Contents

Chapter One

Introduction

1.1 FROM THE NASH SYSTEM TO THE MASTER EQUATION

Game theory formalizes interactions between "rational" decision makers. Its applications are numerous and range from economics and biology to computer science. In this monograph we are interested mainly in noncooperative games, that is, in games in which there is no global planner: each player pursues his or her own interests, which are partly conflicting with those of others.

In noncooperative game theory, the key concept is that of Nash equilibria, introduced by Nash in [82]. A Nash equilibrium is a choice of strategies for the players such that no player can benefit by changing strategies while the other players keep theirs unchanged. This notion has proved to be particularly relevant and tractable in games with a small number of players and action sets. However, as soon as the number of players becomes large, it seems difficult to implement in practice, because it requires that each player knows the strategies the other players will use. Besides, for some games, the set of Nash equilibria is huge and it seems difficult for the players to decide which equilibrium they are going to play: for instance, in repeated games, the Folk theorem states that the set of Nash equilibria coincides with the set of feasible and individually rational payoffs in the one-shot game, which is a large set in general (see [93]).

In view of these difficulties, one can look for configurations in which the notion of Nash equilibria simplifies. As noticed by Von Neumann and Morgenstern [96], one can expect that this is the case when the number of players becomes large and each player individually has a negligible influence on the other players: it "is a well known phenomenon in many branches of the exact and physical sciences that very great numbers are often easier to handle than those of medium size [. . .]. This is of course due to the excellent possibility of applying the laws of statistics and probabilities in the first case" (p. 14). Such *nonatomic games* were analyzed in particular by Aumann [10] in the framework of cooperative games. Schmeidler [91] (see also Mas-Colell [78])) extended the notion of Nash equilibria to that setting and proved the existence of pure Nash equilibria.

In the book we are interested in games with a continuum of players, in continuous time and continuous state space. Continuous time, continuous space games are often called *differential games*. They appear in optimal control problems in which the system is controlled by several agents. Such problems (for a finite number of players) were introduced at about the same time by Isaacs [59] and Pontryagin [87]. Pontryagin derived optimality conditions for these games. Isaacs, working on specific examples of two-player zero-sum differential games, computed explicitly the solution of these games and established the formal connection with the Hamilton–Jacobi equations. The rigorous justification of Isaacs ideas for general systems took some time. The main difficulty arose from from the set of strategies (or from the dependence on the cost of the players with respect to these strategies), which is much more complex than for classical games: indeed, the players have to observe the actions taken by the other players in continuous time and choose their instantaneous actions accordingly. For two-player, zero-sum differential games, the first general existence result of a Nash equilibrium was established by Fleming [39]: in this case the Nash equilibrium is unique and is called the value function (it is a function of time and space). The link between this value function and the Hamilton–Jacobi equations was made possible by the introduction of viscosity solutions by Crandall and Lions [32] (see also [33] for a general presentation of viscosity solutions). The application to zero-sum differential games are due to Evans and Souganidis [35] (for determinist problems) and Fleming and Souganidis [40] (for stochastic ones).

For non-zero-sum differential games, the situation is more complicated. One can show the existence of general Nash equilibria thanks to an adaptation of the Folk theorem: see Kononenko [64] (for differential games of first order) and Buckdahn, Cardaliaguet, and Rainer [23] (for differential games with diffusion). However, this notion of solution does not allow for dynamic programming: it lacks time consistency in general. The existence of time-consistent Nash equilibria, based on dynamic programming, requires the solvability of a strongly coupled system of Hamilton–Jacobi equations. This system, which plays a key role in this book, is here called *the Nash system*. For problems without diffusions, Bressan and Shen explain in [21, 22] that the Nash system is ill-posed in general. However, for systems with diffusions, the Nash system becomes a uniformly parabolic system of partial differential equations. Typically, for a game with N players and with uncontrolled diffusions, this backward in time system takes the form

$$\begin{cases} -\partial_t v^i(t,\boldsymbol{x}) - \operatorname{tr}(a^i(t,\boldsymbol{x})D^2 v^N(t,\boldsymbol{x})) + \mathcal{H}^i(t,\boldsymbol{x},Dv^1(t,\boldsymbol{x}),\dots,Dv^N(t,\boldsymbol{x})) = 0 \\ \qquad\qquad\qquad\qquad \text{in } [0,T]\times(\mathbb{R}^d)^N,\ i\in\{1,\dots,N\}, \\ v^i(T,\boldsymbol{x}) = G^i(\boldsymbol{x}) \qquad \text{in } (\mathbb{R}^d)^N. \end{cases} \tag{1.1}$$

The foregoing system describes the evolution in time of the value function v^i of agent i ($i \in \{1,\dots,N\}$). This value function depends on the positions of all the players $\boldsymbol{x} = (x_1,\dots,x_N)$, x_i being the position of the state of player i. The second-order terms $\operatorname{tr}(a^i(t,\boldsymbol{x})D^2v^N(t,\boldsymbol{x}))$ formalize the noises affecting the dynamics of agent i. The Hamiltonian $\mathcal{H}^i$ encodes the cost player i has to pay to control her state and reaching some goal. This cost depends on the positions of the other players and on their strategies.

The relevance of such a system for differential games has been discussed by Star and Ho [94] and Case [30] (for first-order systems) and by Friedman [43] (1972) (for second-order systems); see also the monograph by Başar and Olsder [11] and the references therein. The well-posedness of this system has been established under some restrictions on the regularity and the growth of the Hamiltonians: See in particular the monograph by Ladyženskaja, Solonnikov, and Ural'ceva [70] and the paper by Bensoussan and Frehse [14].

As for classical games, it is natural to investigate the limit of differential games as the number of players tends to infinity. The hope is that in this limit configuration the Nash system simplifies. This notion makes sense only for time-consistent Nash equilibria, because no simplification occurs in the framework of Folk's theorem, where the player who deviates is punished by all the other players.

Games in continuous space with infinitely many players were first introduced in the economic literature (in discrete time) under the terminology of heterogeneous models. The aim was to formalize dynamic general equilibria in macroeconomics by taking into account not only aggregate variables—GDP, employment, the general price level, for example—but also the distributions of variables, say the joint distribution of income and wealth or the size distribution of firms, and to try to understand how these variables interact. We refer in particular to the pioneering works by Aiyagari [6], Huggett [58], and Krusell and Smith [65], as well as the presentation of the continuous-time counterpart of these models in [5].

In the mathematical literature, the theory of differential games with infinitely many players, known as mean field games (MFGs), started with the works of Lasry and Lions [71, 72, 74]; Huang, Caines, and Malhamé [53–57] presented similar models under the name of the certainty equivalence principle. Since then the literature has grown very quickly, not only for the theoretical aspects, but also for the numerical methods and the applications: we refer to the monographs [16, 48] or the survey paper [49].

This book focuses mainly on the derivation of the MFG models from games with a finite number of players. In classical game theory, the rigorous link between the nonatomic games and games with a large but finite number of agents is quite well-understood: one can show (1) that limits of Nash equilibria as the number of agents tends to infinity is a Nash equilibrium of the nonatomic game (Green [50]), and (2) that any optimal strategy in the nonatomic game provides an ϵ-Nash equilibrium in the game with finitely many players, provided the number of players is sufficiently large (Rashid [90]).

