CONTENTS

5. William Thomson and the Infinitely Long Telegraph Cable Equation

6. Epilogue

CHAPTER 1
Mathematics and Physics

1.1 Introduction

When Isaac Newton showed the intimate connection between celestial mechanics and math by extending the inverse-square law of gravity from mere earthly confines to the entire universe, and when James Clerk Maxwell used math to join together the erstwhile separate subjects of magnetism and electricity, they gave science two examples of the *mutual embrace* (to use Maxwell's words) of math and physics. They had performed what are today called the first two *unifications* of mathematical physics.

Two centuries separated those two unifications, but the next one came much faster, with Albert Einstein's connection of space and time in the special theory of relativity and then, soon after, together with gravity in the general theory, less than a century after Maxwell. Again, it was mathematics that was the glue that sealed the union, but now there was a significant difference. With Newton and Maxwell, the required math was already known to physicists beforehand; but with Einstein, it was not. Einstein was an excellent *applied* mathematician, but he was not a creator of new math and so, in the early 1900s, he was in a semi-desperate state.

As Einstein himself put it, "I didn't become a mathematician because mathematics was so full of beautiful and difficult problems that one might waste one's power in pursuing them without finding the central problem."[1] When he needed tensor calculus to codify the physical principles of general relativity, he had to plead for aid from an old friend, a former fellow student who had helped him pass his college math exams.[2] As one of Einstein's recent biographers has memorialized this interesting situation, when he realized he didn't have the necessary math to express his insights into the *physics* of gravity, Einstein exclaimed "Grossman, you've got to help me or I will go crazy."[3] And, good friend that he was, Einstein's pal Marcel *did* help. And that's how Einstein learned how to mathematically express what he knew *physically*, and thus were born the beautiful, coupled, nonlinear partial differential equations of general relativity that generations of theoretical physicists have wrestled with now for over a century.

I tell you all this for two reasons. First, it's *not* to tease the memory of Einstein (who was, I surely don't have to tell you, a once-in-a-century genius), but rather to heap praise on the mathematicians—people like the German Bernhard Riemann (1826–1866) and the Italians Gregorio Ricci-Curbastro (1853–1925) and Tullio Levi-Civita (1873–1941)—who had developed the math needed by Einstein long before Einstein knew he would need it. And second, because there is an earlier, equally dramatic but not so well-known occurrence of this sort of anticipatory good fortune in mathematical physics. It is that earlier story that has inspired this book.

1.2 Fourier and *The Analytical Theory of Heat*

Sometime around 1804 or so (perhaps even as early as 1801), the French mathematical physicist and political activist Jean Baptiste Joseph Fourier (1768–1830)—who came perilously close to being separated from his world-class brain by the guillotine during the Terror of the French Revolution[4]—began his studies on how heat energy propagates in solid matter. In other words, it was then that he started pondering the physics of *hot molecules in bulk* (and so now you can see where the first half of the title of this book comes from). In the opening of his masterpiece, *The Analytical Theory of Heat* (1822), about which I'll say more in just a bit, Fourier tells us *why* he decided to study heat: "Heat, like gravity, penetrates every substance of the universe. . . . The object of our work is to set forth the mathematical laws [of heat]. The theory of heat will hereafter form one of the most important branches of general physics."

A few years after beginning his studies (1807), he had progressed far enough to write a long report of his work called *On the Propagation of Heat in Solid Bodies*, which received some pretty severe criticism. The critics weren't quacks, but rather included such scientific luminaries as Joseph-Louis Lagrange (1736–1813) and Pierre-Simon Laplace

(1749–1827), who, while certainly pretty smart fellows themselves, nonetheless stumbled over the sheer novelty of Fourier's math. Fourier, you see, didn't hesitate to expand arbitrary periodic functions of space and time in the form of infinite sums of trigonometric terms (what we today call *Fourier series*). Lagrange and Laplace just didn't think that was possible. Fourier, of course, was greatly disappointed by the skepticism. But fortunately, he was not discouraged by the initial lack of enthusiasm. He didn't give up, and continued his studies of heat in matter.[5]

In 1817 Fourier's talent was formally recognized, and he was elected to the French Academy of Sciences, becoming in 1822 the secretary to the mathematical section. That same year finally saw the official publication of his work on heat, a work that is still an impressive read today. In *The Analytical Theory of Heat*, Fourier included his unpublished 1807 effort, plus much more on the representation of periodic functions as infinite sums of trigonometric terms. His *mathematical* discoveries on how to write such series were crucial in his additional discoveries on how to solve the fundamental *physics* equation of heat flow, the aptly named *heat equation*, which is (just to be precise) a second-order partial differential equation. (This will prove to be *not* so scary as it might initially sound.)

