

Evidently, for any continuous function $f: \mathbb{T}(p, r) \rightarrow \mathbb{C}$,

$$\int_{\mathbb{T}(p,r)} f(z) dz = ir \int_0^{2\pi} f(p + re^{it}) e^{it} dt.$$

As a special case, consider a continuous complex-valued function f defined on the **unit circle** $\mathbb{T} = \mathbb{T}(0, 1)$. The integral of f as a scalar function can be expressed as a complex integral:

$$\int_0^{2\pi} f(e^{it}) dt = \int_{\mathbb{T}} f(z) \frac{dz}{iz}.$$

More generally, the **Fourier coefficients** of f , defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} dt \quad (n \in \mathbb{Z}),$$

can be expressed as the complex integrals

$$\hat{f}(n) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(z)}{z^{n+1}} dz.$$

The following elementary observation will be useful:

LEMMA 1.27 (Continuous dependence on vertices). *Let $f: U \rightarrow \mathbb{C}$ be continuous and $T = \triangle abc$ be a closed triangle in U . Then, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|a - a'|$, $|b - b'|$, and $|c - c'|$ are all less than δ , then $T' = \triangle a'b'c' \subset U$ and*

$$\left| \int_{\partial T} f(z) dz - \int_{\partial T'} f(z) dz \right| < \varepsilon.$$

PROOF. Since the integral along the oriented boundary of a triangle is the sum of three integrals along oriented segments, it suffices to prove continuous dependence for oriented segments. Fix $[a, b] \subset U$ and let V be any open neighborhood of $[a, b]$ whose closure \bar{V} is a compact subset of U . Given $\varepsilon > 0$, use uniform continuity of f on V to find $0 < \delta < \varepsilon$ such that $|f(z) - f(w)| < \varepsilon$ whenever $z, w \in V$ and $|z - w| < \delta$. We can also arrange that $[a', b'] \subset V$ whenever $|a - a'| < \delta$ and $|b - b'| < \delta$. Let $[a', b']$ be any such segment and note that if $\gamma(t) = (1 - t)a + tb$ and $\eta(t) = (1 - t)a' + tb'$, then

$$|\gamma(t) - \eta(t)| \leq (1 - t)|a - a'| + t|b - b'| < \delta,$$

$$|\gamma'(t) - \eta'(t)| = |(b - a) - (b' - a')| \leq |b - b'| + |a - a'| < 2\delta$$

for all $0 \leq t \leq 1$. Hence,

$$\begin{aligned} \left| \int_{[a,b]} f(z) dz - \int_{[a',b']} f(z) dz \right| &= \left| \int_0^1 [f(\gamma(t)) \gamma'(t) - f(\eta(t)) \eta'(t)] dt \right| \\ &= \left| \int_0^1 [(f(\gamma(t)) - f(\eta(t)))\gamma'(t) + f(\eta(t))(\gamma'(t) - \eta'(t))] dt \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \int_0^1 |f(\gamma(t)) - f(\eta(t))| |\gamma'(t)| dt + \int_0^1 |f(\eta(t))| |\gamma'(t) - \eta'(t)| dt \\
 &\leq |b - a| \int_0^1 |f(\gamma(t)) - f(\eta(t))| dt + 2\delta \int_0^1 |f(\eta(t))| dt \\
 &\leq |b - a| \varepsilon + 2\delta \sup_{z \in V} |f(z)| \leq (|b - a| + 2 \sup_{z \in V} |f(z)|) \varepsilon,
 \end{aligned}$$

which proves the asserted continuity. \square

A primitive is what students of calculus call “antidetrivative.”

DEFINITION 1.28. A function $F \in \mathcal{O}(U)$ is called a **primitive** of a continuous function $f : U \rightarrow \mathbb{C}$ if $F'(z) = f(z)$ for all $z \in U$.

Suppose F is a primitive of f and $\gamma : [0, 1] \rightarrow U$ is a piecewise C^1 curve. By the chain rule, the relation $(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t)$ holds for all but finitely many $t \in [0, 1]$ (see problem 6). Since $F'(\gamma(t))\gamma'(t)$ is piecewise continuous on $[0, 1]$ with at worst jump discontinuities, the fundamental theorem of calculus shows that

$$\begin{aligned}
 \int_{\gamma} f(z) dz &= \int_0^1 f(\gamma(t)) \gamma'(t) dt = \int_0^1 F'(\gamma(t)) \gamma'(t) dt \\
 &= \int_0^1 (F \circ \gamma)'(t) dt = F(\gamma(1)) - F(\gamma(0)).
 \end{aligned}$$

THEOREM 1.29. A continuous function $f : U \rightarrow \mathbb{C}$ has a primitive in U if and only if $\int_{\gamma} f(z) dz = 0$ for every closed curve γ in U .

PROOF. First suppose f has a primitive F . If $\gamma : [0, 1] \rightarrow U$ is a closed curve, then $\gamma(0) = \gamma(1)$, so

$$\int_{\gamma} f(z) dz = F(\gamma(1)) - F(\gamma(0)) = 0.$$

Conversely, suppose f integrates to zero along every closed curve in U . To show f has a primitive, it suffices to consider the case when U is connected (and therefore path-connected); the general case follows by applying this case to each connected component of U . If γ, η are two curves in U with the same initial and end points, then the product $\gamma \cdot \eta^{-}$ is a closed curve. Hence, by additivity (1.16) and our assumption,

$$\int_{\gamma} f(\zeta) d\zeta - \int_{\eta} f(\zeta) d\zeta = \int_{\gamma} f(\zeta) d\zeta + \int_{\eta^{-}} f(\zeta) d\zeta = \int_{\gamma \cdot \eta^{-}} f(\zeta) d\zeta = 0.$$

Now fix a point $p \in U$. For any $z \in U$ use path-connectivity of U to find a curve γ in U from p to z and define

$$F(z) = \int_{\gamma} f(\zeta) d\zeta.$$

By the above remark, the right-hand side is independent of the choice of γ and yields a well-defined function $F: U \rightarrow \mathbb{C}$. Let us show that F is a primitive of f . Fix $z_0 \in U$ and choose $r > 0$ small enough so that $\mathbb{D}(z_0, r) \subset U$. Let $z \in \mathbb{D}(z_0, r)$ and let γ be any curve in U from p to z_0 . The product $\gamma \cdot [z_0, z]$ is then a curve in U from p to z . By additivity,

$$F(z) - F(z_0) = \int_{\gamma \cdot [z_0, z]} f(\zeta) d\zeta - \int_{\gamma} f(\zeta) d\zeta = \int_{[z_0, z]} f(\zeta) d\zeta,$$

so if $z \neq z_0$,

$$(1.19) \quad \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0, z]} (f(\zeta) - f(z_0)) d\zeta.$$

Since f is continuous at z_0 , for each $\varepsilon > 0$ we can find a $0 < \delta < r$ such that $|f(\zeta) - f(z_0)| < \varepsilon$ whenever $|\zeta - z_0| < \delta$. Since $|z - z_0| < \delta$ implies $|\zeta - z_0| < \delta$ for every $\zeta \in [z_0, z]$, the ML-inequality (1.18) applied to the right side of (1.19) gives

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \frac{1}{|z - z_0|} \cdot \varepsilon \cdot \text{length}([z_0, z]) = \varepsilon$$

whenever $0 < |z - z_0| < \delta$. Thus, $F'(z_0)$ exists and is equal to $f(z_0)$. Since $z_0 \in U$ was arbitrary, we conclude that F is a primitive of f in U . \square

EXAMPLE 1.30. For every integer $n \neq -1$, the power function $f(z) = z^n$ has a primitive $F(z) = z^{n+1}/(n+1)$. It follows from Theorem 1.29 that $\int_{\gamma} z^n dz = 0$ if γ is any closed curve in the punctured plane $\mathbb{C} \setminus \{0\}$ and $n \neq -1$, or if γ is any closed curve in \mathbb{C} and $n \geq 0$.