For MFGs, the situation is completely different. If the equivalent of question (2) is pretty well understood, problem (1) turns out to be surprisingly difficult. Indeed, passing from the MFG equilibria to the differential game with finitely many problem relies mostly on known techniques in mean field theory: this has been developed since the beginning of the theory in [54] and well studied since then (see also, for instance, [25, 62]). On the contrary, when one considers a sequence of solutions to the Nash systems with N players and one wants to let N tend to infinity, the problem becomes extremely intricate. The main reason is that, in classical game theory, this convergence comes from compactness properties of the problem; this compactness is completely absent for differential games. This issue is related to the difficulty of building time-consistent solutions for these games. A less technical way to see this is to note that there is a change of nature between the Nash system and its conjectured limit, the MFG. In the Nash system, the players observe each other, and the deviation of a single player could a priori change entirely the outcome of the game. On the contrary, in the MFG, players react only to the evolving population density and therefore the deviation of a single player has no impact at all on the system. The main purpose of this book is to explain why this limit holds despite this change of nature.

1.1.1 Statement of the Problem

To explain our result further, we first need to specify the Nash system we are considering. We assume that players control their own state and interact only through their cost function. Then the Nash system (1.1) takes the more specific form:

$$\begin{cases} -\partial_t v^{N,i}(t,\boldsymbol{x}) - \sum_{j=1}^{N} \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) - \beta \sum_{j,k=1}^{N} \mathrm{Tr} D^2_{x_j,x_k} v^{N,i}(t,\boldsymbol{x}) \\ \quad + H(x_i, D_{x_i} v^{N,i}(t,\boldsymbol{x})) + \sum_{j\neq i} D_p H(x_j, D_{x_j} v^{N,j}(t,\boldsymbol{x})) \cdot D_{x_j} v^{N,i}(t,\boldsymbol{x}) \\ \quad = F^{N,i}(\boldsymbol{x}) \qquad \text{in } [0,T]\times(\mathbb{R}^d)^N, \\ v^{N,i}(T,\boldsymbol{x}) = G^{N,i}(\boldsymbol{x}) \qquad \text{in } (\mathbb{R}^d)^N. \end{cases} \tag{1.2}$$

As before, the above system is stated in $[0,T]\times(\mathbb{R}^d)^N$, where a typical element is denoted by $(t,\boldsymbol{x})$ with $\boldsymbol{x} = (x_1,\ldots,x_N) \in (\mathbb{R}^d)^N$. The unknowns are the N maps $(v^{N,i})_{i\in\{1,\ldots,N\}}$ (the value functions). The data are the Hamiltonian $H : \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$, the maps $F^{N,i}, G^{N,i} : (\mathbb{R}^d)^N \to \mathbb{R}$, the nonnegative parameter β, and the horizon $T \geqslant 0$. In the second line, the symbol $\cdot$ denotes the inner product in $\mathbb{R}^d$.

System (1.2) describes the Nash equilibria of an N-player differential game (see Section 1.2 for a short description). In this game, the set of "optimal trajectories" solves a system of N coupled stochastic differential equations (SDEs):

$$dX_{i,t} = -D_pH\big(X_{i,t}, Dv^{N,i}(t,\boldsymbol{X}_t)\big)dt + \sqrt{2}\,dB^i_t + \sqrt{2\beta}\,dW_t, \quad t\in[0,T],\ i\in\{1,\ldots,N\}, \tag{1.3}$$

where $v^{N,i}$ is the solution to (1.2) and the $((B^i_t)_{t\in[0,T]})_{i=1,\ldots,N}$ and $(W_t)_{t\in[0,T]}$ are d-dimensional independent Brownian motions. The Brownian motions $((B^i_t)_{t\in[0,T]})_{i=1,\ldots,N}$ correspond to the *individual noises*, while the Brownian motion $(W_t)_{t\in[0,T]}$ is the same for all the equations and, for this reason, is called the *common noise*. Under such a probabilistic point of view, the collection of random processes $((X_{i,t})_{t\in[0,T]})_{i=1,\dots,N}$ forms a dynamical system of interacting particles.

The aim of this book is to understand the behavior, as N tends to infinity, of the value functions $v^{N,i}$. Another, but closely related, objective of our book is to study the mean field limit of the $((X_{i,t})_{t\in[0,T]})_{i=1,\dots,N}$ as N tends to infinity.

1.1.2 Link with the Mean Field Theory

Of course, there is no chance to observe a mean field limit for (1.3) under a general choice of the coefficients in (1.2). Asking for a mean field limit certainly requires that the system has a specific symmetric structure in such a way that the players in the differential game are somewhat exchangeable (when in equilibrium). For this purpose, we suppose that, for each $i \in \{1,\dots,N\}$, the maps $(\mathbb{R}^d)^N \ni \boldsymbol{x} \mapsto F^{N,i}(\boldsymbol{x})$ and $(\mathbb{R}^d)^N \ni \boldsymbol{x} \mapsto G^{N,i}(\boldsymbol{x})$ depend only on x_i and on the empirical distribution of the variables $(x_j)_{j\neq i}$:

$$F^{N,i}(\boldsymbol{x}) = F(x_i, m_{\boldsymbol{x}}^{N,i}) \qquad \text{and} \qquad G^{N,i}(\boldsymbol{x}) = G(x_i, m_{\boldsymbol{x}}^{N,i}), \tag{1.4}$$

where $m_{\boldsymbol{x}}^{N,i} = \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}$ is the empirical distribution of the $(x_j)_{j\neq i}$ and where $F, G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ are given functions, $\mathcal{P}(\mathbb{R}^d)$ being the set of Borel probability measures on $\mathbb{R}^d$. Under this assumption, the solution of the Nash system indeed enjoys strong symmetry properties, which imply in particular the required exchangeability property. Namely, $v^{N,i}$ can be written in a form similar to (1.4):

$$v^{N,i}(t,\boldsymbol{x}) = v^N(t, x_i, m_{\boldsymbol{x}}^{N,i}), \quad t \in [0,T], \quad \boldsymbol{x} \in (\mathbb{R}^d)^N, \tag{1.5}$$

for a function $v^N(t,\cdot,\cdot)$ taking as arguments a state in $\mathbb{R}^d$ and an empirical distribution of size $N-1$ over $\mathbb{R}^d$.

In any case, even under the foregoing symmetry assumptions, it is by no means clear whether the system (1.3) can exhibit a mean field limit. The reason is that the dynamics of the particles $(X_{1,t},\dots,X_{N,t})_{t\in[0,T]}$ are coupled through the unknown solutions $v^{N,1},\dots,v^{N,N}$ to the Nash system (1.2), whose symmetry properties (1.5) may not suffice to apply standard results from the theory of propagation of chaos. Obviously, the difficulty is that the function v^N on the right-hand side of (1.5) precisely depends on N. Part of the challenge in the text is thus to show that the interaction terms in (1.3) get closer and closer, as N tends to the infinity, to some interaction terms with a much more tractable and much more explicit shape.

To get a picture of the ideal case under which the mean-field limit can be taken, one can choose for a while $\beta = 0$ in (1.3) and then assume that the function v^N in the right-hand side of (1.5) is independent of N. Equivalently, one can replace in (1.3) the interaction function $(\mathbb{R}^d)^N \ni \boldsymbol{x} \mapsto D_pH(x_i, v^{N,i}(t,\boldsymbol{x}))$

by $(\mathbb{R}^d)^N \ni \boldsymbol{x} \mapsto b(x_i, m_{\boldsymbol{x}}^{N,i})$, for a map $b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \mapsto \mathbb{R}^d$. In such a case, the coupled system of SDEs (1.3) turns into

$$dX_{i,t} = b\Big(X_{i,t}, \frac{1}{N-1}\sum_{j\neq i} \delta_{X_{j,t}}\Big) dt + \sqrt{2}\, dB_t^i, \quad t \in [0,T],\ i \in \{1,\dots,N\}, \tag{1.6}$$

the second argument in b being nothing but the empirical measure of the particle system at time t. Under suitable assumptions on b (e.g., if b is bounded and Lipschitz continuous in both variables, the space of probability measures being equipped with the Wasserstein distance) and on the initial distribution of the $((X_{i,t})_{i=1,\dots,N})_{t\in[0,T]}$, both the marginal law of $(X_t^1)_{t\in[0,T]}$ (or of any other player) and the empirical distribution of the whole system converge to the solution of the McKean–Vlasov equation:

$$\partial_t m - \Delta m + \operatorname{div}\big(m\, b(\cdot, m)\big) = 0.$$

(see, among many other references, McKean [77], Sznitman [92], Méléard [79]). The standard strategy for establishing the convergence consists in a coupling argument. Precisely, if one introduces the system of N independent equations

$$dY_{i,t} = b\big(Y_{i,t}, \mathcal{L}(Y_{i,t})\big)\ dt + \sqrt{2}\, dB_t^i, \qquad t \in [0,T],\ i \in \{1,\dots,N\},$$