In the following chapters of the first part of this book, we'll develop Fourier's math, then derive the heat equation from first principles (conservation of energy), and then use Fourier's math to solve the heat equation and to numerically evaluate some interesting special cases (including a calculation of the age of the Earth). Then, in the penultimate chapter of the book, I'll show you how the man who did that calculation of the age of the Earth—the Irish-born Scottish mathematical physicist and engineer William Thomson (1824–1907)—discovered a quarter-century after Fourier's death that the heat equation is also the defining physics, under certain circumstances, of a very long submarine telegraph cable (in particular, the famous trans-Atlantic electric cables of the mid-19th century).

Thomson, who was knighted by Queen Victoria in 1866 for his cable work (and later, in 1892, was elevated to the peerage to become the famous Lord Kelvin), directly used and credited Fourier's mathematics in his pioneering study of electric communication cables. The Atlantic cables, in particular, lay deep (in places, up to 15,000 feet beneath the surface) in the cold waters of the Atlantic. And since electric current is caused by the motion of electrons, you now see where the second half of the title of this book comes from.

Telegraphy was the very first commercial application of electricity, being introduced in England by railroad operators as early as 1837. This date is doubly impressive when it is remembered that the electric battery (*voltaic pile*) had been invented by the Italian Alessandro Volta (1745–1827) less than 40 years before. Then, less than 70 years after the battery, messages were being routinely sent through a submarine cable thousands of miles long lying nearly 3 miles beneath the stormy Atlantic Ocean, an accomplishment that struck the imaginations of all but the dullest like a thunderbolt—*it was nothing less than a miracle*—and the men behind the creation of the trans-Atlantic cable became scientific and engineering superstars. What you'll read in this book is the mathematical physics of what those men did, based on the mathematical physics of Fourier's theory of the flow of heat energy in matter.

The technical problems discussed in this book are routinely attacked today by electrical engineers using a powerful mathematical technique generally called the *operational calculus* (specifically, the *Laplace transform*). The transform had been around in mathematics long before the engineers became aware of it in the 1930s, but it was *not* the tool Fourier and Thomson used to solve the equations they encountered. They instead used the classical mathematical techniques of their day, what is called *time domain* mathematics, rather than the modern transform domain approach of engineers. Fourier and Thomson were enormously clever analysts, and since my intention in this book is to weave the historical with the technical, everything you read here is just how either man might have written this book. There are *lots* of other books available that discuss the transform approach, and I'll let you look one up if you're curious.[6]

Now, before we do anything else, let me first show you a little math exercise that is embedded, with little fanfare, in *The Analytical Theory of Heat*, one that uses nothing but high school AP-calculus. It illustrates how an ingenious mind can extract amazing results from what, to less clever minds, appears to be only routine, everyday stuff. What I am about to show you is a mere half page in *Analytical Theory*, but I'm going to elaborate (that is, inflate) it all just a bit to make sure I cover all bets. My reference is the classic 1878 English translation from the original French by Alexander Freeman (1838–1897), who was a Fellow at St. John's College, Cambridge. (You can find exactly what Fourier wrote on page 153 of the 1955 Dover Publications reprint—itself now a minor classic—of Freeman's translation.) Going through this preliminary exercise will sharpen your appreciation for the genius of Fourier.

1.3 A First Peek into Fourier's Mathematical Mind

We'll start with something you could just look up in a math handbook, but, since I want to impress you with how a good high school student (certainly a college freshman) could do all that follows with nothing but a stick to write with on a sandy beach, let's begin by deriving the indefinite integration formula

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C,$$

where C is an arbitrary constant.

