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In support of the theoretical calculations performed in this book, numerical "confirmations" are occasionally provided by using software developed by The MathWorks, Inc., of Natick, MA. Specifically, *MATLAB* 8.9 (release 2019b) running on a Windows 10 PC. I've done this because I support the position advocated by Victor Moll (professor of mathematics at Tulane University) in his book, *Experimental Mathematics in Action* (CRC Press, 2007, pp. 4–5): "Given an interesting identity buried in a long and complicated paper on an unfamiliar subject, which would give you more confidence in its correctness: staring at the proof, or confirming computationally that it is correct to 10,000 decimal places?" The calculations I have *MATLAB* do in this book rarely exceed seven decimal places, but the idea is the same. *MATLAB* is the registered trademark of The MathWorks, Inc. The MathWorks, Inc. does not warrant the accuracy of the text in this book. This book's use or discussion of *MATLAB* does not constitute an endorsement or sponsorship by The MathWorks, Inc., of a particular pedagogical approach or particular use of the *MATLAB* software.

CHAPTER 1

Euler's Problem

1.1 Introducing Euler

The title of this book has been carefully crafted to attract the interest of all those who love mathematics, which would seem to be an obvious thing to do for the author of a book like this one. However, the subtitle of this book seems likely to provoke controversy among professional mathematicians, which, at the other extreme, might seem to be a rather odd thing for an author to do. My primary goal is clear, I think, as everybody likes a good hunt involving puzzles, a fact that explains the attraction of mystery novels, adventure video games, and *Indiana Jones* movies like *Raiders of the Lost Ark*. Hardly anybody, I think, would quibble with that. But how, I can hear each mathematician on the planet grumbling as he/she reads this, can I claim that the puzzle of zeta-3—I'll tell you what *that* is, in just a bit—is the world's most puzzling unsolved math problem? After all, as each of my critics would emphatically state, even while (perhaps) vigorously pounding a desktop or thumping a finger into my chest, "it's quite clear that the problem that's holding *my work* up is the world's most puzzling unsolved problem!"

My selection criteria for choosing which math problem is assigned the label as the most puzzling problem are quite simple: (1) the problem is (obviously!) unsolved; (2) people have been trying (and failing) for centuries to solve it; (3) it has at least some connection to the real world of physics and engineering; and most important of all, (4) despite (1), (2), and (3), a grammar school student who knows how to do elementary arithmetic can instantly understand the problem. The first three criteria are satisfied by lots of really hard problems in math, but if it takes a degree in math to simply understand the question, then such problems clearly fail the fourth test (this eliminates the famous problem of the Riemann hypothesis, about which I'll say more in the next section). At the end of the next chapter, I'll return to this issue, that of selecting the most puzzling math problem.[1]

But for now, let me set the stage for all that follows by introducing the personality most closely associated with the problem of zeta-3, the great (perhaps the *greatest* in history) mathematician, Leonhard Euler (1707–1783). The son of a rural Swiss pastor, Euler trained for the ministry at the University of Basel and at age 17, received a graduate degree from the Faculty of Theology. While a student at Basel, however, he also studied with the famous mathematician Johann Bernoulli (1667–1748), and despite his years-long immersion in religious thought, it was mathematics that captured his soul. Euler never lost his belief in God and in an afterlife, but while he was in *this* world, it was mathematics that had his supreme devotion.

It seemed that there was nothing that could keep him from doing mathematics, not even blindness from a botched cataract operation.

1. Until it was solved in 1995 by Andrew Wiles, perhaps Fermat's Last Theorem would be the problem that would have occurred to most people as the "world's most puzzling math problem," even though many professional mathematicians would have disagreed: for example, the great German mathematician Carl Friedrich Gauss (1777–1855), perhaps as great as Euler, refused to work on the Fermat problem, because he simply found it uninteresting. And, unlike the zeta-3 problem, the Fermat problem makes no appearance (as far as I know) in either science or engineering. Finally, the 1995 solution has been examined and understood by only a few world-class mathematicians. Everybody else simply accepts their thumbs-up verdict that Wiles' proof is correct (it's certainly far beyond AP-calculus!).

(Can you imagine enduring, with no anesthetic, such an operation in the 18th century?) Euler had a marvelous memory (it was said he knew the thousands of lines in the *Aeneid* by heart) and so, for the last 17 years of his life after losing his vision, he simply did monstrously complicated calculations in his head and dictated the results to an aide. Many years after his death, the 19th-century French astronomer Dominique Arago said of him, "Euler calculated without apparent effort, as men breathe or as eagles sustain themselves in the wind." By the time he died, he had written more brilliant mathematics than had any other mathematician in history, and that claim remains true to this day.

Here's one of Euler's accomplishments. Some of the great problems of mathematics involve the prime numbers, which since Euclid's day (more than three centuries before Christ) have been known to be infinite in number. Euclid's proof of that is a gem, commonly taught in high school (see the box), and it wasn't until 1737 that Euler found a second, totally different (but equally beautiful) proof of the infinity of the primes that I'll show you later in this chapter. But what Euler wasn't able to prove (and nobody else since has either) is if the *twin primes* are infinite in number. With the lone exception of 2, all the primes are odd numbers, and two primes form a twin pair if they are consecutive odd numbers (3 and 5, or 17 and 19, for example). Mathematicians would be absolutely astounded if the twin primes are not infinite in number, but there is still no proof of that.[2]

2. This just goes to show that there will never be an end to wonderful math problems, because, if in the (most unlikely) event that the twin primes are someday shown to be finite in number, the hunt would then immediately begin for the largest pair! In 1919, the Norwegian mathematician Viggo Brun (1885–1978) showed that the sum of the reciprocals of the twin primes is finite:

$\left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\cdots \approx 1.90216\ldots$, a number called *Brun's constant*. This small value does not, however, prove that there are a finite number of twin primes, but only that they thin out pretty fast. In 2013 the Chinese-born American mathematician Yitang Zhang showed (when at the University of New Hampshire, just down the hall from my old office in Kingsbury Hall) that there is an infinity of pairs of primes such that each pair is separated by no more than 70 million. In 2014 that rather large gap was reduced to 246. If it could be reduced to 2 (or shown it *couldn't* be so reduced), then the twin prime problem would be resolved.

Euclid proved the primes are infinite in number by showing that a listing of any finite number n of primes must necessarily be incomplete, and so there must instead be an infinite number of primes. Here's how the logic goes. Let the n listed primes be labeled $p_1, p_2, p_3, \ldots, p_n$. Then, consider the number $N = p_1 p_2 p_3 \ldots p_n + 1$, which is obviously not equal to any of the primes on the list. Now, N is either prime or it isn't. If it is then we have directly found a prime not on the list. If, however, N is not a prime, that means it can be factored into a product of two (or more) primes. Equally obvious, however, is that p_1 doesn't divide N (because of that +1), and in fact none of the rest of the primes on the list divides N either, for the same reason. The immediate conclusion is that there must be at least *two* more primes that are not on the list. Since this argument holds for any listing of finite length, there must, in fact, be an infinite number of primes. Done! You're not going to find many proofs in math more elegantly concise than that.