(where $\mathcal{L}(Y_{i,t})$ is the law of $Y_{i,t}$) with the same (chaotic) initial condition as that of the processes $((X_{i,t})_{t\in[0,T]})_{i=1,\dots,N}$, then it is known that (under appropriate integrability conditions; see Fournier and Guillin [42])

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X_{1,t} - Y_{1,t}|\right] \leqslant CN^{-\frac{1}{\max(2,d)}}\big(\mathbf{1}_{\{d\neq 2\}} + \ln(1+N)\mathbf{1}_{\{d=2\}}\big).$$

In comparison with (1.6), all the equations in (1.3) are subject to the common noise $(W_t)_{t\in[0,T]}$, at least when $\beta \neq 0$. This makes a first difference between our limit problem and the above McKean–Vlasov example of interacting diffusions, but, for the time being, it is not clear how deeply this may affect the analysis. Indeed, the presence of a common noise does not constitute a real challenge in the study of McKean–Vlasov equations, the foregoing coupling argument working in that case as well, provided that the distribution of Y is replaced by its conditional distribution given the realization of the common noise. However, the key point here is precisely that our problem is not formulated as a McKean–Vlasov equation, as the drifts in (1.3) are not of the same explicit mean field structure as they are in (1.6) because of the additional dependence on N in the right-hand side of (1.5): obviously this is the second main difference between (1.3) and (1.6). This makes rather difficult any attempt to guess the precise impact of the common noise on the analysis. Certainly, as we already pointed out, the major issue in analyzing (1.3) stems from the complex nature of the underlying interactions. As the equations depend on one another through the nonlinear system (1.2), the evolution with N of the coupling between all of them is indeed much more intricate than in (1.6). And once again, on the top of that, the common noise adds another layer of difficulty. For these reasons, the convergence of both (1.2) and (1.3) has been an open question since Lasry and Lions' initial papers on MFGs [71, 72].

1.1.3 The Mean Field Game System

If one tries, at least in the simpler case $\beta = 0$, to describe—in a heuristic way—the structure of a differential game with infinitely many indistinguishable players, i.e., a "nonatomic differential game," one finds a problem in which each (infinitesimal) player optimizes his payoff, depending on the collective behavior of the others, and, meanwhile, the resulting optimal state of each of them is exactly distributed according to the state of the population. This is the "mean field game system" (MFG system):

$$\begin{cases} -\partial_t u - \Delta u + H(x, D_x u) = F(x, m(t)) & \text{in } [0,T] \times \mathbb{R}^d, \\ \partial_t m - \Delta m - \operatorname{div}(m D_p H(x, D_x u)) = 0 & \text{in } [0,T] \times \mathbb{R}^d, \\ u(T,x) = G(x, m(T)),\ m(0,\cdot) = m_{(0)} & \text{in } \mathbb{R}^d, \end{cases} \tag{1.7}$$

where $m_{(0)}$ denotes the initial state of the population. The system consists in a coupling between a (backward) Hamilton–Jacobi equation, describing the dynamics of the value function of any of the players, and a (forward) Kolmogorov equation, describing the dynamics of the distribution of the population. In that framework, H reads as a Hamiltonian, F is understood as a running cost, and G as a terminal cost. Since its simultaneous introduction by Lasry and Lions [74] and by Huang, Caines, and Malhamé [53], this system has been thoroughly investigated: its existence, under various assumptions, can be found in [15, 25, 54–56, 62, 74, 76]. Concerning uniqueness of the solution, two regimes were identified in [74]. Uniqueness holds under Lipschitz type conditions when the time horizon T is short (or, equivalently, when H, F, and G are "small"), but, as for finite-dimensional two-point boundary value problems, it may fail when the system is set over a time interval of arbitrary length. Over long time intervals, uniqueness is guaranteed under the quite fascinating condition that *F and G are monotone*; i.e., if, for any measures m, m', the following holds:

$$\begin{aligned} &\int_{\mathbb{R}^d} (F(x,m) - F(x,m')\, d(m - m')(x) \geqslant 0 \\ \text{and} \quad &\int_{\mathbb{R}^d} (G(x,m) - G(x,m')\, d(m - m')(x) \geqslant 0. \end{aligned} \tag{1.8}$$

The interpretation of the monotonicity condition is that the players dislike congested areas and favor configurations in which they are more scattered; see Remark 2.3.1 for an example. Generally speaking, condition (1.8) plays a key role throughout the text, as it guarantees not only uniqueness but also stability of the solutions to (1.7).

As observed, a solution to the MFG system (1.7) can indeed be interpreted as a Nash equilibrium for a differential game with infinitely many players: in that framework, it plays the role of the Schmeidler noncooperative equilibrium. A standard strategy to make the connection between (1.7) and differential games consists in inserting the optimal strategies from the Hamilton–Jacobi equation in (1.7) into finitely many player games in order to construct approximate Nash equilibria: see [54], as well as [25, 55, 56, 62]. However, although it establishes the interpretation of the system (1.7) as a differential game with infinitely many players, this says nothing about the convergence of (1.2) and (1.3).

When β is positive, the system describing Nash equilibria within a population of infinitely many players subject to the same common noise of intensity β cannot be described by a deterministic system of the same form as (1.7). Owing to the theory of propagation of chaos for systems of interacting particles (see the short remark earlier), the unknown m in the forward equation is then expected to represent the conditional law of the optimal state of any player given the realization of the common noise. In particular, it must be random. This turns the forward Kolmogorov equation into a forward stochastic Kolmogorov equation. As the Hamilton–Jacobi equation depends on m, it renders u random as well. At any rate, a key fact from the theory of stochastic processes is that the solution to an SDE must be adapted to the underlying observation, as its values at some time t cannot anticipate the future of the noise after t. At first sight, it seems to be very demanding, as u is also required to match, at time T, $G(\cdot, m(T))$, which depends on the whole realization of the noise up until T. The correct formulation to accommodate both constraints is given by the theory of backward SDEs, which suggests penalizing the backward dynamics by a martingale in order to guarantee that the solution is indeed adapted. We refer the reader to the monograph [84] for a complete account on the finite dimensional theory and to the paper [85] for an insight into the infinite dimensional case. Denoting by W "the common noise" (here, a d-dimensional Brownian motion) and by $m_{(0)}$ the initial distribution of the players at time t_0, the MFG system with common noise then takes the form (in which the unknowns are now (u_t, m_t, v_t))

$$\begin{cases} d_t u_t = \big[-(1+\beta)\Delta u_t + H(x, D_x u_t) - F(x, m_t) - \sqrt{2\beta}\,\mathrm{div}(v_t)\big]\,dt \\ \qquad\qquad + v_t \cdot dW_t, \qquad\qquad \text{in } [0,T]\times\mathbb{R}^d, \\ d_t m_t = \big[(1+\beta)\Delta m_t + \mathrm{div}\big(m_t D_p H(x, D_x u_t)\big)\big]\,dt \\ \qquad\qquad - \mathrm{div}\big(m_t \sqrt{2\beta}\,dW_t\big), \qquad\qquad \text{in } [0,T]\times\mathbb{R}^d, \\ u_T(x) = G(x, m_T),\ m_0 = m_{(0)}, \qquad \text{in } \mathbb{R}^d \end{cases} \tag{1.9}$$

where we used the standard convention from the theory of stochastic processes that consists in indicating the time parameter as an index in random functions. As suggested immediately above, the map v_t is a random vector field that forces the solution u_t of the backward equation to be adapted to the filtration generated by $(W_t)_{t\in[0,T]}$. As far as we know, the system (1.9) has never been investigated and part of this book will be dedicated to its analysis (see, however, [27] for an informal discussion). Below, we call the system (1.9) the *MFG system with common noise.*