The case $n = -1$ is completely different: For any $r > 0$,

$$\int_{\mathbb{T}(0,r)} \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{re^{it}} rie^{it} dt = 2\pi i \neq 0.$$

Note that the result is independent of the radius r . It follows from Theorem 1.29 that *the function $z \mapsto 1/z$ does not have a primitive in any punctured neighborhood of 0.*

1.4 Cauchy's theory in a disk

Our primary goal in this section is to prove that every holomorphic function in a disk has a primitive. Somewhat surprisingly, all the local properties of holomorphic functions are consequences of this central fact of Cauchy's theory. The special case of a disk will be enough for our purposes here; general domains and global issues will be dealt with in chapter 2.

According to Theorem 1.29, the existence of a primitive is equivalent to having vanishing integrals along all closed curves. Convexity of the disk allows us to replace the latter with something far simpler in terms of triangles.

THEOREM 1.31. *Let $D \subset \mathbb{C}$ be an open disk and $f : D \rightarrow \mathbb{C}$ be continuous. Suppose $\int_{\partial T} f(z) dz = 0$ for every closed triangle $T \subset D$. Then f has a primitive in D .*

PROOF. Let p be the center of D and define

$$F(z) = \int_{[p,z]} f(\zeta) d\zeta \quad \text{for } z \in D.$$

We show that F is a primitive of f . Take distinct points $z_0, z \in D$ and apply the condition $\int_{\partial T} f(\zeta) d\zeta = 0$ to the closed triangle T with vertices p, z, z_0 to obtain

$$F(z) - F(z_0) = \int_{[p,z]} f(\zeta) d\zeta - \int_{[p,z_0]} f(\zeta) d\zeta = \int_{[z_0,z]} f(\zeta) d\zeta.$$

The rest of the argument, that is, dividing by $z - z_0$ and letting $z \rightarrow z_0$ to show that $F'(z_0) = f(z_0)$, is identical to the proof of Theorem 1.29. \square

The problem of constructing primitives in D is thus reduced to showing that every $f \in \mathcal{O}(D)$ satisfies the triangle condition of Theorem 1.31. If we knew that the derivative f' is continuous (which is true but we have not yet proved it), this would be an easy consequence of Green's theorem. To see this, suppose $f \in \mathcal{O}(D)$ and assume f' is continuous in D . Then the partial derivatives of $u = \operatorname{Re}(f)$ and $v = \operatorname{Im}(f)$ are continuous in D and Green's theorem together with the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$ shows that for every closed triangle $T \subset D$,

$$\int_{\partial T} (u dx - v dy) = \iint_T (-v_x - u_y) dx dy = 0$$

and

$$\int_{\partial T} (v dx + u dy) = \iint_T (u_x - v_y) dx dy = 0.$$

Hence, by (1.15), $\int_{\partial T} f(z) dz = 0$.

It was Goursat's key observation that the triangle condition for a holomorphic function can be proved directly without any reference to Green's theorem and continuity of the derivative.

THEOREM 1.32 (Goursat, 1900). *If $f \in \mathcal{O}(U)$, then $\int_{\partial T} f(z) dz = 0$ for every closed triangle $T \subset U$.*

PROOF. Fix a closed triangle $T \subset U$ and set $I = \int_{\partial T} f(z) dz$. Connect the midpoints of the edges of T to form four congruent triangles, each having half the diameter of T . It is easy to see that I is the sum of the integrals of f along the oriented boundaries of these four triangles (see Fig. 1.2). Hence, one of these triangles, which we call T_1 , satisfies

$$\left| \int_{\partial T_1} f(z) dz \right| \geq \frac{1}{4} |I|.$$

Goursat's formulation of Theorem 1.32 was in fact more complicated. It was A. Pringsheim who in 1901 realized it suffices to consider triangles.

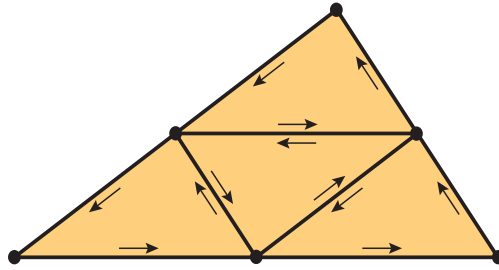


Figure 1.2. The integral along the oriented boundary of the large triangle is equal to the sum of the integrals along the oriented boundaries of the four smaller ones because each internal edge is traversed twice in opposite directions, so its net contribution to the integral is zero.

Replacing T by T_1 in the above construction and continuing inductively, we obtain a nested sequence $T \supset T_1 \supset T_2 \supset T_3 \supset \dots$ of closed triangles with the properties

$$\text{diam}(T_n) = 2^{-n} \text{diam}(T) \quad \text{and} \quad \left| \int_{\partial T_n} f(z) dz \right| \geq 4^{-n} |I|.$$

Here “diam” denotes the Euclidean diameter.

The nested intersection $\bigcap_{n=1}^{\infty} T_n$ is a single point $p \in U$. By the assumption, $f'(p)$ exists, so given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z) - f(p) - f'(p)(z - p)| \leq \varepsilon |z - p| \quad \text{whenever} \quad |z - p| < \delta.$$

Choose n large enough that $\text{diam}(T_n) < \delta$. If $z \in \partial T_n$, then $|z - p| \leq \text{diam}(T_n)$, so

$$|f(z) - f(p) - f'(p)(z - p)| \leq \varepsilon \text{diam}(T_n).$$

Observe that by Theorem 1.29,

$$\int_{\partial T_n} (f(p) + f'(p)(z - p)) dz = 0$$

since the integrand has a primitive $f(p)z + (1/2)f'(p)(z - p)^2$. Hence, by the *ML*-inequality (1.18),

$$\begin{aligned} 4^{-n} |I| &\leq \left| \int_{\partial T_n} f(z) dz \right| = \left| \int_{\partial T_n} (f(z) - f(p) - f'(p)(z - p)) dz \right| \\ &\leq \varepsilon \text{diam}(T_n) \text{length}(\partial T_n) \\ &= \varepsilon 2^{-n} \text{diam}(T) \cdot 2^{-n} \text{length}(\partial T), \end{aligned}$$

which implies

$$|I| \leq \varepsilon \text{diam}(T) \text{length}(\partial T).$$

Since this is true for every $\varepsilon > 0$, we must have $I = 0$. □

Theorems 1.31 and 1.32 put together now imply the following

THEOREM 1.33. *Let $D \subset \mathbb{C}$ be an open disk and $f \in \mathcal{O}(D)$. Then f has a primitive in D .*



Edouard Jean-Baptiste Goursat (1858–1936)

Combining Theorem 1.29 and Theorem 1.33, we arrive at

THEOREM 1.34 (Cauchy's theorem in a disk, 1825). *Let $D \subset \mathbb{C}$ be an open disk and $f \in \mathcal{O}(D)$. Then for every closed curve γ in D ,*

$$\int_{\gamma} f(z) dz = 0.$$

REMARK 1.35. Here is a minor technical point that will be exploited in the next result: Cauchy's Theorem 1.34 remains true under the apparently weaker assumption that f is continuous in D and holomorphic in $D \setminus \{p\}$ for some $p \in D$. To see this, it suffices to show that $\int_{\partial T} f(z) dz = 0$ for every closed triangle $T \subset D$. If $T \subset D \setminus \{p\}$, this follows from Theorem 1.32, so assume $p \in T$. First consider the case where p is on the boundary of T . By slightly moving a vertex of T , we can find a triangle T' , arbitrarily close to T , for which $p \notin T'$. Since $\int_{\partial T'} f(z) dz = 0$ and since by Lemma 1.27 the integral along the boundary of a triangle depends continuously on vertices, we conclude that $\int_{\partial T} f(z) dz = 0$. If p belongs to the interior of $T = \triangle abc$, write $\int_{\partial T} f(z) dz$ as the sum of the integrals along the boundaries of $\triangle abp$, $\triangle bcp$, and $\triangle cap$, and reduce to the previous case.

