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1

The Study and Status of Mathematics

In this chapter, we look at some of the earliest references to the study and status of mathematics in ancient Greek literature. Plato is one of our earliest authorities, and in the *Republic* he proposes a division of mathematics into four subjects: arithmetic, geometry, astronomy, and music. This grouping of subjects later became known as the *quadrivium*, a term first coined by Boethius, and continued into the European Middle Ages as the basis of an exemplary education. We also include selections from Aristotle, demonstrating his own knowledge of mathematics. Several of these reveal Aristotle's desire that mathematics be based on firm principles of logical reasoning. We also include a fascinating excerpt from his *Physics* in which he discusses the idea of infinity. Furthermore, we incorporate several passages that reveal various views on the status of mathematics. Archytas, associate of Plato, links mathematics to the advancement of justice in society; Philo and Nicomachus, invoking interesting analogies, explore its connection with philosophical inquiry and wisdom. Proclus, expressing both his own views and that of earlier astronomer Geminus, considers the foundational principles of mathematics along with its applications and classification. Mathematics also featured in other genres such as the comic plays of Aristophanes. Through such passages we can get insight into the broader public perceptions of mathematics in Athens.

1.1 Plato, *Republic*

Plato (429–347 BCE) was an extraordinarily original and influential philosopher. He was born into an aristocratic family in Athens but was eventually repelled by both the oligarchs and the democrats of Athens. As a result, he withdrew his active participation in politics and instead devoted himself to philosophy. In about 387, he began to gather around himself numerous scholars to discuss diverse topics of interest. Plato's discussions took place in a property he had inherited, named after the legendary Athenian hero Akademos; and after Plato's death, thinkers continued to gather in the gardens, which, as the Academy, became the intellectual center of Greek life and lasted in some form for about three hundred years. Plato is generally considered the father of Western philosophy; in fact, according to Alfred North Whitehead, "All of Western philosophy is but a footnote to Plato." Although not a mathematician, Plato was evidently well versed in mathematics and invited a steady stream of Greek mathematicians to study and teach with him at the Academy. His works were written in the form of dialogues, often with Socrates, a real historical figure and close associate of Plato, being the principal interlocutor. Since the real Socrates died in 399, however, it is believed that most of the ideas discussed in the dialogues come from Plato himself.

In the *Republic*, Plato's best-known dialogue, Socrates discusses the meaning of justice and, in particular, how to create a just city-state. This state would be ruled by philosopher-kings who had studied for many years before being able to rule. Mathematics featured prominently in his proposed curriculum; in the selection from book 7 below, Socrates, in a discussion with Glaucon, elaborates the mathematics that the would-be philosopher-kings must study. The four subjects mentioned—arithmetic, geometry (plane and solid), astronomy, and music—later became known as the *quadrivium*, a major part of the medieval European liberal arts curriculum.

Socrates: If simple unity could be adequately perceived by the sight or by any other sense, then, . . . there would be nothng to attract towards being; but when there is some contradiction always present, and one is the reverse of one and involves the conception of plurality, then thought begins to be aroused within us, and the soul perplexed and wanting to arrive at a decision asks, "What is absolute unity?" This is the way in which the study of the one has a power of drawing and converting the mind to the contemplation of true being.

Glaucon: And surely, this occurs notably in the case of one, for we see the same thing to be both one and infinite in multitude?

Socrates: Yes, and this being true of one must be equally true of all number?

Glaucon: Yes.

Socrates: And they appear to lead the mind towards truth?

Glaucon: Yes, in a very remarkable manner.

Socrates: Then this is a knowledge of the kind for which we are seeking, having a double use, military and philosophical; for the man of war must learn the art of number or he will not know how to array his troops, and the philosopher also, because he has to rise out of the sea of change and lay hold of true being, and therefore he must be an arithmetician.

Glaucon: That is true.

Socrates: And our guardian is both warrior and philosopher?

Glaucon: Certainly.

Socrates: Then this is a kind of knowledge which legislation may fitly prescribe; and we must endeavor to persuade those who are to be the principal men of our State to go and learn arithmetic, not as amateurs, but they must carry on the study until they see the nature of numbers with the mind only; nor again, like merchants or retail-traders, with a view to buying or selling, but for the sake of their military use, and of the soul herself; and because this will be the easiest way for her to pass from becoming to truth and being.

Glaucon: That is excellent.

Socrates: Yes, and now having spoken of it, I must add how charming the science is and in how many ways it conduces to our desired end, if pursued in the spirit of a philosopher, and not of a shopkeeper!

Glaucon: How do you mean?

Socrates: I mean that arithmetic has a very great and elevating effect, compelling the soul to reason about abstract number, and rebelling against the introduction of visible or tangible objects into the argument. You know how steadily the masters of the art repel and ridicule anyone who attempts to divide absolute unity when he is calculating, and if you divide, they multiply, taking care that one shall continue as one and not become lost in fractions.

Glaucon: That is very true.

Socrates: Now, suppose a person were to say to them: O my friends, what are these wonderful numbers about which you are reasoning, in which, as you say, there is a unity such as you demand, and each unit is equal, invariable, indivisible—what would they answer?

Glaucon: They would answer that they were speaking of those numbers which can only be realized in thought.

Socrates: Then you see that this knowledge may be truly called necessary, necessitating as it clearly does the use of the pure intelligence in the attainment of pure truth?

Glaucon: Yes, that is a marked characteristic of it.

Socrates: And have you further observed, that those who have a natural talent for calculation are generally quick at every other kind of knowledge; and even the dull, if they have had an arithmetical training, although they may derive no other advantage from it, always become much quicker than they would otherwise have been?

Glaucon: Very true.

Socrates: And indeed, you will not easily find a more difficult study, and not many as difficult.

Glaucon: You will not.

Socrates: And, for all these reasons, arithmetic is a kind of knowledge in which the best natures should be trained, and which must not be given up.

Glaucon: I agree.

Socrates: Let this then be made one of our subjects of education. And next, shall we inquire whether the kindred science also concerns us?

Glaucon: You mean geometry?

Socrates: Exactly so.

Glaucon: Clearly we are concerned with that part of geometry which relates to war; for in pitching a camp, or taking up a position, or closing or extending the lines of an army, or any other military maneuver, whether in actual battle or on a march, it will make all the difference whether a general is or is not a geometrician.

Socrates: Yes, but for that purpose a very little of either geometry or calculation will be enough; the question relates rather to the greater and more advanced part of geometry—whether that tends in any degree to make more easy the vision of the idea of good; and thither, as I was saying, all things tend which compel the soul to turn her gaze towards that place, where is the full perfection of being, which she ought, by all means, to behold.

Glaucon: True.

Socrates: Then if geometry compels us to view being, it concerns us; if becoming only, it does not concern us?

Glaucon: Yes, that is what we assert.

Socrates: Yet anybody who has the least acquaintance with geometry will not deny that such a conception of the science is in flat contradiction to the ordinary language of geometricians.

Glaucon: How so?

Socrates: They have in view practice only, and are always speaking, in a narrow and ridiculous manner, of squaring and extending and applying and the like—they confuse the necessities of geometry with those of daily life; whereas knowledge is the real object of the whole science.

Glaucon: Certainly.

Socrates: Then must not a further admission be made?

Glaucon: What admission?

Socrates: That the knowledge at which geometry aims is knowledge of the eternal, and not of something perishing and transient.

Glaucon: That may be readily allowed, and is true.

Socrates: Then, my noble friend, geometry will draw the soul towards truth, and create the spirit of philosophy, and raise up that which is now unhappily allowed to fall down.

Glaucon: Nothing will be more likely to have such an effect.

Socrates: Then nothing should be more sternly laid down than that the inhabitants of your fair city should by all means learn geometry. Moreover the science has indirect effects, which are not small.

Glaucon: Of what kind?

Socrates: These are the military advantages of which you spoke; and in all departments of knowledge, as experience proves, anyone who has studied geometry is infinitely quicker of apprehension than one who has not.

Glaucon: Yes indeed, there is an infinite difference between them.

Socrates: Then shall we propose this as a second branch of knowledge which our youth will study?

Glaucon: Let us do so.

Socrates: And suppose we make astronomy the third—what do you say?

Glaucon: I am strongly inclined to it; the observation of the seasons and of months and years is as essential to the general as it is to the farmer or sailor.

Socrates: I am amused at your fear of the world, which makes you guard against the appearance of insisting upon useless studies; and I quite admit the difficulty of believing that in every man there is an eye of the soul which, when by other pursuits lost and dimmed, is by these purified and rekindled; and is more precious far than ten thousand bodily eyes, for by it alone is truth seen. Now there are two classes of persons, one class of those who will agree with you and will take your words as a revelation; another class to whom they will be utterly unmeaning, and who will naturally deem them to be idle tales, for they see no sort of profit which is to be obtained from them. And therefore you had better decide at once with which of the two you are proposing to argue. You will very likely say with neither, and that your chief aim in carrying on the argument is your own improvement; at the same time you do not begrudge to others any benefit which they may receive.

Glaucon: I think that I should prefer to carry on the argument mainly on my own behalf.

Socrates: Then take a step backward, for we have gone wrong in the order of the sciences.

Glaucon: What was the mistake?