Note that the aggregate equations (1.7) and (1.9) (see also the master equation (1.10)) are the continuous-time analogues of equations that appear in the analysis of dynamic stochastic general equilibria in heterogeneous agent models (Aiyagari [6], Bewley [19], and Huggett [58]). In this setting, the factor β describes the intensity of "aggregate shocks," as discussed by Krusell and Smith in the seminal paper [65]. In some sense, the limit problem studied in the text is an attempt to deduce the macroeconomic models, describing the dynamics of a typical (but heterogeneous) agent in an equilibrium configuration, from the microeconomic ones (the Nash equilibria).

1.1.4 The Master Equation

Although the MFG system has been widely studied since its introduction in [74] and [53], it has become increasingly clear that this system was not sufficient to take into account the entire complexity of dynamic games with infinitely many players. A case in point is that the original system (1.7) becomes much more complex in the presence of a common noise (i.e., when $\beta > 0$); see the stochastic version (1.9). In the same spirit, we may notice that the original MFG system (1.7) does not accommodate MFGs with a major player and infinitely many small players; see [52]. And, last but not least, the main limitation is that, so far, the formulation based on the system (1.7) (or (1.9) when $\beta > 0$) has not allowed establishment of a clear connection with the Nash system (1.2).

These issues led Lasry and Lions [76] to introduce an infinite dimensional equation—the so-called "master equation"—that directly describes, at least formally, the limit of the Nash system (1.2) and encompasses the foregoing complex situations. Before writing down this equation, let us explain its main features. One of the key observations has to do with the symmetry properties, to which we already alluded, that are satisfied by the solution of the Nash system (1.2). Under the standing symmetry assumptions (1.4) on the $(F^{N,i})_{i=1,\dots,N}$ and $(G^{N,i})_{i=1,\dots,N}$, (1.5) says that the $(v^{N,i})_{1,\dots,N}$ can be written into a form similar to (1.4), namely $v^{N,i}(t,\boldsymbol{x}) = v^N(t, x_i, m_{\boldsymbol{x}}^{N,i})$ (where the empirical measures $m_{\boldsymbol{x}}^{N,i}$ are defined as in (1.4)), but with the obvious but major restriction that the function v^N that appears on the right-hand side of the equality now depends on N. With such a formulation, the value function to player i reads as a function of the private state of player i and of the empirical distribution formed by the others. Then, one may guess, at least under the additional assumption that such a structure is preserved as $N \to +\infty$, that the unknown in the limit problem takes the form $U = U(t, x, m)$, where x is the position of the (typical) small player at time t and m is the distribution of the (infinitely many) other agents.

The question is then to write down the dynamics of U. Plugging $U = U(t, x_i, m_{\boldsymbol{x}}^{N,i})$ into the Nash system (1.2), one obtains—at least formally—an equation stated in the space of measures (see Section 1.2 for a heuristic discussion). This is the so-called master equation. It takes the form

$$\begin{cases} -\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U) - (1+\beta)\int_{\mathbb{R}^d} \mathrm{div}_y\,[D_m U]\, dm(y) \\ \quad + \int_{\mathbb{R}^d} D_m U \cdot D_p H\big(y, D_x U(\cdot, y, \cdot)\big) dm(y) \\ \quad -2\beta \int_{\mathbb{R}^d} \mathrm{div}_x\,[D_m U]\, dm(y) - \beta \int_{\mathbb{R}^{2d}} \mathrm{Tr}\,\big[D^2_{mm} U\big]\, dm^{\otimes 2}(y, y') \\ = F(x, m) \qquad \text{in } [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d), \\ U(T, x, m) = G(x, m) \qquad \text{in } \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d), \end{cases} \tag{1.10}$$

where $\partial_t U$, $D_x U$, and $\Delta_x U$ are understood as $\partial_t U(t, x, m)$, $D_x U(t, x, m)$, and $\Delta_x U(t, x, m)$; $D_x U(\cdot, y, \cdot)$ is understood as $D_x U(t, y, m)$; and $D_m U$ and $D^2_{mm} U$ are understood as $D_m(t, x, m, y)$ and $D^2_{mm} U(t, x, m, y, y')$.

In Eq. (1.10), $\partial_t U$, $D_x U$, and $\Delta_x U$ stand for the usual time derivative, space derivatives, and Laplacian with respect to the local variables (t, x) of the unknown U, while $D_m U$ and $D^2_{mm} U$ are the first- and second-order derivatives with respect to the measure m. The precise definition of these derivatives is postponed to Chapter 2. For the time being, let us just note that it is related to the derivatives in the space of probability measures described, for instance, by Ambrosio, Gigli, and Savaré in [7] and by Lions in [76]. It is worth mentioning that the master equation (1.10) is not the first example of an equation studied in the space of measures—by far: for instance, Otto [83] gave an interpretation of the porous medium equation as an evolution equation in the space of measures, and Jordan, Kinderlehrer, and Otto [60] showed that the heat equation was also a gradient flow in that framework; notice also that the analysis of Hamilton–Jacobi equations in metric spaces is partly motivated by the specific case in which the underlying metric space is the space of measures (see in particular [8,36] and the references therein). The master equation is, however, the first one to combine at the same time the issue of being nonlocal, nonlinear, and of second order and, moreover, without maximum principle.

Besides the discussion in [76], the importance of the master equation (1.10) has been acknowledged by several contributions: see, for instance, the monograph [16] and the companion papers [17] and [18], in which Bensoussan, Frehse, and Yam generalize this equation to mean field type control problems and reformulate it as a partial differential equation (PDE) set on an L^2 space, and [27], where Carmona and Delarue interpret this equation as a decoupling field of forward–backward SDE in infinite dimension.

If the master equation has been discussed and manipulated thoroughly in the aforementioned references, it is mostly at a formal level: the well-posedness of the master equation has remained, to a large extent, open until now. Besides, even if the master equation has been introduced to explain the convergence of the Nash system, the rigorous justification of the convergence has not been understood.

The aim of this book is to provide an answer to both questions.

1.1.5 Well-posedness of the Master Equation

The largest part of this book is devoted to the proof of the existence and uniqueness of a classical solution to the master equation (1.10), where, by classical, we mean that all the derivatives in (1.10) exist and are continuous. To avoid issues related to boundary conditions or conditions at infinity, we work for simplicity with periodic data: the maps H, F, and G are periodic in the space variable. The state space is therefore the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $m_{(0)}$ belongs to $\mathcal{P}(\mathbb{T}^d)$, the set of Borel probability measures on $\mathbb{T}^d$. We also assume that $F, G : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ satisfy the monotonicity conditions (1.8) and are sufficiently "differentiable" with respect to both variables and, of course, periodic with respect to the state variable. Although the periodicity condition is rather restrictive, the extension to maps defined on the full space or to Neumann boundary conditions is probably not a major issue. At any rate, it would certainly require further technicalities.

