Evidently, for any continuous function $f : \mathbb{T}(p,r) \to \mathbb{C}$,

$$\int_{\mathbb{T}(p,r)} f(z)\,dz = ir \int_0^{2\pi} f(p + re^{it})e^{it}\,dt.$$

As a special case, consider a continuous complex-valued function f defined on the ***unit circle*** $\mathbb{T} = \mathbb{T}(0,1)$. The integral of f as a scalar function can be expressed as a complex integral:

$$\int_0^{2\pi} f(e^{it})\,dt = \int_{\mathbb{T}} f(z)\,\frac{dz}{iz}.$$

More generally, the ***Fourier coefficients*** of f, defined by

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\,e^{-int}\,dt \qquad (n \in \mathbb{Z}),$$

can be expressed as the complex integrals

$$\hat{f}(n) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{z^{n+1}}\,dz.$$

The following elementary observation will be useful:

LEMMA 1.27 (Continuous dependence on vertices). *Let $f : U \to \mathbb{C}$ be continuous and $T = \triangle abc$ be a closed triangle in U. Then, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|a - a'|$, $|b - b'|$, and $|c - c'|$ are all less than δ, then $T' = \triangle a'b'c' \subset U$ and*

$$\left| \int_{\partial T} f(z)\,dz - \int_{\partial T'} f(z)\,dz \right| < \varepsilon.$$

PROOF. Since the integral along the oriented boundary of a triangle is the sum of three integrals along oriented segments, it suffices to prove continuous dependence for oriented segments. Fix $[a,b] \subset U$ and let V be any open neighborhood of $[a,b]$ whose closure $\overline{V}$ is a compact subset of U. Given $\varepsilon > 0$, use uniform continuity of f on V to find $0 < \delta < \varepsilon$ such that $|f(z) - f(w)| < \varepsilon$ whenever $z, w \in V$ and $|z - w| < \delta$. We can also arrange that $[a', b'] \subset V$ whenever $|a - a'| < \delta$ and $|b - b'| < \delta$. Let $[a', b']$ be any such segment and note that if $\gamma(t) = (1-t)a + tb$ and $\eta(t) = (1-t)a' + tb'$, then

$$|\gamma(t) - \eta(t)| \leq (1-t)|a - a'| + t|b - b'| < \delta,$$
$$|\gamma'(t) - \eta'(t)| = |(b - a) - (b' - a')| \leq |b - b'| + |a - a'| < 2\delta$$

for all $0 \leq t \leq 1$. Hence,

$$\left| \int_{[a,b]} f(z)\,dz - \int_{[a',b']} f(z)\,dz \right| = \left| \int_0^1 \big[f(\gamma(t))\,\gamma'(t) - f(\eta(t))\,\eta'(t)\big]\,dt \right|$$
$$= \left| \int_0^1 \Big[\big(f(\gamma(t)) - f(\eta(t))\big)\gamma'(t) + f(\eta(t))\big(\gamma'(t) - \eta'(t)\big)\Big]\,dt \right|$$

$$\leq \int_0^1 |f(\gamma(t)) - f(\eta(t))|\,|\gamma'(t)|\,dt + \int_0^1 |f(\eta(t))|\,|\gamma'(t) - \eta'(t)|\,dt$$

$$\leq |b-a| \int_0^1 |f(\gamma(t)) - f(\eta(t))|\,dt + 2\delta \int_0^1 |f(\eta(t))|\,dt$$

$$\leq |b-a|\,\varepsilon + 2\delta \sup_{z\in V} |f(z)| \leq \left(|b-a| + 2\sup_{z\in V} |f(z)|\right)\varepsilon,$$

which proves the asserted continuity. □

A primitive is what students of calculus call "antidetrivative."

DEFINITION 1.28. A function $F \in \mathscr{O}(U)$ is called a ***primitive*** of a continuous function $f : U \to \mathbb{C}$ if $F'(z) = f(z)$ for all $z \in U$.

Suppose F is a primitive of f and $\gamma : [0,1] \to U$ is a piecewise C^1 curve. By the chain rule, the relation $(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t)$ holds for all but finitely many $t \in [0,1]$ (see problem 6). Since $F'(\gamma(t))\gamma'(t)$ is piecewise continuous on $[0,1]$ with at worst jump discontinuities, the fundamental theorem of calculus shows that

$$\int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt = \int_0^1 F'(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_0^1 (F\circ\gamma)'(t)\,dt = F(\gamma(1)) - F(\gamma(0)).$$

THEOREM 1.29. *A continuous function $f : U \to \mathbb{C}$ has a primitive in U if and only if $\int_\gamma f(z)\,dz = 0$ for every closed curve γ in U.*

