

Contents

<i>List of Color Plates</i>	ix
<i>Preface</i>	xi
<i>Prologue: Cambridge, England, 1993</i>	1
1 Mesopotamia, 1800 BCE	4
Sidebar 1: Did the Egyptians Know It?	13
2 Pythagoras	17
3 Euclid's <i>Elements</i>	32
Sidebar 2: The Pythagorean Theorem in Art, Poetry, and Prose	45
4 Archimedes	50
5 Translators and Commentators, 500–1500 CE	57
6 François Viète Makes History	76
7 From the Infinite to the Infinitesimal	82
Sidebar 3: A Remarkable Formula by Euler	94
8 371 Proofs, and Then Some	98
Sidebar 4: The Folding Bag	115
Sidebar 5: Einstein Meets Pythagoras	117
Sidebar 6: A Most Unusual Proof	119
9 A Theme and Variations	123
Sidebar 7: A Pythagorean Curiosity	140
Sidebar 8: A Case of Overuse	142
10 Strange Coordinates	145
11 Notation, Notation, Notation	158

viii ❖ Contents

12	From Flat Space to Curved Spacetime	168
	Sidebar 9: A Case of Misuse	177
13	Prelude to Relativity	181
14	From Bern to Berlin, 1905–1915	188
	Sidebar 10: Four Pythagorean Brainteasers	197
15	But Is It Universal?	201
16	Afterthoughts	208
	 <i>Epilogue: Samos, 2005</i>	 213
	 Appendixes	
A.	How did the Babylonians Approximate $\sqrt{2}$?	219
B.	Pythagorean Triples	221
C.	Sums of Two Squares	223
D.	A Proof that $\sqrt{2}$ is Irrational	227
E.	Archimedes' Formula for Circumscribing Polygons	229
F.	Proof of some Formulas from Chapter 7	231
G.	Deriving the Equation $x^{2/3} + y^{2/3} = 1$	235
H.	Solutions to Brainteasers	237
I.	A Most Unusual Proof	241
	 <i>Chronology</i>	 245
	<i>Bibliography</i>	251
	<i>Illustrations Credits</i>	255
	<i>Index</i>	257

Mesopotamia, 1800 BCE

We would more properly have to call
“Babylonian” many things which the Greek
tradition had brought down to us as
“Pythagorean.”

—Otto Neugebauer, quoted in Bartel van der Waerden,
Science Awakening, p. 77

The vast region stretching from the Euphrates and Tigris Rivers in the east to the mountains of Lebanon in the west is known as the Fertile Crescent. It was here, in modern Iraq, that one of the great civilizations of antiquity rose to prominence four thousand years ago: Mesopotamia. Hundreds of thousands of clay tablets, found over the past two centuries, attest to a people who flourished in commerce and architecture, kept accurate records of astronomical events, excelled in the arts and literature, and, under the rule of Hammurabi, created the first legal code in history. Only a small fraction of this vast archaeological treasure trove has been studied by scholars; the great majority of tablets lie in the basements of museums around the world, awaiting their turn to be deciphered and give us a glimpse into the daily life of ancient Babylon.

Among the tablets that have received special scrutiny is one with the unassuming designation “YBC 7289,” meaning that it is tablet number 7289 in the Babylonian Collection of Yale University (fig. 1.1). The tablet dates from the Old Babylonian period of the Hammurabi dynasty, roughly 1800–1600 BCE. It shows a tilted square and its two diagonals, with some marks engraved along one side and under the horizontal diagonal. The marks are in cuneiform (wedge-shaped) characters, carved with a stylus into a piece of soft clay which was then dried in the sun or baked in an oven. They turn out to be numbers, written in the peculiar Babylonian numeration system that used the base 60. In this *sexagesimal system*, numbers up to 59 were written in essentially our modern base-ten numeration system, but without a zero. Units were written as vertical Y-shaped notches, while tens were marked with similar notches written horizontally. Let us denote these symbols by | and —, respectively. The number 23, for example, would be written as — — |||. When a number exceeded 59,

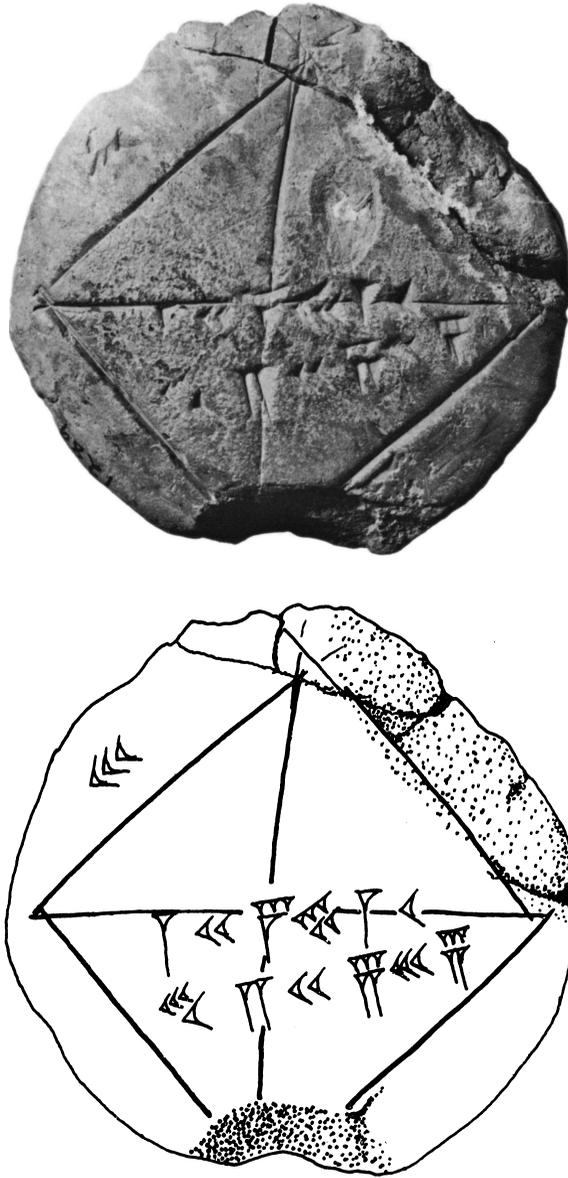


Figure 1.1. YBC 7289

6 ❖ Chapter 1

it was arranged in groups of 60 in much the same way as we bunch numbers into groups of ten in our base-ten system. Thus, 2,413 in the sexagesimal system is $40 \times 60 + 13$, which was written as $\text{— — — — —} \text{—} |||$ (often a group of several identical symbols was stacked, evidently to save space).