So far, the existence of classical solutions to the master equation has been known in more restricted frameworks. Lions discussed in [76] a finite dimensional analogue of the master equation and derived conditions for this hyperbolic system to be well posed. These conditions correspond precisely to the monotonicity property (1.8), which we here assume to be satisfied by the coupling functions F and G. This parallel strongly indicates—but this should not come as a surprise—that the monotonicity of F and G should play a key role in the unique strong solvability of (1.10). Lions also explained in [76] how to get the well-posedness of the master equation without noise (no Laplacian in the equation) by extending the equation to a (fixed) space of random variables under a convexity assumption in space of the data. In [24] Buckdahn, Li, Peng, and Rainer studied equation (1.10), by means of probabilistic arguments, when there is no coupling or common noise ($F = G = 0$, $\beta = 0$) and proved the existence of a classical solution in this setting; in a somewhat similar spirit, Kolokoltsov, Li, and Yang [62] and Kolokoltsov, Troeva, and Yang [63] investigated the tangent process to a flow of probability measures solving a McKean–Vlasov equation. Gangbo and Swiech [45] analyzed the first-order master equation in short time (no Laplacian in the equation) for a particular class of Hamiltonians and of coupling functions F and G (which are required to derive from a potential in the measure argument). Chassagneux, Crisan, and Delarue [31] obtained, by a probabilistic approach similar to that used in [24], the existence and uniqueness of a solution to (1.10) without common noise (when $\beta = 0$) under the monotonicity condition (1.8) in either the nondegenerate case (as we do here) or in the degenerate setting provided that F, H, and G satisfy additional convexity conditions in the variables (x, p). The complete novelty of our result, regarding the specific question of solvability of the master equation, is the existence and uniqueness of a classical solution to the problem with common noise.

The technique of proof in [24, 31, 45] consists in finding a suitable representation of the solution: indeed a key remark in Lions [76] is that the master equation is a kind of transport equation in the space of measures and that its characteristics are, when $\beta = 0$, the MFG system (1.7). Using this idea, the main difficulty is then to prove that the candidate is smooth enough to perform the computation showing that it is a classical solution of (1.10). In [24,31] this is obtained by linearizing systems of forward–backward SDEs, while [45] relies on a careful analysis of the characteristics of the associated first-order PDE.

Our starting point is the same: we use a representation formula for the master equation. When $\beta = 0$, the characteristics are just the solution to the MFG system (1.7). When β is positive, these characteristics become random under the action of the common noise and are then given by the solution of the MFG system with common noise (1.9).

The construction of a solution U to the master equation then relies on the method of characteristics. Namely, we *define* U by letting $U(t_0, x, m_0) := u_{t_0}(x)$, where the pair $(u_t, m_t)_{t\in[t_0,T]}$ is the solution to (1.9) when the forward equation is initialized at $m_{(0)} \in \mathcal{P}(\mathbb{T}^d)$ at time t_0, that is,

$$\begin{cases} d_t u_t = \big[-(1+\beta)\Delta u_t + H(x, D_x u_t) - F(x, m_t) - \sqrt{2\beta}\mathrm{div}(v_t)\big]\, dt \\ \qquad\qquad +v_t \cdot dW_t \qquad\qquad \text{in } [t_0, T] \times \mathbb{T}^d, \\ d_t m_t = \big[(1+\beta)\Delta m_t + \mathrm{div}\big(m_t D_p H(x, D_x u_t)\big)\big]\, dt \\ \qquad\qquad -\mathrm{div}\big(m_t \sqrt{2\beta}\, dW_t\big) \qquad\qquad \text{in } [t_0, T] \times \mathbb{T}^d, \\ u_T(x) = G(x, m_T),\ m_{t_0} = m_{(0)} \qquad \text{in } \mathbb{T}^d. \end{cases} \tag{1.11}$$

There are two main difficult steps in the analysis. The first one is to establish the smoothness of U and the second one is to show that U indeed satisfies the master equation (1.10). To proceed, the cornerstone is to make a systematic use of the monotonicity properties of the maps F and G: basically, monotonicity prevents the emergence of singularities in finite time. Our approach seems to be very powerful, although the reader might have a different feeling because of the length of the arguments. As a matter of fact, part of the technicalities in the proof are caused by the stochastic aspect of the characteristics (1.11). As a result, we spend much effort to handle the case with a common noise (for which almost nothing has been known so far), but, in the simpler case $\beta = 0$, our strategy to handle the first-order master equation provides a much shorter proof than in the earlier works [24,31,45]. For this reason, we decided to display the proof in this simple context separately (Section 3).

It is worth mentioning that, although our result is the first one to address the MFG system (1.11) in the case $\beta > 0$, the existence and uniqueness of equilibria to MFGs with a common noise were already studied in the paper [29] by

Carmona, Delarue, and Lacker. Therein, the strategy is completely different, as the existence is investigated first by combining purely probabilistic arguments together with Kakutani–Fan–Glicksberg's theorem for set-valued mappings. As a main feature, existence of equilibria is proved by means of a discretization procedure of the common noise, which consists in focusing first on the case when the common noise has a finite number of outcomes. This constraint on the noise is relaxed in a second step. However, it must be stressed that the limiting solutions that are obtained in this way (for the MFG driven by the original noise) are *weak equilibria* only, which means that they may not be adapted with respect to the common source of noise. This fact is completely reminiscent of the construction of weak solutions to SDEs. Remarkably, Yamada-Watanabe's principle for weak existence and strong uniqueness to SDEs extends to mean field games with a common noise: provided that a form of strong uniqueness holds for the MFG, any weak solution is in fact strong. Generally speaking, it is shown in [29] that strong uniqueness indeed holds true for MFGs with a common noise whenever the aforementioned monotonicity condition (1.8) is satisfied. In this regard, the result of [29] is completely consistent with the one we obtain here for the solvability of (1.11), as we prove that the solutions to (1.11) are indeed adapted with respect to $(W_t)_{t\in[0,T]}$. The main difference with [29] is that we take a short cut to get the result as we directly benefit from the monotone structure (1.8) to apply a fixed-point argument with uniqueness (instead of a fixed-point argument without uniqueness like Kakutani–Fan–Glicksberg's theorem). As a result, we here get in the same time existence and uniqueness of a solution to (1.11).

1.1.6 The Convergence Result

Although most of the book is devoted to the construction of a solution to the master equation, our main (and *primary*) motivation remains to justify the mean field limit. Namely, we show that the solution of the Nash system (1.2) converges to the solution of the master equation. The main issue here is the complete lack of estimates on the solutions to this large system of Hamilton–Jacobi equations: this prevents the use of any compactness method to prove the convergence. So far, this question has been almost completely open. The convergence has been known in very few specific situations. For instance, it was proved for the ergodic MFGs (see Lasry-Lions [71], revisited by Bardi-Feleqi [13]). In this case, the Nash equilibrium system reduces to a coupled system of N equations in $\mathbb{T}^d$ (instead of N equations in $\mathbb{T}^{Nd}$ as (1.2)) and estimates of the solutions are available. Convergence is also known in the "linear-quadratic" setting, where the Nash system has explicit solutions: see Bardi [12]. Let us finally quote the nice results by Fischer [38] and Lacker [69] on the convergence of *open loop Nash equilibria* for the N-player game and the characterization of the possible limits. Therein, the authors overcome the lack of strong estimates on the solutions to the N-player game by using the notion of *relaxed controls* for which weak compactness criteria are available. The problem addressed here—concerning *closed loop Nash* *equilibria*—differs in a substantial way from [38, 69]: indeed, we underline the striking fact that the Nash system (1.2), which concerns equilibria in which the players observe each other, converges to an equation in which the players only need to observe the evolution of the distribution of the population. This is striking because it allows for a drastic gain of complexity: without common noise, limiting equilibria are deterministic and hence can be precomputed; in particular, the limiting strategies are distributed in the sense that players just need to update their own state to compute the equilibrium strategy; this is in contrast with the equilibrium given by (1.2), as the latter requires updating the states of all the players in the equilibrium feedback function.