Socrates: After plane geometry, we proceeded at once to solids in revolution, instead of taking solids in themselves; whereas after the second dimension the third, which is concerned with cubes and dimensions of depth, ought to have followed.

Glaucon: That is true, Socrates, but so little seems to be known as yet about these subjects.

Socrates: Why, yes, and for two reasons. In the first place, no government values them; this leads to a want of energy in the pursuit of them, and they are difficult; in the second place, students cannot learn them unless they have a director. But then a director can hardly be found, and even if he could, as matters now stand, the students, who are very conceited, would not attend to him. That, however, would be otherwise if the whole State became the director of these studies and gave honor to them; then disciples would want to come, and there would be continuous and earnest search, and discoveries would be made. Since even now, disregarded as they are by the world and although none of their investigators can tell the use of them, still these studies force their way by their natural charm, and very likely, if they had the help of the State, they would some day emerge into light.

Glaucon: Yes, there is a remarkable charm in them. But I do not clearly understand the change in the order. First you began with a geometry of plane surfaces?

Socrates: Yes.

Glaucon: And you placed astronomy next, and then you made a step backward?

Socrates: Yes, and I have delayed you by my hurry; the ludicrous state of solid geometry, which, in natural order, should have followed, made me pass over this branch and go on to astronomy, or motion of solids.

Glaucon: True.

Socrates: Then assuming that the science now omitted would come into existence if encouraged by the State, let us go on to astronomy, which will be fourth.

Glaucon: The right order. And now, Socrates, as you rebuked the vulgar manner in which I praised astronomy before, my praise shall be given in your own spirit. For everyone, as I think, must see that astronomy compels the soul to look upwards and leads us from this world to another.

Socrates: Everyone but myself; to everyone else this may be clear, but not to me.

Glaucon: And what then would you say?

Socrates: I should rather say that those who elevate astronomy into philosophy appear to me to make us look downwards and not upwards.

Glaucon: What do you mean?

Socrates: You have in your mind a truly sublime conception of our knowledge of the things above. And I dare say that if a person were to throw his head back and study ornaments on a ceiling, you would still think that he is looking at them with his understanding and not his eyes. And you are very likely right, and I may be a simpleton; but, in my opinion, that knowledge only which is of being and of the unseen can make the soul look upwards, and whether a man gapes at the heavens or blinks on the ground, seeking to learn some particular of sense, I would deny that he can learn, for nothing of that sort is a matter of science; his soul is looking downwards, not upwards, whether his way to knowledge is by water or by land, whether he floats, or only lies on his back.

Glaucon: I acknowledge the justice of your rebuke. Still, I should like to ascertain how astronomy can be learned in any manner more conducive to that knowledge of which we are speaking?

Socrates: I will tell you. The starry heaven which we behold is wrought upon a visible ground, and therefore, although the fairest and most perfect of visible things, must necessarily be deemed inferior far to the true motions of absolute swiftness and absolute slowness, which are relative to each other, and carry with them that which is contained in them, in the true number and in every true figure. Now, these are to be apprehended by reason and intelligence, but not by sight.

Glaucon: True.

Socrates: The spangled heavens should be used as a pattern and with a view to that higher knowlege; their beauty is like the beauty of figures or pictures excellently wrought by the hand of Daedalus, or some other great artist, which we may chance to behold; any geometrician who saw them would appreciate the exquisiteness of their workmanship, but he would never dream of thinking that in them he could find the true equal or the true double, or the truth of any other proportion.

Glaucon: No, such an idea would be ridiculous.

Socrates: And will not a true astronomer have the same feeling when he looks at the movements of the stars? Will he not think that heaven and the things in heaven are framed by the Creator of them in the most perfect manner? But he will never imagine that the proportions of night and day, or of both to the month, or of the month to the year, or of the stars to these and to one another, and any other things that are material and visible can also be eternal and subject to no deviation—that would be absurd; and it is equally absurd to take so much pains in investigating their exact truth.

Glaucon: I quite agree, though I never thought of this before.

Socrates: Then in astronomy, as in geometry, we should employ problems, and let the heavens alone if we would approach the subject in the right way and so make the natural gift of reason to be of any real use.

Glaucon: That is a work infinitely beyond our present astronomers.

Socrates: Yes, and there are many other things which must also have a similar extension given to them, if our legislation is to be of any value. But can you tell me of any other suitable study?

Glaucon: No, not without thinking.

Socrates: Motion has many forms, and not one only; two of them are obvious enough even to wits no better than ours; and there are others, as I imagine, which may be left to wiser persons.

Glaucon: But where are the two?

Socrates: There is a second which is the counterpart of the one already named.

Glaucon: And what may that be?

Socrates: The second would seem relatively to the ears to be what the first is to the eyes; for I conceive that as the eyes are designed to look up at the stars, so are the ears to hear harmonious motions; and these are sister sciences—as the Pythagoreans say, and we, Glaucon, agree with them?

Glaucon: Yes.

Socrates: But this is a laborious study, and therefore we had better go and learn of them; and they will tell us whether there are any other applications of these sciences. At the same time, we must not lose sight of our own higher object.

Glaucon: What is that?

Socrates: There is a perfection which all knowledge ought to reach, and which our pupils ought also to attain, and not to fall short of, as I was saying that they did in astronomy. For in the science of harmony, as you probably know, the same thing happens. The teachers of harmony compare the sounds and consonances which are heard only, and their labor, like that of the astronomers, is in vain.

Glaucon: Yes, by heaven; and it is as good as a play to hear them talking about their condensed notes, as they call them. They put their ears close alongside of the strings like persons catching a sound from their neighbor's wall—one set of them declaring that they distinguish an intermediate note and have found the least interval which should be the unit of measurement; the others insisting that the two sounds have passed into the same—either party setting their ears before their understanding.

Socrates: You mean, those gentlemen who tease and torture the strings and stretch them on the pegs of the instrument. I might carry on the metaphor and speak after their manner of the blows which the pick gives, and make accusations against the strings that are too responsive or too unresponsive. But this would be tedious, and therefore I will only say that these are not the men, and that I am referring to the Pythagoreans, of whom I was just now proposing to inquire about harmony. For they too are in error, like the astronomers; they investigate the numbers of the harmonies which are heard, but they never attain to problems, that is to say, they never reach the natural harmonies of number, or reflect why some numbers are harmonious and others not.

Glaucon: That is a thing of more than mortal knowledge.

Socrates: A thing which I would rather call useful; that is, if sought after with a view to the beautiful and good; but if pursued in any other spirit, useless.

Glaucon: Very true.

Socrates: Now, when all these studies reach the point of intercommunion and connection with one another, and come to be considered in their mutual affinities, then, I think, but not until then, will the pursuit of them have a value for our objects; otherwise there is no profit in them.

1.2 Plato, *Gorgias*

In this dialogue of Plato, Socrates debates with several sophists, including Gorgias, seeking the true meaning of "rhetoric." We have included only the brief section where Socrates discusses the arts of arithmetic and calculation and whether these should be called "rhetoric" as well.

Socrates: As to the arts generally, they are for the most part concerned with doing, and require little or no speaking; in painting, and statuary, and many other arts, the work may proceed in silence; and of such arts I suppose you would say that they do not come within the province of rhetoric.

Gorgias: You perfectly conceive my meaning, Socrates.

Socrates: But there are other arts which work wholly through the medium of language, and require either no action or very little, as, for example, the arts of arithmetic, of calculation, of geometry, and of playing draughts; in some of these speech is pretty nearly co-extensive with action, but in most of them the verbal element is greater—they depend wholly on words for their efficacy and power; and I take your meaning to be that rhetoric is an art of this latter sort?

Gorgias: Exactly.

Socrates: And yet I do not believe that you really mean to call any of these arts rhetoric; although the precise expression which you used was, that rhetoric is an art which works and takes effect only through the medium of discourse; and an adversary who wished to be captious might say, "And so, Gorgias, you call arithmetic rhetoric." But I do not think that you really call arithmetic rhetoric any more than geometry would be so called by you.

Gorgias: You are quite right, Socrates, in your apprehension of my meaning.

Socrates: Well, then, let me now have the rest of my answer: Seeing that rhetoric is one of those arts which works mainly by the use of words, and there are other arts which also use words, tell me what is that quality in words with which rhetoric is concerned. Suppose that a person asks me about some of the arts which I was mentioning just now; he might say, "Socrates, what is arithmetic?" and I should reply to him, as you replied to me, that arithmetic is one of those arts which take effect through words. And then he would proceed to ask: "Words about what?" and I should reply, Words about odd and even numbers, and how many there are of each. And if he asked again: "What is the art of calculation?" I should say, That also is one of the arts which is concerned wholly with words. And if he further said, "Concerned with what?" I should say, like the clerks in the assembly, "as aforesaid" of arithmetic, but with a difference, the difference being that the art of calculation considers not only the quantities of odd and even numbers, but also their numerical relations to themselves and to one another. And suppose, again, I were to say that astronomy is only words, he would ask, "Words about what, Socrates?" and I should answer, that astronomy tells us about the motions of the stars and sun and moon, and their relative swiftness.

Gorgias: You would be quite right, Socrates.