Because the Babylonians did not have a symbol for the “empty slot”—our modern zero—there is often an ambiguity as to how the numbers should be grouped. In the example just given, the numerals $\text{— — — — —} \text{—} |||$ could also stand for $40 \times 60^2 + 13 \times 60 = 144,780$; or they could mean $40/60 + 13 = 13.666$, or any other combination of powers of 60 with the coefficients 40 and 13. Moreover, had the scribe made the space between — — — — — and $\text{—} |||$ too small, the number might have erroneously been read as $\text{— — — — —} \text{—} |||$, that is, $50 \times 60 + 3 = 3,003$. In such cases the correct interpretation must be deduced from the context, presenting an additional challenge to scholars trying to decipher these ancient documents.

Luckily, in the case of YBC 7289 the task was relatively easy. The number along the upper-left side is easily recognized as 30. The one immediately under the horizontal diagonal is 1;24,51,10 (we are using here the modern notation for writing Babylonian numbers, in which commas separate the sexagesimal “digits,” and a semicolon separates the integral part of a number from its fractional part). Writing this number in our base-10 system, we get $1 + 24/60 + 51/60^2 + 10/60^3 = 1.414213$, which is none other than the decimal value of $\sqrt{2}$, accurate to the nearest one hundred thousandth! And when this number is multiplied by 30, we get 42.426389, which is the sexagesimal number 42;25,35—the number on the second line below the diagonal. The conclusion is inescapable: the Babylonians knew the relation between the length of the diagonal of a square and its side, $d = a\sqrt{2}$. But this in turn means that they were familiar with the Pythagorean theorem—or at the very least, with its special case for the diagonal of a square ($d^2 = a^2 + a^2 = 2a^2$)—more than a thousand years before the great sage for whom it was named.

Two things about this tablet are especially noteworthy. First, it proves that the Babylonians knew how to compute the square root of a number to a remarkable accuracy—in fact, an accuracy equal to that of a modern eight-digit calculator.¹ But even more remarkable is the probable purpose of this particular document: by all likelihood, it was intended as an example of how to find the diagonal of *any* square: simply multiply the length of the side by 1;24,51,10. Most people, when given this task, would follow the “obvious” but more tedious route: start with 30, square it, double the result, and take the square root: $d = \sqrt{30^2 + 30^2} = \sqrt{1800} = 42.4264$, rounded to four places. But suppose you had to do this over and over for squares of different sizes; you would have to repeat the process each time with a new number, a rather tedious task. The anonymous scribe who carved these numbers into a clay tablet nearly four thousand years ago showed us a simpler way: just multiply the side of the square by $\sqrt{2}$ (fig. 1.2). Some simplification!

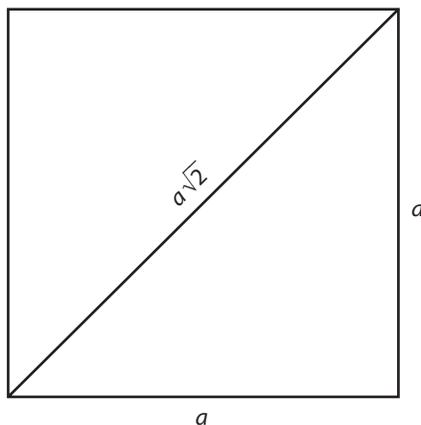


Figure 1.2. A square and its diagonal

But there remains one unanswered question: why did the scribe choose a side of 30 for his example? There are two possible explanations: either this tablet referred to some particular situation, perhaps a square field of side 30 for which it was required to find the length of the diagonal; or—and this is more plausible—he chose 30 because it is one-half of 60 and therefore lends itself to easy multiplication. In our base-ten system, multiplying a number by 5 can be quickly done by halving the number and moving the decimal point one place to the right. For example, $2.86 \times 5 = (2.86/2) \times 10 = 1.43 \times 10 = 14.3$ (more generally, $a \times 5 = \frac{a}{2} \times 10$). Similarly, in the sexagesimal system multiplying a number by 30 can be done by halving the number and moving the “sexagesimal point” one place to the right ($a \times 30 = \frac{a}{2} \times 60$).

Let us see how this works in the case of YBC 7289. We recall that $1;24,51,10$ is short for $1 + 24/60 + 51/60^2 + 10/60^3$. Dividing this by 2, we get $\frac{1}{2} + \frac{12}{60} + \frac{25\frac{1}{2}}{60^2} + \frac{5}{60^3}$, which we must rewrite so that each coefficient of a power of 60 is an integer. To do so, we replace the $1/2$ in the first and third terms by $30/60$, getting $\frac{30}{60} + \frac{12}{60} + \frac{25 + \frac{30}{60}}{60^2} + \frac{5}{60^3} = \frac{42}{60} + \frac{25}{60^2} + \frac{35}{60^3} = 0;42,25,35$. Finally, moving the sexagesimal point one place to the right gives us $42;25,35$, the length of the diagonal. It thus seems that our scribe chose 30 simply for pragmatic reasons: it made his calculations that much easier.



If YBC 7289 is a remarkable example of the Babylonians’ mastery of elementary geometry, another clay tablet from the same period goes even further: it shows that they were familiar with algebraic procedures as well.² Known as

8 ❖ Chapter 1



Figure 1.3. Plimpton 322

Plimpton 322 (so named because it is number 322 in the G. A. Plimpton Collection at Columbia University; see fig. 1.3), it is a table of four columns, which might at first glance appear to be a record of some commercial transaction. A close scrutiny, however, has disclosed something entirely different: the tablet is a list of *Pythagorean triples*, positive integers (a, b, c) such that $a^2 + b^2 = c^2$. Examples of such triples are $(3, 4, 5)$, $(5, 12, 13)$, and $(8, 15, 17)$. Because of the Pythagorean theorem,³ every such triple represents a right triangle with sides of integer length.