Our main contribution is a general convergence result, in large time, for MFGs with common noise, as well as an estimate of the rate of convergence. The convergence holds in the following sense: for any $\boldsymbol{x} \in (\mathbb{T}^d)^N$, let $m^N_{\boldsymbol{x}} := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$; then

$$\sup_{i=1,\cdots,N} \left| v^{N,i}(t_0, \boldsymbol{x}) - U(t_0, x_i, m^N_{\boldsymbol{x}}) \right| \leqslant CN^{-1}, \tag{1.12}$$

for a constant C independent of N, t_0, and $\boldsymbol{x}$. We also prove a mean field result for the optimal solutions (1.3): if the initial conditions of the $((X_{i,\cdot}))_{i=1,\dots,N}$ are i.i.d. and with the same law $m_{(0)} \in \mathcal{P}(\mathbb{T}^d)$, then

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_{i,t} - Y_{i,t}| \Big] \leqslant CN^{-\frac{1}{d+8}}, \tag{1.13}$$

where the $((Y_{i,t})_{i=1,\dots,N})_{t\in[0,T]}$ are the solutions to the McKean–Vlasov SDE

$$\begin{aligned} dY_{i,t} &= -D_pH\big(Y_{i,t}, D_xU\big(t, Y_{i,t}, \mathcal{L}(Y_{i,t}|W)\big)\big)dt \\ &\quad + \sqrt{2}dB^i_t + \sqrt{2\beta}\, dW_t, \quad t \in [t_0, T], \end{aligned}$$

with the same initial condition as the $((X_{i,t})_{i=1,\dots,N})_{t\in[0,T]}$. Here U is the solution of the master equation and $\mathcal{L}(Y_{i,t}|W)$ is the conditional law of $Y_{i,t}$ given the realization of the whole path W. Since the $((Y_{i,t})_{t\in[0,T]})_{i=1,\dots,N}$ are conditionally independent given W, (1.13) shows that (conditional) propagation of chaos holds for the N-Nash equilibria.

The technique of proof consists in testing the solution U of the master equation (1.10) as nearly a solution to the N-Nash system (1.2). On the model of (1.4), a natural candidate for being an approximate solution to the N-Nash system is indeed

$$u^{N,i}(t, \boldsymbol{x}) = U\big(t, x_i, m^{N,i}_{\boldsymbol{x}}\big), \quad t \in [0,T], \ \boldsymbol{x} \in (\mathbb{T}^d)^N.$$

Taking advantage of the smoothness of U, we then prove that the "proxies" $(u^{N,i})_{i=1,\dots,N}$ almost solve the N-Nash system (1.2) up to a remainder term that vanishes as N tends to ∞. As a byproduct, we deduce that the $(u^{N,i})_{i=1,\dots,N}$ get closer and closer to the "true solutions" $(v^{N,i})_{i=1,\dots,N}$ when N tends to ∞, which yields (1.12). As the reader may notice, the convergence property (1.12) holds in supremum norm, which is a very strong fact.

It is worth mentioning that the monotonicity properties (1.4) play no role in our proof of the convergence. However, surprisingly, the uniform parabolicity of the MFG system is a key ingredient of the proof. On the one hand, in the uniformly parabolic setting, the convergence holds under the sole assumption that the master equation has a classical solution (plus structural Lipschitz continuity conditions on the coefficients). On the other hand, we do not know if one can dispense with the parabolicity condition.

1.1.7 Conclusion and Further Prospects

The fact that the existence of a classical solution to the master equation suffices to prove the convergence of the Nash system demonstrates the deep interest of the master equation, when regarded as a mathematical concept in its own right. Considering the problem from a more abstract point of view, the master equation indeed captures the evolution of the time-dependent semigroup generated by the Markov process formed, on the space of probability measures, by the forward component of the MFG system (1.11). Such a semigroup is said to be *lifted* as the corresponding Markov process has $\mathcal{P}(\mathbb{T}^d)$ as state space. In other words, the master equation is a nonlinear PDE driven by a Markov generator acting on functions defined on $\mathcal{P}(\mathbb{T}^d)$. The general contribution of our book is thus to show that any classical solution to the master equation accommodates a given perturbation of the lifted semigroup and that the information enclosed in such a classical solution suffices to determine the distance between the semigroup and its perturbation. Obviously, as a perturbation of a semigroup on the space of probability measures, we are here thinking of a system of N interacting particles, exactly as that formed by the Nash equilibrium of an N-player game.

Identifying the master equation with a nonlinear PDE driven by the Markov generator of a lifted semigroup is a key observation. As already pointed out, the Markov generator is precisely the operator, acting on functions from $\mathcal{P}(\mathbb{T}^d)$ to $\mathbb{R}$, generated by the forward component of the MFG system (1.11). Put differently, the law of the forward component of the MFG system (1.11), which resides in $\mathcal{P}(\mathcal{P}(\mathbb{T}^d))$, satisfies a forward Kolmogorov equation, also referred to as a "master equation" in physics. This says that "our master equation" is somehow the *dual* (in the sense that it is driven by the adjoint operator) of the "master equation" that would describe, according to the terminology used in physics, the law of the Nash equilibrium for a game with infinitely many players (in which case the Nash equilibrium itself is a distribution). We stress that this interpretation is very close to the point of view developed by Mischler and Mouhot, [80] in order to investigate Kac's program (except that, differently from ours, Mischler and Mouhot's work investigates uniform propagation of chaos over an infinite time horizon; we refer to the companion paper by Mischler, Mouhot, and Wennberg [81] for the analysis, based on the same technology, of mean field models in finite time). Therein, the authors introduce the evolution equation satisfied by the (*lifted*) semigroup, acting on functions from $\mathcal{P}(\mathbb{R}^d)$ to $\mathbb{R}$, generated by the d-dimensional Boltzmann equation. According to our terminology, such an evolution equation is a "master equation" on the space of probability measures, but it is linear and of the first order while ours is nonlinear and of the second order (meaning second order on $\mathcal{P}(\mathbb{T}^d)$).

In this perspective, we also emphasize that our strategy for proving the convergence of the N-Nash system relies on a similar idea to that used in [80] to establish the convergence of Kac's jump process. Whereas our approach consists in inserting the solution of the master equation into the N-Nash system, Mischler and Mouhot's point of view is to compare the semigroup generated by the N-particle Kac's jump process, which operates on symmetric functions from $(\mathbb{R}^d)^N$ to $\mathbb{R}$ (or equivalently on empirical distributions of size N), with the *limiting lifted* semigroup, when acting on the same class of symmetric functions from $(\mathbb{R}^d)^N$ to $\mathbb{R}$. Clearly, the philosophy is the same, except that, in our setting, the "limiting master equation" is nonlinear and of second order (which renders the analysis more difficult) and is set over a finite time horizon only (which does not ask for uniform in time estimates). It is worth mentioning that similar ideas have been explored by Kolokoltsov in the monograph [61] and developed, in the McKean–Vlasov framework, in the subsequent works [62] and [63] in collaboration with his coauthors.

Of course, these parallels raise interesting questions, but we refrain from comparing these different works in a more detailed way: this would require to address more technical questions regarding, for instance, the topology used on the space of probability measures and the regularity of the various objects in hand; clearly, this would distract us from our original objective. We thus feel better to keep the discussion at an informal level and to postpone a more careful comparison to future works on the subject.