Later we will see that such a point p is not really exceptional, so under the above assumptions $f \in \mathcal{O}(D)$ (compare Example 1.40 or Theorem 3.5).

THEOREM 1.36 (Cauchy's integral formula in a disk). *Let $D \subset \mathbb{C}$ be an open disk and $f \in \mathcal{O}(D)$. If $\mathbb{D}(p, r) \subset D$, then*

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{for } z \in \mathbb{D}(p, r).$$

In particular, the values of f on the circle $\mathbb{T}(p, r)$ uniquely determine the values of f inside the disk $\mathbb{D}(p, r)$.

PROOF. Fix $z \in \mathbb{D}(p, r)$ and define $g : D \rightarrow \mathbb{C}$ by

$$g(\zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z. \end{cases}$$

Evidently g is continuous in D and holomorphic in $D \setminus \{z\}$. Hence by Remark 1.35, $\int_{\mathbb{T}(p,r)} g(\zeta) d\zeta = 0$. This gives

$$\frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \cdot \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{1}{\zeta - z} d\zeta.$$

To finish the proof, we need to show that the integral on the right is $2\pi i$. Take the parametrization of $\mathbb{T}(p, r)$ defined by $\gamma(t) = z + \rho(t)e^{it}$ for $t \in [0, 2\pi]$, where $\rho(t)$ is

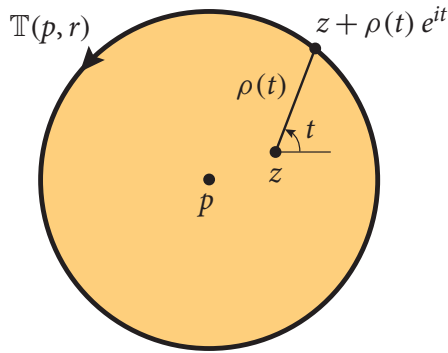


Figure 1.3. Parametrizing the oriented circle $\mathbb{T}(p, r)$ as seen from an off-center point z , used in the proof of Theorem 1.36.

the unique positive number which satisfies $|z + \rho(t)e^{it} - p| = r$ (see Fig. 1.3). It is easy to check that $t \mapsto \rho(t)$ is continuously differentiable. Hence

$$\begin{aligned} \int_{\mathbb{T}(p,r)} \frac{1}{\zeta - z} d\zeta &= \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t) - z} dt = \int_0^{2\pi} \frac{(\rho'(t) + i\rho(t))e^{it}}{\rho(t)e^{it}} dt \\ &= \int_0^{2\pi} \frac{\rho'(t)}{\rho(t)} dt + 2\pi i \\ &= \log(\rho(2\pi)) - \log(\rho(0)) + 2\pi i = 2\pi i, \end{aligned}$$

where the last equality holds since $\rho(2\pi) = \rho(0)$. □

More general versions of Theorems 1.34 and 1.36 will be proved in chapter 2. For now, let us collect some corollaries of these basic results. The first one is the converse of Theorem 1.20:

THEOREM 1.37 (Holomorphic implies complex analytic). *Every $f \in \mathcal{O}(U)$ is complex analytic in U : In every disk $\mathbb{D}(p, r) \subset U$ there is a power series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - p)^n$$

where the coefficients $\{a_n\}$ are given by

$$(1.20) \quad a_n = \frac{f^{(n)}(p)}{n!} = \frac{1}{2\pi i} \int_{\mathbb{T}(p,s)} \frac{f(\zeta)}{(\zeta - p)^{n+1}} d\zeta$$

for any $0 < s < r$.

PROOF. Fix $0 < s < r$ and a point $z \in \mathbb{D}(p, s)$. For any $\zeta \in \mathbb{T}(p, s)$,

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - p) \left[1 - \left(\frac{z - p}{\zeta - p} \right) \right]} = \frac{1}{\zeta - p} \sum_{n=0}^{\infty} \left(\frac{z - p}{\zeta - p} \right)^n.$$

Here the geometric series converges uniformly in ζ since its general term has absolute value $|z - p|/s < 1$ independent of ζ . Thus, we can integrate this series term-by-term on the circle $\mathbb{T}(p, s)$. By Theorem 1.36, we obtain

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}(p,s)} \sum_{n=0}^{\infty} \frac{f(\zeta)(z-p)^n}{(\zeta-p)^{n+1}} d\zeta = \sum_{n=0}^{\infty} a_n (z-p)^n,$$

where the a_n are given by (1.20). This proves that f can be represented by the power series $\sum_{n=0}^{\infty} a_n (z-p)^n$ in $\mathbb{D}(p, s)$. Since this holds for every $s < r$, Theorem 1.20(iii) shows that the power series with the same coefficients must converge to $f(z)$ for all $z \in \mathbb{D}(p, r)$. \square

It follows from Theorem 1.20 that

COROLLARY 1.38. *If $f \in \mathcal{O}(U)$, then $f' \in \mathcal{O}(U)$. Therefore, the k -th derivative $f^{(k)}$ exists and belongs to $\mathcal{O}(U)$ for every $k \geq 1$.*

In particular, by Theorem 1.7, a differentiable map $f : U \rightarrow \mathbb{R}^2$ which satisfies the Cauchy-Riemann equation $f_{\bar{z}} = 0$ throughout U is automatically C^∞ -smooth.

The following converse of Theorem 1.32 is a useful criterion for deciding when a continuous function is holomorphic:

THEOREM 1.39 (Morera, 1886). *Suppose $f : U \rightarrow \mathbb{C}$ is continuous and $\int_{\partial T} f(z) dz = 0$ for every closed triangle $T \subset U$. Then $f \in \mathcal{O}(U)$.*

PROOF. Let $D \subset U$ be a disk. By Theorem 1.31, f has a primitive F in D . Since $F \in \mathcal{O}(D)$ and since the derivative of a holomorphic function is holomorphic by Corollary 1.38, it follows that $f = F' \in \mathcal{O}(D)$. As this holds for every disk $D \subset U$, we conclude that $f \in \mathcal{O}(U)$. \square



Giacinto Morera
(1856–1909)

EXAMPLE 1.40 (Lines are removable). Let $U \subset \mathbb{C}$ be open and L be a straight line which intersects U . Suppose $f : U \rightarrow \mathbb{C}$ is a continuous function which is holomorphic in $U \setminus L$. We prove that f is holomorphic in U by showing that $\int_{\partial T} f(z) dz = 0$ for every triangle $T \subset U$. First assume that the interior of T is disjoint from L . Then, by moving the vertices of T slightly, we can find a triangle $T' \subset U \setminus L$, arbitrarily close to T . By Goursat's Theorem 1.32, $\int_{\partial T'} f(z) dz = 0$. Since the integral along the boundary of a triangle depends continuously on vertices by Lemma 1.27, we must have $\int_{\partial T} f(z) dz = 0$. If the interior of T meets L , write T as the union of at most three triangles with pairwise disjoint interiors, each meeting L along a vertex or an edge, and reduce to the previous case.