The problem of determining the size of the set of the twin primes is an unsolved problem that definitely fits most (if not all, as perhaps some extra explanation would be required for a grammar school student[3]) of my selection criteria. So, why (you ask) doesn't the twin prime problem deserve to have the label of being the most puzzling math problem? Well, maybe it does, but I'm making a judgment call here, with the following reason for why I've come down on the side of zeta-3. The twin prime problem appears to stand mostly alone, with few peripheral connections to the rest of math and science. In contrast, the zeta-3 problem is at the center of all sorts of other problems. (You'll see some of them, starting in the next section when I'll *finally* tell you what the zeta-3 problem is!) It's this issue, of the

3. For example, to understand the nature of the primes, it is necessary to first study the so-called *unique factorization theorem*, which says that every integer can be factored into a product of primes in exactly one way. This is not terribly difficult to show, but it is a step beyond mere arithmetic.

relative connectedness to the rest of math, that makes our ignorance of the nature of zeta-3 the more exasperating (hence, the more puzzling and mysterious) in comparison to the problem of the infinity (or not) of the twin primes.

Challenge Problem 1.1.1: In a 1741 letter to a friend, Euler made the following claim:

$\frac{2^{\sqrt{-1}}+2^{-\sqrt{-1}}}{2} \approx \frac{10}{13}$, a claim that must have appeared to his friend to be like something he would have found in a book of magical incantations. Calculate each side of this "almost equality" out to several decimal places and so verify Euler's claim. Hint: You may find what today is called *Euler's identity* to be of great help: $e^{ix} = \cos(x) + i\sin(x)$, where $i = \sqrt{-1}$. You can find an entire book on this identity in my *Dr. Euler's Fabulous Formula* (Princeton University Press, 2017), but you do not have to read that book to do this problem. Simply notice that $2^i = e^{\ln(2^i)} = e^{i\ln(2)}$ (and similarly for 2^{-i}). Then apply Euler's identity.

1.2 The Harmonic Series and the Riemann Zeta Function

As Euler entered the second half of his third decade, he was known to his local contemporaries as a talented mathematician, but to become a famous mathematician, it was necessary (as it is today) to be the first to solve a *really hard* problem. There are always numerous such problems in mathematics, but in the 1730s, there was one that was particularly challenging, one that satisfies all of my selection criteria. This was the problem of summing the infinite series of the reciprocals of the squares of the positive integers. That is, the calculation of

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = ? \qquad (1.2.1)$$

It's important to understand what is really being asked for in (1.2.1). The numerical value of the sum is a calculation in arithmetic (but it's not a trivial one, if one wants, for example, the first 1,000 correct digits), and almost from the very day the problem was first posed, it was known that the value is about 1.6 or so. But that's not what mathematicians wanted. They wanted an exact *symbolic expression* involving integers (and roots of integers), simple functions (like the exponential, logarithmic, factorial, and trigonometric), and known constants like π and e. The simpler that expression, the better, and in fact, Euler found such an expression in 1734. A little later I'll show you his brilliant solution (and not to keep you in suspense, run $\pi^2/6$ through your calculator). For now, my central point is that, from 1734 on, Euler was a superstar in mathematics whose fame extended from one end of Europe to the other. The origin of the problem in (1.2.1) played a big, continuing role in both Euler's life and the zeta-3 problem (which I admit I still have yet to tell you about, but I will, soon!).

In the 14th century, a similar problem had bedeviled mathematicians: summing the infinite series of the reciprocals of the positive integers. That is, calculating

$$\sum_{k=1}^{\infty}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=? \tag{1.2.2}$$

Then, about 1350, the French mathematician and philosopher Nicole Oresme (c. 1320–1382) showed that the answer is infinity! That is, as mathematicians put it, the sum in (1.2.2) diverges. Oresme's claim, without exception, surprises (greatly!) students when they first are told this, because the individual terms continually get smaller and smaller (indeed, they are approaching zero). It just seems impossible that, eventually, if you keep adding these ever-decreasing terms, the so-called *partial sum* will exceed any value you wish. That is, no matter how large a number N that you name, there is a finite value for q such that

$$h(q)=\sum_{k=1}^{q}\frac{1}{k}>N. \tag{1.2.3}$$

The symbol h is used in (1.2.3) because the sum of the reciprocals of the positive integers is called the *harmonic series*. The $h(q)$ function will occur over and over in this book. Some of Euler's most beautiful discoveries after 1734 involve $h(q)$, and it continues to inspire researchers to this day.

Oresme's proof of (1.2.3) is an elegant example of the power of mathematical reasoning, even at the high school level. One simply makes clever use of brackets to group the terms as follows:

$$\sum_{k=1}^{\infty}\frac{1}{k}=1+\left\{\frac{1}{2}\right\}+\left\{\frac{1}{3}+\frac{1}{4}\right\}+\left\{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right\}+\cdots$$

followed by replacing each term in each pair of curly brackets with the last (*smallest*) term in that pair. Notice that this last term is always of the form $\frac{1}{2^m}$ where m is some integer ($m = 1$ in the first pair, $m = 2$ in the second pair, $m = 3$ in third pair, and so on), and that there are 2^{m-1} terms in a bracket pair. The process gives a *lower* bound on the sum, and so we have

$$\sum_{k=1}^{\infty}\frac{1}{k}>1+\left\{\frac{1}{2}\right\}+\left\{\frac{1}{4}+\frac{1}{4}\right\}+\left\{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right\}+\cdots=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots.$$

That is, the lower bound is 1 plus an infinity of $\frac{1}{2}$'s, which obviously gives a sum that "blows up" (diverges) to infinity, just as claimed in the Preface:

$$\lim_{q\to\infty} h(q)=\infty. \tag{1.2.4}$$

The explanation for (1.2.3) and (1.2.4) is that while it is clearly necessary for the terms in an infinite series in which every term is positive to continually decrease toward zero if the sum is to be finite (for the sum to *converge*, as mathematicians put it), a decrease alone is not a sufficient condition for a finite sum. Not only must the terms decrease toward zero, but that decrease has to be a sufficiently fast one. The terms of the harmonic series simply don't go to zero fast enough. *Almost* fast enough, to be sure, but not quite fast enough, which results in the divergence of the harmonic series being astonishingly slow. For $h(q) > 15$, for example, we must have $q > 1.6 \times 10^6$ terms, while $h(q) > 100$ requires $q > 1.5 \times 10^{43}$ terms.