Look at Figure 1.3.1, which shows a right triangle with perpendicular sides 1 and x, and so geometry (the Pythagorean theorem) says the hypotenuse is $\sqrt{1+x^2}$. The base angle is denoted by θ, and so we have, by construction,

$$x = \tan(\theta). \tag{1.3.1}$$

If we differentiate (1.3.1) with respect to x, we'll get

$$1 = \frac{d}{dx}\tan(\theta) = \frac{d}{dx}\left\{\frac{\sin(\theta)}{\cos(\theta)}\right\} = \frac{\cos^2(\theta)\dfrac{d\theta}{dx} + \sin^2(\theta)\dfrac{d\theta}{dx}}{\cos^2(\theta)}$$

or,

$$1 = \frac{d\theta}{dx}\,\frac{\cos^2(\theta)+\sin^2(\theta)}{\cos^2(\theta)}$$

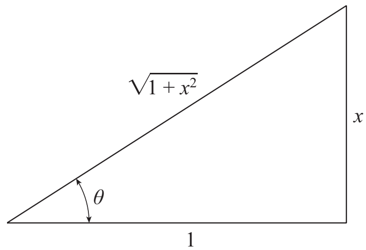

FIGURE 1.3.1. A right triangle, with x any positive value.

or, recalling the identity $\cos^2(\theta)+\sin^2(\theta)=1$,

$$1=\frac{d\theta}{dx}\frac{1}{\left\{\frac{1}{\sqrt{1+x^2}}\right\}^2}=\frac{d\theta}{dx}(1+x^2).$$

That is,

$$\frac{d\theta}{dx}=\frac{1}{1+x^2}. \tag{1.3.2}$$

But from (1.3.1), we have

$$\theta=\tan^{-1}(x)$$

and so, putting that into (1.3.2), we have

$$\frac{d}{dx}\tan^{-1}(x)=\frac{1}{1+x^2},$$

which, when integrated indefinitely, instantly gives us our result:

$$\tan^{-1}(x)+C=\int\frac{dx}{1+x^2}, \tag{1.3.3}$$

where C is some (any) constant.

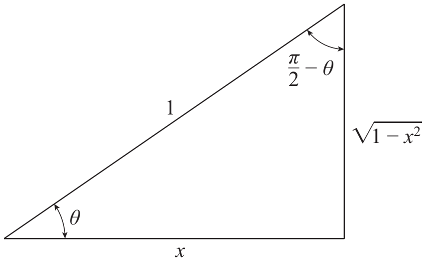

FIGURE 1.3.2. Another right triangle.

Okay, put (1.3.3) aside for now, and look at Figure 1.3.2, which shows a right triangle with a hypotenuse of 1, and one of the perpendicular sides as x. Thus, the other perpendicular side is $\sqrt{1-x^2}$. Since Euclid tells us that the two acute angles sum to $\frac{\pi}{2}$ radians, we can immediately write

$$\frac{\pi}{2} = \tan^{-1}\left\{\frac{\sqrt{1-x^2}}{x}\right\} + \tan^{-1}\left\{\frac{x}{\sqrt{1-x^2}}\right\}. \tag{1.3.4}$$

If we write

$$u = \frac{\sqrt{1-x^2}}{x},$$

then (1.3.4) becomes the identity

$$\frac{\pi}{2} = \tan^{-1}\{u\} + \tan^{-1}\left\{\frac{1}{u}\right\}. \tag{1.3.5}$$

Okay, put (1.3.5) aside for now, and turn your attention to the claim

$$\frac{1}{1+u^2} = 1 - u^2 + u^4 - u^6 + \cdots. \tag{1.3.6}$$

Do you see where this comes from? If not, just multiply through by $1+u^2$, and watch how (1.3.6) reduces to $1=1$, which is pretty hard to deny! (Or, just do the long division on the left of (1.3.6) directly.) Now, integrate both sides of (1.3.6) which, recalling (1.3.3), says

$$\int \frac{du}{1+u^2} = \tan^{-1}(u) + C = u - \frac{1}{3}u^3 + \frac{1}{5}u^5 - \frac{1}{7}u^7 + \cdots.$$

When $u=0$, it's clear that the infinite sum on the right is zero, and since $\tan^{-1}(0)=0$, then $C=0$. That is,

$$\tan^{-1}(u) = u - \frac{1}{3}u^3 + \frac{1}{5}u^5 - \frac{1}{7}u^7 + \cdots. \tag{1.3.7}$$