PROOF. First suppose f has a primitive F. If $\gamma : [0,1] \to U$ is a closed curve, then $\gamma(0) = \gamma(1)$, so

$$\int_\gamma f(z)\,dz = F(\gamma(1)) - F(\gamma(0)) = 0.$$

Conversely, suppose f integrates to zero along every closed curve in U. To show f has a primitive, it suffices to consider the case when U is connected (and therefore path-connected); the general case follows by applying this case to each connected component of U. If γ, η are two curves in U with the same initial and end points, then the product $\gamma \centerdot \eta^-$ is a closed curve. Hence, by additivity (1.16) and our assumption,

$$\int_\gamma f(\zeta)\,d\zeta - \int_\eta f(\zeta)\,d\zeta = \int_\gamma f(\zeta)\,d\zeta + \int_{\eta^-} f(\zeta)\,d\zeta = \int_{\gamma\centerdot\eta^-} f(\zeta)\,d\zeta = 0.$$

Now fix a point $p \in U$. For any $z \in U$ use path-connectivity of U to find a curve γ in U from p to z and define

$$F(z) = \int_\gamma f(\zeta)\,d\zeta.$$

By the above remark, the right-hand side is independent of the choice of γ and yields a well-defined function $F : U \to \mathbb{C}$. Let us show that F is a primitive of f. Fix $z_0 \in U$ and choose $r > 0$ small enough so that $\mathbb{D}(z_0, r) \subset U$. Let $z \in \mathbb{D}(z_0, r)$ and let γ be any curve in U from p to z_0. The product $\gamma \centerdot [z_0, z]$ is then a curve in U from p to z. By additivity,

$$F(z) - F(z_0) = \int_{\gamma \centerdot [z_0,z]} f(\zeta)\, d\zeta - \int_{\gamma} f(\zeta)\, d\zeta = \int_{[z_0,z]} f(\zeta)\, d\zeta,$$

so if $z \neq z_0$,

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0,z]} (f(\zeta) - f(z_0))\, d\zeta. \tag{1.19}$$

Since f is continuous at z_0, for each $\varepsilon > 0$ we can find a $0 < \delta < r$ such that $|f(\zeta) - f(z_0)| < \varepsilon$ whenever $|\zeta - z_0| < \delta$. Since $|z - z_0| < \delta$ implies $|\zeta - z_0| < \delta$ for every $\zeta \in [z_0, z]$, the ML-inequality (1.18) applied to the right side of (1.19) gives

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \frac{1}{|z - z_0|} \cdot \varepsilon \cdot \text{length}([z_0, z]) = \varepsilon$$

whenever $0 < |z - z_0| < \delta$. Thus, $F'(z_0)$ exists and is equal to $f(z_0)$. Since $z_0 \in U$ was arbitrary, we conclude that F is a primitive of f in U. □

EXAMPLE 1.30. For every integer $n \neq -1$, the power function $f(z) = z^n$ has a primitive $F(z) = z^{n+1}/(n+1)$. It follows from Theorem 1.29 that $\int_\gamma z^n\, dz = 0$ if γ is any closed curve in the punctured plane $\mathbb{C} \smallsetminus \{0\}$ and $n \neq -1$, or if γ is any closed curve in $\mathbb{C}$ and $n \geq 0$.

The case $n = -1$ is completely different: For any $r > 0$,

$$\int_{\mathbb{T}(0,r)} \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{re^{it}}\, rie^{it}\, dt = 2\pi i \neq 0.$$

Note that the result is independent of the radius r. It follows from Theorem 1.29 that *the function* $z \mapsto 1/z$ *does not have a primitive in any punctured neighborhood of* 0.

1.4 Cauchy's theory in a disk

Our primary goal in this section is to prove that every holomorphic function in a disk has a primitive. Somewhat surprisingly, all the local properties of holomorphic functions are consequences of this central fact of Cauchy's theory. The special case of a disk will be enough for our purposes here; general domains and global issues will be dealt with in chapter 2.

According to Theorem 1.29, the existence of a primitive is equivalent to having vanishing integrals along all closed curves. Convexity of the disk allows us to replace the latter with something far simpler in terms of triangles.

THEOREM 1.31. *Let $D \subset \mathbb{C}$ be an open disk and $f : D \to \mathbb{C}$ be continuous. Suppose $\int_{\partial T} f(z)\, dz = 0$ for every closed triangle $T \subset D$. Then f has a primitive in D.*

PROOF. Let p be the center of D and define

$$F(z) = \int_{[p,z]} f(\zeta)\, d\zeta \qquad \text{for } z \in D.$$

We show that F is a primitive of f. Take distinct points $z_0, z \in D$ and apply the condition $\int_{\partial T} f(\zeta)\, d\zeta = 0$ to the closed triangle T with vertices p, z, z_0 to obtain

$$F(z) - F(z_0) = \int_{[p,z]} f(\zeta)\, d\zeta - \int_{[p,z_0]} f(\zeta)\, d\zeta = \int_{[z_0,z]} f(\zeta)\, d\zeta.$$

The rest of the argument, that is, dividing by $z - z_0$ and letting $z \to z_0$ to show that $F'(z_0) = f(z_0)$, is identical to the proof of Theorem 1.29. □

The problem of constructing primitives in D is thus reduced to showing that every $f \in \mathscr{O}(D)$ satisfies the triangle condition of Theorem 1.31. If we knew that the derivative f' is continuous (which is true but we have not yet proved it), this would be an easy consequence of Green's theorem. To see this, suppose $f \in \mathscr{O}(D)$ and *assume* f' is continuous in D. Then the partial derivatives of $u = \operatorname{Re}(f)$ and $v = \operatorname{Im}(f)$ are continuous in D and Green's theorem together with the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$ shows that for every closed triangle $T \subset D$,

$$\int_{\partial T} (u\, dx - v\, dy) = \iint_T (-v_x - u_y)\, dx\, dy = 0$$

and

$$\int_{\partial T} (v\, dx + u\, dy) = \iint_T (u_x - v_y)\, dx\, dy = 0.$$

Hence, by (1.15), $\int_{\partial T} f(z)\, dz = 0$.