Once Oresme had solved the problem of summing the harmonic series, the question of summing the reciprocals *squared* stepped forward, with its explicit statement attributed to the Italian Pietro Mengoli (1625–1686) in 1644. And once Euler had solved that problem in 1734, you can surely understand the curiosity that drove mathematicians to next turn their attention to summing the reciprocals *cubed*. To their dismay, they couldn't do it. Even Euler couldn't do it. And so, at last, we have the zeta-3 problem: What is

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots = ? \tag{1.2.5}$$

The numerical value is easily calculated to be 1.2020569 . . . but, unlike the sum of the reciprocals squared, there is no known simple symbolic expression. The search for such an expression is, today, an ongoing effort involving many of the best mathematicians in the world.

This search is not an idle one of mere curiosity, either, as the value of zeta-3 appears in physics (as you'll see later) as well as in mathematics.

The pressure on modern academics to solve problems is, as it was in Euler's day, enormous, and in fact, that pressure is relentless. That is, after solving a tough problem, the successful analyst certainly gets a pat on the back but then, almost immediately after, is asked "So, what are you going to do next?" Having a good answer to that question may be more a matter of professional pride for a tenured senior professor, but for a young untenured assistant professor, it is, quite literally, a matter of survival. The famous Hungarian mathematician Paul Erdös (1913–1996) wrote a little witticism that nicely sums up this situation:

> A theorem a day
> Means promotion and pay.
> A theorem a year
> And you're out on your ear!

To that, in the spirit of this book, I would add these two lines:

> But if your next theorem computes zeta-3
> Then acclaimed tenured full prof you'll instantly be!

Erdös, who received the 1983 Wolf Prize, never held an academic position, but instead endlessly traveled the world, living temporarily with mathematician friends, then moving on to his next stop. At each stay, he and his host would write a joint paper (his co-authors numbered in the hundreds): his motto was "Another roof, another proof."

The reason for the name *zeta* is that in 1737, Euler considered the general problem of summing the reciprocals of the *s*th power of the positive integers:

$$\sum_{k=1}^{\infty}\frac{1}{k^s}=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots, \tag{1.2.6}$$

which is today written (with the Greek letter zeta) as $\zeta(s)$, and so (1.2.1) is $\zeta(2)$ and (1.2.5) is $\zeta(3)$. That is, zeta(2) and zeta(3), pronounced "zeta-2" and "zeta-3." Euler took s to be a positive integer, subject only to the constraint that $s > 1$ to ensure convergence of (1.2.6) ($s = 1$ gives, of course, the divergent harmonic series). What makes the failure to solve the zeta-3 problem particularly puzzling is that not only did Euler solve the zeta-2 problem, but he also solved all of the zeta-$2n$ problems. That is, he found symbolic expressions for the sum of the reciprocals of *any even* power of the integers. The first few of these solutions are:

$$\text{Zeta-2} = \zeta(2) = \frac{\pi^2}{6},$$

$$\text{Zeta-4} = \zeta(4) = \frac{\pi^4}{90},$$

$$\text{Zeta-6} = \zeta(6) = \frac{\pi^6}{945},$$

$$\text{Zeta-8} = \zeta(8) = \frac{\pi^8}{9{,}450},$$

$$\text{Zeta-10} = \zeta(10) = \frac{\pi^{10}}{93{,}555}.$$

Starting with $\zeta(3)$, however, not even one of the $\zeta(2n + 1)$ problems has been solved. Lots of results that dance around $\zeta(2n + 1)$ have been found since Euler—in 1979, for example, the French mathematician Roger Apéry (1916–1994) showed that, whatever $\zeta(3)$ is, it is irrational (which confirmed what every mathematician since Euler has always believed, but having a proof is, of course, the Holy Grail of mathematics).[4] A simple symbolic expression for $\zeta(3)$ remains as elusive today as it was for Euler.

4. The irrationality of zeta-2 wasn't proven until 1796, decades after Euler calculated $\zeta(2)$, when the French mathematician Adrien-Marie Legendre (1752–1833) proved that π^2 is irrational (the Swiss mathematician Johann Lambert (1728–1777) proved that π is irrational in 1761, but that does not prove that π^2 is irrational). Can you think of an irrational number whose square is rational? This should take you, at most, two (big hint here) seconds!

In 1859 the German mathematician Bernhard Riemann (1826–1866) extended Euler's $\zeta(s)$ to complex values of s (which, of course, includes the integers as special cases). Today, $\zeta(s)$ is called the *Riemann zeta function*, although its origin is with Euler. For highly technical reasons, beyond the level of this book, there are many important problems in mathematics (including the theory of primes) that are connected to what are called the *zeros* of $\zeta(s)$. That is, to the solutions of the equation $\zeta(s) = 0$. All the even, negative integer values of s are zeros, but the situation for complex zeros is far from resolved. After calculating just the first three (!) complex zeros, Riemann conjectured, but was unable to prove (and nobody else since has, either), that all the infinite number of complex zeros of $\zeta(s)$ are "very likely" of the form $s = \frac{1}{2} + b\sqrt{-1}$ for an infinite number of values for $b > 0$. That is, what has become known as the *Riemann hypothesis* is that all of the complex zeros are on the vertical line (called the *critical line*) in the complex plane with its real part equal to $\frac{1}{2}$. (In Chapter 3, I'll tell you a lot more about the critical line.) In 1914 the English mathematician G. H. Hardy (1877–1947) proved that $\zeta(s)$ has an infinite number of complex zeros on the critical line, but that does not prove that *all* the complex zeros are there. In 1989 it was shown that at least two-fifths of the complex zeros are on the critical line. Again, that does not prove that all the complex zeros are there. In 2011, 22 years later, that 40% value was increased to 41.05%, a small increase for two decades of work that hints at just how difficult a challenge the Riemann hypothesis is. Using high-speed electronic computers, billions upon billions of the complex zeros have been calculated as the parameter b is steadily increased and, so far, every last one of them does indeed have a real part of exactly $\frac{1}{2}$. But that does not say anything about all of the complex zeros being on the critical line. If just *one* complex zero is ever found off the critical line, then the Riemann hypothesis will be instantly swept into the wastebasket of history (and the discoverer of that rogue zero will become an instant superstar in the world of mathematics).

Euler's results for $\zeta(2n)$ all have the form of

$$\zeta(2n)=\frac{a}{b}\pi^{2n},$$

where a and b are positive integers. That is, for k an even integer, $\zeta(k)$ is a rational number, times pi to the kth power. This suggests that, for some integers a and b,

$$\zeta(3)=\frac{a}{b}\pi^{3}$$

but that suggestion has not been realized (nobody has ever found integers a and b that give the known numerical value of $\zeta(3)$). In 1740 Euler conjectured that, instead,

$$\zeta(3) = N\pi^3$$

where N somehow involves ln(2), but that hasn't resulted in any progress, either.