We complete the introduction by pointing out possible generalizations of our results. For simplicity of notation, we work in the autonomous case, but the results remain unchanged if H or F is time dependent provided that the coefficients F, G, and H, and their derivatives (whenever they exist), are continuous in time and that the various quantitative assumptions we put on F, G, and H hold uniformly with respect to the time variable. We can also remove the monotonicity condition (1.8) provided that the time horizon T is assumed to be small enough. The reason is that the analysis of the smoothness of U relies on the solvability and stability properties of the forward–backward system (1.11) and of its linearized version: as for finite-dimensional two-point boundary value problems, Lipschitz type conditions on the coefficients (and on their derivatives since we are also dealing with the linearized version) are sufficient whenever T is small enough.

As already mentioned, we also choose to work in the periodic framework. We expect similar results under other type boundary conditions, like the entire space $\mathbb{R}^d$ or Neumann boundary conditions.

Notice also that our results can be generalized without much difficulty to the *stationary setting*, corresponding to infinite horizon problems. This framework is particularly meaningful for economic applications. In this setting the Nash system takes the form

$$\begin{cases} rv^{N,i}(\boldsymbol{x}) - \sum_{j=1}^{N} \Delta_{x_j} v^{N,i}(\boldsymbol{x}) - \beta \sum_{j,k=1}^{N} \mathrm{Tr} D^2_{x_j,x_k} v^{N,i}(\boldsymbol{x}) + H(x_i, D_{x_i} v^{N,i}(\boldsymbol{x})) \\ \qquad + \sum_{j \neq i} D_p H(x_j, D_{x_j} v^{N,j}(\boldsymbol{x})) \cdot D_{x_j} v^{N,i}(\boldsymbol{x}) = F^{N,i}(\boldsymbol{x}) \qquad \text{in } (\mathbb{R}^d)^N, \end{cases}$$

where $r > 0$ is interpreted as a discount factor. The corresponding master equation is

$$\begin{cases} rU - (1+\beta)\Delta_x U + H(x, D_x U) \\ \quad -(1+\beta) \int_{\mathbb{R}^d} \mathrm{div}_y \left[D_m U\right] dm(y) + \int_{\mathbb{R}^d} D_m U \cdot D_p H\big(y, D_x U(y, \cdot)\big) dm(y) \\ \quad -2\beta \int_{\mathbb{R}^d} \mathrm{div}_x \left[D_m U\right] dm(y) - \beta \int_{\mathbb{R}^{2d}} \mathrm{Tr}\left[D^2_{mm} U\right] dm^{\otimes 2}(y, y') = F(x, m) \\ \qquad\qquad \text{in } \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d), \end{cases}$$

where the unknown is the map $U = U(x, m)$, and with the same convention of notation as in (1.10). One can again solve this system by using the method of (infinite dimensional) characteristics, paying attention to the fact that these characteristics remain time dependent. The MFG system with common noise takes the form (in which the unknown are now (u_t, m_t, v_t))

$$\begin{cases} d_t u_t = \big[ru_t - (1+\beta)\Delta u_t + H(x, D_x u_t) - F(x, m_t) - 2\beta \mathrm{div}(v_t)\big] dt \\ \qquad\qquad + v_t \cdot \sqrt{2\beta}\, dW_t \qquad\qquad \text{in } [0, +\infty) \times \mathbb{R}^d \\ d_t m_t = \big[(1+\beta)\Delta m_t + \mathrm{div}\big(m_t D_p H(m_t, D_x u_t)\big)\big] dt - \mathrm{div}\big(m_t \sqrt{2\beta}\, dW_t\big), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0, +\infty) \times \mathbb{R}^d \\ m_0 = m_{(0)} \qquad \text{in } \mathbb{R}^d,\ (u_t)_t \text{ bounded a.s.} \end{cases}$$

Lastly, we point out that, even though we do not address this question in the book, our work could be used later for numerical purposes. Solving numerically MFGs is indeed a delicate issue and, so far, numerical methods have been regarded mostly in the case without common noise: We refer to the works of Achdou and his coauthors; see, for instance [1–3] for discretization schemes of the MFG system (1.7). Owing to obvious issues of complexity, the case with common noise seems especially challenging. A case in point is that the system (1.11) is an infinite-dimensional fully coupled forward–backward system, which could be thought, at the discrete level, as an infinite-dimensional equation expanding along all the possible branches of the tree generated by a discrete random walk. Although our work does not provide any clue for bypassing these complexity issues, we guess that our theoretical results—both the representation of the equilibria in the form of the MFG system system (1.11) and through the master equation (1.10) and their regularity—could be useful for a numerical analysis.

1.1.8 Organization of the Text and Reading Guide

We present our main results in Chapter 2, where we also explain the notation, state the assumption, and rigorously define the notion of derivative on the space of measures. The well-posedness of the master equation is proved in Chapter 3 when $\beta = 0$. Unique solvability of the MFG system with common noise is discussed in Chapter 4. Results obtained in Chapter 4 are implemented in Chapter 5 to derive the existence of a classical solution to the master equation in the general case. The last chapter is devoted to the convergence of the Nash system. In the Appendix, we revisit the notion of derivative on the space of probability measures and discuss some useful auxiliary properties.

We strongly recommend that the reader starts with Section 1.2 and with Chapters 2 and 3. Section 1.2 provides heuristic arguments for the construction of a solution to the master equation; this might be really helpful to understand the key results of the book. The complete proof of existence for the first-order case (i.e., without common noise) is the precise aim of Chapter 3; we feel it really accessible.

The reader who is more interested in the analysis of the convergence problem than in the study of the case with common noise may directly skip to Chapter 6; to make things easier, she/he may follow the computations of Chapter 6 by letting $\beta = 0$ therein (i.e., no common noise). In fact, we suggest that, even if she/he is interested in the case with common noise, the reader also follow this plan, especially if she/he is not keen on probability theory and stochastic calculus; at a second time, she/he can go back to Chapters 4 and 5, which are more technical. In these latter two chapters, the reader who is really interested in MFGs with common noise will find new results: The analysis of the MFG system with common noise is mostly the aim of Chapter 4; if needed, the reader may return to Section 3.1, Proposition 3.1.1, for a basic existence result in the case without common noise. The second-order master equation (with common noise) is investigated in Chapter 5, but requires the well-posedness of the MFG system with common noise as stated in Theorem 4.3.1.

The reader should be aware of some basics of stochastic calculus (mostly Itô's formula) to follow the computations of Chapter 6. Chapters 4 and 5 are partly inspired from the theory of backward stochastic differential equations; although this might not be necessary, the reader may have a look at the two monographs [84, 97] for a complete overview of the subject and at the textbook [88] for an introduction.

Of course, the manuscript borrows considerably from the PDE literature and in particular from the theory of Hamilton–Jacobi equations; the reason is that a solution to an MFG is defined as a fixed point of a mapping taking as inputs the optimal trajectories of a family of optimal stochastic control problems. As for the connection between stochastic optimal control problems and Hamilton–Jacobi equations, we refer the reader to the monographs [41, 66]. Some PDE regularity estimates are used quite often in the text, especially for linear and nonlinear second-order parabolic equations; most of them are taken from well-known books on the subject, among which are [70] and [75].

Lastly, the reader will also find in the book results that may be useful for other purposes: Derivatives in the space of measures are discussed in Section 2.2 (definition and basic results) and in Section A.1 of the Appendix (link with Lions' approach); a chain rule (Itô's formula) for functions defined on the space of measures, when taken along the solution of a stochastic Kolmogorov equation, is derived in Section A.3 of the Appendix.

1.2 INFORMAL DERIVATION OF THE MASTER EQUATION

Before stating our main results, it is worthwhile explaining the meaning of the Nash system and the heuristic derivation of the master equation from the Nash system and its main properties. We hope that this (by no means rigorous) presentation might help the reader to be acquainted with our notation and the main ideas of proof. To emphasize the informal aspect of the discussion, we state all the ideas in $\mathbb{R}^d$, without bothering about the boundary issues (whereas in the rest of the text we always work with periodic boundary conditions).