Unfortunately, the left edge of the tablet is partially missing, but traces of modern glue found on the edges prove that the missing part broke off after the tablet was discovered, raising the hope that one day it may show up on the antiquities market. Thanks to meticulous scholarly research, the missing part has been partially reconstructed, and we can now read the tablet with relative ease. Table 1.1 reproduces the text in modern notation. There are four columns, of which the rightmost, headed by the words “its name” in the original text, merely gives the sequential number of the lines from 1 to 15. The second and third columns (counting from right to left) are headed “solving number of the diagonal” and “solving number of the width,” respectively; that is, they give the length of the diagonal and of the short side of a rectangle, or equivalently, the length of the hypotenuse and the short leg of a right triangle. We will label these columns with the letters c and b , respectively. As

TABLE 1.1
Plimpton 322

$(c/a)^2$	b	c	
[1,59,0,]15	1,59	2,49	1
[1,56,56,]58,14,50,6,15	56,7	3,12,1	2
[1,55,7,]41,15,33,45	1,16,41	1,50,49	3
[1,]5[3,1]0,29,32,52,16	3,31,49	5,9,1	4
[1,]48,54,1,40	1,5	1,37	5
[1,]47,6,41,40	5,19	8,1	6
[1,]43,11,56,28,26,40	38,11	59,1	7
[1,]41,33,59,3,45	13,19	20,49	8
[1,]38,33,36,36	9,1	12,49	9
1,35,10,2,28,27,24,26,40	1,22,41	2,16,1	10
1,33,45	45	1,15	11
1,29,21,54,2,15	27,59	48,49	12
[1,]27,0,3,45	7,12,1	4,49	13
1,25,48,51,35,6,40	29,31	53,49	14
[1,]23,13,46,40	56	53	15

Note: The numbers in brackets are reconstructed.

an example, the first line shows the entries $b = 1,59$ and $c = 2,49$, which represent the numbers $1 \times 60 + 59 = 119$ and $2 \times 60 + 49 = 169$. A quick calculation gives us the other side as $a = \sqrt{169^2 - 119^2} = \sqrt{14400} = 120$; hence (119, 120, 169) is a Pythagorean triple. Again, in the third line we read $b = 1,16,41 = 1 \times 60^2 + 16 \times 60 + 41 = 4601$, and $c = 1,50,49 = 1 \times 60^2 + 50 \times 60 + 49 = 6649$; therefore, $a = \sqrt{6649^2 - 4601^2} = \sqrt{23\,040\,000} = 4800$, giving us the triple (4601, 4800, 6649).

The table contains some obvious errors. In line 9 we find $b = 9,1 = 9 \times 60 + 1 = 541$ and $c = 12,49 = 12 \times 60 + 49 = 769$, and these do not form a Pythagorean triple (the third number a not being an integer). But if we replace the 9,1 by 8,1 = 481, we do indeed get an integer value for a : $a = \sqrt{769^2 - 481^2} = \sqrt{360\,000} = 600$, resulting in the triple (481, 600, 769). It seems that this error was simply a “typo”; the scribe may have been momentarily distracted and carved nine marks into the soft clay instead of eight; and once the tablet dried in the sun, his oversight became part of recorded history.

10 ❖ Chapter 1

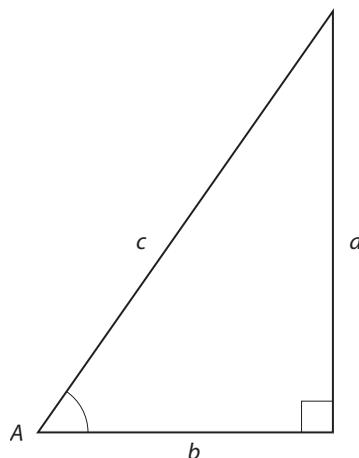


Figure 1.4. The cosecant of an angle: $\csc A = c/a$

Again, in line 13 we have $b = 7,12,1 = 7 \times 60^2 + 12 \times 60 + 1 = 25\,921$ and $c = 4,49 = 4 \times 60 + 49 = 289$, and these do not form a Pythagorean triple; but we may notice that 25 921 is the square of 161, and the numbers 161 and 289 do form the triple (161, 240, 289). It seems the scribe simply forgot to take the square root of 25 921. And in row 15 we find $c = 53$, whereas the correct entry should be twice that number, that is, $106 = 1,46$, producing the triple (56, 90, 106).⁴ These errors leave one with a sense that human nature has not changed over the past four thousand years; our anonymous scribe was no more guilty of negligence than a student begging his or her professor to ignore “just a little stupid mistake” on the exam.⁵

The leftmost column is the most intriguing of all. Its heading again mentions the word “diagonal,” but the exact meaning of the remaining text is not entirely clear. However, when one examines its entries a startling fact comes to light: this column gives the square of the ratio c/a , that is, the value of $\csc^2 A$, where A is the angle opposite side a and \csc is the cosecant function studied in trigonometry (fig. 1.4). Let us verify this for line 1. We have $b = 1,59 = 119$ and $c = 2,49 = 169$, from which we find $a = 120$. Hence $(c/a)^2 = (169/120)^2 = 1.983$, rounded to three places. And this indeed is the corresponding entry in column 4: $1;59,0,15 = 1 + 59/60 + 0/60^2 + 15/60^3 = 1.983$. (We should note again that the Babylonians did not use a symbol for the “empty slot” and therefore a number could be interpreted in many different ways; the correct interpretation must be deduced from the context. In the example just cited, we assume that the leading 1 stands for units rather than sixties.) The reader may check other entries in this column and confirm that they are equal to $(c/a)^2$.

Several questions immediately arise: Is the order of entries in the table random, or does it follow some hidden pattern? How did the Babylonians find

those particular numbers that form Pythagorean triples? And why were they interested in these numbers—and in particular, in the ratio $(c/a)^2$ —in the first place? The first question is relatively easy to answer: if we compare the values of $(c/a)^2$ line by line, we discover that they decrease steadily from 1.983 to 1.387, so it seems likely that the order of entries was determined by this sequence. Moreover, if we compute the square root of each entry in column 4—that is, the ratio $c/a = \csc A$ —and then find the corresponding angle A , we discover that A increases steadily from just above 45° to 58° . It therefore seems that the author of this text was not only interested in finding Pythagorean triples, but also in determining the ratio c/a of the corresponding right triangles. This hypothesis may one day be confirmed if the missing part of the tablet shows up, as it may well contain the missing columns for a and c/a . If so, Plimpton 322 will go down as history's first trigonometric table.

As to how the Babylonian mathematicians found these triples—including such enormously large ones as (4601, 4800, 6649)—there is only one plausible explanation: they must have known an algorithm which, 1,500 years later, would be formalized in Euclid's *Elements*: Let u and v be any two positive integers, with $u > v$; then the three numbers

$$a = 2uv, \quad b = u^2 - v^2, \quad c = u^2 + v^2 \quad (1)$$

form a Pythagorean triple. (If in addition we require that u and v are of opposite parity—one even and the other odd—and that they do not have any common factor other than 1, then (a, b, c) is a *primitive* Pythagorean triple, that is, a , b , and c have no common factor other than 1.) It is easy to confirm that the numbers a , b , and c as given by equations (1) satisfy the equation $a^2 + b^2 = c^2$:

$$\begin{aligned} a^2 + b^2 &= (2uv)^2 + (u^2 - v^2)^2 \\ &= 4u^2v^2 + u^4 - 2u^2v^2 + v^4 \\ &= u^4 + 2u^2v^2 + v^4 \\ &= (u^2 + v^2)^2 = c^2. \end{aligned}$$

The converse of this statement—that *every* Pythagorean triple can be found in this way—is a bit harder to prove (see Appendix B).