1.3 Archytas, *On Things Scientific*

Archytas (428–350 BCE) was born in Tarentum, in southern Italy, became a member of the Tarentine elite, and was a friend and associate of Plato. He evidently wrote numerous works on such subjects as music theory, geometry, number theory, and optics, but none of his works survive. There are various fragments in books by others that purport to quote Archytas, and it is from these fragments and references in the works of others that we can deduce that he was a superb mathematician as well as a political leader. The following fragment dealing with the use of calculation is taken from Iamblichus (third century CE), *On General Mathematical Science II*, in which the notion that mathematics underpins justice is explored.

Wherefore Archytas says in the *On Things Scientific*:

For it is necessary to come to know those things which you did not know, either by learning from another or by discovering yourself. Learning is from another and belongs to another, while discovery is through oneself and belongs to oneself. Discovery, while not seeking, is difficult and infrequent but, while seeking, easy and frequent, but, if one does not know how to calculate, it is impossible to seek.

Once calculation was discovered, it stopped discord and increased concord. For people do not want more than their share, and equality exists, once this has come into being. For by means of calculation we will seek reconciliation in our dealings with others. Through this, then, the poor receive from the powerful, and the wealthy give to the needy, both in the confidence that they will have what is fair on account of this. It serves as a standard and a hindrance to the unjust. It stops those who know how to calculate, before they commit injustice, persuading them that they will not be able to go undetected, whenever they appeal to it. It hinders those who do not know how to calculate from committing injustice, having revealed them as unjust by means of it.

1.4 Aristotle, *Metaphysics*

Aristotle (384–322 BCE) was the most prolific of the Greek philosophers. Having studied with Plato as a young man, he later was hired by Philip II of Macedon to undertake the education of his son Alexander, who soon after he acceded to the throne in 335 began his successful conquest of the Mediterranean world. Meanwhile, Aristotle returned to Athens and founded his own school, the Lyceum, where he spent the rest of his days writing, lecturing, and holding discussions with his associates. His writings cover many subjects including physics, biology, zoology, metaphysics, logic, ethics, aesthetics, poetry, theater, music, rhetoric, psychology, linguistics, economics, politics, and government. Even though many of his scientific findings have been overhauled and updated, intellectuals in Europe and the Islamic world through early modern times began many of their inquiries by trying to support or refute his ideas. Although Aristotle was not a mathematician, he was clearly cognizant of the mathematical ideas circulating in Greece during his lifetime. We consider here excerpts of some of his writings that refer to mathematics or that have influenced the development of mathematics, in particular his ideas on methods of proof.

In the following excerpts from the *Metaphysics*, Aristotle tells us why mathematics began in Egypt, why mathematicians believe in proof, and why the Pythagoreans thought that the principles of mathematics were the principles of all things.

Book 1

Chapter 1

At first, he who invented any art whatever that went beyond the common perceptions of man was naturally admired by men, not only because there was something useful in the inventions, but because he was thought wise and superior to the rest. But as more arts were invented, and some were directed to the necessities of life, others to recreation, the inventors of the latter were naturally always regarded as wiser than the inventors of the former, because their branches of knowledge did not aim at utility. Hence when all such inventions were already established, the sciences which do not aim at giving pleasure or at the necessities of life were discovered, and first in the places where men first began to have leisure. This is why the mathematical arts were founded in Egypt; for there the priestly caste was allowed to be at leisure.

Chapter 2

For all men begin, as we said, by wondering that things are as they are, as they do about self-moving marionettes, or about the solstices or the incommensurability of the diagonal of a square with the side; for it seems wonderful to all who have not yet seen the reason, that there is a thing which cannot be measured even by the smallest unit. But we must end in the contrary and, according to the proverb, the better state, as is the case in these instances too when men learn the cause; for there is nothing which would surprise a geometer so much as if the diagonal turned out to be commensurable.

Chapter 5

Contemporaneously with these philosophers[1] and before them, the so-called Pythagoreans, who were the first to take up mathematics, not only advanced this study, but also having been brought up in it they thought its principles were the principles of all things. Since of these principles numbers are by nature the first, and in numbers they seemed to see many resemblances to the things that exist and come into being—more than in fire and earth and water (such and such a modification of numbers being justice, another being soul and reason, another being opportunity—and similarly almost all other things being numerically expressible); since, again, they saw that the modifications and the ratios of the musical scales were expressible in numbers; since, then, all other things seemed in their whole nature to be modeled on numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number. And all the properties of numbers and scales which they could show to agree with the attributes and parts and the whole arrangement of the heavens, they collected and fitted into their scheme; and if there was a gap anywhere, they readily made additions so as to make their whole theory coherent. For example, as the number ten is thought to be perfect and to comprise the whole nature of numbers, they say that the bodies which move through the heavens are ten, but as the visible bodies are only nine, to meet this they invent a tenth, the “counter-earth.”

1.5 Aristotle, *Prior Analytics*

In these excerpts from the *Prior Analytics*, Aristotle discusses the idea of a syllogism. He then refers to the proof that the diagonal of a square is incommensurable with

[1] Anaxagoras of Clazomenae and Empedocles of Acragas, both of the fifth century BCE.

the side. He also gives hints of a proof of the theorem that the base angles of an isosceles triangle are equal. In his writings about proof, he expresses his belief that logical arguments should always be built out of syllogisms, even though later Greek writers on mathematics generally used the ideas of propositional logic to formulate such arguments. Nevertheless, the basic ideas about proofs are embedded in Aristotle's writings.

Book 1

Chapter 1

A syllogism is discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so.[2] I mean by the last phrase that they produce the consequence, and by this, that no further term is required from without in order to make the consequence necessary. I call that a perfect syllogism which needs nothing other than what has been stated to make plain what necessarily follows; a syllogism is imperfect, if it needs either one or more propositions, which are indeed the necessary consequences of the terms set down, but have not been expressly stated as premises.

Chapter 23

For all who effect an argument *per impossible* infer syllogistically what is false, and prove the original conclusion hypothetically when something impossible results from the assumption of its contradictory; e.g., that the diagonal of the square is incommensurate with the side, because odd numbers are equal to evens if it is supposed to be commensurate. One infers syllogistically that odd numbers come out equal to evens, and one proves hypothetically the incommensurability of the diagonal, since a falsehood results through contradicting this.

Chapter 24

In every syllogism one of the premises must be affirmative, and universality must be present; unless one of the premises is universal either a syllogism will not be possible, or it will not refer to the subject proposed, or the original position will be begged. Suppose we have to prove that pleasure in music is good. If one should claim as a premise that pleasure is good without adding "all," no syllogism will be possible; if one should claim that some pleasure is good, then if it is different from pleasure in music, it is not relevant to the subject proposed; if it is this very pleasure, one is assuming that which was proposed at the outset to be proved. This is more obvious in geometrical proofs, e.g., that the angles at the base of an isosceles triangle are equal. Suppose the lines A and B have been drawn to the center. If then one should assume that the angle AC is equal to the angle BD, without claiming generally that angles of semicircles are equal; and again if one should assume

[2] As a simple example, we may take the following argument: All monkeys are primates, and all primates are mammals; therefore, all monkeys are mammals.

that the angle C is equal to the angle D, without the additional assumption that every angle of a segment is equal to every other angle of the same segment; and further if one should assume that when equal angles are taken from the whole angles, which are themselves equal, the remainders E and F are equal, he will beg the thing to be proved, unless he also states that when equals are taken from equals the remainders are equal.[3]

1.6 Aristotle, *Posterior Analytics*

In the *Posterior Analytics*, Aristotle discusses some of the basic ideas of a proof. The excerpts here explore the notion that syllogistic reasoning enables one to use “old knowledge” to impart new. Of course, he notes, one must begin with some basic truths that are assumed without proof.

Book 1

Chapter 1

All instruction given or received by way of argument proceeds from pre-existent knowledge. This becomes evident upon a survey of all the species of such instruction. The mathematical sciences and all other speculative disciplines are acquired in this way, and so are the two forms of dialectical reasoning, syllogistic and inductive; for each of these latter make use of old knowledge to impart new, the syllogism assuming an audience that accepts its premises, induction exhibiting the universal as implicit in the clearly known particular. Again, the persuasion exerted by rhetorical arguments is in principle the same, since they use either example, a kind of induction, or enthymeme,[4] a form of syllogism.

The pre-existent knowledge required is of two kinds. In some cases admission of the fact must be assumed, in others comprehension of the meaning of the term used, and sometimes both assumptions are essential. Thus, we assume that every predicate can be either truly affirmed or truly denied of any subject, and that “triangle” means so and so; as regards “unit” we have to make the double assumption of the meaning of the word and the existence of the thing. The reason is that these several objects are not equally obvious to us. Recognition of a truth may in some cases contain as factors both previous knowledge and also knowledge acquired simultaneously with that recognition—knowledge, this latter, of the particulars actually falling under the universal and therein already virtually known.

Chapter 2

What I now assert is that at all events we do know by demonstration. By demonstration I mean a syllogism productive of scientific knowledge, a syllogism, that is, the grasp

[3] For a possible explanation of Aristotle’s proof, see Thomas L. Heath, *The Thirteen Books of Euclid’s Elements with Introduction and Commentary*, vol. 1 (New York: Dover, 1956), 252–53.