In particular, if we set $u=1$, then (1.3.7) reduces to the beautiful (if computationally useless way to compute π, because convergence is *extremely* slow)

$$\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

a result discovered by other means in 1682 by the German mathematician Gottfried Wilhelm von Leibniz (1646–1716). It is interesting to note, in passing, the similar-looking series discovered in 1668 by the Danish-born mathematician Nicolaus Mercator (1620–1687):

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots.$$

Since (1.3.7) is an identity in u, it remains true if we replace every u with $\frac{1}{u}$, and so

$$\tan^{-1}\left(\frac{1}{u}\right) = \frac{1}{u} - \frac{1}{3}\left(\frac{1}{u}\right)^3 + \frac{1}{5}\left(\frac{1}{u}\right)^5 - \frac{1}{7}\left(\frac{1}{u}\right)^7 + \cdots. \tag{1.3.8}$$

Thus, from (1.3.5), (1.3.7), and (1.3.8), we have

$$\frac{\pi}{2} = \left(u + \frac{1}{u}\right) - \frac{1}{3}\left(u^3 + \frac{1}{u^3}\right) + \frac{1}{5}\left(u^5 + \frac{1}{u^5}\right) - \cdots.$$

Then, as Fourier wrote, "if we now write $e^{x\sqrt{-1}}$[$= e^{ix}$, in modern notation, where $i = \sqrt{-1}$] instead of u . . . we shall have"

$$\frac{\pi}{2} = (e^{ix} + e^{-ix}) - \frac{1}{3}(e^{i3x} + e^{-i3x}) + \frac{1}{5}(e^{i5x} + e^{-i5x}) - \cdots.$$

Using a famous identity (sometimes called a "fabulous formula")[7] due to the Swiss mathematician Leonhard Euler (1707–1783), who published it in 1748,

$$e^{i\theta} = \cos(\theta) + i\,\sin(\theta),$$

it follows that

$$e^{ix} + e^{-ix} = 2\cos(x),\ e^{i3x} + e^{-i3x} = 2\cos(3x),\ e^{i5x} + e^{-i5x} = 2\cos(5x), \ldots,$$

and so on. Using this, Fourier then immediately wrote

$$\frac{\pi}{4} = \cos(x) - \frac{1}{3}\cos(3x) + \frac{1}{5}\cos(5x) - \frac{1}{7}\cos(7x) + \cdots, \tag{1.3.9}$$

which is one of his famous infinite sums of trigonometric functions that Lagrange so objected to in 1807. This gives Leibniz's sum when $x = 0$, but now we see that Fourier has gone far beyond Leibniz, declaring that the sum is $\frac{\pi}{4} \approx 0.785$ for *lots* of other values of x as well. This is, I think you'll agree, a pretty remarkable claim!

With the invention of electronic computers and easy-to-use programming languages, it is a simple matter to experiment numerically with (1.3.9), and Figure 1.3.3 shows what (1.3.9) looks like, where the right-hand-side of (1.3.9) has been calculated from the first 100 terms of the sum (that is, up to and including $\frac{1}{199}\cos(199x)$) for each of 20,000 values of x uniformly distributed over the interval $-10 < x < 10$.

Suppose a function $f(t)$ is written in the form of a power series. That is,

$$f(t)=c_0+c_1t+c_2t^2+\cdots+c_nt^n+\cdots.$$

It's a freshman calculus exercise to show that all the coefficients follow from the general rule

$$c_n=\frac{1}{n!}\left(\left.\frac{d^n f}{dt^n}\right|_{t=0}\right),\ n\geq 1,$$

that is, by taking successive derivatives of $f(t)$, and after each differentiation, setting $t=0$. (The $n=0$ case means, literally, *don't* differentiate, just set $t=0$.) In this way, it is found, for example, that

$$\sin(t)=t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\cdots,$$
$$\cos(t)=1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\cdots,$$
$$e^t=1+t+\frac{1}{2!}t^2+\frac{1}{3!}t^3+\frac{1}{4!}t^4+\frac{1}{5!}t^5+\cdots.$$

Now, in the last series, set $t=ix$. Then

$$\begin{aligned}e^{ix}&=1+ix+\frac{1}{2!}(ix)^2+\frac{1}{3!}(ix)^3+\frac{1}{4!}(ix)^4+\frac{1}{5!}(ix)^5+\cdots\\&=1+ix-\frac{1}{2!}x^2-i\frac{1}{3!}x^3+\frac{1}{4!}x^4+i\frac{1}{5!}x^5+\cdots\\&=\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\cdots\right)+i\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots\right)\\&=\cos(x)+i\sin(x).\end{aligned}$$

This is an identity in x, and so continues to hold if we replace every x with a θ to give us our result: $e^{i\theta}=\cos(\theta)+i\sin(\theta)$.