Plimpton 322 thus shows that the Babylonians were not only familiar with the Pythagorean theorem, but that they knew the rudiments of number theory and had the computational skills to put the theory into practice—quite remarkable for a civilization that lived a thousand years before the Greeks produced their first great mathematician.

Notes and Sources

1. For a discussion of how the Babylonians approximated the value of $\sqrt{2}$, see Appendix A.
2. The text that follows is adapted from *Trigonometric Delights* and is based on

12 ❖ Chapter 1

Otto Neugebauer, *The Exact Sciences in Antiquity* (1957; rpt. New York: Dover, 1969), chap. 2. See also Eves, pp. 44–47.

3. More precisely, its *converse*: if the sides of a triangle satisfy the equation $a^2 + b^2 = c^2$, the triangle is a right triangle.

4. This, however, is not a *primitive triple*, since its members have the common factor 2; it can be reduced to the simpler triple (28, 45, 53). The two triples represent similar triangles.

5. A fourth error occurs in line 2, where the entry 3,12,1 should be 1,20,25, producing the triple (3367, 3456, 4825). This error remains unexplained.

Note: Arabic names with the prefix *al* are listed alphabetically according to their main name, preceded by *al-*; for example, al-Biruni is to be found under the letter B.

- Abbott, Edwin Abbott (1838–1926), 157 n.7
Absolute value (of vectors). *See* Magnitude
Acoustics, 18, 138
Action at a distance, 192
A'h-mose (Ahmes, ca. 1650 BCE), 14
Alexander III of Macedonia ("the Great," 356–323 BCE), 33
Alexandria (Egypt), 33, 60; Great Library of, 33, 34, 60, 68
Algebra, geometric, 23; symbolic, 23
Alhambra (Granada, Spain), 72
Almagest. *See* Ptolemy (Claudius Ptolemaeus)
Analytic geometry. *See* Geometry, analytic
Annairizi of Arabia (ca. 900 CE), 114 n.6
Anthony, Mark (Marcus Antonius, ca. 83–30 BCE), 216
Antiderivative, 84
Apéry, Roger (1916–1994), 96 n.4
Apollonius of Perga (ca. 262–ca. 190 BCE), 57, 60
Arago, Dominique François Jean (1786–1853), *quoted*, 94
Arc length: of astroid, 90–91; of catenary, 92; of cycloid, 89–90; of logarithmic spiral, 86–88, 89; of parabola, 86; on a cylinder, 171–172, 176 n.2; on Mercator's map, 180; on a sphere, 170–171. *See also* Metric (in differential geometry)
Archimedes of Syracuse (287–212 BCE), 50–51, 54–56, 57, 60, 71, 77, 81, 132; formulas of, 51–54, 229–230; *Measurement of a Circle*, 51–56; *The Method*, 51, 55 n.2; spiral of, 181
Architas of Tarentum (fl. ca. 400 BCE), 32
Arecibo (Puerto Rico), radio message, 205–206; radio telescope, 205
Aristarchus of Samos (ca. 310–230 BCE), 213, 216, 217 n.1
Arithmetic, fundamental theorem of, 201, 227
Astroid, 90–91, 153, 157 n.5, n.6, 235–236; rectification of, 90–91, 153, 233–234
Athens (Greece), 32, 213; Academy of, 33, 60, 61, 213
Aubrey, John (1626–1697), *quoted*, 47
Augustus, Gaius Julius Caesar Octavianus (63 BCE–14 CE), 216
Axioms (Euclid), 34–35
Babylonians, the, xi, xiv–xv, 3, 4, 6–7, 10–11, 13, 155 (note), 219–220
Baghdad (Iraq), 68–69, 70
Basel (Switzerland), 181–182
Baudhayana (fl. 600? BCE), 66–67
Beckmann, Petr (1924–1993), *quoted*, 76
Beethoven, Ludwig van (1770–1827), Violin Concerto in D major, Op. 61, 174
Berkeley, George (1685–1753), *quoted*, 82
Bernoulli, Jakob (1654–1705), 77, 92, 94, 181
Bernoulli, Johann (1667–1748), 92, 94
Bessel, Friedrich Wilhelm (1784–1846), 212 n.1; functions, 211, 212 n.1
Besso, Michele Angelo (1873–1955), 186
Bhaskara ("the Learned," 1114–ca. 1185), 64; *Lilavati*, 199
Billingsley, Sir Henry (d. 1606), 73
al-Biruni, Mohammed ibn Ahmed, Abul Rihan, (973?–1048), 131
Bogomolny, Alexander (website of), 111, 114 n.5
Börne, Karl Ludwig (1786–1837), *quoted*, 46
Boyer, Carl Benjamin (1906–1976), *quoted*, 139 n.8
Broken Bamboo, The, 64–66
Bronowski, Jacob (1908–1974), *quoted*, xi, 46
Calandri, Filippo (fl. 15th century), 73
Calculus, 82, 145, 202, 208, 211; differential, 83–84; fundamental theorem of, 84; integral, 84, 87
Carroll, Lewis. *See* Dodgson, Charles Lutwidge