[4] A syllogism where one of the premises is not explicitly stated. Such a syllogism is often used in rhetorical arguments.

of which is *eo ipso*[5] such knowledge. Assuming then that my thesis as to the nature of scientific knowing is correct, the premises of demonstrated knowledge must be true, primary, immediate, better known than and prior to the conclusion, which is further related to them as effect to cause. Unless these conditions are satisfied, the basic truths will not be "appropriate" to the conclusion. Syllogism there may indeed be without these conditions, but such syllogism, not being productive of scientific knowledge, will not be demonstration. The premises must be true, for that which is non-existent cannot be known—we cannot know, e.g., that the diagonal of a square is commensurate with its side. The premises must be primary and indemonstrable; otherwise they will require demonstration in order to be known, since to have knowledge, if it be not accidental knowledge, of things which are demonstrable, means precisely to have a demonstration of them. The premises must be the causes of the conclusion, better known than it, and prior to it; its causes, since we possess scientific knowledge of a thing only when we know its cause; prior, in order to be causes; antecedently known, this antecedent knowledge being not our mere understanding of the meaning, but knowledge of the fact as well.

Chapter 10

I call the basic truths of every genus those elements in it the existence of which cannot be proved. As regards both these primary truths and the attributes dependent on them the meaning of the name is assumed. The fact of their existence as regards the primary truths must be assumed; but it has to be proved of the remainder, the attributes. Thus we assume the meaning alike of unity, straight, and triangular; but while as regards unity and magnitude we assume also the fact of their existence, in the case of the remainder proof is required.

Of the basic truths used in the demonstrative sciences some are peculiar to each science, and some are common, but common only in the sense of analogous, being of use only in so far as they fall within the genus constituting the province of the science in question.

Peculiar truths are, e.g., the definitions of line and straight; common truths are such as "take equals from equals and equals remain." Only so much of these common truths is required as falls within the genus in question; for a truth of this kind will have the same force even if not used generally but applied by the geometer only to magnitudes, or by the arithmetician only to numbers. Also peculiar to a science are the subjects the existence as well as the meaning of which it assumes, and the essential attributes of which it investigates, e.g., in arithmetic units, in geometry points and lines. Both the existence and the meaning of the subjects are assumed by these sciences; but of their essential attributes only the meaning is assumed. For example arithmetic assumes the meaning of odd and even, square and cube, geometry that of incommensurable, or of deflection or verging of lines, whereas the existence of these attributes is demonstrated by means of the axioms and from previous conclusions as premises.

[5] A Latin phrase that means "by that very fact."

1.7 Aristotle, *Physics*

In this selection from the *Physics*, Aristotle discourses at length about the existence of the infinite, eventually concluding that although there cannot be an infinite "body," infinity does exist—mathematically—in potential. Note that Aristotle's concept of physics is not the same as ours; in general, his physics is concerned with the notion of change, including change of place (motion), quantitative change, qualitative change, and substantial change, as in coming into or out of existence.

Book 3

Chapter 4

Belief in the existence of the infinite comes mainly from five considerations:

1. From the nature of time—for it is infinite.
2. From the division of magnitudes—for the mathematicians also use the notion of the infinite.
3. If coming to be and passing away do not give out, it is only because that from which things come to be is infinite.
4. Because the limited always finds its limit in something, so that there must be no limit, if everything is always limited by something different from itself.
5. Most of all, a reason which is peculiarly appropriate and presents the difficulty that is felt by everybody—not only number but also mathematical magnitudes and what is outside the heaven are supposed to be infinite because they never give out in our thought.

The last fact (that which is outside is infinite) leads people to suppose that body also is infinite, and that there is an infinite number of worlds. Why should there be body in one part of the void rather than in another? Grant only that mass is anywhere and it follows that it must be everywhere. Also, if void and place are infinite, there must be infinite body too, for in the case of eternal things what may be must be. But the problem of the infinite is difficult; many contradictions result whether we suppose it to exist or not to exist. If it exists, we have still to ask how it exists; as a substance or as the essential attribute of some entity? Or in neither way, yet none the less is there something which is infinite or some things which are infinitely many?

The problem, however, which specially belongs to the physicist is to investigate whether there is a sensible magnitude which is infinite. We must begin by distinguishing the various senses in which the term "infinite" is used.

1. What is incapable of being gone through, because it is not in its nature to be gone through (the sense in which the voice is "invisible").
2. What admits of being gone through, the process however having no termination, or
3. What scarcely admits of being gone through.
4. What naturally admits of being gone through, but is not actually gone through or does not actually reach an end.

Further, everything that is infinite may be so in respect of addition or division or both.

In chapter 5, Aristotle endeavors to prove that an infinite body is an impossibility. He then continues:

Chapter 6

But on the other hand to suppose that the infinite does not exist in any way leads obviously to many impossible consequences: there will be a beginning and an end of time, a magnitude will not be divisible into magnitudes, number will not be infinite. If, then, in view of the above considerations, neither alternative seems possible, an arbiter must be called in; and clearly there is a sense in which the infinite exists and another in which it does not. We must keep in mind that the word "is" means either what potentially is or what fully is. Further, a thing is infinite either by addition or by division. Now, as we have seen, magnitude is not actually infinite. But by division it is infinite. (There is no difficulty in refuting the theory of indivisible lines.) The alternative then remains that the infinite has a potential existence.

But the phrase "potential existence" is ambiguous. When we speak of the potential existence of a statue we mean that there will be an actual statue. It is not so with the infinite. There will not be an actual infinite. The word "is" has many senses, and we say that the infinite "is" in the sense in which we say "it is day" or "it is the games," because one thing after another is always coming into existence. For of these things too the distinction between potential and actual existence holds. We say that there are Olympic games, both in the sense that they may occur and that they are actually occurring. . . .

In a way the infinite by addition is the same thing as the infinite by division. In a finite magnitude, the infinite by addition comes about in a way inverse to that of the other. For in proportion as we see division going on, in the same proportion we see addition being made to what is already marked off. For if we take a determinate part of a finite magnitude and add another part *determined by the same ratio* (not taking in the same amount of the original whole), and so on, we shall not traverse the given magnitude. But if we increase the ratio of the part, so as always to take in the same amount, we shall traverse the magnitude, for every finite magnitude is exhausted by means of any determinate quantity however small.

The infinite, then, exists in no other way, but in this way it does exist, potentially and by reduction. It exists fully in the sense in which we say "it is day" or "it is the games"; and potentially as matter exists, not independently as what is finite does.

By addition then, also, there is potentially an infinite, namely, what we have described as being in a sense the same as the infinite in respect of division. For it will always be possible to take something *ab extra*. Yet the sum of the parts taken will not exceed every determinate magnitude, just as in the direction of division every determinate magnitude is surpassed in smallness and there will be a smaller part. But in respect of addition there cannot be an infinite which even potentially exceeds every assignable magnitude, unless it has the attribute of being actually infinite, as the physicists hold to be true of the body which is outside the world, whose essential nature is air or something of the kind. But if there cannot be in this way a sensible body which is infinite in the full sense, evidently there can no more be a body which is potentially infinite in respect of addition, except as the inverse of the infinite by division, as we have said. It is for this reason that Plato also makes the infinites two in number, because it is supposed to be possible to exceed all limits and to proceed *ad infinitum* in the direction both of increase and of reduction. Yet though he makes the infinites two, he does not use them. For in the numbers the infinite in the direction of reduction is not present, as the monad is the smallest; nor is the infinite in the direction of increase, for the parts number only up to the decad.

The infinite turns out to be the contrary of what it is said to be. It is not what has nothing outside it that is infinite, but what always has something outside it.

Chapter 7

It is reasonable that there should not be held to be an infinite in respect of addition such as to surpass every magnitude, but that there should be thought to be such an infinite in the direction of division. For the matter and the infinite are contained inside what contains them, while it is the form which contains. It is natural too to suppose that in number there is a limit in the direction of the minimum, and that in the other direction every assigned number is surpassed. In magnitude, on the contrary, every assigned magnitude is surpassed in the direction of smallness, while in the other direction there is no infinite magnitude. The reason is that what is one is indivisible whatever it may be, e.g., a man is one man, not many. Number on the other hand is a plurality of “ones” and a certain quantity of them. Hence number must stop at the indivisible; for “two” and “three” are merely derivative terms, and so with each of the other numbers. But in the direction of largeness it is always possible to think of a larger number; for the number of times a magnitude can be bisected is infinite. Hence this infinite is potential, never actual; the number of parts that can be taken always surpasses any assigned number. But this number is not separable from the process of bisection, and its infinity is not a permanent actuality but consists in a process of coming to be, like time and the number of time.

With magnitudes the contrary holds. What is continuous is divided *ad infinitum*, but there is not infinite in the direction of increase. For the size which it can potentially be, it can also actually be. Hence since no sensible magnitude is infinite, it is impossible to exceed every assigned magnitude; for if it were possible there would be something bigger than the heavens. . . . Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversible. In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish. It is possible to have divided in the same ratio as the largest quantity another magnitude of any size you like. Hence, for the purposes of proof, it will make no difference to them to have such an infinite instead, while its existence will be in the sphere of real magnitudes.

