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Chapter One

Introduction

THE FIRST PART is notes of a course given by Peter Scholze on local Shimura varieties and contains very recent results of Scholze's. They are part of the Langlands program and deal with the local Langlands conjecture.

Local Langlands conjecture predicts a correspondence between representations of the Weil group of the field of p-adic numbers $\mathbb{Q}_p$ and (possibly, infinite-dimensional) representations of p-adic reductive groups. The Weil group $W_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ consists of all elements in the absolute Galois group of $\mathbb{Q}_p$ whose restriction to the maximal unramified extension of $\mathbb{Q}_p$ equals an integral power of Frobenius.

To be more precise, fix a reductive algebraic group G over $\mathbb{Q}_p$ and a prime number ℓ different from p. On one side of the local Langlands correspondence, one considers irreducible smooth representations π over the field $\bar{\mathbb{Q}}_\ell$ of the p-adic group $G(\mathbb{Q}_p)$. On the other side of the correspondence, one considers L-parameters of G, which are continuous homomorphisms from $W_{\mathbb{Q}_p}$ to the Langlands dual group ${}^LG$ over $\bar{\mathbb{Q}}_\ell$ that satisfy certain properties. In particular, if G is split, then an L-parameter is the same as a continuous homomorphism from $W_{\mathbb{Q}_p}$ to the connected component ${}^LG^\circ$ of the Langlands dual group (up to conjugation). Local Langlands correspondence associates conjecturally to π its L-parameter $LLC(\pi)$. This association should satisfy certain special properties.

First, consider the group $G = \boldsymbol{GL}_n$. In this case, the local Langlands conjecture was proven by Harris-Taylor and Henniart. The group $\boldsymbol{GL}_n$ is split and ${}^LG^\circ = \boldsymbol{GL}_n$. Thus in this case, an L-parameter is the same as an n-dimensional representation of $W_{\mathbb{Q}_p}$ over $\bar{\mathbb{Q}}_\ell$. A linking bridge between representations of $\boldsymbol{GL}_n(\mathbb{Q}_p)$ and of $W_{\mathbb{Q}_p}$ is provided by the following general construction: let H be an (infinite-dimensional) $\bar{\mathbb{Q}}_\ell$-vector space together with commuting actions of $W_{\mathbb{Q}_p}$ and $\boldsymbol{GL}_n(\mathbb{Q}_p)$. Then for any π as above, one obtains a $W_{\mathbb{Q}_p}$-representation $\mathrm{Hom}_{\boldsymbol{GL}_n(\mathbb{Q}_p)}(\pi, H)$. (Actually, this will not yet be the n-dimensional representation $LLC(\pi)$ of $W_{\mathbb{Q}_p}$, but it allows one to reconstruct $LLC(\pi)$ with the help of the (non-conjectural) Jacquet-Langlands correspondence.)

Of course, one needs to make a very particular choice of H so that it gives the local Langlands correspondence. According to Harris and Taylor, the needed H is given by ℓ-adic cohomology of the Lubin-Tate space. This is a (pro-)rigid analytic space over the p-adic completion L of the unramified extension of $\mathbb{Q}_p$. The Lubin-Tate space has the following modular sense: it parametrizes deformations of the one-dimensional formal group law of height n over $\bar{\mathbb{F}}_p$ considered up to quasi-isogenies with a p-adic level structure, that is, with a trivialization of the p-adic Tate module of a deformation.

Now let G be a reductive group over $\mathbb{Q}_p$ and, for simplicity, assume that G is split. Then the L-parameter $LLC(\pi)$ is given by a homomorphism $W_{\mathbb{Q}_p} \to {}^L G^\circ$. Thus it is natural to detect $LLC(\pi)$ by describing its compositions $r \circ LLC(\pi)$ with various (irreducible, finite-dimensional, and algebraic) representations r of ${}^L G^\circ$ over $\bar{\mathbb{Q}}_\ell$. Similarly as above, one constructs the representations $r \circ LLC(\pi)$ of $W_{\mathbb{Q}_p}$ with the help of cohomology of certain p-adic spaces endowed with the actions of both $W_{\mathbb{Q}_p}$ and $G(\mathbb{Q}_p)$.