258 ❖ Index

- Catenary, 91–92; area under; 92; rectification of, 92
- Chamisso, Adelbert von (1781?–1838), *quoted*, 46
- Chao Pei Suan Ching*, 62, 74 n.8
- Characteristic triangle, 82–83, 85, 87
- Cheops, Great Pyramid of, 14
- Chinese, the, xiii, 25, 45, 62–66, 75 n.20
- Chiu Chang Suan Shu*, 64, 198
- Circle(s), xii, 35, 75 n.19, 88–89, 94, 96 n.3, 123–126, 225–226; generated by its tangent lines, 148, 150, 153; inscribed and circumscribed by regular polygons, 51–56, 77–81, 229–230; line equation of, 154; used in proofs of the Pythagorean theorem, 102–104, 108–110, 111–112. *See also* Great circle; Latitude, circles of; Longitude, circles of
- Circular functions. *See* Trigonometric functions
- Circumcircle, 123, 133
- Clairaut, Alexis Claude (1713–1765), 134, 139 n.8
- Clavius, Christopher (1537–1612), 77
- Cleopatra (queen of Egypt, 69–30 BCE), 216
- Columbus, Christopher (1451–1506), 72
- Condit, Ann (age 16, 1938), proof of the Pythagorean theorem, 102, 106–108
- Congruence modulo m , 223–224
- Constantine I (280?–337 CE), 60
- Constantinople (Turkey), 60, 72
- Construction with straightedge and compass. *See* Straightedge and compass, construction with
- Coolidge, E. A. (blind girl, 1888), 102
- Coolidge, Julian Lowell (1873–1954), *quoted*, 195
- Copernicus, Nicolaus (1473–1543), 20, 58, 216
- Cosine function. *See* Trigonometric functions, cosine
- Curved space, 175, 176 n.4
- Cycloid, 86, 87, 88–90; area under, 88–89; rectification of, 89–90, 232–233
- Cylinder, surface of, 171–172
- Dantzig, Tobias (1884–1956), *quoted*, 32, 39
- Darius (king of Persia, 558?–486 BCE), 32
- Dee, John (1527–1608), 73
- Derivative, 82–83, 92
- Desargues, Gérard (1593–1662), 48
- Descartes, René (1596–1650), 28, 34, 73, 76, 88, 133, 139 n.11, 145, 158, 168, 204
- Differential geometry. *See* Geometry, differential
- Differentials, 83
- Differentiation, 83
- Diophantus of Alexandria (fl. ca. 250–275 CE), 2, 57–58, 60, 224
- Dirac, Paul Adrien Maurice (1902–1984), *quoted*, 28
- Distance formula, 85, 133–134, 139 n.8, 159, 161, 166, 172, 180, 190–191, 192, 204, 211
- Dodgson, Charles Lutwidge (Lewis Carroll, 1832–1898), xi–xii, 45; *quoted*, 45–46
- Dot product. *See* Vectors, dot product of
- Double-angle formula for sine, 79–80, 81 n.3
- Drake, Frank (1930–), 206–207; *quoted*, 206, 207
- Duality, principle of, 146–148, 149–151
- Dudeney, Henry Ernest (1857–1930), 197, 198; *quoted*, 197
- e (base of natural logarithms), 86, 92, 212
- $E = mc^2$, 117, 189, 195 n.3
- Early European universities, 71
- Eddington, Sir Arthur Stanley (1882–1944), 193
- Einstein, Albert (1879–1955), 28, 117–118, 186, 188–196; *quoted*, 117, 191; summation notation, 164; “thought experiments,” 184, 193, 204
- Electromagnetism, 184–185, 188, 202
- Elementary functions, 92, 93 n.8, 211
- Elements, The* (Euclid), xi, xiii, 25, 30 n.2, 34–36, 43, 58, 60, 61, 69, 71, 73, 82, 204, 206 n.5; Propositions: *I* 1, 35; *I* 5, 45; *I* 38, 36–37; *I* 47, xi, 25, 36–43, 59, 92, 94, 113, 140; *I* 48, 42–43; *II* 9, 48; *II* 12, 127–128; *II* 13, 127–128; *II* 14, 75 n.19; *III* 20, 78, 96 n.3; *III* 31, 123; *III* 35, xii–xiii, 11, 108; *III* 36, 48, 108, 110; *VI* 31, 6, 25, 41–43, 115, 123; *IX* 20, 201
- Envelope of a curve, 148, 235
- Epicycle, 20
- Equivalence, principle of, 193
- Eratosthenes of Cyrene (ca. 275–ca. 194 BCE), 58
- Escher, Maurits Cornelis (1898–1972), 72
- Ether, 185–186
- Euclid (fl. 300 BCE), 25, 30 n.3, 33–36, 39, 41–43, 44 n.11, 113; *quoted*, 33, 34, 61. *See also* *Elements, The* (Euclid)
- Euclidean geometry. *See* Geometry, Euclidean
- Eudemus of Rhodes (fl. ca. 335 BCE), 30 n.2, 61–62; *Summary* of, *see* Proclus, *Eudemean Summary*
- Eudoxus of Cnidus (ca. 408–ca. 355 BCE), 25, 42, 55

- Euler, Leonhard (1707–1783), 134, 168, 181, 202; and Fermat primes, 155 n.1; and Fermat's Last Theorem, 2; formula of ($e^{i\pi} + 1 = 0$), xii; infinite product, 81 n.4; infinite series, 94–95, 96 n.1, n.4; and perfect numbers, 30 n.3
- Eupalinus (fl. 6th century BCE), 215; Eupalinus Tunnel, 215, 217 n.4
- Eves, Howard W. (1911–2004), *quoted*, 31 n.9, 43 n.4
- Fermat, Pierre de (1601–1665), 1–2, 85; Fermat's Last Theorem (FLT), 1–3, 212; Fermat primes, 155 n.1, 212
- Fibonacci, Leonardo ("Pisano," ca. 1170–ca. 1250), 58
- Five (number), 20–21
- Fluent(s), 82, 202
- Fluxion, 82
- Fréchet, Maurice (1878–1973), 167 n.1
- Frey, Gerhard (1944–), 2
- From Here to the Moon* (Jules Verne), 203, 206 nn. 3 and 4
- Function space(s), 165–166
- Galileo Galilei (1564–1642), 58, 88, 91
- Garfield, James Abram (20th U.S. President; 1831–1881): 106; proof of the Pythagorean theorem by, 106–107
- Gauss, Carl Friedrich (1777–1855), 30 n.4, 146, 155 n.1, 168, 173, 175–176, 203, 206 n.4
- Geodesic, 237. *See also* Great circle
- Geometer (spherical ruler), 169
- Geometry: analytic, 27, 145, 204, 208; differential, 168, 192; Euclidean, 35, 145–146, 172, 174–175, 211; non-Euclidean, 168, 174–175, 211; projective, 146–148, 157 n.4, 168; variable, 173–175
- Gherardo of Cremona (1114–1187), 71
- Gibbs, Josiah Willard (1839–1903), 159
- Gilbert and Sullivan, *The Pirates of Penzance*, 47
- Gillings, Richard J., *quoted*, 13–14
- GIMPS (Great Internet Mersenne Prime Search), 202. *See also* Mersenne Marin, primes
- Goethe, Johann Wolfgang von (1749–1832), *quoted*, 158
- Goldbach, Christian (1690–1764), 202; conjecture, 202
- Golden section, 49 n.10
- Gravity, 192–195
- Great circle, 169, 175, 240 n.1. *See also* Geodesic
- Greek, cosmology, 20, 28
- Greek mathematics: geometry, 30 n.2, n.5, 59, 75 n.20, 145, 204; infinity, their fear of, 76, 80, 93, 138
- Gregory, James (1638–1675), 96 n.2
- Gregory XIII (Pope, 1502–1585), 77
- Grossmann, Marcel (1878–1936), 192
- Guldin, Paul (1577–1643), 58, 74 n.3
- Gutenberg, Johannes (1400?–1468), 73
- Hakim, Joy, *quoted*, 13
- Half-angle formulas, 77–80, 81 n.3
- Hamilton, Sir William Rowan (1805–1865), 158–159
- Han dynasty (China; 206 BCE–221 CE), 62, 64
- Harmonics (overtones), 138, 212 n.2
- Harmonic series, 19, 94
- Heath, Sir Thomas Little (1861–1940), *quoted*, 15, 31 n.9
- Henry IV (French king, 1553–1610), 76
- Heraion (temple in Samos), 216
- Heron (ruler of Sicily, 2nd century BCE), 50
- Heron (Hero) of Alexandria (1st century CE?), 132; formula of, 131–133
- Higher-dimensional spaces, 173–176, 190–191, 192. *See also* Spacetime
- Hilbert, David (1862–1943): Hilbert Space, 166–167
- Hindus, the, 25, 66–68, 75 n.17, n.20
- Hipparchus of Nicaea (ca. 190–ca. 120 BCE), 56 n.3, 58
- Hippasus (5th century BCE), 28
- Hippocrates of Chios (fl. 440 BCE), 125
- Hjelmslev, J. (1873–1950), 157 n.2
- Hobbs, Thomas (1588–1679), 47
- Hoffmann, Banesh (1906–1986), 117
- Hsiang Chieh Chiu Chang Suan Fa*, 64
- Hundred-Year War (1338–1453), 71
- Huygens, Christiaan (1629–1695), 88, 102
- Hypatia (ca. 370–415 CE), 60–61
- Hyperbolic cosine. *See* Hyperbolic functions
- Hyperbolic functions, 92, 93 n.5, n.7
- Hyperbolic sine. *See* Hyperbolic functions
- Hyperboloid, 89–90
- Hypotenuse, middle point of, 107; origin of word, xiii; perpendicular to, 127, 153; theorem, *see* Pythagorean theorem
- $i(\sqrt{-1})$, 166, 190–191
- Iamblicus of Apamea (ca. 245–ca. 325 CE), *quoted*, 213
- Incircle, 126, 133
- Incommensurable (numbers), 227–228