1.8 Philo of Alexandria, *On Mating with the Preliminary Studies*

Philo of Alexandria (fl. 10–55 CE) was one of the most important Jewish authors at the time of Roman rule in Judea. Jews had been moving to Alexandria ever since the founding of the city, but during Philo’s lifetime, an estimated 200,000 Jews lived there, the largest Jewish community outside Palestine. Philo himself was from a prominent and wealthy family and was given a strong Greek education. His many works of philosophy, written in Greek, revolve around interpreting the Bible, often in allegorical terms. In *On Mating with the Preliminary Studies*, Philo presents an interpretation of Abraham's having a child with Hagar, Sarah's handmaiden. Ideally, Abraham should have a child with Sarah, who represents virtue. But before this can happen, he must partake of "intermediate instruction," learning the subjects of the Greek *trivium* and *quadrivium*, all subjects specified by Plato for the education of Greek citizens.[6]

[13] For, says she [Sarah], "The Lord has closed me up so, that I may not bear children." . . . [14] "Therefore," says she, "go thou in to my handmaiden," that is to say, to the intermediate instruction of the intermediate and encyclical branches of knowledge, "that you may first have children by her"; for hereafter you shall be able to enjoy a connection with her mistress, tending to the procreation of legitimate children. [15] For grammar, by teaching you the histories which are to be found in the works of poets and historians, will give you intelligence and abundant learning; and, moreover, will teach you to look with contempt on all the vain fables which erroneous opinions invent, on account of the ill success which history tells us that the heroes and demigods who are celebrated among those writers, meet with. [16] And music will teach what is harmonious in the way of rhythm, and what is ill arranged in harmony, and, rejecting all that is out of tune and all that is inconsistent with melody, will guide what was previously discordant to concord. And geometry, sowing the seeds of equality and just proportion in the soul, which is fond of learning, will, by means of the beauty of continued contemplation, implant in you an admiration of justice. [17] And rhetoric, having sharpened the mind for contemplation in general, and having exercised and trained the faculties of speech in interpretation and explanations, will make man really rational, taking care of that peculiar and especial duty which nature has bestowed upon it, but upon no other animal whatever. [18] And dialectic science, which is the sister, the twin sister of rhetoric, as some persons have called it, separating true from false arguments, and refuting the plausibilities of sophistical arguments, will cure the great disease of the soul, deceit. It is profitable, therefore, to aide among these and other sciences resembling them, and to devote one's especial attention to them. For perhaps, I say, as has happened to many, we shall become known to the queenly virtues by means of their subjects and handmaidens. . . .

[146] And yet even this is not unknown to anyone, namely, that philosophy has bestowed upon all the particular sciences their first principles and seeds, from which speculations respecting them appear to arise. For it is geometry which invented equilateral and scalene triangles, and circles, and polygons, and all kinds of other figures. But it was no longer geometry that discovered the nature of a point, and line, and a surface, and a solid, which are the roots and foundations of the aforementioned figures. [147] For from when could it define and pronounce that a point is that which has no parts, that a line is length without breadth; that a surface is that which has only length and breadth; that a solid is that which has the three properties, length, breadth, and depth? For these discoveries belong to philosophy, and the consideration of these definitions belongs wholly to the philosopher.

[6] The *trivium* consists of logic, grammar, and rhetoric, while the *quadrivium* consists of arithmetic, geometry, music, and astronomy. These subjects became the bases for a liberal education starting in Greece and extending into the medieval period.

1.9 Nicomachus, *Introduction to Arithmetic*

Nicomachus was probably born in Gerasa—now Jarash, Jordan—late in the first century CE. He became a leading member of the neo-Pythagorean school. The *Introduction to Arithmetic* is an elementary treatise in two books. It is written in a very informal style, and it does not give formal proofs. Often, the work just gives statements of results and examples; sometimes, in fact, the stated theorems are not generally true. Still, Nicomachus's work proved very popular and was used, in a Latin version by Boethius, as a textbook well into the medieval period. In this selection, we present his views on the study of mathematics. Note that Nicomachus quotes Archytas of Tarentum in support of his views on the study of mathematics. Excerpts from the remainder of the treatise are in chapter 2 of this *Sourcebook*.

Book 1

Chapter 1

[1] The ancients, who under the leadership of Pythagoras first made science systematic, defined philosophy as the love of wisdom. Indeed the name itself means this, and before Pythagoras all who had knowledge were called "wise" indiscriminately—a carpenter, for example, a cobbler, a helmsman, and in a word anyone who was versed in any art or handicraft. Pythagoras, however, restricting the title so as to apply to the knowledge and comprehension of reality, and calling the knowledge of the truth in this the only wisdom, naturally designated the desire and pursuit of this knowledge philosophy, as being desire for wisdom.

Chapter 2

[3] If we crave for the goal that is worthy and fitting for man, namely, happiness of life—and this is accomplished by philosophy alone and by nothing else, and philosophy, as I said, means for us desire for wisdom, and wisdom the science of the truth in things, and of things some are properly so called, others merely share the name—it is reasonable and most necessary to distinguish and systematize the accidental qualities of things. [4] Things, then, both those properly so called and those that simply have the name, are some of them unified and continuous, for example, an animal, the universe, a tree, and the like, which are properly and peculiarly called "magnitudes"; others are discontinuous, in a side-by-side arrangement, and, as it were, in heaps, which are called "multitudes," a flock, for instance, a people, a heap, a chorus, and the like. [5] Wisdom, then, must be considered to be the knowledge of these two forms. Since, however, all multitude and magnitude are by their own nature of necessity infinite—for multitude starts from a definite root and never ceases increasing; and magnitude, when division beginning with a limited whole is carried on, cannot bring the dividing process to an end, but proceeds therefore to infinity—-and since sciences are always sciences of limited things, and never of infinites, it is accordingly evident that a science dealing either with magnitude, per se, or with multitude, per se, could never be formulated, for each of them is limitless in itself, multitude in the direction of the more, and magnitude in the direction of the less. A science, however, would arise to deal

with something separated from each of them, with quantity, set off from multitude, and size, set off from magnitude.

Chapter 3

[1] Again, to start afresh, since of quantity one kind is viewed by itself, having no relation to anything else, as "even," "odd," "perfect," and the like, and the other is relative to something else and is conceived of together with its relationship to another thing, like "double," "greater," "smaller," "half," "one and one-half times," "one and one-third times," and so forth, it is clear that two scientific methods will lay hold of and deal with the whole investigation of quantity; arithmetic, absolute quantity, and music, relative quantity. [2] And once more, inasmuch as part of "size" is in a state of rest and stability, and another part in motion and revolution, two other sciences in the same way will accurately treat of "size," geometry the part that abides and is at rest, astronomy that which moves and revolves. [3] Without the aid of these, then, it is not possible to deal accurately with the forms of being nor to discover the truth in things, knowledge of which is wisdom, and evidently not even to philosophize properly, for "just as painting contributes to the menial arts toward correctness of theory, so in truth lines, numbers, harmonic intervals, and the revolutions of circles bear aid to the learning of the doctrines of wisdom," says the Pythagorean Androcydes.[7] [4] Likewise Archytas of Tarentum, at the beginning of his treatise *On Harmony*, says the same thing, in about these words: "It seems to me that they do well to study mathematics, and it is not at all strange that they have correct knowledge about each thing, what it is. For if they knew rightly the nature of the whole, they were also likely to see well what is the nature of the parts. About geometry, indeed, and arithmetic and astronomy, they have handed down to us a clear understanding, and not least also about music. For these seem to be sister sciences; for they deal with sister subjects, the first two forms of being."

Chapter 4

[1] Which then of these four methods must we first learn? Evidently, the one which naturally exists before them all, is superior and takes the place of origin and root and, as it were, of mother to the others. [2] And this is arithmetic, not solely because we said that it existed before all the others in the mind of the creating God like some universal and exemplary plan, relying upon which as a design and archetypal example the creator of the universe sets in order his material creations and makes them attain to their proper ends; but also because it is naturally prior in birth, inasmuch as it abolishes other sciences with itself, but it not abolished together with them. For example, "animal" is naturally antecedent to "man," for abolish "animal" and "man" is abolished; but if "man" be abolished, it no longer follows that "animal" is abolished at the same time. And again, "man" is antecedent to "schoolteacher"; for if "man" does not exist, neither does "schoolteacher," but if "schoolteacher" is nonexistent, it is still possible for "man" to be. Thus since it has the property of abolishing the other ideas with itself, it is likewise the older. . . . [4] So it

[7] Androcydes was a Pythagorean philosopher whose work survives only in some fragments. He lived before the first century BCE, possibly as early as the fourth.

is with the foregoing sciences; if geometry exists, arithmetic must also needs be implied, for it is with the help of this latter that we can speak of triangle, quadrilateral, octahedron, icosahedron, double, eightfold, or one and one-half times, or anything else of the sort which is used as a term by geometry, and such things cannot be conceived of without the numbers that are implied with each one. For how can "triple" exist, or be spoken of, unless the number 3 exists beforehand, or "eightfold" without 8? But on the contrary 3, 4, and the rest might be without the figures existing to which they give names. [5] Hence arithmetic abolishes geometry along with itself, but is not abolished by it, and while it is implied by geometry, it does not itself imply geometry.