It was Goursat's key observation that the triangle condition for a holomorphic function can be proved directly without any reference to Green's theorem and continuity of the derivative.

Goursat's formulation of Theorem 1.32 was in fact more complicated. It was A. Pringsheim who in 1901 realized it suffices to consider triangles.

THEOREM 1.32 (Goursat, 1900). *If $f \in \mathscr{O}(U)$, then $\int_{\partial T} f(z)\, dz = 0$ for every closed triangle $T \subset U$.*

PROOF. Fix a closed triangle $T \subset U$ and set $I = \int_{\partial T} f(z)\, dz$. Connect the midpoints of the edges of T to form four congruent triangles, each having half the diameter of T. It is easy to see that I is the sum of the integrals of f along the oriented boundaries of these four triangles (see Fig. 1.2). Hence, one of these triangles, which we call T_1, satisfies

$$\left| \int_{\partial T_1} f(z)\, dz \right| \geq \frac{1}{4} |I|.$$

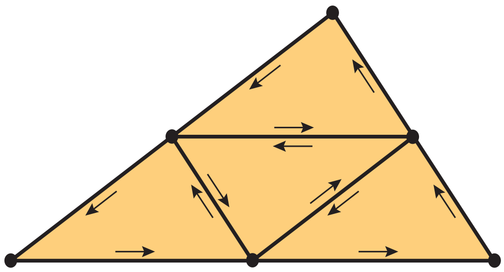

Figure 1.2. The integral along the oriented boundary of the large triangle is equal to the sum of the integrals along the oriented boundaries of the four smaller ones because each internal edge is traversed twice in opposite directions, so its net contribution to the integral is zero.

Replacing T by T_1 in the above construction and continuing inductively, we obtain a nested sequence $T \supset T_1 \supset T_2 \supset T_3 \supset \cdots$ of closed triangles with the properties

$$\operatorname{diam}(T_n) = 2^{-n} \operatorname{diam}(T) \qquad \text{and} \qquad \left| \int_{\partial T_n} f(z)\, dz \right| \geq 4^{-n} |I|.$$

Here "diam" denotes the Euclidean diameter.

The nested intersection $\bigcap_{n=1}^{\infty} T_n$ is a single point $p \in U$. By the assumption, $f'(p)$ exists, so given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z) - f(p) - f'(p)(z-p)| \leq \varepsilon |z-p| \quad \text{whenever} \quad |z-p| < \delta.$$

Choose n large enough that $\operatorname{diam}(T_n) < \delta$. If $z \in \partial T_n$, then $|z-p| \leq \operatorname{diam}(T_n)$, so

$$|f(z) - f(p) - f'(p)(z-p)| \leq \varepsilon \operatorname{diam}(T_n).$$

Observe that by Theorem 1.29,

$$\int_{\partial T_n} (f(p) + f'(p)(z-p))\, dz = 0$$

since the integrand has a primitive $f(p)z + (1/2)f'(p)(z-p)^2$. Hence, by the *ML*-inequality (1.18),

$$\begin{aligned} 4^{-n}|I| \leq \left| \int_{\partial T_n} f(z)\, dz \right| &= \left| \int_{\partial T_n} (f(z) - f(p) - f'(p)(z-p))\, dz \right| \\ &\leq \varepsilon \operatorname{diam}(T_n) \operatorname{length}(\partial T_n) \\ &= \varepsilon\, 2^{-n} \operatorname{diam}(T) \cdot 2^{-n} \operatorname{length}(\partial T), \end{aligned}$$

which implies

$$|I| \leq \varepsilon \operatorname{diam}(T) \operatorname{length}(\partial T).$$

Since this is true for every $\varepsilon > 0$, we must have $I = 0$. □

Edouard Jean-Baptiste Goursat (1858–1936)

Theorems 1.31 and 1.32 put together now imply the following

THEOREM 1.33. *Let $D \subset \mathbb{C}$ be an open disk and $f \in \mathscr{O}(D)$. Then f has a primitive in D.*