260 ❖ Index

- Infinite product: Euler's, 81 n.4; Viète's, 77, 79–81
- Infinite series, 94–97, 120–121, 137–138, 211
- Infinitesimals, 83
- Infinity, 76, 80, 93, 137–138, 157 n.4, 162, 165
- Inner product, 162, 166
- Integral(s), 84, 165, 166, 208, 210–211
- Integration, 84, 86
- Irrational: double meaning of the word, 26; numbers, 26–28, 227–228
- Islamic Empire, 68–71
- Isoperimetric problem, 58
- Jashemski, Stanley (age 19, 1934), 116
- Joseph, George Gheverghese, 75 n.18; *quoted*, 66
- Justinian I (483–565 CE), 60
- Kaku, Michio (1947–), *quoted*, 168, 201
- Kapoor, Anish (1954–), *Cloud Gate* (sculpture), 176 n.4, plate 3
- Karlovasi (Samos, Greece), 216
- al-Kashi, Jemshid ibn Mes'ud ibn Mahmud, Giyat ed-din, (d. 1429 or 1436), 71
- Katyayana (fl. ca. 400? BCE), 67
- Kepler, Johannes (1571–1630), 20, 28, 47, 58; *quoted*, 47
- Kerkis, Mt. (Samos, Greece), 216–217
- al-Khowarizmi, Mohammed ibn Musa (ca. 780–ca. 850), 69
- Kolaios (fl. 7th century BCE), 215
- Kou-ku* theorem, 64, 66
- La Hyre, Laurent de (1606–1656), 47–48, 49 n.12; *Allegory of Geometry* (painting), 48, 49 n.12. *See also* cover illustration
- Lanczos, Cornelius (1893–1974), *quoted*, 181
- Landau, Edmund (1877–1938), 119–121; *quoted*, 119
- Latitude, circles of (“parallels”), 168–171, 176 nn.1–3, 178–180, 204
- Law of Cosines, 127–130, 131, 172
- Lederman, Leon (1922–), *quoted*, 46–47, 123
- Legendre, Adrien-Marie (1752–1833), 102, 118
- Leibniz, Gottfried Wilhelm Freiherr von (1646–1716), 82–85, 92, 96 n.2, 102
- Length, 161–162, 163, 165, 191; of radius vector, 190. *See also* Arc length; Distance formula
- Leonardo da Vinci. *See* Vinci, Leonardo da
- Levi-Civita, Tullio (1873–1941), 192
- Light, 184–186; aberration of, 194; bending of, 193–194; speed of, 188–189, 191, 192
- Lilavati, the* (Bhaskara), 199
- Line: equation, 153–154, 235; coordinates, 148–154, 236; designs, 154, 156 “Little Pythagorean theorem,” 127
- Littrow, Joseph Johann von (1781–1840), 206 n.4
- Logarithmic spiral, 86–88, 183; rectification of, 86–88, 89, 231–233
- Longitude, 177, 204; circles of (meridians), 168–171, 178–180
- Loomis, Elias (1811–1899), 98, 99
- Loomis, Elisha Scott (1852–1940), xiii, 98–102, 106, 107, 116 n.1, 119; *quoted*, 98, 99, 106, 107, 111, 116, 119, 140; *The Pythagorean Proposition*, xiii, xvi, 98, 99–110, 116, 117, 119, 140–141
- Loomis Joseph (17th century), 98, 99
- Lorentz, Hendrik Antoon (1853–1928), 189; transformation, 189–190
- Lune of Hippocrates, 125–126
- M-13 (star cluster in Hercules), 205, 206 n.6
- Magnitude, 159, 161
- al-Mamun (reigned 809–833), 69
- al-Mansur (712?–775), 69
- Map projections, 178–180
- Marcellus, Marcus Claudius (268?–208 BCE), 50, 51
- Mascheroni, Lorenzo (1750–1800), 146, 157 n.2
- Mathematics, nature of, 204–206, 208
- Maxwell, James Clerk (1831–1879), equations of, xii, 184–185
- Mercator, Gerhard (Gerardus, 1512–1594), 177–180; projection, 177–180
- Mercer, John (“Johnny”) Herndon (1909–1976), *quoted*, 45
- Meridians. *See* Longitude, circles of
- Mersenne Marin (1588–1648), 30 n.3, 202; primes of, 30 n.3, 202, 212
- Mesopotamia, 4, 13. *See also* Babylonians, the
- Meter (in music), 174
- Method of exhaustion, 42, 55, 71
- Metric (in differential geometry), 168, 172, 173–175, 195, 211
- Michelson, Albert Abraham (1852–1931), 185–186
- Michelson-Morley experiment, 185–186, 188
- Miletus (Asia Minor), 17
- Miller, Walter James, *quoted*, 206 n.4