Chapter 5

[3] So then we have rightly undertaken first the systematic treatment of this, as the science naturally prior, more honorable, and more venerable, and, as it were, mother and nurse of the rest; and here we will take our start for the sake of clearness.

1.10 Iamblichus, *On the General Science of Mathematics*

Iamblichus was a neo-Pythagorean philosopher who lived in the third century CE and taught at a school in Apamea, Syria, an ancient Greek and Roman city. The treatise, *On the General Science of Mathematics*, from which these excerpts are drawn, sets out the basic ideas of Pythagoras and his school toward mathematics, at least as Iamblichus understood them some eight hundred years after Pythagoras lived. He relates the familiar anecdote of Thales, a story picked up by Proclus in the fifth century, and also notes that Pythagoras studied with teachers in Egypt and Babylonia—although he refers to these teachers as "barbarians." There is not much specific detail here on Pythagoras's mathematics, only general principles detailing the close connection of mathematics with ethics and theology. Iamblichus also emphasizes the practical relevance of mathematics for the pursuit of the good life. In addition, he sets out in a general way the Pythagorean methods of mathematical instruction.

Chapter 7

What the special object of knowledge is which is the basis of each branch of mathematics, and how it is possible by division to make a general distinction of them, so as to know both the one and the manifold in mathematics, its character, and how it should be defined.

Since one must also, for each of the mathematical sciences, determine the proper object of knowledge that underlies each, let us distinguish, taking our start from a process of division, the species of studies with which they deal. In that way we may most easily understand the unity and the plurality of mathematical science, its character, and by what sort of differences it is distinguished. Let us, then, take our start from this point.

The nature of the continuous and the discrete for all that is, that is to say for the whole structure of the cosmos, may be conceived in two ways: there is the discrete through juxtaposition and through piling up, and the continuous through unification and through conjunction. Strictly speaking, continuity and unification should be called magnitude, being adjacent, and discrete plurality. In accordance with the essence of magnitude the cosmos would be conceived as one and would be called solid, spherical, and fused together, extended and conjoined; but, again, according to the form and concept of plurality the ordering, disposition, and joining together of the whole would be thought of as, we may say, being constructed of so many oppositions and similarities of elements, spheres, stars, kinds, animals, and plants. But in the case of the unified, division from the totality is without limit, while its increase is to a limited point; while conversely, in the case of plurality, increase is unlimited, but division limited. Naturally, indeed, and conceptually, both are unlimited and therefore indefinable by sciences; "there will be no principle of object of understanding, since all things are unlimited," according to Philolaus.[8]

Since it is necessary that the nature of a science be discerned in things thus given exactitude by divine providence, certain sciences have cut off part from each and limited their content; from plurality they developed the concept of quantity, which was already familiar, while from magnitude in the same way they developed that of extension. Both of these kinds they subsumed under sciences according to their own forms—under arithmetic, quantity, under geometry, extension. But, since these were not of a single form, each of them permitted still further subdivision. Thus of quantity part is absolute, freed from any relationship to something else, as for example the even, the odd, the whole, and the like; while some is relative in some way or other, which is termed relative quantity in the strict sense, such as the equal, unequal, a multiple, the superparticular, the superpartient, and the similar; and again some quantity both is and is thought of as unchanging, other as changing and moving. For this reason it is to be expected that two other sciences share in and are attached to the objects of each of the two above-mentioned sciences. For in connection with arithmetic, to which specially belong the study of quantity as such, music is allotted a role in the technical study of relative quantity; for its study of harmonics and concords professes nothing other than the classification of relations and ratios to each other of sounds, and the quantity of their relative height and depth; while for geometry, which is concerned with persistent and static extension, spherical astronomy arose as a collaborator and as arbiter of moving quantity, clearly of that which is most perfect and exhibits orderly and uniform motion. Therefore, since the sciences are themselves concerned with twin objects, it is sensible to think of these sciences as sisters, so as not foolishly to discount the saying of Archytas: "For these studies seem to be sisters," and it is sensible to consider them as connected together like links in a chain and joined by a common bond, in the words of the most divine Plato. The uniform kinship of these studies should manifest itself to him who studies them in the right way; and him who has received all of them in the way that Plato himself proposes he calls the most truly wise, and playfully insists on it, and exhorts those eager to be philosophers that these studies, whether difficult or easy, should be pursued and chosen above all. This is entirely sensible, since a grasp of the continuous and the discrete comes about in these ways alone, while the cosmos and all that is therein contained consists of the continuous and the discrete. The accurate grasp of these is wisdom, and philosophy is the struggle for wisdom. But philosophy is the absolutely single one of all skills and sciences that is

[8] Philolaus of Croton was a fifth-century BCE Pythagorean philosopher.

concerned with man's own natural end and leads to the state of flourishing which is proper to him alone among animals, and is naturally sought after as the goal most worth his seeking.

Chapter 18

The special methods of the Pythagorean presentation of the mathematical sciences, how and in what matters they made use of them, and for whom; also that they always took due account both of the subject matter and of learners.

Indeed the special methods of Pythagorean teaching of mathematics were marvelously accurate and greatly surpassed the skill of those who were engaged in the teaching of mathematics on the technical level. So let us present an outline of it, so far as is possible to speak about it in general terms.

So let this one thing be agreed, that by starting from above from first principles they provided the first structuring of mathematical theorems, undertaking their reasoning on the basis, as it were, of the theorems' primal being itself, and finally leading the whole mathematical enterprise back up to this focus. Further, as a consequence of this, they were accustomed to exhibit first the discoveries of the theorems and to take nothing for granted, but in all cases to show how what was being demonstrated in mathematics came to be the case. They also shared another procedure, the mathematical use of symbols, e.g. the pentad as a symbol of justice, because it signifies symbolically all the forms of justice. This form was useful to them in all philosophy, since they performed the majority of their teaching through symbols, and also believed that this procedure was appropriate to the gods and befitted nature. But, indeed, it is clear from the other mathematical sciences that they provided the first principles and discoveries of mathematics, and it is plain also from their methods in number theory. For they teach clearly how each kind and species of number first comes to be and how it is discovered by us, on the principle that the study of numbers was not scientific unless one should grasp them by starting from above.

Furthermore, they always assimilated the theorems of mathematics to true beings and everything divine, to the conditions and powers of the soul, to the heavenly phenomena and the orbits of the stars, to all the elements of bodies in becoming and the things composed of them, and to matter and the things that come to be from it, likening mathematical theorems both in general, and taking from each its own points of similarity to each part of true being. They applied mathematics to things either through their having the same reason-principles in common or through some dim impression, or through a likeness either close or distant, or through some association of images, or through some dominant cause as in the role of a paradigm, or in some other way. They also join mathematics together with things in many other ways, since things can be likened to mathematics and also mathematics has a nature to be likened to things, and both can be assimilated to each other.

So they were not at all pleased by the elaboration of terminology and the wealth of procedures, since it was rather too logic-chopping and divorced from the true nature of the facts, but they particularly welcomed recognition of the problems themselves, as contributing to the knowledge and discovery of reality. They emphasized the discovery of truths, and focus on realities rather than the shrewdness and acuity of reasonings about the facts. Therefore they did not set a high value on dialectical argumentation in mathematics, but they evidently esteemed more highly that which contributed to the discovery of realities.

So these methods and the like were those they used for mathematical teaching. But they used them scientifically, in conjunction with the theoretical philosophy of true beings, and aiming at the Good. For they thought that they ought always to give precedence to and honor the limited and the most succinct, while selecting from these whatever was useful for themselves and their companions and for complete knowledge of true beings. Also, however, in their instruction they aimed in one way at the facts, their ordering and the mutual relationships; for they marked off their first and second theorems with reference to such a sequence; but in another way they also turned their attention to their pupils, their capacity and how they could be helped by them, and what should be taught to beginners and what to the more advanced, and what sciences were esoteric and what exoteric, which could be spoken of and which could not, and to whom instruction should be communicated with scientific understanding of the facts and to whom only mathematically. For accuracy in all these matters was pursued by them not without purpose, but in order that mathematical study should hold on to one thing, the fine and good, and should be aimed at one thing, knowledge of reality and assimilation to the Good itself. So then, in this way not only a bare knowledge of mathematics was handed down, but also a suitable way of life was joined to it, and an ascent to the most honorable ends was suitably provided by it. Therefore it is worthwhile to occupy oneself in accordance with the Pythagorean practice in mathematics, as excelling and preferable to all other mathematical skills.

Chapter 19

The Pythagorean division of mathematical science as a whole into genera and the most important species, which creates a common study of them.

But since we must survey not only the merit of mathematics as a whole but also its kinds and species, their number and which should be chosen, let us make our teaching about them general and capable of extending both to the whole and equally to each branch of mathematics.