Combining Theorem 1.29 and Theorem 1.33, we arrive at

THEOREM 1.34 (Cauchy's theorem in a disk, 1825). *Let $D \subset \mathbb{C}$ be an open disk and $f \in \mathscr{O}(D)$. Then for every closed curve γ in D,*

$$\int_\gamma f(z)\, dz = 0.$$

REMARK 1.35. Here is a minor technical point that will be exploited in the next result: Cauchy's Theorem 1.34 remains true under the apparently weaker assumption that f is continuous in D and holomorphic in $D \smallsetminus \{p\}$ for some $p \in D$. To see this, it suffices to show that $\int_{\partial T} f(z)\, dz = 0$ for every closed triangle $T \subset D$. If $T \subset D \smallsetminus \{p\}$, this follows from Theorem 1.32, so assume $p \in T$. First consider the case where p is on the boundary of T. By slightly moving a vertex of T, we can find a triangle T', arbitrarily close to T, for which $p \notin T'$. Since $\int_{\partial T'} f(z)\, dz = 0$ and since by Lemma 1.27 the integral along the boundary of a triangle depends continuously on vertices, we conclude that $\int_{\partial T} f(z)\, dz = 0$. If p belongs to the interior of $T = \triangle abc$, write $\int_{\partial T} f(z)\, dz$ as the sum of the integrals along the boundaries of $\triangle abp$, $\triangle bcp$, and $\triangle cap$, and reduce to the previous case.

Later we will see that such a point p is not really exceptional, so under the above assumptions $f \in \mathscr{O}(D)$ (compare Example 1.40 or Theorem 3.5).

THEOREM 1.36 (Cauchy's integral formula in a disk). *Let $D \subset \mathbb{C}$ be an open disk and $f \in \mathscr{O}(D)$. If $\overline{\mathbb{D}}(p, r) \subset D$, then*

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \text{for } z \in \mathbb{D}(p, r).$$

In particular, the values of f on the circle $\mathbb{T}(p, r)$ uniquely determine the values of f inside the disk $\mathbb{D}(p, r)$.

PROOF. Fix $z \in \mathbb{D}(p, r)$ and define $g : D \to \mathbb{C}$ by

$$g(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z. \end{cases}$$

Evidently g is continuous in D and holomorphic in $D \smallsetminus \{z\}$. Hence by Remark 1.35, $\int_{\mathbb{T}(p,r)} g(\zeta)\, d\zeta = 0$. This gives

$$\frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z) \cdot \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{1}{\zeta - z}\, d\zeta.$$

To finish the proof, we need to show that the integral on the right is $2\pi i$. Take the parametrization of $\mathbb{T}(p, r)$ defined by $\gamma(t) = z + \rho(t) e^{it}$ for $t \in [0, 2\pi]$, where $\rho(t)$ is

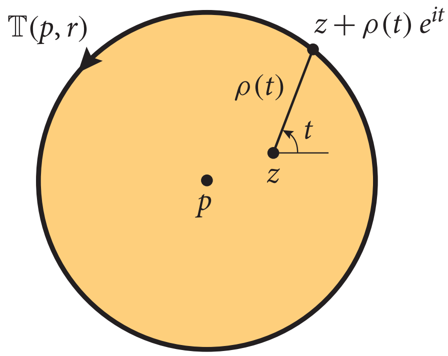

Figure 1.3. Parametrizing the oriented circle $\mathbb{T}(p,r)$ as seen from an off-center point z, used in the proof of Theorem 1.36.

the unique positive number which satisfies $|z+\rho(t)e^{it}-p| = r$ (see Fig. 1.3). It is easy to check that $t \mapsto \rho(t)$ is continuously differentiable. Hence

$$\begin{aligned}\int_{\mathbb{T}(p,r)} \frac{1}{\zeta - z}\, d\zeta &= \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)-z}\, dt = \int_0^{2\pi} \frac{(\rho'(t)+i\rho(t))\, e^{it}}{\rho(t)\, e^{it}}\, dt \\ &= \int_0^{2\pi} \frac{\rho'(t)}{\rho(t)}\, dt + 2\pi i \\ &= \log(\rho(2\pi)) - \log(\rho(0)) + 2\pi i = 2\pi i,\end{aligned}$$

where the last equality holds since $\rho(2\pi) = \rho(0)$. □

More general versions of Theorems 1.34 and 1.36 will be proved in chapter 2. For now, let us collect some corollaries of these basic results. The first one is the converse of Theorem 1.20:

THEOREM 1.37 (Holomorphic implies complex analytic). *Every $f \in \mathcal{O}(U)$ is complex analytic in U: In every disk $\mathbb{D}(p,r) \subset U$ there is a power series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n\, (z-p)^n$$

where the coefficients $\{a_n\}$ are given by

$$a_n = \frac{f^{(n)}(p)}{n!} = \frac{1}{2\pi i} \int_{\mathbb{T}(p,s)} \frac{f(\zeta)}{(\zeta - p)^{n+1}}\, d\zeta \tag{1.20}$$

for any $0 < s < r$.