Why ln(2)? Why not ln(17) or ln(3)? Perhaps because Euler had shown, before he solved the zeta-2 problem, that $\zeta(2)=\sum_{k=1}^{\infty}\frac{1}{k^2 2^{k-1}}+\{\ln(2)\}^2$. This was helpful in calculating the *numerical* value of $\zeta(2)$ because this sum converges much more rapidly than does the original sum in the definition of $\zeta(2)$, and he knew the value of ln(2) to many decimal places. The Russian mathematician Andrei Markov (1856–1922) did the same for $\zeta(3)$ when he showed, in 1890, that $\zeta(3)=\frac{5}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k^3\binom{2k}{k}}=\frac{5}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(k!)^2}{k^3(2k)!}$, where $\binom{n}{k}$ is the binomial coefficient $\frac{n!}{(n-k)!k!}$. I'll show you the details of how Euler derived his fast-converging series expression for $\zeta(2)$ (it's all just AP-calculus) in the next chapter. Markov's analysis is, as you might suspect, *just a bit* more advanced.

Another formula, one known to Euler (and which we'll derive later), is particularly tantalizing:

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{\pi^3}{32}.$$

Why the sum of the reciprocals of the *odd* integers cubed, with alternating signs, has such a nice form, a form that matches that of $\zeta(2n)$, while $\zeta(3)$ seems not to, is an exasperating puzzle for mathematicians. What worries many of today's mathematicians is that if *Euler*—a genius of the first rank (if not even higher)—couldn't solve for zeta-3, even after decades of trying, well, maybe there simply isn't an exact symbolic expression. What a dreary thought! Why would the world be made that way? It seems so . . . inelegant. And yet, such things *do* happen. The ancient geometric construction problems of angle trisection, cube doubling, and circle squaring, for example, all stumped mathematicians for thousands of years until all were eventually proven to have (using only a straightedge and a compass) no solutions (see Appendix 1 for one way to sidestep this perhaps shocking conclusion).

Challenge Problem 1.2.1: The older brother of Euler's mentor in Basel (Johann Bernoulli) was Jacob Bernoulli (1654–1705), also a talented mathematician. He was highly skilled in summing infinite series,[5] but the problem of $\zeta(2)$ utterly defeated him. When Johann learned of his former student's success, he wrote "If only my brother were alive!" Jacob did have his successes, however. For example, three interesting series he evaluated are:

5. Johann was fascinated by infinite series, too. The mysterious integral $\int_0^1 x^x dx$ was done by him in 1697, when he showed the answer is $1-\frac{1}{2^2}+\frac{1}{3^3}-\frac{1}{4^4}+\frac{1}{5^5}-\cdots=0.7834\ldots$. This result, called by Bernoulli his "series mirabili" ("marvelous series")—as well as the perhaps even more intimidating $\int_0^1 x^{x^2} dx=1-\frac{1}{3^2}+\frac{1}{5^3}-\frac{1}{7^4}+\frac{1}{9^5}-\cdots=0.8964\ldots$, or its "twin" $\int_0^1 x^{\sqrt{x}} dx=1-\left(\frac{2}{3}\right)^2+\left(\frac{2}{4}\right)^3-\left(\frac{2}{5}\right)^4+\left(\frac{2}{6}\right)^5-\cdots=0.6585\ldots$ —is derived in my *Inside Interesting Integrals* (2nd edition, Springer 2020, pp. 227–229).

$$\sum_{k=1}^{\infty}\frac{k}{2^k}=2,\quad \sum_{k=1}^{\infty}\frac{k^2}{2^k}=6,\quad \sum_{k=1}^{\infty}\frac{k^3}{2^k}=26.$$

Can you discover a general way to sum series like these? If so, confirm Jacob's results, and then do the next obvious sum: $\sum_{k=1}^{\infty}\frac{k^4}{2^k}=?$ Hint: Try differentiating a certain geometric series.

1.3 Euler's Constant, the Zeta Function, and Primes

In 1731 Euler made a curious observation. Writing the harmonic series, (1.2.3), we have

$$h(q)=\sum_{k=1}^{q}\frac{1}{k}=\sum_{k=1}^{q-1}\frac{1}{k}+\frac{1}{q}=h(q-1)+\frac{1}{q}. \tag{1.3.1}$$

If $h(q)$ were a *continuous* function of q (which it isn't, but just suppose), then we could write

$$\frac{dh}{dq}=\lim_{\Delta q\to 0}\frac{h(q)-h(q-\Delta q)}{\Delta q}.$$

Of course, we are stuck with $\Delta q = 1$, but suppose we ignore that and, using (1.3.1), we write

$$h(q)-h(q-1)=\frac{1}{q}$$

and so

$$\frac{dh}{dq}\approx\frac{1}{q} \tag{1.3.2}$$

and argue that (1.3.2) gets "better and better" as $q \to \infty$.

This is all very casual, of course, but in fact it is fairly typical of how Euler found inspiration. We then integrate (1.3.2) indefinitely to get

$$h(q) = \ln(q) + C$$

where C is the constant of indefinite integration. Well, you ask, what *is* C? For Euler,

$$C = \lim_{q\to\infty}\{h(q) - \ln(q)\}. \tag{1.3.3}$$

Now, as Oresme showed, $h(q)$ blows up in (1.2.4) as $q \to \infty$, but so does $\ln(q)$, and so (perhaps pondered Euler) might their difference approach a finite limit? This is what in fact happens, and C (now usually written as the Greek gamma, γ) has become famous in mathematics as *Euler's constant*. After π and e, γ is perhaps the most important constant in mathematics. We can get an idea of the value of γ by simply plotting (1.3.3), and this is done in the semi-log plot of Figure 1.3.1 as q varies from 1 to 10,000. The plot is certainly not a proof that there is such a limit (maybe for physicists or engineers,

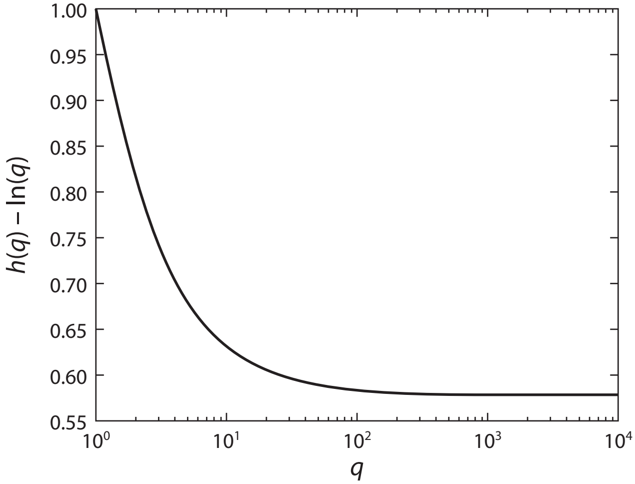

FIGURE 1.3.1.