Already in the case $G = \boldsymbol{GL}_n$, one can search for representations of $\boldsymbol{GL}_n$ other than the tautological one. The case of wedge powers of the tautological representation is treated with the help of Rapoport-Zink spaces, which generalize Lubin-Tate spaces. More precisely, for the d-th wedge power of the tautological representation of $\boldsymbol{GL}_n$, one takes ℓ-adic cohomology of the Rapoport-Zink space that has the following modular sense: it parametrizes deformations of the isoclinic p-divisible group of height n and dimension d over $\bar{\mathbb{F}}_p$ considered up to quasi-isogenies with a p-adic level structure on a deformation.

Generalizing this to an arbitrary reductive group G, one replaces a p-divisible group over $\bar{\mathbb{F}}_p$ by purely group-theoretic datum called a local Shimura datum, which consists of a conjugacy class $\bar{\mu}$ of a cocharacter $\mu\colon \mathbb{G}_m \to G$ (possibly, defined over an extension of $\mathbb{Q}_p$) and a Frobenius-twisted conjugacy class b in $G(L)$. The cocharacter μ and the conjugacy class b are assumed to be compatible in a way. Also, one requires μ to be minuscule.

It was predicted by Kottwitz that for each local Shimura datum $(G, b, \bar{\mu})$, there exists a so-called local Shimura variety, which is a (pro-)rigid analytic space. This is now an unpublished theorem, by the work of Fargues, Kedlaya-Liu, and Caraiani-Scholze. Furthermore, ℓ-adic cohomology of the local Shimura variety conjecturally provides a way to reconstruct the representation $r_\mu \circ LLC(\pi)$, where r_μ is the highest weight representation of ${}^L G^\circ$ associated naturally with the cocharacter μ of G.

If the cocharacter μ is not minuscule, then one does not expect the existence of such local Shimura variety in the category of (pro-)rigid analytic spaces. Thus the above approach deals only with minuscule representations, which is a small part of all representations r_μ of ${}^L G^\circ$. Hence it does not allow one to reconstruct the conjectural L-parameter $LLC(\pi)$. To solve this problem, one needs to introduce new geometric objects instead of rigid analytic spaces.

The new geometric objects live in the world of perfectoid spaces. Namely, they are diamonds, that is, algebraic spaces with respect to the pro-étale topology on perfectoid spaces (in particular, rigid analytic spaces are diamonds). Even more, perfectoid spaces allow one to interpret local Langlands correspondence as a geometric Langlands correspondence on a perfectoid curve called a Fargues-Fontaine curve. This opens a new powerful approach to constructing the local Langlands correspondence. Though this program is not yet entirely implemented, substantial progress has been achieved in recent works by Fontaine, Fargues, Scholze, Weinstein, Caraiani, and V. Lafforgue.

The second part, a course by Umberto Zannier, deals with a rather classical theme but from a modern point of view. His course is on hyperelliptic continued fractions and generalized Jacobians. The starting point is the classical Pell equation which he considers over the ring of polynomials over $\mathbb{C}$.

The classical Pell equation is the Diophantine equation $x^2 - dy^2 = 1$, to be solved in integers x and y, where d is some fixed integer. The problem reduces to the case of positive non-square integers d, in which case there are always infinitely many solutions, each of which can be generated from a minimal one. As is well-known today, Pell's equation is strongly related to the theory of continued fraction expansions of real quadratic numbers, as the solutions appear as numerators and denominators of convergents of $\sqrt{d}$. It was Lagrange who proved that indeed there always exists a solution, showing that the sequence of partial quotients of a real number a is eventually periodic if and only if a is algebraic of degree two.

Now let $D \in \mathbb{C}[t]$ be a complex polynomial of even degree and consider the "polynomial Pell equation," $x(t)^2 - D(t)y(t)^2 = 1$. We call D *Pellian* if there are nonzero polynomials $x, y \in \mathbb{C}[t]$ which solve the equation. Similarly, as in the classical setting, one may define the continued fraction expansion of $\sqrt{D(t)}$, viewed as a Laurent series at infinity, proceeding analogously to the classical algorithm, but now replacing the integer part by its polynomial part.

One may ask whether there is again a correspondence between the solvability of the polynomial Pell equation and the continued fraction expansion of $\sqrt{D(t)}$. Indeed, it was already known by Abel and Chebyshev that D is Pellian if and only if the sequence of partial quotients in the continued fraction of $\sqrt{D(t)}$ (which are polynomials now) is eventually periodic. On the other hand, this turns out to be a quite rare phenomenon and, in contrary to the case of real numbers, periodic continued fraction expansions are not limited to square roots of polynomials.