PROOF. Fix $0 < s < r$ and a point $z \in \mathbb{D}(p,s)$. For any $\zeta \in \mathbb{T}(p,s)$,

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - p)\left[1 - \left(\frac{z-p}{\zeta - p}\right)\right]} = \frac{1}{\zeta - p} \sum_{n=0}^{\infty} \left(\frac{z-p}{\zeta - p}\right)^n.$$

Here the geometric series converges uniformly in ζ since its general term has absolute value $|z-p|/s < 1$ independent of ζ. Thus, we can integrate this series term-by-term on the circle $\mathbb{T}(p,s)$. By Theorem 1.36, we obtain

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}(p,s)} \sum_{n=0}^{\infty} \frac{f(\zeta)(z-p)^n}{(\zeta - p)^{n+1}}\, d\zeta = \sum_{n=0}^{\infty} a_n\,(z-p)^n,$$

where the a_n are given by (1.20). This proves that f can be represented by the power series $\sum_{n=0}^{\infty} a_n\,(z-p)^n$ in $\mathbb{D}(p,s)$. Since this holds for every $s < r$, Theorem 1.20(iii) shows that the power series with the same coefficients must converge to $f(z)$ for all $z \in \mathbb{D}(p,r)$. □

It follows from Theorem 1.20 that

COROLLARY 1.38. *If $f \in \mathscr{O}(U)$, then $f' \in \mathscr{O}(U)$. Therefore, the k-th derivative $f^{(k)}$ exists and belongs to $\mathscr{O}(U)$ for every $k \geq 1$.*

In particular, by Theorem 1.7, a differentiable map $f : U \to \mathbb{R}^2$ which satisfies the Cauchy-Riemann equation $f_{\bar{z}} = 0$ throughout U is automatically C^∞-smooth.

The following converse of Theorem 1.32 is a useful criterion for deciding when a continuous function is holomorphic:

Giacinto Morera (1856–1909)

THEOREM 1.39 (Morera, 1886). *Suppose $f : U \to \mathbb{C}$ is continuous and $\int_{\partial T} f(z)\, dz = 0$ for every closed triangle $T \subset U$. Then $f \in \mathscr{O}(U)$.*

PROOF. Let $D \subset U$ be a disk. By Theorem 1.31, f has a primitive F in D. Since $F \in \mathscr{O}(D)$ and since the derivative of a holomorphic function is holomorphic by Corollary 1.38, it follows that $f = F' \in \mathscr{O}(D)$. As this holds for every disk $D \subset U$, we conclude that $f \in \mathscr{O}(U)$. □

EXAMPLE 1.40 (Lines are removable). Let $U \subset \mathbb{C}$ be open and L be a straight line which intersects U. Suppose $f : U \to \mathbb{C}$ is a continuous function which is holomorphic in $U \smallsetminus L$. We prove that f is holomorphic in U by showing that $\int_{\partial T} f(z)\, dz = 0$ for every triangle $T \subset U$. First assume that the interior of T is disjoint from L. Then, by moving the vertices of T slightly, we can find a triangle $T' \subset U \smallsetminus L$, arbitrarily close to T. By Goursat's Theorem 1.32, $\int_{\partial T'} f(z)\, dz = 0$. Since the integral along the boundary of a triangle depends continuously on vertices by Lemma 1.27, we must have $\int_{\partial T} f(z)\, dz = 0$. If the interior of T meets L, write T as the union of at most three triangles with pairwise disjoint interiors, each meeting L along a vertex or an edge, and reduce to the previous case.

This shows in particular that points are removable: If f is continuous in U and holomorphic in $U \setminus \{p\}$, then $f \in \mathscr{O}(U)$. More general removability results are discussed in Theorem 3.5 and in chapter 10.

REMARK 1.41. Morera's theorem holds if we replace triangles with other special families of closed sets with nice boundaries. A typical example, which turns out to be more convenient in some situations, is the family of closed rectangles, or even squares. See problem 25.

THEOREM 1.42 (Cauchy's estimates, 1835). *Suppose f is continuous on $\overline{\mathbb{D}}(p,r)$ and holomorphic in $\mathbb{D}(p,r)$. Then,*

$$|f^{(n)}(p)| \leq \frac{n!}{r^n} \sup_{|z-p|=r} |f(z)| \qquad (n \geq 0). \tag{1.21}$$

The example $f(z) = z^n$ in the unit disk $\mathbb{D}$ shows that the bound in (1.21) is optimal for each n.

PROOF. Take $0 < s < r$ and represent f by a power series $\sum_{n=0}^{\infty} a_n (z-p)^n$ in $\mathbb{D}(p,s)$. By (1.20),

$$|f^{(n)}(p)| = n!\,|a_n| = \frac{n!}{2\pi} \left| \int_{\mathbb{T}(p,s)} \frac{f(z)}{(z-p)^{n+1}}\, dz \right|,$$

which by the ML-inequality implies

$$|f^{(n)}(p)| \leq \frac{n!}{2\pi} \cdot \sup_{|z-p|=s} \frac{|f(z)|}{|z-p|^{n+1}} \cdot 2\pi s = \frac{n!}{s^n} \sup_{|z-p|=s} |f(z)|.$$

Letting $s \to r$, we obtain (1.21). □

Cauchy's estimates lead to various quantitative results on holomorphic functions which have no counterpart in the smooth category. Here we prove two basic but important statements of this type.