This shows in particular that points are removable: If f is continuous in U and holomorphic in $U \setminus \{p\}$, then $f \in \mathcal{O}(U)$. More general removability results are discussed in Theorem 3.5 and in chapter 10.

REMARK 1.41. Morera's theorem holds if we replace triangles with other special families of closed sets with nice boundaries. A typical example, which turns out to be more convenient in some situations, is the family of closed rectangles, or even squares. See problem 25.

THEOREM 1.42 (Cauchy's estimates, 1835). *Suppose f is continuous on $\overline{\mathbb{D}}(p, r)$ and holomorphic in $\mathbb{D}(p, r)$. Then,*

$$(1.21) \quad |f^{(n)}(p)| \leq \frac{n!}{r^n} \sup_{|z-p|=r} |f(z)| \quad (n \geq 0).$$

The example $f(z) = z^n$ in the unit disk \mathbb{D} shows that the bound in (1.21) is optimal for each n .

PROOF. Take $0 < s < r$ and represent f by a power series $\sum_{n=0}^{\infty} a_n (z-p)^n$ in $\mathbb{D}(p, s)$. By (1.20),

$$|f^{(n)}(p)| = n! |a_n| = \frac{n!}{2\pi} \left| \int_{\mathbb{T}(p,s)} \frac{f(z)}{(z-p)^{n+1}} dz \right|,$$

which by the *ML*-inequality implies

$$|f^{(n)}(p)| \leq \frac{n!}{2\pi} \cdot \sup_{|z-p|=s} \frac{|f(z)|}{|z-p|^{n+1}} \cdot 2\pi s = \frac{n!}{s^n} \sup_{|z-p|=s} |f(z)|.$$

Letting $s \rightarrow r$, we obtain (1.21). \square

Cauchy's estimates lead to various quantitative results on holomorphic functions which have no counterpart in the smooth category. Here we prove two basic but important statements of this type.

THEOREM 1.43. *If a holomorphic function f maps the disk $\mathbb{D}(p, r)$ into the disk $\mathbb{D}(q, R)$, then $|f'(p)| \leq R/r$.*

Note that we have *not* assumed $q = f(p)$. In particular, if $f : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic, then $|f'(0)| \leq 1$. This is a basic version of the so-called "Schwarz lemma" which has deep applications and will be discussed at length in chapters 4, 11, and 13.

PROOF. Take $0 < s < r$ and apply (1.21) to the function $g = f - q$:

$$|f'(p)| = |g'(p)| \leq \frac{1}{s} \sup_{|z-p|=s} |g(z)| \leq \frac{R}{s}.$$

Letting $s \rightarrow r$ proves the result. \square

Liouville's theorem was known to Cauchy in 1844.

THEOREM 1.44 (Liouville, 1847). *Every bounded entire function is constant.*

PROOF. Let $f \in \mathcal{O}(\mathbb{C})$ and $|f(z)| < M$ for all $z \in \mathbb{C}$. Then f maps any disk $\mathbb{D}(p, r)$ into $\mathbb{D}(0, M)$, so by Theorem 1.43, $|f'(p)| \leq M/r$. Letting $r \rightarrow +\infty$, we obtain $f'(p) = 0$. Since this holds for every $p \in \mathbb{C}$, f must be constant. \square



Joseph Liouville
(1809–1882)

EXAMPLE 1.45 (The fundamental theorem of algebra). Let $P: \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial of degree $d \geq 1$, so $\lim_{z \rightarrow \infty} P(z) = \infty$. If $P(z) \neq 0$ for all z , then $f(z) = 1/P(z)$ is entire and $\lim_{z \rightarrow \infty} f(z) = 0$. Hence there is an $R > 0$ such that $|f(z)| \leq 1$ whenever $|z| \geq R$. Since by continuity f is bounded on the closed disk $\overline{\mathbb{D}}(0, R)$, it follows that f is bounded on the plane. Liouville's theorem then implies that f is constant, which is a contradiction. Thus, P has at least one root z_1 and we can write $P(z) = (z - z_1)P_1(z)$ for some polynomial P_1 of degree $d - 1$. If $d - 1 = 0$ so P_1 is constant, stop. Otherwise repeat the argument with P_1 in place of P to find a root z_2 of P_1 , and so on. This process stops after d steps and shows that P factors as $P(z) = a(z - z_1)(z - z_2) \cdots (z - z_d)$ for some $a, z_1, \dots, z_d \in \mathbb{C}$. Thus, every complex polynomial of degree $d \geq 1$ has precisely d roots counting multiplicities.

We end this section with a useful theorem which, roughly speaking, says that the integral of a function which depends holomorphically on a parameter is a holomorphic function of that parameter, and differentiation under the integral sign is legitimate. We formulate a simple version of the theorem which will be sufficient for our purposes. One should note, however, that the result holds in much more general settings (see problem 27).

THEOREM 1.46. *Let $U \subset \mathbb{C}$ be open and $\varphi: U \times [a, b] \rightarrow \mathbb{C}$ be a continuous function such that for each $t \in [a, b]$, $z \mapsto \varphi(z, t)$ is holomorphic in U with derivative $\varphi'(z, t)$. Then, the function $f: U \rightarrow \mathbb{C}$ defined by*

$$f(z) = \int_a^b \varphi(z, t) dt$$

is holomorphic and we can differentiate under the integral sign:

$$f'(z) = \int_a^b \varphi'(z, t) dt \quad \text{for all } z \in U.$$

PROOF. Fix $p \in U$ and take $r > 0$ such that $\overline{\mathbb{D}}(p, r) \subset U$. Let $0 < |z - p| < r/2$. By Theorem 1.36,

$$\varphi(z, t) - \varphi(p, t) = \frac{1}{2\pi i} \int_{\mathbb{T}(p, r)} \varphi(\zeta, t) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - p} \right) d\zeta,$$

so

$$\frac{\varphi(z, t) - \varphi(p, t)}{z - p} = \frac{1}{2\pi i} \int_{\mathbb{T}(p, r)} \frac{\varphi(\zeta, t)}{(\zeta - z)(\zeta - p)} d\zeta.$$

Since $|\zeta - z| > r/2$ whenever $|\zeta - p| = r$, we obtain the following estimate using the *ML*-inequality:

$$\left| \frac{\varphi(z, t) - \varphi(p, t)}{z - p} \right| \leq \frac{1}{2\pi} \cdot M \cdot \frac{2}{r^2} \cdot 2\pi r = \frac{2M}{r}.$$

Here M is the supremum of $|\varphi|$ on the compact set $\overline{\mathbb{D}}(p, r) \times [a, b]$. If $\{z_n\}$ is any sequence in $U \setminus \{p\}$ which tends to p , then

$$g_n(t) = \frac{\varphi(z_n, t) - \varphi(p, t)}{z_n - p}$$

is a sequence of continuous functions on $[a, b]$ which converges pointwise to $\varphi'(p, t)$ and is bounded by $2M/r$ for all large n . Hence, by Lebesgue's dominated convergence theorem, the function $t \mapsto \varphi'(p, t)$ is integrable on $[a, b]$ and

$$\lim_{n \rightarrow \infty} \frac{f(z_n) - f(p)}{z_n - p} = \lim_{n \rightarrow \infty} \int_a^b g_n(t) dt = \int_a^b \varphi'(p, t) dt.$$

Since this holds for every sequence $z_n \rightarrow p$, we conclude that $f'(p)$ exists and equals $\int_a^b \varphi'(p, t) dt$. \square

REMARK 1.47. Under the assumptions of the above theorem, the derivative $(z, t) \mapsto \varphi'(z, t)$ is in fact continuous on $U \times [a, b]$ (see problem 26). Thus, the result holds when $\varphi(z, t)$ is replaced with $\varphi'(z, t)$, and a simple induction proves the formula

$$f^{(n)}(z) = \int_a^b \varphi^{(n)}(z, t) dt \quad \text{for all } z \in U,$$

where $\varphi^{(n)}(z, t)$ is the n -th derivative of $\varphi(z, t)$ with respect to z .