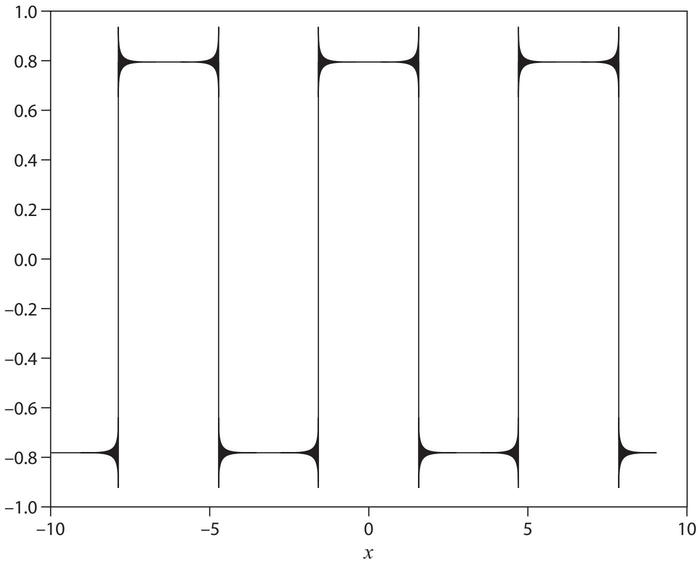

FIGURE 1.3.3. A computer-generated plot of Fourier's equation (1.3.9).

As you can see from Figure 1.3.3, the numerical "evidence" does appear to suggest that Fourier's math is correct—*sort of* (this numerical experiment means, admittedly, very little, if anything, to a pure mathematician, but it is quite compelling for physicists and engineers). The sum flips back and forth between two values,[8] $\frac{\pi}{4}$ and $-\frac{\pi}{4}$, with the swings occurring at odd multiples of $\frac{\pi}{2}$ for x. As Fourier states in *Analytical Theory* (page 144 in Freeman's translation), "the [sum] is $\frac{1}{4}\pi$ if [x] is included between 0 and $\frac{1}{2}\pi$, but . . . is $-\frac{1}{4}\pi$, if [x] is included between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$." Figure 1.3.3 also suggests that Fourier's comment implies that he did *not* actually make such a plot, because if he had, he would certainly have noticed all the dramatic oscillatory behavior around the transition points. In fact, Fourier made no comment at all on this hard-to-miss feature. For some reason, making such a plot wasn't done until 1848(!), when the twenty-two-year-old Henry Wilbraham (1825–1883) finally did so; his plots clearly show the oscillations.

After publishing his discovery in the *Cambridge and Dublin Mathematical Journal*, Wilbraham, a recent graduate of Trinity College, Cambridge, authored a few more mathematical papers and then, for some unknown reason, disappeared from the world of mathematics. When the oscillations were rediscovered again, many years later, they were named after somebody else: they are now called the *Gibbs phenomenon*, after the American mathematical physicist J. W. Gibbs (1839–1903), who briefly commented on them in an 1899 letter to the British science journal *Nature*. The oscillations occur in any Fourier series that represents a *discontinuous* function.[9] Mathematicians have known since 1906 that such a Fourier series converges to the *average* of the function's values on each side of the discontinuity when the series is evaluated *at* the point of discontinuity.