Computer determination of Euler's constant as a limit.

but not for rigorous mathematicians), but it does strongly suggest that $\gamma \approx 0.57$.

To get our hands on the actual value of γ, we need an analytical expression, and here's one possible way to do that, using the power series expansion for $\ln(1 + x)$, where x is in the interval -1 to 1. This expression was derived by the Danish mathematician Nikolaus Mercator (1620–1687) in 1668, and we'll do it here as follows. We start by observing that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots, \tag{1.3.4}$$

which you can confirm by either doing the long division or by doing the multiplication $(1 + x)(1 - x + x^2 - x^3 + x^4 - \ldots)$ and seeing that all the terms cancel except for the leading 1. Then, integrating, (1.3.4) term-by-term yields

$$\int \frac{1}{1+x}\,dx = \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots + K$$

where K is the constant of indefinite integration. Setting $x = 0$ gives $\ln(1) = 0 = K$ and so

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots$$

or, rearranging,

$$x = \ln(1+x) + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{5}x^5 + \cdots. \tag{1.3.5}$$

Now, successively substitute the values of $x = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{q}$ into (1.3.5), which gives the following sequence of expressions:

$$1 = \ln(2) + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$$

$$\frac{1}{2}=\ln\left(\frac{3}{2}\right)+\frac{1}{2}\left(\frac{1}{2^2}\right)-\frac{1}{3}\left(\frac{1}{2^3}\right)+\frac{1}{4}\left(\frac{1}{2^4}\right)-\frac{1}{5}\left(\frac{1}{2^5}\right)+\cdots$$
$$\frac{1}{3}=\ln\left(\frac{4}{3}\right)+\frac{1}{2}\left(\frac{1}{3^2}\right)-\frac{1}{3}\left(\frac{1}{3^3}\right)+\frac{1}{4}\left(\frac{1}{3^4}\right)-\frac{1}{5}\left(\frac{1}{3^5}\right)+\cdots$$
$$\frac{1}{4}=\ln\left(\frac{5}{4}\right)+\frac{1}{2}\left(\frac{1}{4^2}\right)-\frac{1}{3}\left(\frac{1}{4^3}\right)+\frac{1}{4}\left(\frac{1}{4^4}\right)-\frac{1}{5}\left(\frac{1}{4^5}\right)+\cdots$$
$$\cdots$$
$$\frac{1}{q}=\ln\left(\frac{q+1}{q}\right)+\frac{1}{2}\left(\frac{1}{q^2}\right)-\frac{1}{3}\left(\frac{1}{q^3}\right)+\frac{1}{4}\left(\frac{1}{q^4}\right)-\frac{1}{5}\left(\frac{1}{q^5}\right)+\cdots.$$

What do we do with all of these expressions? There are a lot of things we *could* do, but let's do the simplest thing and simply add them. On the left, we immediately get

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{q}=h(q).$$

Next, we'll add the right-hand sides of the expressions in two steps. First, adding all the logarithmic terms gives us

$$\ln(2)+\ln\left(\frac{3}{2}\right)+\ln\left(\frac{4}{3}\right)+\ln\left(\frac{5}{4}\right)+\cdots+\ln\left(\frac{q+1}{q}\right)$$
$$=\ln(2)+\{\ln(3)-\ln(2)\}+\{\ln(4)-\ln(3)\}$$
$$+\{\ln(5)-\ln(4)\}+\cdots+\{\ln(q+1)-\ln(q)\}=\ln(q+1),$$

because all the terms but the penultimate one cancel (as mathematicians put it, the series *telescopes*). Next, adding the rest of the terms on the right-hand sides of the sequences of expressions together in the highly suggestive way they present themselves (in columns), we see that

$$h(q)-\ln(q+1)=\frac{1}{2}\left\{1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\frac{1}{q^2}\right\}-\frac{1}{3}\left\{1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots+\frac{1}{q^3}\right\}$$
$$+\frac{1}{4}\left\{1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots+\frac{1}{q^4}\right\}-\frac{1}{5}\left\{1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\cdots+\frac{1}{q^5}\right\}+\cdots$$

or, from (1.3.3)—with its $\ln(q)$ term replaced with $\ln(q + 1)$, which hardly matters, since we are about to let $q \to \infty$ anyway—we have (writing γ now, instead of C)

$$\gamma = \lim_{q\to\infty} \{h(q) - \ln(q+1)\} = \sum_{s=2}^{\infty} \frac{(-1)^s}{s} \left\{ \sum_{q=1}^{\infty} \frac{1}{q^s} \right\}$$

or, amazingly and seemingly out of nowhere,

$$\gamma = \sum_{s=2}^{\infty} \frac{(-1)^s}{s} \zeta(s). \tag{1.3.6}$$

The intimate connection between Euler's constant and the zeta function is on full display in (1.3.6) but, alas, the sum doesn't converge very rapidly.[6] Still, using just the first 10 terms gives $\gamma \approx 0.5338$ (using the first 100 terms gives $\gamma \approx 0.5723$), which is consistent with Figure 1.3.1. (The actual value is $\gamma = 0.5772156649\ldots$.) There is still a lot of mystery to γ. Euler was able to correctly calculate the first few digits (an impressive feat, in its own right), and electronic computers have extended that out to millions of digits. Despite all that, however, it is still not known if γ is rational or not, although every mathematician in the Solar System would be astonished if it turned out to be rational. If it is rational, then it is known that the denominator integer b in $\gamma = a/b$ would have to have hundreds of thousands of digits! As a practical matter, the fraction 228/395 correctly gives the first six digits, which is almost certainly (as engineers like to put it) "good enough for government work."

6. The ultimate convergence of (1.3.6) is, however, guaranteed by the following beautiful little theorem from first-year calculus: an *alternating* series in which the successive terms continually decrease in magnitude toward zero always converges. The issue of the rapidity of the decrease no longer appears. Since (1.2.6) tells us that $\lim_{s\to\infty} \zeta(s) = 1$, then from (1.3.6), we see that $\frac{\zeta(s)}{s} \to 0$ as $s \to \infty$, and so the theorem's requirements are satisfied. At the end of this chapter, I'll show you a generalization of (1.3.6) that converges *much* faster.