The primary study of every mathematician, including the private individual, in each matter, whatsoever it be, is the theological, by harmonizing it in rank and activities with the being and the power of the gods by some fitting analogy. This, indeed, is thought to deserve the greatest concern from men, as for example, in the case of numbers, which sorts of numbers are akin to which sorts of gods and of like nature with them, and in the cases of other branches of mathematics they are accustomed to have the same conception. Now after that among these men mathematics endeavors to exercise itself concerning truly real and intelligible being, the intelligible circle and the formal number, and they pursue other similar branches of mathematics in conformity with purest being. Again they bring the practice of mathematics under the same head as the study of self-moving being and the eternal reason-principles by defining that same self-moving number and by discovering certain measures of ratios through certain mathematical symmetries.

Much mathematical science also studies the heavens and all the heavenly orbits, both the fixed and those of the planets; it also works out not only the complicated motions of the spheres but also those of them that are uniform. It also concerns itself already with reason-principles in matter and enmattered forms, what they are like and how they were initially brought about; for such is that part of mathematics which separates conceptually form and figures from bodies. Further it endeavors to give an account, in accord with principles of natural philosophy, of things in the world of becoming, studying both the simple elements and the reason-principles concerned with bodies.

So the Pythagorean school uses all these parts of their procedure in each and every branch of mathematics, and by them produces order and purification. For as in mathematics second-level principles are made known from more basic ones, so in the case of the powers of the soul the ascent to more complete lives and activities comes about through mathematics. However, they neither neglect nor omit anything intermediate which goes to complete such a science nor leave unexplored the extremes. They traverse the whole without omission, and thus this science hands down, in the case of the most important and primary topics, that division which the science of division demonstrated. From this division it is possible to find out also the subdivisions of mathematics, of which we shall make mention as we proceed in the discussion specifically devoted to this.

Chapter 21

Who were the originators of Pythagorean mathematics, and what according to him are the salient points of such a science; how, according to him, one should organize mathematics. A general survey.

But since we are especially enquiring into Pythagorean mathematics, but it is not possible to give a complete account of it unless one comes to see its earliest beginning, for this reason it is necessary to include also those who were the predecessors of Pythagoras in such a study for the purposes of our present enquiry. For our examination of it would become as complete as possible if we were to establish it by beginning from its first roots.

Now they say that Thales first discovered a good many things in geometry and passed them on to Pythagoras. So we should rightly assimilate the mathematical investigations of Thales which we have received from him to Pythagorean mathematics. After Thales, Pythagoras spent a long time with the Egyptians and he derived from them much of value for mathematical science. Therefore we should not be acting improperly in including alongside as well many things from Egyptian mathematics. But since he also later associated with the Assyrians and those among them who are called Chaldeans (for this is what their mathematicians are called), it is necessary for us to accept much from them into our approach to mathematics.

Chapter 22

The distinctive practice of mathematical science according to Pythagoras, and how many uses of it he envisaged for the soul and for mankind; also how they practiced it throughout the whole of their own life.

But this is not after all a sufficient account. Rather, since Pythagoras added much from his own store to the mathematics he had received from the barbarians, we must both include such beginnings and add also an account of the special character of his mathematics. For he studied much of mathematics in a philosophical way and adapted it to his own focus, even though it was handed down by others; also he imposed the proper order on it and made fitting lines of inquiry about it, and he always throughout provided coherence, so as nowhere to transgress the bounds of consequentiality.

So we should conform to these principles in following in the track of Pythagorean mathematics. But let us take select parts from it as being common elements, so that we may learn from them the symbolic and unfamiliar use of mathematical terms. For since he was aiming at realities and truths he therefore applied their natural names in mathematics. He made from them a starting point of his teaching, capable of guiding his students if by sufficient experience they understood the relevant terms sufficiently. Indeed, by the clarity of his demonstrations, their elegance and accuracy, he excels all study by others in the same field; he employs much clarity, and he starts from what is familiar. But the finest aspect of his work is actually the loftiness of thought which leads up to the primary causes, which pursues its studies for the sake of the facts, which seizes purely on realities, and which sometimes conjoins mathematical insight with theological. One might put forward this much for the present as select common elements of such a science.

Now as regards how one should engage in the pursuit of this science, it is worth making the following summary remarks, in conformity with what has been passed down by these men. So then, since the great bulk of this doctrine was actively present to their minds, and was therefore preserved unwritten in memories which now no longer remain, none of it can be easily vouched for or discovered either from writings or from hearsay. So something like this must be done. We must start from small clues and continually build them up and increase them; we must take them up to suitable starting points, interpolate what is missing, and speculate so far as possible as to their doctrinal positions, what they would say if it were possible for one of them to inform us. But as it is, from pursuing the logical consequences, we are able to uncover their teachings suitably from what they have indisputably handed down to us. For such methods of investigation will enable us either to arrive at the actual Pythagorean mathematical science or to come very close to it, to the highest possible degree. I have reached the conclusion that in this way I was agreeing with its practice as it was carried on according to its own originator. For it was in every way peculiar to him, and it stands out from other disciplines, since it pays attention to the soul and to the purification of the eye of the soul, provoking the discovery of the primary forms and causes of the being of mathematics and harmonizing it with the nature of things themselves. It assimilates it to the intelligible forms and teaches its kinship with the Good and the community of the branches of mathematics with each other.

Since, then, the mathematical discipline was of this character, it searched out earnestly and keenly the theorems in its province without intermission. It contributed to the soul clarity of knowledge and subtlety in reasoning, accuracy of argument, contact with the incorporeal entities involved in it, well-adjustment, harmony, and conversion to the world of being. Also for mankind it brings order into life, peace from passions, fineness of behavior, and discovery of the other things that are of advantage to human life. They practiced it throughout their own life, weaving the gain from it both into their activities and into the conduct of their souls, the organization of their cities, and the management of their homes; also into skilled work and the equipment for war or peace, and altogether they introduced mathematics into every aspect of life, suitably to the objects of activity, profitably to the agents, and with great care for both of these and correspondingly for everything else.

One must then follow in these footsteps not simply by practicing mathematics as such; for the mathematics now popular relies too much on perception and imagination, is a stranger to truth, and has unnaturally turned rather to the world of becoming. But if we should wish to study mathematics in a Pythagorean manner we should follow eagerly its divine route which raises, purifies, and perfects.

1.11 Proclus, *Commentary on the First Book of Euclid's Elements*

Proclus (411–485) was a Greek philosopher originally from Lycia, on the southern coast of Asia Minor. He studied in Alexandria and then moved to Athens, where he became a member of the Neoplatonic Academy, probably founded in the fifth century but with no direct connection to Plato's Academy. During Proclus's time in Athens, the Academy was headed by Plutarch (350–431) and later Syrianus (d. 437). Eventually, in fact, Proclus became head of the Academy. Among his many writings is his *Commentary on the First Book of Euclid's Elements*, excerpts of which are presented here and alongside the selections from the *Elements* themselves. Here, in the first part of his prologue, Proclus discusses his mathematical philosophy, much of it inspired by the works of Plato and Aristotle. He also deals with the classification of the mathematical sciences, taken, he says, from the Pythagoreans and Geminus.

PROLOGUE

Part I

Chapter 2

The Common Principles of Mathematical Being. The Limit and the Unlimited

To find the principles of mathematical being as a whole, we must ascend to those all-pervading principles that generate everything from themselves: namely, the Limit and the Unlimited. For these, the two highest principles after the indescribable and utterly incomprehensible causation of the One, give rise to everything else, including mathematical beings. From these principles proceed all other things collectively and transcendentally, but as they come forth, they appear in appropriate divisions and take their place in an ordered procession, some coming into being first, others in the middle, and others at the end. The objects of Nous, by virtue of their inherent simplicity, are the first partakers of the Limit and the Unlimited.[9] Their unity, their identity, and their stable and abiding existence they derive from the Limit; but for their variety, their generative fertility, and their divine otherness and progression they draw upon the Unlimited. Mathematicals

[9] These speculations have their source in Plato's *Philebus*. The idea is that the human mind needs these categories to understand what is real.

are the offspring of the Limit and the Unlimited, but not of the primary principles alone, nor of the hidden intelligible cause, but also of secondary principles that proceed from them and, in cooperation with one another, suffice to generate the intermediate orders of things and the variety that they display. This is why in these orders of being there are ratios proceeding to infinity, but controlled by the principle of the Limit. For number, beginning with unity, is capable of indefinite increase, yet any number you choose is finite; magnitudes likewise are divisible without end, yet the magnitudes distinguished from one another are all bounded, and the actual parts of a whole are limited. If there were no infinity, all magnitudes would be commensurable and there would be nothing inexpressible or irrational, features that are thought to distinguish geometry from arithmetic; nor could numbers exhibit the generative power of the monad, nor would they have in them all the ratios—such as multiple and superparticular[10]—that are in things. For every number that we examine has a different ratio to unity and to the number just before it. And if the Limit were absent, there would be no commensurability or identity of ratios in mathematics, no similarity and equality of figures, nor anything else that belongs in the column of the better. There would not even be any sciences dealing with such matters, nor any fixed and precise concepts. Thus mathematics needs both these principles as do the other realms of being. As for the lowest realities, those that appear in matter and are molded by nature, it is quite obvious at once that they partake of both principles, of the Unlimited as the ground that underlies their forms and of the Limit by virtue of their ratios, figures, and shapes. It is clear, then, that the principles primary in mathematics are those that preside over all things.