However, Zannier discovered that some periodicity still remains in full generality: The sequence of the degrees of the partial quotients of $\sqrt{D(t)}$ is eventually periodic. In the case that D is squarefree, the proof relies on certain divisor relations in the associated Jacobian variety of the underlying projective curve corresponding to $u^2 = D(t)$ and the application of a variant of the theorem of Skolem-Mahler-Lech for algebraic groups. The case of nonsquarefree D, on the other hand, involves the study of so-called generalized Jacobians.

Among the periodicity of the partial quotients of $\sqrt{D(t)}$, the Pellianity of certain families of polynomials $D_\lambda \in \mathbb{C}(\lambda)[t]$ is investigated as well, such as $D_\lambda(t) = t^4 + \lambda t^2 + t + 1$. One may ask for which specializations of the parameter $\lambda \in \mathbb{C}$ the equation in question has a nontrivial solution. Again, the study relies on a criterion which links the solvability to certain points on the associated (generalized) Jacobians. Anyhow, there are also various further phenomena to observe in the context of continued fraction expansions of Laurent series as well as connections to other related problems, such as *unlikely intersections* in families of Jacobians of hyperelliptic curves or *Padé approximations.*

The notes are organized as follows. After the topic is motivated and some historical background is given in the first section, Sections 3.2–3.4 are dedicated to continued fractions. We recall the method and some related results for the standard setup in Section 3.2. The third section generalizes continued fractions to more general settings, giving some illustrative examples, whereas Subsection 4 explains the procedure for Laurent series in detail. Section 3.5 deals with the continued fraction expansion of $\sqrt{D}$ and gives related results, including the main theorem. In Section 3.6 a criterion for D to be Pellian in terms of a special point of the Jacobian of the underlying curve in the case of squarefree D is given, thereupon Section 3.7 treats the case of D not being squarefree. Various pencils of polynomials, for squarefree and nonsquarefree D_λ, are analyzed as to their Pellianity in these two subsections. Sections 3.8 and 3.9 are dedicated to the mentioned version of the Skolem-Mahler-Lech theorem and the proof of the periodicity of the partial quotients of $\sqrt{D(t)}$. In the appendix, the reader shall find solutions to the exercises given during the lecture.

The theme of the third course by Shou-Wu Zhang originates in the famous Chowla-Selberg formula which was taken up by Gross and Zagier in 1984 to relate values of the L-function for elliptic curves with the height of Heegner points on the curves. Only in recent years has a very significant step been taken by P. Colmez relating L-values for abelian varieties with complex multiplication to the Faltings height of the abelian variety. Building on this work, X. Yuan, Shou-Wu Zhang, and Wei Zhang succeeded in proving the Gross-Zagier formula on Shimura curves and shortly later they verified the Colmez conjecture on average. In the course Zhang presents new interesting aspects of the formula.

Let K be a number field and let A be an abelian variety over K of dimension $g \geq 1$. In the first part, we will define the stable Faltings height $h(A)$ of A. It is a real number associated to A which is invariant under base change. Faltings introduced this invariant in his paper on the Mordell conjecture, Shafarevich conjecture, and Tate conjecture in order to study isogenies of abelian varieties over number fields. The stable Faltings height has since gained momentum as a tool to answer other questions in arithmetic geometry, and has a deep connection with the abc-conjecture, which is a consequence of the following conjecture.

Conjecture 1.1 (Generalized Szpiro Conjecture). *Let K be a number field and let $g \geq 1$ be an integer. Then any abelian variety A over K of dimension g satisfies*

$$h(A) \leq \tfrac{\alpha}{[K:\mathbb{Q}]} (\log N_A + \log \Delta_K) + \beta, \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}$ are constants depending only on $[K:\mathbb{Q}]$ and g. Here Δ_K denotes the absolute discriminant of $K/\mathbb{Q}$ and N_A denotes the norm of the conductor ideal of A/K.

A strong form of this conjecture says that for each real number $\epsilon > 0$ one can take here $\alpha = \frac{g}{2} + \epsilon$ and β depending only on g and ϵ. The generalized Szpiro conjecture has many striking consequences. In particular, it implies an effective version of the Mordell conjecture (assuming α and β are effectively computable), and "no Landau-Siegel zero" which follows from the strong form of the conjecture for CM elliptic curves. We shall discuss several applications of the generalized Szpiro conjecture and we shall also consider a function field analogue: The so-called "Arakelov inequality" which was proved by Arakelov, Faltings, and Parshin.