The following corollary of the above theorem will be used repeatedly:

COROLLARY 1.48. *Let $\gamma : [0, 1] \rightarrow \mathbb{C}$ be a piecewise C^1 curve and $g : |\gamma| \rightarrow \mathbb{C}$ be a continuous function. Then, for each integer $n \geq 1$, the function*

$$f(z) = \int_\gamma \frac{g(\zeta)}{(\zeta - z)^n} d\zeta$$

is holomorphic in $\mathbb{C} \setminus |\gamma|$, and

$$f'(z) = n \int_\gamma \frac{g(\zeta)}{(\zeta - z)^{n+1}} d\zeta \quad \text{for all } z \in \mathbb{C} \setminus |\gamma|.$$

PROOF. This follows from Theorem 1.46 applied to $\varphi : (\mathbb{C} \setminus |\gamma|) \times [0, 1] \rightarrow \mathbb{C}$ defined by

$$\varphi(z, t) = \frac{g(\gamma(t)) \gamma'(t)}{(\gamma(t) - z)^n}.$$

(Technically, we need to break up $[0, 1]$ into finitely many intervals in which γ' is continuous and add up the corresponding integrals, but that is a trivial matter.) \square

EXAMPLE 1.49 (Cauchy's integral formula for higher derivatives). A special case of the above corollary is Cauchy's integral formula. If $f \in \mathcal{O}(U)$ and $\overline{\mathbb{D}}(p, r) \subset U$, then

$$\frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{\zeta - z} d\zeta$$

defines a holomorphic function in $\mathbb{C} \setminus \mathbb{T}(p, r)$. By Theorem 1.36, this function coincides with f inside the disk $\mathbb{D}(p, r)$. Differentiation under the integral sign then gives

$$f'(z) = \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \quad \text{for } z \in \mathbb{D}(p, r).$$

It follows by induction that for every $n \geq 0$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \quad \text{for } z \in \mathbb{D}(p, r).$$

Observe that for $z = p$ this is the formula (1.20) that we derived earlier.

1.5 Mapping properties of holomorphic functions

DEFINITION 1.50. Suppose $f \in \mathcal{O}(U)$ and f is not identically zero in the disk $\mathbb{D}(p, r) \subset U$. Let $f(z) = \sum_{n=0}^{\infty} a_n (z - p)^n$ be the power series representation of f in $\mathbb{D}(p, r)$. The smallest integer m with the property $a_m \neq 0$ is called the **order** of p and is denoted by $\text{ord}(f, p)$. Thus, $\text{ord}(f, p) \geq 1$ if and only if $f(p) = 0$. We call p a **simple zero** of f if $\text{ord}(f, p) = 1$.

Alternatively, $\text{ord}(f, p)$ can be described as the unique integer $m \geq 0$ for which f can be factored as

$$f(z) = (z - p)^m f_1(z)$$

with $f_1 \in \mathcal{O}(U)$ and $f_1(p) \neq 0$. The function f_1 is given by $(z - p)^{-m} f(z)$ in $U \setminus \{p\}$. It is holomorphic in U since it is represented by the power series $\sum_{n=m}^{\infty} a_n (z - p)^{n-m}$ in $\mathbb{D}(p, r)$.

EXAMPLE 1.51 (Holomorphic L'Hôpital's rule). Suppose f and g are holomorphic in some neighborhood of p , with $\text{ord}(f, p) = \text{ord}(g, p) = m \geq 1$. Write $f(z) = (z - p)^m f_1(z)$ and $g(z) = (z - p)^m g_1(z)$, where f_1 and g_1 are non-zero and holomorphic near p . Since

$$f_1(p) = \frac{f^{(m)}(p)}{m!} \quad \text{and} \quad g_1(p) = \frac{g^{(m)}(p)}{m!},$$

it follows that

$$\lim_{z \rightarrow p} \frac{f(z)}{g(z)} = \frac{f_1(p)}{g_1(p)} = \frac{f^{(m)}(p)}{g^{(m)}(p)}.$$

Let us call $U \subset \mathbb{C}$ a **domain** if U is non-empty, open, and connected.

LEMMA 1.52. *Suppose $U \subset \mathbb{C}$ is a domain and $f \in \mathcal{O}(U)$. If the zero-set $f^{-1}(0) = \{z \in U : f(z) = 0\}$ has an accumulation point in U , then $f = 0$ everywhere in U .*

Connectivity of U is essential here: If U is the disjoint union of non-empty open sets U_1 and U_2 , and if $f = 0$ in U_1 and $f = 1$ in U_2 , then $f \in \mathcal{O}(U)$ and $f^{-1}(0) = U_1$ has accumulation points in U , but f is not identically zero in U .

PROOF. Let E be the non-empty set of accumulation points of $f^{-1}(0)$ in U . Then E is closed in U , and $E \subset f^{-1}(0)$ by continuity of f . Suppose $p \in E$ and there is a disk $\mathbb{D}(p, r) \subset U$ in which f is not identically zero. Then we can write $f(z) = (z - p)^m f_1(z)$, where $m = \text{ord}(f, p) \geq 1$, $f_1 \in \mathcal{O}(U)$, and $f_1(p) \neq 0$. By continuity, f_1 does not vanish in some neighborhood of p . It follows that p is the only zero of f in this neighborhood, contradicting the fact that $p \in E$. Thus, if $\mathbb{D}(p, r) \subset U$, then f is identically zero in $\mathbb{D}(p, r)$ and therefore $\mathbb{D}(p, r) \subset E$. This shows that E is an open set. Since U is connected, we must have $E = f^{-1}(0) = U$. \square

Since every domain $U \subset \mathbb{C}$ is a countable union of open disks, it is clear that every uncountable subset of U must have an accumulation point in U . It follows from the above lemma that *a non-constant holomorphic function in a domain has at most countably many zeros, all of which are isolated*. Another immediate corollary is

THEOREM 1.53 (The identity theorem). *Suppose $U \subset \mathbb{C}$ is a domain, $f, g \in \mathcal{O}(U)$, and the set $\{z \in U : f(z) = g(z)\}$ has an accumulation point in U . Then $f = g$ everywhere in U .*

EXAMPLE 1.54. The complex cosine and sine are the entire functions defined by

$$\begin{aligned} \cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}. \end{aligned}$$

They extend the usual cosine and sine functions defined on the real line. It follows from Theorem 1.53 that any trigonometric identity between cosine and sine that holds on \mathbb{R} must continue

to hold in \mathbb{C} . For example, the identities $\cos^2 z + \sin^2 z = 1$, $\sin(2z) = 2 \sin z \cos z$, and $\cos(2z) = \cos^2 z - \sin^2 z$ remain valid for all $z \in \mathbb{C}$.