THEOREM 1.43. *If a holomorphic function f maps the disk $\mathbb{D}(p,r)$ into the disk $\mathbb{D}(q,R)$, then $|f'(p)| \leq R/r$.*

Note that we have *not* assumed $q = f(p)$. In particular, if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then $|f'(0)| \leq 1$. This is a basic version of the so-called "Schwarz lemma" which has deep applications and will be discussed at length in chapters 4, 11, and 13.

PROOF. Take $0 < s < r$ and apply (1.21) to the function $g = f - q$:

$$|f'(p)| = |g'(p)| \leq \frac{1}{s} \sup_{|z-p|=s} |g(z)| \leq \frac{R}{s}.$$

Letting $s \to r$ proves the result. □

Liouville's theorem was known to Cauchy in 1844.

THEOREM 1.44 (Liouville, 1847). *Every bounded entire function is constant.*

PROOF. Let $f \in \mathscr{O}(\mathbb{C})$ and $|f(z)| < M$ for all $z \in \mathbb{C}$. Then f maps any disk $\mathbb{D}(p, r)$ into $\mathbb{D}(0, M)$, so by Theorem 1.43, $|f'(p)| \leq M/r$. Letting $r \to +\infty$, we obtain $f'(p) = 0$. Since this holds for every $p \in \mathbb{C}$, f must be constant. □

Joseph Liouville (1809–1882)

EXAMPLE 1.45 (The fundamental theorem of algebra). Let $P: \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 1$, so $\lim_{z\to\infty} P(z) = \infty$. If $P(z) \neq 0$ for all z, then $f(z) = 1/P(z)$ is entire and $\lim_{z\to\infty} f(z) = 0$. Hence there is an $R > 0$ such that $|f(z)| \leq 1$ whenever $|z| \geq R$. Since by continuity f is bounded on the closed disk $\overline{\mathbb{D}}(0, R)$, it follows that f is bounded on the plane. Liouville's theorem then implies that f is constant, which is a contradiction. Thus, P has at least one root z_1 and we can write $P(z) = (z - z_1)P_1(z)$ for some polynomial P_1 of degree $d - 1$. If $d - 1 = 0$ so P_1 is constant, stop. Otherwise repeat the argument with P_1 in place of P to find a root z_2 of P_1, and so on. This process stops after d steps and shows that P factors as $P(z) = a(z - z_1)(z - z_2) \cdots (z - z_d)$ for some $a, z_1, \ldots, z_d \in \mathbb{C}$. Thus, *every complex polynomial of degree $d \geq 1$ has precisely d roots counting multiplicities.*

We end this section with a useful theorem which, roughly speaking, says that the integral of a function which depends holomorphically on a parameter is a holomorphic function of that parameter, and differentiation under the integral sign is legitimate. We formulate a simple version of the theorem which will be sufficient for our purposes. One should note, however, that the result holds in much more general settings (see problem 27).

THEOREM 1.46. *Let $U \subset \mathbb{C}$ be open and $\varphi: U \times [a, b] \to \mathbb{C}$ be a continuous function such that for each $t \in [a, b]$, $z \mapsto \varphi(z, t)$ is holomorphic in U with derivative $\varphi'(z, t)$. Then, the function $f: U \to \mathbb{C}$ defined by*

$$f(z) = \int_a^b \varphi(z, t)\, dt$$

is holomorphic and we can differentiate under the integral sign:

$$f'(z) = \int_a^b \varphi'(z, t)\, dt \qquad \text{for all } z \in U.$$

PROOF. Fix $p \in U$ and take $r > 0$ such that $\overline{\mathbb{D}}(p, r) \subset U$. Let $0 < |z - p| < r/2$. By Theorem 1.36,

$$\varphi(z, t) - \varphi(p, t) = \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \varphi(\zeta, t) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - p} \right) d\zeta,$$

so

$$\frac{\varphi(z, t) - \varphi(p, t)}{z - p} = \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{\varphi(\zeta, t)}{(\zeta - z)(\zeta - p)}\, d\zeta.$$

Since $|\zeta - z| > r/2$ whenever $|\zeta - p| = r$, we obtain the following estimate using the *ML*-inequality:

$$\left| \frac{\varphi(z,t) - \varphi(p,t)}{z-p} \right| \leq \frac{1}{2\pi} \cdot M \cdot \frac{2}{r^2} \cdot 2\pi r = \frac{2M}{r}.$$

Here M is the supremum of $|\varphi|$ on the compact set $\overline{\mathbb{D}}(p,r) \times [a,b]$. If $\{z_n\}$ is any sequence in $U \smallsetminus \{p\}$ which tends to p, then

$$g_n(t) = \frac{\varphi(z_n,t) - \varphi(p,t)}{z_n - p}$$

is a sequence of continuous functions on $[a,b]$ which converges pointwise to $\varphi'(p,t)$ and is bounded by $2M/r$ for all large n. Hence, by Lebesgue's dominated convergence theorem, the function $t \mapsto \varphi'(p,t)$ is integrable on $[a,b]$ and

$$\lim_{n\to\infty} \frac{f(z_n) - f(p)}{z_n - p} = \lim_{n\to\infty} \int_a^b g_n(t)\, dt = \int_a^b \varphi'(p,t)\, dt.$$

Since this holds for every sequence $z_n \to p$, we conclude that $f'(p)$ exists and equals $\int_a^b \varphi'(p,t)\, dt$. □

REMARK 1.47. Under the assumptions of the above theorem, the derivative $(z,t) \mapsto \varphi'(z,t)$ is in fact continuous on $U \times [a,b]$ (see problem 26). Thus, the result holds when $\varphi(z,t)$ is replaced with $\varphi'(z,t)$, and a simple induction proves the formula

$$f^{(n)}(z) = \int_a^b \varphi^{(n)}(z,t)\, dt \qquad \text{for all } z \in U,$$

where $\varphi^{(n)}(z,t)$ is the n-th derivative of $\varphi(z,t)$ with respect to z.