In the second part, we will consider the case of CM abelian varieties. Let E be a CM field of degree $[E:\mathbb{Q}] = 2g$ and assume that A has CM type $(\mathcal{O}_E, \Phi)$ where Φ is a CM type of E and $\mathcal{O}_E$ is the ring of integers of E. Faltings height computations are especially amenable to these abelian varieties. In particular, Colmez showed that $h(A)$ only depends on the CM type Φ. We may therefore write $h(\Phi)$ to denote this Faltings height. In the case of CM elliptic curves we can in fact say more using the following version of the formula of Lerch-Chowla-Selberg.

Theorem 1.2 (Lerch-Chowla-Selberg). *Suppose that E is an imaginary quadratic field of discriminant $-d < 0$ and let $\eta : (\mathbb{Z}/d\mathbb{Z})^\times \to \{\pm 1\}$ be its quadratic character. Then it holds*

$$h(\Phi) = -\tfrac{1}{2}\frac{L'(\eta, 0)}{L(\eta, 0)} - \tfrac{1}{4}\log d,$$

where $L'(\eta, s)$ is the derivative of the Dirichlet L-function $L(\eta, s)$ of η.

We will discuss several generalizations and reformulations of this formula. In particular we shall consider the following averaged version of a conjecture of Colmez.

Theorem 1.3 (Averaged Colmez Conjecture). *Let F be the maximal totally real subfield of E. Denote by $\Delta_{E/F}$ the norm of the relative discriminant of E/F, and write Δ_F for the absolute discriminant of F. Then it holds*

$$\tfrac{1}{2^g}\sum_{\Phi} h(\Phi) = -\tfrac{1}{2}\frac{L'(\eta, 0)}{L(\eta, 0)} - \tfrac{1}{4}\log(\Delta_{E/F}\Delta_F)$$

with the sum taken over all distinct CM types Φ of E. Here $L'(\eta, s)$ is the derivative of the L-function $L(\eta, s)$ of the quadratic character η of $\mathbb{A}_F^\times / F^\times$ defined by E/F.

The above theorem was proven by Yuan-Zhang and independently by Andreatta-Goren-Howard-Madapusi Pera. The averaged Colmez conjecture played a key role in proving the André-Oort conjecture for large classes of Shimura varieties, including the moduli spaces A_g of principally polarized abelian varieties of arbitrary positive dimension g. We shall explain the main ideas and concepts of the proof of the averaged Colmez conjecture given by Yuan-Zhang. Their proof involves Shimura curves and it uses the method of Yuan-Zhang-Zhang which they developed to prove generalized Gross-Zagier formulas for Shimura curves.

In the last part, we will then discuss the work of Yun-W. Zhang which can be viewed as a simultaneous generalization for function fields of the Chowla-Selberg formula, the Waldspurger formula, and the Gross-Zagier formula. In particular they studied higher order derivatives of certain L-functions at the center and they proved a formula for unramified cuspidal automorphic representation π of PGL_2 over a function field $F=k(X)$, where X is a curve over a finite field k. In fact they express the r-th central derivative of the L-function (base changed along a quadratic extension E of F) in terms of the self-intersection number of the π-isotypic component of the Heegner-Drinfeld cycle Sht_T^r on the moduli stack Sht_G^r.

Theorem 1.4 (Higher Gross-Zagier formula, Yun-W. Zhang). *There is an explicit positive constant $c(\pi)$ such that*

$$([\mathrm{Sht}_T^r]_\pi, [\mathrm{Sht}_T^r]_\pi) = c(\pi) L^{(r)}(\pi_E, 1/2).$$

Here the moduli stack Sht_G^r is closely related to the moduli stack of Drinfeld shtukas of rank two with r modifications. An important feature of Sht_G^r is that it admits a natural fibration $\mathrm{Sht}_G^r \to X^r$ where X^r is the r-fold self-product of X over k. In the number field case, the analogous spaces (currently) only exist when $r \leq 1$. In the case $r=0$, the moduli stack Sht_G^0 is the constant groupoid over k given by $\mathrm{Bun}_G(k) \cong G(F)\backslash G(\mathbb{A}_F)/H$ where $\mathbb{A}_F$ is the ring of adèles of F and where H is a maximal compact open subgroup of $G(\mathbb{A}_F)$. The double coset $G(F)\backslash G(\mathbb{A}_F)/H$ has a meaning when F is a number field. In the case when $r=1$ and $F=\mathbb{Q}$, the counterpart of Sht_G^1 is the moduli stack of elliptic curves which is defined over $\mathbb{Z}$.