- Minkowski, Hermann (1864–1909), 190–191, 195n; *quoted*, 188
- Mohammed (prophet and founder of Islam), 570–632), 68
- Mohr, Georg (1640–1697), 157 n.2
- Moon, the, 203
- Morley, Edward Williams (1838–1923), 185–186
- “Mover’s dilemma, the,” 153
- Mozart, Wolfgang Amadeus (1756–1791), 208–209, 211; Piano Concerto No. 16 in D major, K. 451, 209
- Music, 19, 28, 158, 174, 208–209, 211, 212 n.2
- Musical, harmony, 19–20, 28, 195; intervals, 19; sound, 138, 139 n.12
- Mykale, Strait of (Greece), 216
- Needham, Joseph Terence Montgomery (1900–1995), 74 n.8; *quoted*, 64
- Nelsen, Roger B., 114 n.6
- Neugebauer, Otto E. (1899–1990), *quoted*, 4
- Newton, Sir Isaac (1642–1727), 73, 82, 84–85, 145, 158, 185, 188, 192, 195 n.6, 202; second law of motion, xii, 161; universal law of gravitation, 211–212
- Newton-Raphson formula, 219–220
- Non-Euclidean geometry. *See* Geometry, non-Euclidean
- Norm, 165–166
- Notation, 158–167
- Nowak, Martin, 30 n.3
- Numbers, 20–23; figurative, 22–23, 30 n.4; irrational, 25, 26–27; perfect, 21–22, 30 n.3, 212; rational, 19, 27, 28; triangular, 30 n.4. *See also* Prime numbers
- Number theory, 19, 20, 202, 208, 212. *See also* Prime numbers
- Orthogonal: coordinates, 172; functions, 166; vectors, 164, 166
- Overtones. *See* Harmonics
- π (pi), 51–55, 71, 77, 80, 81 n.4, 90, 94–95, 96 n.2, 125, 212, 233
- Pacioli, Luca (ca. 1445–1509), 45
- Pappus of Alexandria (fl. ca. 300 CE), 58–59, 74 n.3
- Parabola, 91; arc length of, 86; area of a segment of, 51
- Parallel, postulate (Euclid’s Fifth), 35; lines, 175, 176 n.3
- Parallelogram rule. *See* Triangle rule
- Parallels (circles of latitude). *See* Latitude, circles of
- Pascal, Blaise (1623–1662), 85
- Peloponesian War, 32
- Perfect numbers. *See* Numbers, prefect
- Persia, 32, 216
- Perspective (in art), 146
- Philip (Macedonian king, 382–336 BCE), 33
- Plato (ca. 427–347 BCE), 33, 60; *quoted*, 33
- Plimpton* 322, 7–12, 225
- Plücker, Julius (1801–1868), 148, 154
- Plutarch (46?–120? CE), *quoted*, 50
- Polar coordinates, 86, 151, 168
- Polycrates (ruler of Samos, d. ca. 522 BCE), 17, 215–216
- Polygons, regular: built on the sides of a right triangle, 115, 123–124; circumscribing a circle, 51, 53–54, 56 n.3, 81, 229–230; constructible with straightedge and compass, 146, 155 n.1, 212; inscribed in a circle, 51–53, 54, 56 n.3, 77, 79–81
- Polyhedra, regular, 20–21
- Prime numbers, 30 n.3, 155 n.1, 201–203, 206, 212, 227
- Proclus (412–485 CE), 30 n.2, 42, 61; *Eudemian Summary*, 30 n.2, 42, 61, 62
- Projective geometry. *See* Geometry, projective
- Ptolemy (Claudius Ptolemaeus, ca. 85–165 CE), 58, 60, 102; *Almagest*, 58, 60, 69, 71, 102; theorem, 102–104, 113 n.4
- Ptolemy I (king of Egypt, fl. 306 BCE), 33
- Pyramids, the, 13–14
- Pythagoras of Samos (ca. 572?–ca. 501? BCE), 17–19, 25, 32; general reference to, xi, 28, 32, 36, 43, 62, 93, 99, 100, 123, 138, 186, 195, 213–215, 217; landmarks named for, 214, 215; his proof of his theorem, *see* Pythagorean theorem, proofs of
- Pythagorean cuboids, 134
- “Pythagorean Curiosity,” 101, 140–141
- Pythagorean identities, xiv, 119, 211
- Pythagorean magic squares, 101
- Pythagorean quadruples, 134
- Pythagorean Square (puzzle), 197–198, 237
- Pythagorean theorem:
- Babylonian knowledge of, xi, 6, 145, 155 (note);
 - in China, 25–26, 62–66, 75 n.16;
 - converse of, 12 n.3, 42–43;
 - on a cylinder, 171–172;
 - differential form of, 85;
 - Egyptian knowledge of, 13–16;
 - generalizations of, 58–59, 69, 123–125, 127–130;