The following corollary of the above theorem will be used repeatedly:

COROLLARY 1.48. *Let $\gamma : [0,1] \to \mathbb{C}$ be a piecewise C^1 curve and $g : |\gamma| \to \mathbb{C}$ be a continuous function. Then, for each integer $n \geq 1$, the function*

$$f(z) = \int_\gamma \frac{g(\zeta)}{(\zeta - z)^n}\, d\zeta$$

is holomorphic in $\mathbb{C} \smallsetminus |\gamma|$, and

$$f'(z) = n \int_\gamma \frac{g(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \qquad \textit{for all } z \in \mathbb{C} \smallsetminus |\gamma|.$$

PROOF. This follows from Theorem 1.46 applied to $\varphi : (\mathbb{C} \smallsetminus |\gamma|) \times [0,1] \to \mathbb{C}$ defined by

$$\varphi(z,t) = \frac{g(\gamma(t))\, \gamma'(t)}{(\gamma(t) - z)^n}.$$

(Technically, we need to break up $[0,1]$ into finitely many intervals in which γ' is continuous and add up the corresponding integrals, but that is a trivial matter.) □

EXAMPLE 1.49 (Cauchy's integral formula for higher derivatives). A special case of the above corollary is Cauchy's integral formula. If $f \in \mathscr{O}(U)$ and $\overline{\mathbb{D}}(p,r) \subset U$, then

$$\frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

defines a holomorphic function in $\mathbb{C} \smallsetminus \mathbb{T}(p,r)$. By Theorem 1.36, this function coincides with f inside the disk $\mathbb{D}(p,r)$. Differentiation under the integral sign then gives

$$f'(z) = \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \qquad \text{for } z \in \mathbb{D}(p,r).$$

It follows by induction that for every $n \geq 0$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \qquad \text{for } z \in \mathbb{D}(p,r).$$

Observe that for $z = p$ this is the formula (1.20) that we derived earlier.

1.5 Mapping properties of holomorphic functions

DEFINITION 1.50. Suppose $f \in \mathscr{O}(U)$ and f is not identically zero in the disk $\mathbb{D}(p,r) \subset U$. Let $f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n$ be the power series representation of f in $\mathbb{D}(p,r)$. The smallest integer m with the property $a_m \neq 0$ is called the ***order*** of p and is denoted by $\operatorname{ord}(f,p)$. Thus, $\operatorname{ord}(f,p) \geq 1$ if and only if $f(p) = 0$. We call p a ***simple zero*** of f if $\operatorname{ord}(f,p) = 1$.

Alternatively, $\operatorname{ord}(f,p)$ can be described as the unique integer $m \geq 0$ for which f can be factored as

$$f(z) = (z-p)^m f_1(z)$$

with $f_1 \in \mathscr{O}(U)$ and $f_1(p) \neq 0$. The function f_1 is given by $(z-p)^{-m} f(z)$ in $U \smallsetminus \{p\}$. It is holomorphic in U since it is represented by the power series $\sum_{n=m}^{\infty} a_n (z-p)^{n-m}$ in $\mathbb{D}(p,r)$.

EXAMPLE 1.51 (Holomorphic L'Hôpital's rule). Suppose f and g are holomorphic in some neighborhood of p, with $\operatorname{ord}(f,p) = \operatorname{ord}(g,p) = m \geq 1$. Write $f(z) = (z-p)^m f_1(z)$ and $g(z) = (z-p)^m g_1(z)$, where f_1 and g_1 are non-zero and holomorphic near p. Since

$$f_1(p) = \frac{f^{(m)}(p)}{m!} \qquad \text{and} \qquad g_1(p) = \frac{g^{(m)}(p)}{m!},$$

it follows that

$$\lim_{z \to p} \frac{f(z)}{g(z)} = \frac{f_1(p)}{g_1(p)} = \frac{f^{(m)}(p)}{g^{(m)}(p)}.$$

Let us call $U \subset \mathbb{C}$ a ***domain*** if U is non-empty, open, and connected.

LEMMA 1.52. *Suppose $U \subset \mathbb{C}$ is a domain and $f \in \mathscr{O}(U)$. If the zero-set $f^{-1}(0) = \{z \in U : f(z) = 0\}$ has an accumulation point in U, then $f = 0$ everywhere in U.*

Connectivity of U is essential here: If U is the disjoint union of non-empty open sets U_1 and U_2, and if $f = 0$ in U_1 and $f = 1$ in U_2, then $f \in \mathscr{O}(U)$ and $f^{-1}(0) = U_1$ has accumulation points in U, but f is not identically zero in U.