Chapter 3

The Common Theorems Governing Mathematical Kinds

Just as we have noted these common principles and seen that they pervade all classes of mathematical objects, so let us enumerate the simple theorems that are common to them all, that is, the theorems generated by the single science that embraces alike all forms of mathematical knowledge; and let us see how they fit into all these sciences and can be observed alike in numbers, magnitudes, and motions. Such are the theorems governing proportion, namely, the rules of compounding, dividing, converting, and alternating: likewise the theorems concerning ratios of all kinds, multiple, superparticular, superpartient, and their counterparts; and the theorems about equality and inequality in their most general and universal aspects, not equality or inequality of figures, numbers, or motions, but each of the two by itself as having a nature common to all its forms and capable of more simple apprehension. And certainly beauty and order are common to all branches of mathematics, as are the method of proceeding from things better known to things we seek to know and the reverse path from the latter to the former, the methods called analysis and synthesis. Likeness and unlikeness of ratios are not absent from any branch of mathematics, for we call some figures similar and other dissimilar, and in the same way some numbers like and others unlike. And matters pertaining to powers obviously belong to general mathematics, whether they be roots or squares. All these

[10] These ratios are discussed in the excerpt from Boethius's *On Arithmetic* in chapter 2.

Socrates in the *Republic* puts in the mouths of his loftily-speaking Muses, bringing together in determinate limits the elements common to all mathematical ratios and setting them up in specific numbers by which the periods of fruitful birth and of its opposite, unfruitfulness, can be discerned.

Chapter 8

The Utility of Mathematics

From what we have said it is clear that mathematical science makes a contribution of the greatest importance to philosophy and to its particular branches, which we must also mention. For theology, first of all, mathematics prepares our intellectual apprehension. Those truths about the gods that are difficult for imperfect minds to discover and understand, these the science of mathematics, with the help of likenesses, shows to be trustworthy, evident, and irrefutable. It proves that numbers reflect the properties of beings above being and in the objects studied by the understanding reveals the powers of the intellectual figures. Thus Plato teaches us many wonderful doctrines about the gods by means of mathematical forms, and the philosophy of the Pythagoreans clothes its secret theological teaching in such draperies. . . .

Mathematics also makes contributions of the very greatest value to physical science. It reveals the orderliness of the ratios according to which the universe is constructed and the proportion that binds things together in the cosmos, making, as the *Timaeus* somewhere says, divergent and warring factors into friends and sympathetic companions. It exhibits the simple and primal causal elements as everywhere clinging fast to one another in symmetry and equality, the properties through which the whole heaven was perfected when it took upon itself the figures appropriate to its particular region; and it discovers, furthermore, the numbers applicable to all generated things and to their periods of activity and of return to their starting points, by which it is possible to calculate the times of fruitfulness or the reverse for each of them. All these I believe the *Timaeus* sets forth, using mathematical language throughout in expounding its theory of the nature of the universe. It regulates by numbers and figures the generation of the elements, showing how their powers, characteristics, and activities are derived therefrom and tracing the causes of all change back to the acuteness or obtuseness of their angles, the uniformity or diversity of their sides, and the number or fewness of the elements involved.

How, then, can we deny that mathematics brings many remarkable benefits to what is called political philosophy? By measuring the periods of activity and the varied revolutions of the All, it finds the numbers that are appropriate for generation, that is, those that cause homogeneity or diversity in progeny, those that are fruitful and perfecting and their opposites, those that bring a harmonious life in their train and those that being discord, and in general those that are responsible for prosperity and those that occasion want. . . .

Finally, how much benefit mathematics confers on the other sciences and arts we can learn when we reflect that to the theoretical arts, such as rhetoric and all those like it that function through discourse, it contributes completeness and orderliness, by providing for them a likeness of a whole made perfect through first, intermediate, and concluding parts; that to the poetical arts it stands as a paradigm, furnishing in itself models for the speeches that the authors compose and the meters that they employ;

and that for the practical arts it defines their motion and activity through its own fixed and unchangeable forms. In general, as Socrates says in the *Philebus*, all the arts require the aid of counting, measuring, and weighing, of one or all of them; and these arts are all included in mathematical reasonings and are made definite by them, for it is mathematics that knows the divisions of numbers, the variety of measures, and the differences of weights. These considerations will make evident to the student the utility of general mathematics both to philosophy itself and to the other sciences and arts.

Chapter 12

The Pythagorean Classification of the Mathematical Sciences

We must next distinguish the species of mathematical science and determine what and how many they are; for after its generic and all-inclusive form it is necessary to consider the specific differences between the particular sciences. The Pythagoreans considered all mathematical science to be divided into four parts: one half they marked off as concerned with quantity, the other half with magnitude; and each of these they posited as twofold. A quantity can be considered in regard to its character by itself or in its relation to another quantity, magnitudes as either stationary or in motion. Arithmetic, then, studies quantity as such, music the relations between quantities, geometry magnitude at rest, spherics [that is, astronomy] magnitude inherently moving. The Pythagoreans consider quantity and magnitude not in their generality, however, but only as finite in each case. For they say that the sciences study the finite in abstraction from infinite quantities and magnitudes, since it is impossible to comprehend infinity in either of them. Since this assertion is made by men who have reached the summit of wisdom, it is not for us to demand that we be taught about quantity in sense objects or magnitude that appears in bodies. To examine these matters is, I think the province of the science of nature, not that of mathematics itself. . . .

Chapter 13

Geminus' Classfication of the Mathematical Sciences

But others, like Geminus, think that mathematics should be divided differently; they think of one part as concerned with intelligibles only and of another as working with perceptibles and in contact with them. By intelligibles, of course, they mean those objects that the soul arouses by herself and contemplates in separation from embodied forms. Of the mathematics that deals with intelligibles they posit arithmetic and geometry as the two primary and most authentic parts, while the mathematics that attends to sensibles contains six sciences: mechanics, astronomy, optics, geodesy, canonics, and calculation. Tactics they do not think it proper to call a part of mathematics, as others do, though they admit that it sometimes uses calculation, as in the enumeration of military forces, and sometimes geodesy, as in the division and measurement of encampments. Much less do they think of history and medicine as parts of mathematics, even though writers of history often bring in mathematical theorems in describing the lie of certain regions or in calculating the size, breadth, or perimeters of cities, and physicians often clarify their own doctrines by such methods, for the utility of astronomy to medicine is made clear by Hippocrates and all who speak of seasons and places. So also the master of tactics will use the theorems of mathematics, even though he is not a mathematician, if he should ever want to lay out a circular camp to make his army appear as small as possible, or a square or pentagonal or some other form of camp to make it appear very large.

These, then, are the species of general mathematics. Geometry in its turn is divided into plane geometry and stereometry. There is no special branch of study devoted to points and lines, inasmuch as no figure can be constructed from them without planes or solids; and it is always the function of geometry, whether plane or solid, either to construct figures or to compound or divide figures already constructed. In the same way arithmetic is divided into the study of linear numbers, plane numbers, and solid numbers; for it examines number as such and its various kinds as they proceed from the number one, investigating the generation of plane numbers, both similar and dissimilar, and progressions to the third dimension. Geodesy and calculation are analogous to these sciences, since they discourse not about intelligible but about sensible numbers and figures. For it is not the function of geodesy to measure cylinders or cones, but heaps of earth considered as cones and wells considered as cylinders; and it does not use intelligible straight lines, but sensible ones, sometimes more precise ones, such as rays of sunlight, sometimes coarser ones, such as a rope or a carpenter's rule. Nor does the student of calculation consider the properties of number as such, but of numbers as present in sensible objects; and hence he gives them names from the things being numbered, calling them sheep numbers or cup numbers. He does not assert, as does the arithmetician, that something is least; nevertheless with respect to any given class he assumes a least, for when he is counting a group of men, one man is his unit. Again optics and canonics [that is, music] are offshoots of geometry and arithmetic. The former science uses visual lines and the angles made by them; it is divided into a part specifically called optics, which explains the illusory appearances presented by objects seen at a distance, such as the converging of parallel lines or the rounded appearance of square towers, and general catoptrics, which is concerned with the various ways in which light is reflected. The latter is closely bound up with the art of representation and studies what is called "scene-painting," showing how objects can be represented by images that will not seem disproportionate or shapeless when seen at a distance or on an elevation. The science of canonics deals with the perceptible ratios between notes of the musical scales and discovers the divisions of the monochord, everywhere relying on sense-perception and, as Plato says, "putting the ear ahead of the mind."

In addition to these there is the science called mechanics, a part of the study of perceptible and embodied forms. Under it comes the art of making useful engines of war, like the machines that Archimedes is credited with devising for defense against the besiegers of Syracuse, and also the art of wonder-working, which invents figures moved sometimes by wind, like those written about by Ctesibius[11] and Heron, sometimes by weights, whose imbalance and balance respectively are responsible for movement and rest, as the *Timaeus* shows, and sometimes by cords and ropes in imitation of the tendons and movements of living beings. Under mechanics also falls the science of equilibrium in general and the study of the so-called center of gravity, as well as the art of making spheres imitating the revolutions of the heavens, such as was cultivated by Archimedes,

[11] Ctsebius lived in Alexandria in the third century BCE and was probably the first head of the museum there.

(continued...)

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