Now, for just a moment, let me indulge in a little aside. If you plug $x = 1$ into the power series expansion for $\ln(1 + x)$ you get

$$\ln(2)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots.$$

That is, if we write the harmonic series with alternating signs as on the right-hand side, the sum now converges, just as footnote 6 claims. There is, however, a subtle, perplexing issue with the convergence of the series: if we sum the terms in a different order, we'll get a different sum. For example, suppose we start with the 1, then add the next two negative terms, then the first skipped positive term, then the next two negative terms, then the next skipped positive term, and so on. Thus, we add the same terms that appear in the ln(2) expression, but now in the following order:

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\frac{1}{7}-\frac{1}{14}-\frac{1}{16}+\frac{1}{9}-\cdots.$$

If we group these terms as

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\left(\frac{1}{7}-\frac{1}{14}\right)-\frac{1}{16}+\cdots$$

that gives

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\cdots=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots\right)$$

$$=\frac{1}{2}\ln(2)\neq\ln(2).$$

In 1837 the German mathematician Gustav Dirichlet (1805–1859) proved that for any rearrangement of a series to always converge to the same value, the series must be what mathematicians call *absolutely convergent*. That is, the series must converge even if all its terms are taken as positive (and, as Oresme showed, that is *not* the case for the harmonic series, and that's why we see this curious behavior). In 1854, Riemann observed that the terms of the harmonic series with alternating signs can always be rearranged to converge to any value, positive or negative, that you wish! (For a sketch on how to prove that, see the solution to Challenge Problem 1.3.1.)

In 1737 Euler did something with $\zeta(s)$ that, in some ways, might be even more astounding than is (1.3.6). What he did was show that there is an intimate connection between $\zeta(s)$, a continuous function of s, and the primes (which as integers are the very signature of discontinuity). To start, multiply through (1.2.6), the definition of $\zeta(s)$, by $\frac{1}{2^s}$ to get

$$\frac{1}{2^s}\zeta(s)=\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{10^s}+\frac{1}{12^s}+\cdots, \tag{1.3.7}$$

then subtract (1.3.7) from (1.2.6) to arrive at

$$\zeta(s)-\frac{1}{2^s}\zeta(s)=\left(1-\frac{1}{2^s}\right)\zeta(s)=1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}+\frac{1}{20^s}+\cdots. \tag{1.3.8}$$

Now, multiply (1.3.8) by $\frac{1}{3^s}$ to get

$$\left(1-\frac{1}{2^s}\right)\frac{1}{3^s}\zeta(s)=\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\frac{1}{21^s}+\cdots \tag{1.3.9}$$

and so, if we subtract (1.3.9) from (1.3.8), we have

$$\left(1-\frac{1}{2^s}\right)\zeta(s)-\left(1-\frac{1}{2^s}\right)\frac{1}{3^s}\zeta(s)=\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\zeta(s)=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\cdots. \tag{1.3.10}$$

Next, multiply (1.3.10) by $\frac{1}{5^s}$ to get . . . and on and on we go, and I'm sure you see the pattern. As we repeat this process over and over, multiplying through our last result by $1/p^s$, where p denotes successive primes, we relentlessly subtract out all the multiples of the primes. You may recognize what we're doing here as essentially executing the famous method called *Eratosthenes' sieve*, developed by the third century BC Greek mathematician Eratosthenes of Cyrene as the fundamental algorithmic procedure for finding all of the primes in the first place.

If we imagine doing this multiply-and-subtract process for all primes, then when we are done (after an infinity of such operations), we will have removed every term but the leading 1 on the right-hand side of (1.3.10) exactly once because of the unique factorization theorem (see note 3). Thus, using Π to denote a product, Euler arrived at

$$\left\{\prod_{p\ prime}\left(1-\frac{1}{p^s}\right)\right\}\zeta(s)=1$$

or, as it is more commonly written,

$$\zeta(s)=\prod\nolimits_{p\ prime}\left(1-\frac{1}{p^s}\right)^{-1}, \tag{1.3.11}$$

which is called the Eulerian product form of the zeta function.

In addition to simply being a beautiful expression as it stands, there are two astonishing implications hidden in (1.3.11). One was already known (the infinity of the primes), while the other was new and totally unexpected. To see how Euler had found a new proof for the infinity of the primes, simply notice that if we set $s = 1$, then $\zeta(1)$ is the divergent harmonic series. That is,

$$\prod\nolimits_{p\ prime}\left(1-\frac{1}{p}\right)^{-1}=\infty. \tag{1.3.12}$$

Now, since $p \geq 2$, then every factor in the product is greater than 1, and so the product increases with each additional factor. To increase without bound (that is, to diverge), however, requires that there be an infinite number of factors (that is, an infinity of primes).

This is nice (it's always good to have multiple proofs of a theorem), but it really can't compare with the second, new result Euler extracted from (1.3.11): The sum of the reciprocals of *just the primes, alone*, diverges! It was, after all, a huge surprise when it was realized that the harmonic series, the sum of the reciprocals of all the positive integers, diverges, but to still have divergence even when just the primes are used seems completely and totally unbelievable. Here's how Euler showed, despite that skepticism, that we nevertheless have to believe it. Taking the natural logarithm of (1.3.12), we have (because the log of a product is the sum of the logs, and because $\ln(\infty) = \infty$)

$$-\sum_{p\ prime} \ln\left(1-\frac{1}{p}\right)=\infty. \tag{1.3.13}$$

Next, looking back at (1.3.5), if we set $x = -\frac{1}{p}$ then

$$\ln\left(1-\frac{1}{p}\right)=-\frac{1}{p}-\frac{1}{2}\left(\frac{1}{p^2}\right)-\frac{1}{3}\left(\frac{1}{p^3}\right)-\frac{1}{4}\left(\frac{1}{p^4}\right)-\frac{1}{5}\left(\frac{1}{p^5}\right)-\cdots \tag{1.3.14}$$

and so (1.3.13) becomes

$$\sum_{p\ prime}\left\{\frac{1}{p}+\frac{1}{2}\left(\frac{1}{p^2}\right)+\frac{1}{3}\left(\frac{1}{p^3}\right)+\frac{1}{4}\left(\frac{1}{p^4}\right)+\frac{1}{5}\left(\frac{1}{p^5}\right)+\cdots\right\}=\infty$$

or

$$\sum_{p\ prime}\frac{1}{p}+\sum_{p\ prime}\left\{\frac{1}{2}\left(\frac{1}{p^2}\right)+\frac{1}{3}\left(\frac{1}{p^3}\right)+\frac{1}{4}\left(\frac{1}{p^4}\right)+\frac{1}{5}\left(\frac{1}{p^5}\right)+\cdots\right\}=\infty. \tag{1.3.15}$$