262 ❖ Index

- Pythagorean theorem (*continued*)
general reference to, xi, xii, xiii, 1, 8,
25–26, 30, 32, 39, 48, 51, 73, 77, 81, 87,
92, 98, 99, 111, 117, 119, 133, 142, 148,
172, 195, 198, 208, 211, 214, 215, 217;
in higher dimensions, 134, 137–138;
in Hilbert space, 166;
in India, 25, 66–68;
on Mercator’s map, 180;
names of, xi, 45, 48 nn.1 and 4, 64;
in popular culture, xi, 45–49;
proofs of, xi, xiii, xiv, 98–114; based on
circles, 108–110; Chinese, 26, 39, 61,
63–64; Ann Condit’s, 102, 106–108; by
differentials, 111–112; by dissection, 39,
61, 63–64; Henry E. Dudeney’s, 198–199;
“dynamic,” 39, 43 n.4, 100; Albert
Einstein’s, xi, 117–118, 192; Euclid *I* 47,
36–40, 41, 43, 47, 48, 59; Euclid *VI* 31,
41–42, 43, 117; “Folding Bag,” 115–116;
James A. Garfield’s, xi, 106–107; Hindu,
25, 68; Stanley Jashemski’s, 116; Ed-
mund Landau’s, 119–121; Leonardo da
Vinci’s, 104–106; longest, 102, 103;
Ptolemy’s, 102–104; Pythagoras’s,
24–26, 31 n.9, 42, 61, 99; shortest, 102,
115–116; by tessellation, 112–113;
and SETI, 203, 206 n.4;
special case of 45–45–90–degree triangle,
25, 26, 67;
on a sphere, 169–171, 175;
and the theory of relativity, 186, 190;
universality of, 203–204;
in vector notation, 160, 164.
See also *Elements*, *The*, Propositions *I* 47,
I 48, *VI* 31
- Pythagorean triangles. *See* Pythagorean
triples
- Pythagorean triples, 1, 8–12, 23–24, 67,
126–127, 139 n.3, 223, 225; primitive, 11,
12 n.4, 24, 58, 221–222
- Pythagoreans, the, 17–24, 26–28, 30 n.6, 31
n.8, 32; motto of, 17
- Pythagorio (Samos, Greece), 214–216
- QED, 43 n.3
- Quadrivium, 19, 48
- Quaternions, 99, 158–159
- Radius vector, 160–161, 190
- al-Rashid, Harun (“the Just,” 764?–809), 69
- Rate of change, 82–83
- Rectangular coordinates, 151, 153, 168
- Rectification, 85–93; of the astroid, 90–91,
233–234; of the catenary, 92; of the cycloid,
89–90, 232–233; of the logarithmic spiral,
86–88, 89, 231–232; of the parabola, 86
- Relativity, theory of, 186, 202; general, 28,
117, 176, 186, 192–196; special, 117, 186,
188–191, 192
- Renaissance, the, 47
- Rhind, Alexander Henry (1833–1863), 14;
Rhind Papyrus, 14–15
- Rhine (river), 181–183
- Rhumb line, 177–178
- Ribet, Kenneth Alan, 2
- Ricci-Curbastro, Gregorio (1853–1925), 192
- Riemann, Georg Friedrich Bernhard
(1826–1866), 168, 173–176, 192, 195; hy-
pothesis, 97 n.4
- Roberval, Gilles Persone de (1602–1675), 89
- Roman Empire, 60, 216
- Romans, the, 202–203
- Sagan, Carl Edward (1934–1996), 46,
49 n.9, 201
- Samaina* (Samian boat), 215–216
- Samos (Greece), 17, 213–217; history of,
215–217
- Scalar, 162
- Schmidt, Erhard (1876–1959), 167 n.1
- Schopenhauer, Arthur (1788–1860), 39, 113;
quoted, 39
- Semiperimeter, 131, 133
- SETI (Search for Extraterrestrial
Intelligence), 206
- Sexagesimal system, 4–7, 55, 155 (note)
- Shi Huang-ti (Chinese emperor, 259–210
BCE), 62
- Simha, Evelyn, *quoted*, 16 n.6
- Sine function. *See* Trigonometric functions, sine
- Smith, David Eugene (1860–1944), 48 n.1, 74
n.3, n.20; *quoted*, 48 n.1
- Snake and the Peacock*, *The* (brainteaser),
199–200, 240
- Spacetime, 190–191, 192, 195
- Spain, 71–72
- Sparta (Greece), 32–33
- Sphere, surface of, 168–171, 174–175,
177–178, 203–204, 206 n.5, 240 n.1. *See*
also Great circle
- Spider and the Fly*, *The* (brainteaser),
197, 237
- Square root of two ($\sqrt{2}$), 6, 11 n.1, 26–28, 81;
irrationality of, 26–28, 227–228; Babylon-
ian computation of, 219–220

- Square root spiral, 126
“Squaring”: of a circle, 126; of a rectangle, 67, 75 n.19; of a triangle, 125
St. Paul’s Cathedral (London), 89
Straightedge and compass, construction with, 35, 95, 96 n.3, 125–126, 145–146, 155 n.1, 212
Stravinsky, Igor Fyodorovitch (1882–1971), *The Rite of Spring*, 174
Strings, vibrations of, 18–19, 28, 138, 165–166
Struik, Dirk Jan (1894–2000), 16 n.6; *quoted*, 15
Sulbastura(s), 66–67
Sums of squares, 94, 223–225
Symmetry, xii, 20
Synthetic geometry. *See* Geometry, synthetic
Syracuse (Sicily, Italy), 50, 60

Tabit ibn Qorra ibn Mervan, Abu-Hasan, al-Harrani (826–901), 69–70
Talmud, the, 34
Tangent line(s), 82–83, 148–149, 152, 153, 232, 236
Taniyama, Yutaka (1927–1958), 2
Tarentum (Italy), 32
Taylor, Richard, 2
Tensor, 192, 195, 196 n.8; analysis, 193–194
Thales of Miletus (ca. 640–ca. 546 BCE), 17, 59, 215
Theodosius I (Roman emperor, 346?–395 CE), 60
Theon of Alexandria (fl. ca. 390 CE), 60
Thoreau, Henry David (1817–1862), *quoted*, 115
Tigani (Samos, Greece). *See* Pythagorio
Time, relativity of, 188–189, 195 n.1
Tinseau, D’Amondans Charles de (1748–1822), 139 n.11
Torricelli, Evangelista (1608–1647), 87, 88
Triangle inequality, 160, 166, 211
Triangle rule, 159

Trigonometric functions, 92, 93 n.5, n.8, 120–121, 211; cosine, 77–80, 81 n.3, n.4, 92, 120–121, 211; sine, 71, 79–80, 81 n.3, n.4, 92, 93 n.7, 96 n.1, 113 n.4, 120–121, 138, 166, 211
Trigonometry, xiv, 11, 56 n.3, 76–77, 102, 119, 127, 208, 211, 237; spherical, 174. *See also* Trigonometric functions
Trivium, 48
Twin primes, 201–202
Two (number), 20, 77, 208, 211–212

Ulugh Beg (1393–1449), 71

Van der Waerden, Bartel Leendert (1903–1996), *quoted*, 15, 15 n.5
Vectors, 159–167, 191; addition of, 159–164, 181–184; dot product of, 163–164, 166; orthogonal, 164, 166; unit, 160–161; world, 191
Vector spaces, 162–167
Viète, François (1540–1603), 76–81, 158; product, 77–81
Vinci, Leonardo da (1452–1519), proof of the Pythagorean theorem, 102, 104–106

Wantzel, Pierre-Laurent (1814–1848), 155 n.1
Wells, David (1940–), *quoted*, 208
Wheeler, John Archibald (1911–), *quoted*, 192
Wiles, Andrew John (1953–), 1–3
Winding Vine, The (brainteaser), 198–199, 237–240
World vectors. *See* Vectors, world
Wren, Sir Christopher (1632–1723), 89–90
Wright, Edward (ca. 1560–1615), 180

 $x^2 + y^2$, frequent occurrence of in mathematics, xiv, 208–211

Yang Hui (fl. ca. 1260 CE), 64–66
YBC 7289, 4–7, 219

Zeta function, 96 n.4