given circle, for example. Parts of the book dealt not with geometry but with numbers and ratios, ranging from the most basic facts ('if an odd number is multiplied by an odd number, then the product is odd') to a procedure for finding the mysterious 'perfect' numbers, equal to the sum of their divisors.

Finally, Euclid turned to three-dimensional shapes. The last three books of the *Elements* – books 11, 12 and 13 – were concerned with spheres, cones and cylinders, with cubes and cuboids and with the regular polyhedra. These were beautiful solids whose faces were all the same regular polygon: triangles, squares or pentagons. There were just five regular polyhedra: the tetrahedron (four triangular faces), the cube, the octahedron (eight triangles), the dodecahedron (twelve pentagons) and the icosahedron (twenty triangles). He showed how to make such shapes, starting from, say, a given triangle or a given circle; he showed how to find their surface areas and their volumes. Euclid's explorations in these final books were frequently ingenious, and sometimes applied an almost unbelievable amount of lateral thinking. Despite its gentle beginnings, and its incorporation of a lot that anyone could understand, the *Elements* as a whole was a virtuoso performance; a road that only the keenest of geometrical minds could follow all the way.

It added up to more than 20,000 lines of text in Greek. Euclid was careful, but he was no superman, and both joins and slips were sometimes visible. A few definitions (oblong, rhombus, rhomboid) seemed to have been carried across from older sources but never actually used. Quite a number of terms were used that were, conversely, not defined earlier on. Some words were ambiguous. A surprising amount was taken for granted about the properties of points and lines that Euclid never explicitly set out in his assumptions. Some propositions were merely special cases of others; some propositions were, strictly speaking, unnecessary because they were the mere logical consequences of others. But despite such wrinkles, the *Elements* was a fine, even an awe-inspiring monument to all that had been done in Greek geometry so far.

Euclid was by no means the author of just one book. The order of events is not clear, but it is certain that he wrote more. There were perhaps four other books on special topics in elementary geometry, and there were books on the applications of mathematics – to music, astronomy, optics and more – as well. In total nearly a dozen books are mentioned in the early evidence; eight actual texts survive, though most are disputed by historians.

Back in seething Alexandria, where the building work is still going on and the streets are ever more crowded. By the end of Euclid's life the great lighthouse at Pharos has been built (did the architects consult Euclid? It would be intriguing to think so); the library and Museum are nearing completion and the palace complex is grander than ever. The *Elements* is finished: thirteen rolls of papyrus covered with neat columns of text and diagrams. And Euclid is still teaching, still taking on new students.

One beginner is impatient, like the king before him. After he has understood the first proposition he bursts out, 'What is my profit now that I have learned that?'

A glance of contempt, or perhaps of pity. Euclid calls for a servant. 'Give him threepence, since he must always make a profit out of what he learns.'

Another romantic legend, perhaps: one that circulated later in Greece, during its period of Roman domination. Like the 'royal road' story, it helped to protect Euclid from the whiff of servility, of sycophancy that hung over anyone connected with Ptolemaic Alexandria and its institutions. It preserved and dramatised the idea that geometry was a leisured, cultured pursuit, part of the life of the mind. Not a profitable trade, but pure, truthful and beautiful for its own perfect sake.

Some 350 geometrical propositions in the driest of styles. It's an odd thing, to have become one of the most enduring cultural artefacts of the Greek world. Ptolemaic Alexandria is largely dust today; a few wrecked statues are dug from the ground or pulled from the sea from time to time, but the splendour is all gone. Ptolemy's dynasty died with Cleopatra. The library is scattered. But the books – the *Elements* among them – the books lived.

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