In the second sum on the left, replace every term with a larger one and, in addition, include terms for *every* p (not just for p a prime). Then it is certainly true that

$$\sum_{p\ prime}\left\{\frac{1}{2}\left(\frac{1}{p^2}\right)+\frac{1}{3}\left(\frac{1}{p^3}\right)+\frac{1}{4}\left(\frac{1}{p^4}\right)+\frac{1}{5}\left(\frac{1}{p^5}\right)+\cdots\right\}$$
$$<\sum_{p=2}^{\infty}\left\{\left(\frac{1}{p^2}\right)+\left(\frac{1}{p^3}\right)+\left(\frac{1}{p^4}\right)+\left(\frac{1}{p^5}\right)+\cdots\right\}.$$

The expression in the curly brackets on the right is a geometric series, easily summed to give

$$\sum_{p\ prime}\left\{\frac{1}{2}\left(\frac{1}{p^2}\right)+\frac{1}{3}\left(\frac{1}{p^3}\right)+\frac{1}{4}\left(\frac{1}{p^4}\right)+\frac{1}{5}\left(\frac{1}{p^5}\right)+\cdots\right\}<\sum_{p=2}^{\infty}\frac{1}{p(p-1)}.$$

The sum on the right is easily evaluated, because it telescopes as

$$\sum_{p=2}^{\infty}\frac{1}{p(p-1)}=\sum_{p=2}^{\infty}\left\{\frac{1}{p-1}-\frac{1}{p}\right\}=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)$$
$$-\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)=1$$

and so (1.3.15) becomes

$$\sum_{p\ prime}\frac{1}{p}+(\text{something less than }1)=\infty$$

or, just like that,

$$\sum_{p\ prime}\frac{1}{p}=\infty$$

and so the sum of the reciprocals of nothing but the primes, *alone*, diverges.

That is hard to believe, without a doubt, but it's true. As you won't be surprised to learn, the divergence is excruciatingly slow. We know from (1.3.3) that the harmonic series diverges logarithmically: that is, for large q, $h(q) \approx \ln(q)$, where we ignore the "correction" term of γ, which becomes ever-less significant as q increases. The log function is a slowly increasing function of its argument, and so the obvious question now is: What grows even more slowly than the log? I won't prove it here, but an answer is the *iterated-log*, that is, the log of a log. The divergence of the sum of the reciprocals of the primes is as $\ln\{\ln(q)\}$. How good is this estimate? By actual calculation, when the reciprocals of all the primes in the first 1 million integers are added, the result is slightly less than 2.9. The iterated-log estimate gives us

$$\ln\{\ln(10^6)\} = \ln\{6\ln(10)\} = \ln(13.815) = 2.6,$$

which is, in fact, actually not that far off the mark.

Challenge Problem 1.3.1: Write the harmonic series with alternating signs as, first, the sum of all the positive terms, added to the sum of all the negative terms. That is, as

$$\left\{1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\right\}+\left[-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}+\cdots\right]=A+B.$$

Explain why $A=1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots$ diverges to plus infinity, while $B=-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}+\cdots$ diverges to minus infinity. (Hint: With what you've read in the text, not much more actual math is needed to explain either of these divergences.) Can you use these two conclusions to justify Riemann's observation that there is always some rearrangement of the terms in A and B that will result in the convergence of $A + B$ to *any* value, negative or positive, that you wish?

1.4 Euler's Gamma Function, the Reflection Formula, and the Zeta Function

As 1729 turned to 1730, Euler started the development of what we today call the *gamma function*.[7] This involves a study of the integral

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx,\ n>0, \tag{1.4.1}$$

which has the wonderful property of extending the idea of the factorial function from just the non-negative integers to all real numbers. Here's how that works.

For $n = 1$, it is easy to calculate

$$\Gamma(1)=\int_0^\infty e^{-x}dx=\{-e^{-x}\}\big|_0^\infty=1\,. \tag{1.4.2}$$

If you integrate by parts, (1.4.1) quickly becomes[8]

$$\Gamma(n+1) = n\Gamma(n) \tag{1.4.3}$$

and so, for n a positive integer, we immediately see the connection between $\Gamma(n)$ and the factorial function:

$$\Gamma(2) = 1\Gamma(1) = 1(1) = 1!$$

$$\Gamma(3) = 2\Gamma(2) = 2(1!) = 2!$$

$$\Gamma(4) = 3\Gamma(3) = 3(2!) = 3!$$

and so on, all the way to

$$\Gamma(n) = (n-1)(n-2)! = (n-1)! \tag{1.4.4}$$

7. For an erudite presentation, see Philip J. Davis, "Leonhard Euler's Integral: A Historical Profile of the Gamma Function," *American Mathematical Monthly*, December 1959, pp. 849–869.

8. In $\int_0^\infty u\,dv=\{uv\}\big|_0^\infty-\int_0^\infty v\,du$, let $u = e^{-x}$ and $dv = x^{n-1}\,dx$. Expression (1.4.3) is called the *functional equation* of the gamma function.

Notice, in particular, that setting $n = 1$ in (1.4.4) results in

$$\Gamma(1) = 0!$$

but (1.4.2) then tells us that

$$0! = 1$$

not 0, as most students initially think.[9] To be really emphatic about this,

$$0! \neq 0 \; (!!!!!)$$

In the integral definition of $\Gamma(n)$, in (1.4.1), n does not have to be a positive integer. Indeed, it was the question of how to interpolate the factorial function (for example, $(\frac{1}{2})! = ?$) that motivated Euler to develop the integral definition in the first place. We can also use (1.4.4) to extend n to all real n, including negative values, and so give meaning to objects as strange looking as $(-\frac{1}{2})!$ probably strikes you. Here's how to do that.

First, a specific example. Setting $n = \frac{1}{2}$ in both (1.4.1) and (1.4.4), we have

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = \left(-\frac{1}{2}\right)! \tag{1.4.5}$$

Next, change variable to $x = t^2$, and so $dx = 2t\,dt$. Thus,

$$\left(-\frac{1}{2}\right)! = \int_0^\infty \frac{e^{-t^2}}{t} 2t\,dt = 2\int_0^\infty e^{-t^2}\,dt = 2\left\{\frac{1}{2}\sqrt{\pi}\right\}$$

9. A more direct way to arrive at this result is to write $n! = n(n-1)!$ and then set $n = 1$. Thus, $1! = 1(0!) = 0!$ Since $1! = 1$ we have, again, $0! = 1$.

and so

$$\Gamma\left(\frac{1}{2}\right)=\left(-\frac{1}{2}\right)!=\sqrt{\pi}. \tag{1.4.6}$$

Who would have believed that $(-\frac{1}{2})!$ could mean *anything*, before Euler came along? Until Euler, nobody had even wondered about such a weird thing.[10] (That last integral, $\int_0^\infty e^{-t^2}\,dt$, of course, needs some explaining; see Appendix 2 for a derivation.)