A dramatic calculation, one that also appears in *Analytical Theory*, is based on the integration of (1.3.9), which results in

$$\frac{\pi}{4}x = \sin(x) - \frac{1}{3^2}\sin(3x) + \frac{1}{5^2}\sin(5x) - \frac{1}{7^2}\sin(7x) + \cdots, \tag{1.3.10}$$

where the arbitrary constant of integration is zero (do you see why?—Evaluate (1.3.10) for $x=0$). (Figure 1.3.4, a plot of the right-hand side of (1.3.10), shows that while (1.3.9) is discontinuous, its integral is *continuous*.) If we substitute $x=\frac{\pi}{2}$ in (1.3.10), we get

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots. \tag{1.3.11}$$

Now, a quarter century before Fourier's birth, one of the great unsolved problems that had been taunting mathematicians for *centuries* was the calculation of

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots.$$

The calculation of S had quickly become the next obvious problem to attack after the surprising discovery by the 14th-century French math-

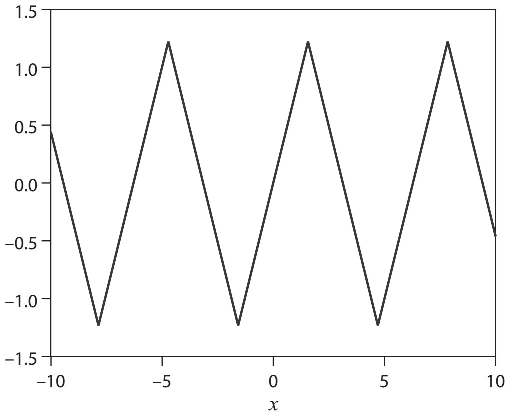

FIGURE 1.3.4. Integral of the right-hand side of (1.3.9).

ematician and philosopher Nicoleo Oresme (1320–1382) that the sum (called the *harmonic series*), defined as

$$H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots,$$

diverges. This is a counterintuitive result for most people, because the terms seem to add *so very slowly* (for the partial sum to exceed 15 takes more than 1.6 million terms, and to reach a partial sum of 100 takes more than 1.5×10^{43} terms).

To prove that $H = \infty$ is not hard. Simply write H as

$$H = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + (\text{next 8 terms}) + (\text{next 16 terms}) + \cdots,$$

and then observe that

$$H > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots,$$

where each new pair of parentheses contains twice as many terms as the previous pair, and each term in a pair is replaced with the last (smallest) term in the pair. The sum of the terms in each modified pair is then $\frac{1}{2}$, and so we have formed a *lower* bound on H that is the sum of an infinite number of $\frac{1}{2}$s. That is, H is "greater than infinity," so to speak, which is just an enthusiastic way of saying H itself blows up.

But that's *not* what happens with S, which can be written as

$$S=\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)+\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots\right). \tag{1.3.12}$$

But since

$$\begin{aligned}\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots&=\frac{1}{(2\times 1)^2}+\frac{1}{(2\times 2)^2}+\frac{1}{(2\times 3)^2}+\cdots\\&=\frac{1}{2^2}\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right]=\frac{1}{4}S,\end{aligned}$$

then using (1.3.11) and (1.3.12), we have

$$S=\frac{\pi^2}{8}+\frac{1}{4}S.$$

This is easily solved to give the finite sum of

$$S=\left(\frac{4}{3}\right)\left(\frac{\pi^2}{8}\right)=\frac{\pi^2}{6}, \tag{1.3.13}$$

a result due (via other, more complicated means) to Euler; this discovery (in 1734, 400 years after Oresme) made Euler a superstar in the world of mathematics.[10] Still, *this* derivation is pretty straightforward, depending essentially on nothing much more than the elementary properties of right triangles. With Fourier's approach to the problem, any college freshman today can do in minutes what it took a genius like Euler *years* to do three centuries ago.

Of particular fascination to Euler and his fellow mathematicians must have been that pi is *squared*. We are used to seeing pi, *alone*, in many "ordinary" applications ($2\pi r$ and πr^2, for example, for the circumference and area, respectively, of a circle with radius r), but π^2 was something new. As the English mathematician Augustus de Morgan (1806–1871) is said to have remarked about pi, *alone*, its appearance in mathematics is so common that one imagines "it comes on many occasions through the window and through the door, sometimes even down the chimney." But not pi *squared*.[11]

Here's another quick calculation using Euler's fabulous formula:

$$(e^{it})^n = \{\cos(t) + i\,\sin(t)\}^n = e^{int} = \cos(nt) + i\,\sin(nt).$$

This result,

$$\{\cos(t) + i\,\sin(t)\}^n = \cos(nt) + i\,\sin(nt), \tag{1.3.14}$$

is called *De Moivre's theorem*,[12] and it is highly useful in both numerical computations and in theoretical analyses. You can find several examples in a previous book of mine[13] of the theorem's value in avoiding lots of grubby numerical work, so let me show you here an application of (1.3.14) in a theoretical context.