PROOF. Let E be the non-empty set of accumulation points of $f^{-1}(0)$ in U. Then E is closed in U, and $E \subset f^{-1}(0)$ by continuity of f. Suppose $p \in E$ and there is a disk $\mathbb{D}(p, r) \subset U$ in which f is not identically zero. Then we can write $f(z) = (z - p)^m f_1(z)$, where $m = \operatorname{ord}(f, p) \geq 1$, $f_1 \in \mathscr{O}(U)$, and $f_1(p) \neq 0$. By continuity, f_1 does not vanish in some neighborhood of p. It follows that p is the only zero of f in this neighborhood, contradicting the fact that $p \in E$. Thus, if $\mathbb{D}(p, r) \subset U$, then f is identically zero in $\mathbb{D}(p, r)$ and therefore $\mathbb{D}(p, r) \subset E$. This shows that E is an open set. Since U is connected, we must have $E = f^{-1}(0) = U$. □

Since every domain $U \subset \mathbb{C}$ is a countable union of open disks, it is clear that every uncountable subset of U must have an accumulation point in U. It follows from the above lemma that *a non-constant holomorphic function in a domain has at most countably many zeros, all of which are isolated.* Another immediate corollary is

THEOREM 1.53 (The identity theorem)**.** *Suppose $U \subset \mathbb{C}$ is a domain, $f, g \in \mathscr{O}(U)$, and the set $\{z \in U : f(z) = g(z)\}$ has an accumulation point in U. Then $f = g$ everywhere in U.*

EXAMPLE 1.54. The complex cosine and sine are the entire functions defined by

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

They extend the usual cosine and sine functions defined on the real line. It follows from Theorem 1.53 that any trigonometric identity between cosine and sine that holds on $\mathbb{R}$ must continue to hold in $\mathbb{C}$. For example, the identities $\cos^2 z + \sin^2 z = 1$, $\sin(2z) = 2\sin z \cos z$, and $\cos(2z) = \cos^2 z - \sin^2 z$ remain valid for all $z \in \mathbb{C}$.

EXAMPLE 1.55. Suppose $f \in \mathscr{O}(\mathbb{C})$ has the power series representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If every coefficient a_n is real, then clearly $f(\mathbb{R}) \subset \mathbb{R}$. Conversely, suppose $f(\mathbb{R}) \subset \mathbb{R}$ and consider the entire function

$$g(z) = \overline{f(\bar{z})} = \sum_{n=0}^{\infty} \overline{a_n}\, z^n.$$

Since $f(z)$ is real when z is real, we have $g = f$ on the real line. By Theorem 1.53, $g = f$ everywhere in $\mathbb{C}$. Uniqueness of power series then shows that every a_n is real.

Our next goal is to prove the fundamental fact that the image of a domain under a non-constant holomorphic function is open (Theorem 1.62). This will follow from a much stronger result on the local behavior of holomorphic functions (Theorem 1.59).

LEMMA 1.56. *If $f \in \mathscr{O}(U)$, the function $g : U \times U \to \mathbb{C}$ defined by*

$$g(\zeta, z) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$

is continuous.

PROOF. Clearly g is continuous off the diagonal $\{(z, z) : z \in U\}$, so it is enough to check continuity of g at a diagonal point (p, p). Let $\varepsilon > 0$ be given. Since f' is continuous at p, there is an $r > 0$ such that

$$|f'(z) - f'(p)| < \varepsilon \qquad \text{whenever } z \in \mathbb{D}(p, r). \tag{1.22}$$

Let $\zeta, z \in \mathbb{D}(p, r)$. If $\zeta = z$, then $|g(\zeta, z) - g(p, p)| = |f'(z) - f'(p)| < \varepsilon$. If $\zeta \neq z$, then

$$\frac{f(\zeta) - f(z)}{\zeta - z} = \frac{1}{\zeta - z} \int_{[z,\zeta]} f'(w)\, dw = \int_0^1 f'(\gamma(t))\, dt,$$

where $\gamma(t) = (1 - t)z + t\zeta$. Hence,

$$\begin{aligned} |g(\zeta, z) - g(p, p)| &= \left| \frac{f(\zeta) - f(z)}{\zeta - z} - f'(p) \right| = \left| \int_0^1 [\, f'(\gamma(t)) - f'(p)\,]\, dt \right| \\ &\leq \int_0^1 |f'(\gamma(t)) - f'(p)|\, dt \leq \varepsilon, \end{aligned}$$

where the last inequality holds since by (1.22), $|f'(\gamma(t)) - f'(p)| < \varepsilon$ for every $t \in [0, 1]$. □

THEOREM 1.57 (Holomorphic inverse function theorem). *Suppose $f \in \mathscr{O}(U)$, $p \in U$, and $f'(p) \neq 0$. Then, there exist open neighborhoods $V \subset U$ of p and $W \subset \mathbb{C}$ of $f(p)$ such*

(continued...)

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