EXAMPLE 1.55. Suppose $f \in \mathcal{O}(\mathbb{C})$ has the power series representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If every coefficient a_n is real, then clearly $f(\mathbb{R}) \subset \mathbb{R}$. Conversely, suppose $f(\mathbb{R}) \subset \mathbb{R}$ and consider the entire function

$$g(z) = \overline{f(\bar{z})} = \sum_{n=0}^{\infty} \overline{a_n} z^n.$$

Since $f(z)$ is real when z is real, we have $g = f$ on the real line. By Theorem 1.53, $g = f$ everywhere in \mathbb{C} . Uniqueness of power series then shows that every a_n is real.

Our next goal is to prove the fundamental fact that the image of a domain under a non-constant holomorphic function is open (Theorem 1.62). This will follow from a much stronger result on the local behavior of holomorphic functions (Theorem 1.59).

LEMMA 1.56. If $f \in \mathcal{O}(U)$, the function $g: U \times U \rightarrow \mathbb{C}$ defined by

$$g(\zeta, z) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$

is continuous.

PROOF. Clearly g is continuous off the diagonal $\{(z, z) : z \in U\}$, so it is enough to check continuity of g at a diagonal point (p, p) . Let $\varepsilon > 0$ be given. Since f' is continuous at p , there is an $r > 0$ such that

$$(1.22) \quad |f'(z) - f'(p)| < \varepsilon \quad \text{whenever } z \in \mathbb{D}(p, r).$$

Let $\zeta, z \in \mathbb{D}(p, r)$. If $\zeta = z$, then $|g(\zeta, z) - g(p, p)| = |f'(z) - f'(p)| < \varepsilon$. If $\zeta \neq z$, then

$$\frac{f(\zeta) - f(z)}{\zeta - z} = \frac{1}{\zeta - z} \int_{[z, \zeta]} f'(w) dw = \int_0^1 f'(\gamma(t)) dt,$$

where $\gamma(t) = (1-t)z + t\zeta$. Hence,

$$\begin{aligned} |g(\zeta, z) - g(p, p)| &= \left| \frac{f(\zeta) - f(z)}{\zeta - z} - f'(p) \right| = \left| \int_0^1 [f'(\gamma(t)) - f'(p)] dt \right| \\ &\leq \int_0^1 |f'(\gamma(t)) - f'(p)| dt \leq \varepsilon, \end{aligned}$$

where the last inequality holds since by (1.22), $|f'(\gamma(t)) - f'(p)| < \varepsilon$ for every $t \in [0, 1]$. \square

THEOREM 1.57 (Holomorphic inverse function theorem). Suppose $f \in \mathcal{O}(U)$, $p \in U$, and $f'(p) \neq 0$. Then, there exist open neighborhoods $V \subset U$ of p and $W \subset \mathbb{C}$ of $f(p)$ such

(continued...)

Index

- Aut(U), 110
- Abel, N., 273
- action
 - free, 367
 - properly discontinuous, 367
 - simply 2-transitive, 111
 - simply 3-transitive, 104
 - simply transitive, 367
- Ahlfors function of a domain, 383
- Ahlfors's theorem, 351
- Ahlfors, L., 9, 64, 130, 184, 336, 343, 351, 383, 397, 407, 419
- Ahlfors-Grunsky conjecture, 336
- analytic arc, 318
 - canonical reflection, 320
 - coordinate function, 319
 - coordinate neighborhood, 319
- analytic content, 299
- analytic continuation
 - along a curve, 306
 - direct, 305
 - homotopy lifting property, 307
- analytic Jordan curve, 290, 333
- angle preservation, 106
- anharmonic group, 114
- anti-holomorphic function, 36, 190, 320
- argument principle, 95
 - generalized, 101
- Artin, E., 64, 65
- Arzelà-Ascoli theorem, 139
- asymptotic value, 363
- automorphism group, 110
 - of \mathbb{C} , 111
 - of $\hat{\mathbb{C}}$, 110
 - of \mathbb{C}^* , 132
 - of \mathbb{D} , 113
 - of \mathbb{H} , 113
 - of punctured spheres, 114
- B_f , 335
- \mathfrak{B} , 335
- Bernstein, S., 188
- Bers, L., 41
- Berteloot, F., 348, 419
- Beurling, A., 184
- Bieberbach's conjecture, 171
- Bieberbach's inequality, 166
- Bieberbach, L., 167, 171
- biholomorphism, 31
- Blaschke product
 - finite, 133, 218, 242, 373
 - infinite, 245
- Blaschke, W., 219
- Bloch's constant, 335
- Bloch's principle, 348
- Bloch's theorem, 335
- Boas, R., 260, 419
- Borel-Carathéodory inequality, 255
- Bouquet, J. C., 8
- branch point, 375
- branched covering, 375
- Briot, C. A., 8
- Brouwer, L. E. J., 31, 54, 74
- \mathbb{C} , 1
- $\mathcal{C}(U, X)$, 135
- $\hat{\mathbb{C}}$, 79
- \mathbb{C}^* , 47
- $\chi(U)$, 379
- $\chi(z, w)$, 148
- canonical products, 239
 - order, 253
- Cantor set, 286
- Carathéodory, C., 111, 158, 178
- Carathéodory's extension theorem, 174
- Cartan, H., 185
- Casorati, F., 77
- Casorati-Weierstrass theorem, 76
- Cauchy transform of a measure, 40
- Cauchy's estimates, 25
- Cauchy's integral formula
 - for higher derivatives, 28
 - in a disk, 22
- Cauchy's theorem
 - homology version, 67
 - in a disk, 22
- Cauchy, A. L., 10, 67, 87
- Cauchy-Pompeiu formula, 73
- Cauchy-Riemann equations, 8
- Cayley map, 107
- chain (analytic continuation), 306
- chain (homology), 62
- chain rule, 2
- Chebyshev polynomials, 188
- chordal distance, 148
- Christoffel, E. B., 317
- Clairaut, A., 189
- compact convergence, 136
 - topology, 135
- compactly bounded family, 144
- complex analytic function, 11
- complex conjugation, 36, 299, 309, 388
- complex derivative, 2
- complex differentiability, 2
- complex Green's theorem, 73
- conformal linear map, 8, 37
- conformal map, 158
- conformal metric, 121
 - density function, 121
 - pull-back, 122
- conformal radius, 185
- conformally isomorphic, 158
- covering map, 358
 - branched, 375
 - Galois, 384
 - universal, 366
- covering space, 358
 - isomorphism, 358
- critical point, 33
- critical value, 33
- cross ratio, 108
 - alternative formulas, 110
 - basic properties, 108
 - cocycle relation, 131
 - hyperbolic distance formula, 134
 - projective invariance, 132
- cross-cut, 178
- curvature, 350
 - conformal invariance, 350
- curve, 14, 42
 - closed, 14, 42
 - constant, 43
 - end point, 14, 42
 - image, 14, 42
 - initial point, 14, 42
 - null-homotopic, 45
 - piecewise C^1 , 14
 - reverse, 15, 43
- curve lifting property, 49, 359
- cycle, 62
 - null-homologous, 64
- \mathbb{D} , 1
- $\mathbb{D}(p, r)$, 1
- Δ , 36, 189
- \mathbb{D}^* , 100
- $\mathbb{D}^*(p, r)$, 76
- $\deg(f)$, 273, 359, 376
- $\deg(f, p)$, 33, 78, 375
- $\dim_{\mathbb{H}}$, 324
- dist_{σ} , 148
- dist_g , 122
- $\mathbf{d}(f, g)$, 137
- d'Alembert, J., 8
- de Branges' theorem, 171
- de Branges, L., 171, 419