In both pure mathematics and physics, the expressions

$$S_1(t) = \sum_{n=1}^{\infty} r^n \cos(nt)$$

and

$$S_2(t) = \sum_{n=1}^{\infty} r^n \sin(nt)$$

often occur, where r is some real number in the interval $0 \le r < 1$. (Do you see *why* this restriction? Think about convergence.) We can find closed-form expressions for these two infinite sums as follows. We start by defining

$$S = \sum_{n=1}^{\infty} z^n = z + z^2 + z^3 + \cdots,$$

with

$$z = r\{\cos(t) + i\,\sin(t)\}.$$

S is simply a geometric series, easily evaluated in the usual way by multiplying through by z. This gives

$$S=\frac{z}{1-z}=\frac{r\cos(t)+i\,r\sin(t)}{1-r\cos(t)-i\,r\sin(t)}.$$

Multiplying top and bottom of the right-hand side by the conjugate[14] of the bottom gives

$$S=\frac{\{r\cos(t)+i\,r\sin(t)\}\{1-r\cos(t)+i\,r\sin(t)\}}{\{1-r\cos(t)\}^2+r^2\sin^2(t)},$$

which reduces to

$$S=\frac{r\cos(t)-r^2+i\,r\sin(t)}{1-2r\cos(t)+r^2}.$$

Now, by De Moivre's theorem, we also have

$$S=\sum_{n=1}^{\infty}z^n=\sum_{n=1}^{\infty}r^n\{\cos(t)+i\sin(t)\}^n=\sum_{n=1}^{\infty}r^n\{\cos(nt)+i\sin(nt)\},$$

and so

$$S=S_1+iS_2.$$

Equating the imaginary parts of our two results for S gives

$$S_2(t)=\sum_{n=1}^{\infty}r^n\sin(nt)=\frac{r\sin(t)}{1-2r\cos(t)+r^2},\ 0\le r<1.$$

And equating real parts, we have

$$S_1(t)=\sum_{n=1}^{\infty}r^n\cos(nt)=\frac{r\cos(t)-r^2}{1-2r\cos(t)+r^2},\ 0\le r<1.$$

With a bit of additional algebra, $S_1(t)$ is often expressed in the alternative form:

$$\frac{r\cos(t)-r^2}{1-2r\cos(t)+r^2}=\frac{1}{2}\left[\frac{2r\cos(t)-2r^2}{1-2r\cos(t)+r^2}\right]=\frac{1}{2}\left[\frac{1-r^2-1+2r\cos(t)-r^2}{1-2r\cos(t)+r^2}\right]$$
$$=\frac{1}{2}\left[\frac{1-r^2}{1-2r\cos(t)+r^2}-\frac{1-2r\cos(t)+r^2}{1-2r\cos(t)+r^2}\right]$$
$$=\frac{1}{2}\left[\frac{1-r^2}{1-2r\cos(t)+r^2}-1\right],$$

and so we have

$$S_1(t)=\sum\nolimits_{n=1}^{\infty}r^n\cos(nt)=\frac{1}{2}\left[\frac{1-r^2}{1-2r\cos(t)+r^2}\right]-\frac{1}{2},\ 0\le r<1.$$

The quantity

$$\frac{1-r^2}{1-2r\cos(t)+r^2}$$

occurs often enough in advanced mathematics that it has been given its own name: *Poisson's kernel*.[15]

Well, okay, all of this is undeniably fun stuff, but it is relatively lightweight compared to what Fourier did mathematically for physics in *Analytical Theory*. As a quick flip through the rest of this book will show you, there are a *lot* of equations in it. At the beginning of *Analytical Theory*, in what he called a "Preliminary Discourse," Fourier explained to his readers why that was equally so in his book, and his words explain why it is true for this book as well. As he wrote, "Profound study of nature is the most fertile source of mathematical discoveries. . . . Mathematical analysis is as extensive as nature itself." So, all the math you'll read here isn't here because I want to make your life difficult—*it's here because that's the way the world is made.*

To lay the foundation of Fourier's mathematics will take a couple more chapters and so, with no further delay, let's get started.

INDEX