Now, let's be more general. If, in (1.4.4), we replace n with $1 - n$ on both sides of the expression, we obtain the interesting result

$$\Gamma(1-n) = (1-n-1)! = (-n)! \tag{1.4.7}$$

Since from (1.4.4) we have

$$n\Gamma(n) = n(n-1)! = n!$$

then

$$n\Gamma(n)\Gamma(1-n) = (n!)(-n)! \tag{1.4.8}$$

That is, if we could evaluate the left-hand side of (1.4.8), we would then have a way to calculate $(-n)!$ from the value of $n!$, for any $n \geq 0$. An evaluation of $n\Gamma(n)\Gamma(1-n)$ can, in fact, be done by working directly with the integral definition of the gamma function, (1.4.1), but that approach has (for us, in this book) the drawback of using some mathematics that is just beyond AP-calculus.[11] What I'll

10. See if you can calculate $(\frac{1}{2})!$ right now. I'll ask you to think about this again a little later in this section and, yet again, at the end of this section as a challenge question.

11. See my *An Imaginary Tale: The Story of* $\sqrt{-1}$ (Princeton University Press, 2016), pp. 182–184. That discussion concludes with an evaluation of the integral $\int_0^\infty \frac{s^{\alpha-1}}{1+s^\beta}\,ds$, which is done using complex function theory (contour integration), a topic developed in that book on its pp. 187–226.

show you next, instead, is a way to calculate $(n!)(-n)!$ which neatly avoids that problem.

To start, let's develop another of Euler's beautiful discoveries, one that we'll need in just a bit. This is his formulation of the sine function as an infinite product:

$$\sin(y) = y\prod_{n=1}^{\infty}\left(1-\frac{y^2}{n^2\pi^2}\right). \tag{1.4.9}$$

This famous expression is normally established with some pretty sophisticated mathematics, but I'll limit my comments here to a series of plausible assertions (but I think you'll find them pretty convincing). If we write $\sin(y)$ as a power series, that is, as

$$\sin(y) = y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \cdots,$$

or, dividing through by y,

$$\frac{\sin(y)}{y} = 1 - \frac{1}{3!}y^2 + \frac{1}{5!}y^4 - \cdots, \tag{1.4.10}$$

then it doesn't seem unreasonable to say (as, in fact, did Euler) that $\frac{\sin(y)}{y}$ is a *polynomial of infinite degree*. Notice, in particular, that y appears in (1.4.10) raised only to ever-increasing even powers.

Now, fall back on your algebraic experience with polynomials of finite degree. If somebody told you she was thinking of a polynomial $P(y)$ of degree s, with non-zero roots $r_1, r_2, \ldots, r_s$, then to within a scale factor of A you'd write that polynomial as the product

$$P(y) = A(y - r_1)(y - r_2)\cdots(y - r_s).$$

If we write each factor as $y - r_k = -(r_k - y)$ and absorb the s minus signs into A, then

$$P(y) = A(r_1 - y)(r_2 - y)\cdots(r_s - y)$$

or, factoring out $r_1, r_2, \ldots, r_s$,

$$P(y) = A\left[r_1\left(1-\frac{y}{r_1}\right)r_2\left(1-\frac{y}{r_2}\right)\cdots r_s\left(1-\frac{y}{r_s}\right)\right].$$

Again, if we absorb the product $r_1 r_2 \ldots r_s$ into A,

(1.4.11) $$P(y) = A\left[\left(1-\frac{y}{r_1}\right)\left(1-\frac{y}{r_2}\right)\cdots\left(1-\frac{y}{r_s}\right)\right].$$

We know the roots of $\sin(y) = 0$ are y equal to any integer multiple of π. That is, the roots are $y = 0, \pm\pi, \pm 2\pi, \pm 3\pi$, and so on, or equivalently, $y^2 = 0, \pi^2, 2^2\pi^2, 3^2\pi^2$, and so on. The situation for $\frac{\sin(y)}{y}$ is the same, with the exception that $y = 0$ is *not* a root of $\frac{\sin(y)}{y} = 0$, because $\lim_{y\to 0}\frac{\sin(y)}{y} = 1$ by L'Hôpital's rule. So, using (1.4.11) as a guide, it seems reasonable to jump from the finite to the infinite (but, to be honest, this is not always legitimate!) and write

(1.4.12) $$\frac{\sin(y)}{y} = A\left[\left(1-\frac{y^2}{\pi^2}\right)\left(1-\frac{y^2}{2^2\pi^2}\right)\left(1-\frac{y^2}{3^2\pi^2}\right)\cdots\right].$$

Notice that (1.4.12) has y raised only to *even* powers of y, just as in (1.4.10). Since the left-hand side of (1.4.12) is 1 at $y = 0$, and the right-hand side is A, we have

$$\frac{\sin(y)}{y} = \prod_{n=1}^{\infty}\left(1-\frac{y^2}{n^2\pi^2}\right)$$

or

$$\sin(y) = y\prod_{n=1}^{\infty}\left(1-\frac{y^2}{n^2\pi^2}\right),$$

which is (1.4.9). (See the following box for how the value for $\zeta(2)$ follows almost immediately from (1.4.9).)

From (1.4.9), and writing $\sin(y)$ as a power series, we have

$$y-\frac{1}{3!}y^3+\frac{1}{5!}y^5-\cdots=y\left(1-\frac{y^2}{\pi^2}\right)\left(1-\frac{y^2}{2^2\pi^2}\right)\left(1-\frac{y^2}{3^2\pi^2}\right)\cdots$$

$$=y-\left(\frac{1}{\pi^2}+\frac{1}{2^2\pi^2}+\frac{1}{3^2\pi^2}+\cdots\right)y^3+\text{higher-order terms.}$$

Equating the coefficients of the y^3 term on each side,

$$-\frac{1}{3!}=-\left(\frac{1}{\pi^2}+\frac{1}{2^2\pi^2}+\frac{1}{3^2\pi^2}+\cdots\right)$$

or,

$$\frac{\pi^2}{3!}=\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\zeta(2).$$

Done!

Okay, back to calculating $(n!)(-n)!$

Let's initially assume n is a positive integer, and so

$$n! = 1\cdot 2\cdot 3\cdots(n-1)n \tag{1.4.13}$$

and then we'll manipulate (1.4.13)—using a method due to the German mathematician Karl Weierstrass (1815–1897)—until we get an expression that makes sense even when n is not an integer, or even positive. So, suppose α is another integer whose particular value doesn't matter, because we are going to be taking the limit $\alpha\to\infty$. Multiplying (1.4.13) by 1, we have

(continued...)

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