- degree
 - of a branched covering, 376
 - of a covering, 359
 - of a rational map, 378
 - of an elliptic function, 273
- derivative norm, 123
 - spherical, 153
- devil's staircase, 9
- Dieudonné, J., 186, 419
- Dirichlet integral, 226
- Dirichlet problem in the disk, 204
 - L^∞ version, 213
- Dixon, J., 68, 419
- do Carmo, M., 122, 419
- dog-on-a-leash lemma, 96
- domain, 29
 - finitely connected, 290
- doubly periodic, 39, 272
- Duval, J., 348, 419
- dyadic square, 325

- E_d , 236
- $\text{ext}(\gamma)$, 57
- Eisenstein series, 277
- elliptic function, 271
- elliptic modular function, 396
- entire function, 8
- equicontinuous family, 138
- escaping
 - curve, 363
 - sequence, 174
- Escher, M. C., 118
- Euler characteristic, 379
- Euler's product formula, 259, 260
- Euler, L., 8, 189
- evenly covered neighborhood, 47, 358
- exhaustion by compact sets, 136
 - nice, 285
- exponent of convergence, 253
- exponential function, 9
 - ubiquity of, 348

- φ_p , 112
- $f^\#$, 153
- Falconer, K., 323, 419
- family
 - compactly bounded, 144
 - equicontinuous, 138
 - normal, 152
 - pointwise bounded, 140
 - precompact, 137
- Farey
 - neighbors, 394, 415
 - sum, 394, 415
- Farey, J., 394
- Fatou's radial limit theorem, 182, 213
- Fatou, P., 215, 378
- Fejér, L., 158

- Fermat equation
 - holomorphic solutions, 341
 - meromorphic solutions, 341
- fiber, 359
- finitely connected domain, 290
- first homology group, 65
- Fisher, Y., 398, 419
- fixed point
 - index, 91
 - multiplicity, 91
 - simple, 91
- Fomenko, O., 171, 419
- Ford circles, 415
- Forster, O., 361, 419
- Fourier coefficients, 17, 85
- Fourier series, 85
- freely homotopic, 46
- full compact set, 280
- Fulton, W., 57, 66, 368, 380, 419
- function element, 305
- fundamental group, 45
- fundamental parallelogram, 271
- fundamental theorem of algebra, 26, 35, 242

- g_2, g_3 , 278
- g_U , 401
- \mathbb{D} , 127
- Gårding, L., 419
- Galois covering, 384
- Gamelin, T., 297, 419
- Garnett, J., 419
- Gauss-Lucas theorem, 35
- genus of an entire function, 263
- Glicksberg, I., 102, 419
- Goursat's theorem, 20
- Goursat, E., 21
- Grönwall's area theorem, 164
- Grönwall, T. H., 165
- Grötzsch, H., 184
- Gray, J., 9, 419
- Green's first identity, 226
- Green's second identity, 226

- $H_1(U)$, 65
- \mathbb{H} , 107
- Hölder condition, 140
- Hadamard's 3-circles theorem, 243
- Hadamard's factorization theorem, 256
- Hadamard's gap theorem, 304
- Hadamard, J., 11, 255
- harmonic conjugate, 224
- harmonic extension by reflection, 205
- harmonic function, 190
 - periods, 193, 292
- harmonic measure, 216
- Harnack distance, 225
- Harnack's inequalities, 206
- Harnack, A., 207
- Hartogs-Rosenthal theorem, 299

- Hatcher, A., 66, 368, 419
- Hausdorff dimension, 324
- Hausdorff measure, 322
- Hausdorff, F., 324
- Hawaiian earring, 383
- holomorphic branch
 - of arbitrary powers, 53
 - of the n -th root, 52
 - of the logarithm, 52
- holomorphic function, 8
- holomorphic implicit function theorem, 101
- holomorphic inverse function theorem, 30
- holomorphically removable set, 321
 - for bounded functions, 333
- homologous cycles, 64
- homology class of a cycle, 64
- homotopic curves, 42
- homotopy, 42
 - free, 46
- homotopy class of a curve, 44
- Hubbard, J., 398, 419
- Hurwitz's theorem, 143
- Hurwitz, A., 143
- hyperbolic domain, 352
 - characterization, 354, 400
- hyperbolic geodesics, 128
- hyperbolic metric
 - of \mathbb{D} , 127
 - of \mathbb{H} , 127
 - of a hyperbolic domain, 401
 - of a round annulus, 402
 - of a strip, 402
 - of the punctured disk, 402

- $\text{ind}(f, p)$, 91
- $\text{int}(\gamma)$, 57
- ι , 81, 104
- ideal triangle, 390
- identity theorem, 29
 - for harmonic functions, 192
- index at a fixed point, 91
- infinite product
 - absolutely convergent, 232
 - compactly convergent, 233
 - convergent, 228
- integral along a curve, 14
- invariance of domain theorem, 31, 74
- invariants of a lattice, 278
- isometric circle, 131

- Jacobian, 36
- Jensen's formula, 241
- Jensen, J., 241
- Jordan curve, 57
 - exterior, 57
 - interior, 57
- Jordan curve theorem, 57
- Jordan, C., 57, 59

- jump principle
 - for Cauchy transforms, 73
 - for the winding number, 57
- K_g , 350
- κ , 36, 309
- Kirchhoff's law, 62
- Koebe function, 161
- Koebe's 1/4-theorem, 171
- Koebe's circle domain theorem, 291
- Koebe's distortion bounds, 171
- Koebe, P., 145, 158, 172, 291, 397, 419
- Krantz, S., 353, 419
- Kraus, W., 188
- Kuz'mina, G., 171, 419
- Λ_α , 322
- ℓ , 216
- length $_g$, 122
- L'Hôpital's rule, 28
- lacunary power series, 304
- Lambert W -function, 372
- Landau, E., 256, 336
- Laplace operator, 36, 189
- Laplace transform, 39
- Laplace, P., 189
- Lattès map, 296
- Laurent series, 83
- Laurent, P., 83
- Lebesgue number of a cover, 49
- Lebesgue point, 210
- lift, 48, 359
 - uniqueness, 49, 359
- Lindelöf's maximum principle, 225
- Liouville's theorem
 - for entire functions, 26
 - for harmonic functions, 207
 - hyperbolic version, 352
- Liouville, J., 25
- Lipschitz function, 333
- local degree, 33, 375
- local isometry, 123
- local mean value property, 195
- locally compact space, 140
- locally connected, 174
 - characterizations, 175
- locally path-connected, 51
- locally powerlike map, 375
- locally simply connected, 363
- logarithmic differentiation, 235
- Looman-Menshov theorem, 9
- $\mathcal{M}(U)$, 78
- $\mathcal{M}(\mathbb{C}, \Lambda)$, 271
- Möb, 103
- $\text{mod}(\cdot)$, 183, 187, 314, 333, 409
- $M_f(r)$, 243
- Möb, 103
- Möbius group, 103
- Möbius map, 103
 - elliptic, 119
 - fixed point multipliers, 116
 - fixed points, 116
 - hyperbolic, 119
 - invariant curves, 118
 - loxodromic, 119
 - parabolic, 119
 - trace-squared, 119
- Möbius, A., 105
- Marty's theorem, 153
- Marty, F., 153
- Mattila, P., 419
- maximum principle
 - for harmonic functions, 196, 197
 - for holomorphic functions, 34
 - for open maps, 34
 - Lindelöf's, 225
- McKean, H., 271, 279, 397, 419
- McMullen, C., 398, 419
- mean value property, 195
- Mergelyan's theorem, 287
- Mergelyan, S., 287
- meromorphic function, 78
- Milnor, J., 93, 115, 241, 419
- minimal geodesic, 122, 403
- Mittag-Leffler's theorem
 - for open sets, 269
 - for the plane, 265
- Mittag-Leffler, G., 240, 265
- ML-inequality, 16
- modular group, 393
 - level 2 congruence subgroup, 413
- modulus
 - of a rectangle, 183, 314
 - of a topological annulus, 409
 - of an annulus, 187, 333
- Moll, V., 271, 279, 397, 419
- monodromy theorem, 308, 370
- Montel's theorem
 - basic version, 145
 - general version, 344, 397
- Montel, P., 145
- Morera's theorem, 24
- Morera, G., 24
- Morris, S., 9, 419
- $\nu(f)$, 253
- $N_f(D, q)$, 95
- $N_f(r)$, 243
- natural boundary, 302
- Nevanlinna, R., 419
- non-degenerate continuum, 290
- non-tangential limit, 227
- normal derivative, 226
- normal family, 152
- normalizer subgroup, 416
- $\mathcal{O}(U)$, 8
- $\omega(z, E)$, 216
- ω_z , 217
- $\text{ord}(f, p)$, 28, 77
- omitted value, 339
- one-point compactification, 385
- open mapping theorem, 33
- order
 - of a pole, 77
 - of a canonical product, 253
 - of a power series, 250
 - of a zero, 28
 - of an elementary factor, 249
 - of an entire function, 247
- Osgood, W., 156, 158, 175
- Ostrowski pair, 303
- Ostrowski's theorem, 303
- Ostrowski, A., 303
- over-convergence phenomenon, 303
- $P(\zeta, z)$, 199
- Π_p , 271
- $\pi_1(X, p)$, 45
- \wp , 267, 274
- $\mathcal{P}[\cdot]$, 202
- $\text{PSL}_2(\mathbb{C})$, 104
- $\text{PSL}_2(\mathbb{R})$, 113
- $\text{PSL}_2(\mathbb{Z})$, 393
- Pólya, G., 188, 419
- Paatero, V., 419
- Painlevé's theorem, 327
- Painlevé, P., 327
- Parseval's formula, 39, 214
- path-connected, 45
- periodic point, 116
- Picard's great theorem, 343
- Picard's little theorem, 339, 397
- Picard, E., 240, 343
- Pick's theorem, 129
- Pick, G., 129
- Poincaré, H., 45, 358, 397
- Poincaré-Hurewicz theorem, 66
- Poisson integral, 202
 - radial limits, 210
- Poisson integral formula, 198
- Poisson kernel, 199
 - basic properties, 200
 - Schwarz's interpretation, 223
- Poisson, S. D., 198
- Poisson-Jensen formula, 262
- Pommerenke, C., 186, 323, 419
- Porter, M. B., 146
- power series, 10
 - order, 250
 - radius of convergence, 11
 - recursive coefficients, 38
- powerlike map, 375

- precompact, 137
 primitive, 18
 principal branch of the logarithm, 52, 231
 principal part, 77
 Pringsheim, A., 20, 331
 product of two curves, 15, 43
 proper map, 373
 Ptolemy's theorem, 106

 R_α , 388
 Ref, 388
 $\text{Ref}(\mathbb{D})$, 389
 $\text{res}(f, p)$, 87
 $\rho(f)$, 247
 Radó, T., 378
 radial limit
 of bounded harmonic functions, 213
 of conformal maps, 182
 ramified neighborhood, 375
 rational fixed point formula, 247
 real analytic function, 311
 real derivative, 5
 real symmetric domain, 311
 rectifiable arc, 324
 reflection
 across an analytic arc, 320
 in a circle, 388
 reflection group, 388
 regular point, 301
 Remmert, R., 8, 41, 337, 339, 341, 419
 residue, 87
 at infinity, 90
 fractional, 100
 residue theorem, 88
 Riemann map, 160
 normalized, 160
 of a semidisk, 160
 Riemann mapping theorem, 158
 Riemann sphere, 79
 Riemann's removable singularity theorem, 76
 Riemann's zeta function, 142
 Riemann, B., 159, 358
 Riemann-Hurwitz formula, 380
 Riesz brothers theorem, 246
 Riesz, F., 158
 Roth, A., 299
 Rouché's theorem, 97
 symmetric version, 102
 Rouché, E., 97
 Rudin, W., 140, 156, 210, 212, 262, 287, 419

 Runge's theorem
 for compact sets, 280
 for open sets, 286
 Runge, C., 280

 S_f , 187
 \mathcal{S} , 161
 \mathcal{S} , 163
 σ , 126, 147
 σ -compact space, 140
 Schönflies theorem, 289
 Schönflies, A. M., 290
 schlicht functions, 161
 compactness, 168
 universal bounds, 168
 Schottky's theorem, 341
 Schottky, F. H., 341
 Schwarz lemma
 Ahlfors's version, 351
 basic version, 111
 for hyperbolic domains, 405
 Pick's version, 129
 Schwarz reflection principle, 181, 313
 Schwarz's interpretation of Poisson kernel, 223
 Schwarz, H. A., 112, 158, 189, 199, 203, 223
 Schwarz-Christoffel formula, 317
 Schwarzian derivative, 187
 Seifert, H., 358, 419
 separable space, 140
 simple pole, 77
 simple zero, 28
 simply connected, 45
 characterizations, 288
 singular point, 301
 singularity
 essential, 76
 isolated, 75
 pole, 76
 removable, 75
 slit, 380
 spherical derivative norm, 153
 spherical metric, 126, 147
 square root trick, 159, 400
 star-shaped domain, 46
 Steinmetz, N., 379, 419
 stereographic projection, 79, 130, 147
 Stirling's formula, 251
 Stone-Weierstrass theorem, 287
 subordination principle, 184
 Swiss cheese set, 299
 Szegő, G., 188, 419

 \mathbb{T} , 17
 $\mathbb{T}(p, r)$, 16
 $\tau(f)$, 119
 Taylor, E., 175
 tessellation by ideal triangles, 391
 Threlfall, W., 358, 419
 tile, 390
 generation, 392
 reflection group, 390
 Titchmarsh, E., 256, 419
 topological annulus, 408
 degenerate hyperbolic, 409
 Euclidean, 408
 non-degenerate hyperbolic, 409
 transcendental entire function, 247

 uniformization theorem, 397
 universal covering, 366
 unramified disk, 335

 Valiron, G., 338
 Veech, W., 420
 Veita's formula, 261
 Vitali, G., 146
 Vitali-Porter theorem, 146

 $W(\gamma, p)$, 54
 Wall, C. T. C., 57, 420
 Wallis's formula, 261
 Weierstrass σ -function, 294
 Weierstrass \wp -function, 267, 274
 basic properties, 275
 critical values, 278
 differential equation, 276, 278
 Laurent series, 277
 Weierstrass M -test, 141
 Weierstrass convergence theorem, 140
 Weierstrass elementary factors, 236
 order, 249
 Weierstrass factorization theorem, 239
 Weierstrass product theorem
 for open sets, 240
 for the plane, 238
 Weierstrass, K., 141
 Weyl, H., 358
 winding number, 54
 jump principle for, 57
 Wittner, B., 398, 419

 Zalcman's theorem, 345
 Zalcman, L., 175, 331, 345, 348, 420
 Zhukovskii map, 163, 166, 188, 382
 Zhukovskii, N. E., 163