1.2.1 The Differential Game

The Nash system (1.2) arises in differential game theory. Differential games are just optimal control problems with many (here N) players. In this game, player i ($i = 1, \dots, N$) controls her/his state $(X_{i,t})_{t\in[0,T]}$ through her/his control $(\alpha_{i,t})_{t\in[0,T]}$. The state $(X_{i,t})_{t\in[0,T]}$ evolves according to the SDE:

$$dX_{i,t} = \alpha_{i,t}dt + \sqrt{2}\, dB_t^i + \sqrt{2\beta}\, dW_t, \qquad X_{t_0} = x_{i,0}. \tag{1.14}$$

Recall that the d-dimensional Brownian motions $((B_t^i)_{t\in[0,T]})_{i=1,\dots,N}$ and $(W_t)_{t\in[0,T]}$ are independent, $(B_t^i)_{t\in[0,T]}$ corresponding to the *individual noise* (or *idiosyncratic noise*) of player i and $(W_t)_{t\in[0,T]}$ being the *common noise*, which affects all the players. Controls $((\alpha_{i,t})_{t\in[0,T]})_{i=1,\dots,N}$ are required to be progressively measurable with respect to the filtration generated by all the noises. Given an initial condition $\boldsymbol{x}_0 = (x_{1,0}, \dots, x_{N,0}) \in (\mathbb{R}^d)^N$ for the whole system at time t_0, each player aims at minimizing the cost functional:

$$J_i^N\big(t_0, \boldsymbol{x}_0, (\alpha_{j,\cdot})_{j=1,\ldots,N}\big)$$
$$= \mathbb{E}\left[\int_{t_0}^T \big(L(X_{i,s}, \alpha_{i,s}) + F^{N,i}(\boldsymbol{X}_s)\big)\, ds + G^{N,i}(\boldsymbol{X}_T)\right],$$

where $\boldsymbol{X}_t = (X_{1,t}, \ldots, X_{N,t})$ and where $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $F^{N,i} : \mathbb{R}^{Nd} \to \mathbb{R}$ and $G^{N,i} : \mathbb{R}^{Nd} \to \mathbb{R}$ are given Borel maps. For each player i, in order to assume that the other players are indistinguishable, we shall suppose, as in (1.4), that $F^{N,i}$ and $G^{N,i}$ are of the form

$$F^{N,i}(\boldsymbol{x}) = F(x_i, m_{\boldsymbol{x}}^{N,i}) \qquad \text{and} \qquad G^{N,i}(\boldsymbol{x}) = G(x_i, m_{\boldsymbol{x}}^{N,i}).$$

In the above expressions, $F, G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$, where $\mathcal{P}(\mathbb{R}^d)$ is the set of Borel measures on $\mathbb{R}^d$. The Hamiltonian of the problem is related to L by the formula

$$\forall (x,p) \in \mathbb{R}^d \times \mathbb{R}^d, \qquad H(x,p) = \sup_{\alpha \in \mathbb{R}^d} \{-\alpha \cdot p - L(x,\alpha)\}.$$

Let now $(v^{N,i})_{i=1,\ldots,N}$ be the solution to (1.2). By Itô's formula, it is easy to check that $(v^{N,i})_{i=1,\ldots,N}$ corresponds to an optimal solution of the problem in the sense of Nash, i.e., a *Nash equilibrium* of the game. Namely, the feedback strategies

$$\big(\alpha_i^*(t, \boldsymbol{x}) := -D_p H(x_i, D_{x_i} v^{N,i}(t, \boldsymbol{x}))\big)_{i=1,\ldots,N} \tag{1.15}$$

provide a feedback Nash equilibrium for the game:

$$v^{N,i}\big(t_0, \boldsymbol{x}_0\big) = J_i^N\big(t_0, \boldsymbol{x}_0, (\alpha_{j,\cdot}^*)_{j=1,\ldots,N}\big) \leqslant J_i^N\big(t_0, \boldsymbol{x}_0, \alpha_{i,\cdot}, (\hat{\alpha}_{j,\cdot}^*)_{j\neq i}\big)$$

for any $i \in \{1, \ldots, N\}$ and any control $\alpha_{i,\cdot}$, progressively measurable with respect to the filtration generated by $((B_t^j)_{j=1,\ldots,N})_{t\in[0,T]}$ and $(W_t)_{t\in[0,T]}$. In the left-hand side, $\alpha_{j,\cdot}^*$ is an abuse of notation for the process $(\alpha_j^*(t, X_{j,t}))_{t\in[0,T]}$, where $(X_{1,t}, \ldots, X_{N,t})_{t\in[0,T]}$ solves the system of SDEs (1.14) when $\alpha_{j,t}$ is precisely given under the implicit form $\alpha_{j,t} = \alpha_j^*(t, X_{j,t})$. Similarly, in the right-hand side, $\hat{\alpha}_j^*$, for $j \neq i$, denotes $(\alpha_j^*(t, X_{j,t}))_{t\in[0,T]}$, where $(X_{1,t}, \ldots, X_{N,t})_{t\in[0,T]}$ now solves the system of SDEs (1.14) for the given $\alpha_{i,\cdot}$, the other $(\alpha_{j,t})_{j\neq i}$'s being given under the implicit form $\alpha_{j,t} = \alpha_j^*(t, X_{j,t})$. In particular, system (1.3), in which all the players play the optimal feedback (1.15), describes the dynamics of the optimal trajectories.

1.2.2 Derivatives in the Space of Measures

To describe the limit of the maps $(v^{N,i})$, let us introduce—in a completely informal manner—a notion of derivative in the space of measures $\mathcal{P}(\mathbb{R}^d)$. A rigorous description of the notion of derivative used in this book is given in Section 2.2.

In the following discussion, we argue as if all the measures had a density. Let $U : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$. *Restricting* the function U to the elements m of $\mathcal{P}(\mathbb{R}^d)$ that have a density in $L^2(\mathbb{R}^d)$ and assuming that U is defined in a neighborhood $\mathcal{O} \subset L^2(\mathbb{R}^d)$ of $\mathcal{P}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we can use the Hilbert structure on $L^2(\mathbb{R}^d)$. We denote by $\delta U/\delta m$ the gradient of U in $L^2(\mathbb{R}^d)$, namely

$$\frac{\delta U}{\delta m}(p)(q) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(U(p+\varepsilon q) - U(p)\Big), \quad p \in \mathcal{O},\ q \in L^2(\mathbb{R}^d).$$

Of course, we can identify $[\delta U/\delta m](p)$ with an element of $L^2(\mathbb{R}^d)$, which we denote by $\mathbb{R}^d \ni y \mapsto [\delta U/\delta m](p,y) \in \mathbb{R}$. Then, the duality product $[\delta U/\delta m](p)(q)$ reads as the inner product $\langle [\delta U/\delta m](p,\cdot), q(\cdot)\rangle_{L^2(\mathbb{R}^d)}$. Similarly, we denote by $\delta^2 U/\delta m^2(p)$ the second-order derivative of U at $p \in L^2(\mathbb{R}^d)$ (which can be identified with a symmetric bilinear form on $L^2(\mathbb{R}^d)$ and hence with a symmetric function $\mathbb{R}^d \times \mathbb{R}^d \ni (y,y') \mapsto [\delta^2 U/\delta m^2](p,y,y') \in \mathbb{R}$ in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$):

$$\frac{\delta U}{\delta m}(p)(q,q') = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(\frac{\delta U}{\delta m}(p+\varepsilon q)(q') - \frac{\delta U}{\delta m}(p)(q')\Big), \quad p \in \mathcal{O},\ q,q' \in L^2(\mathbb{R}^d).$$

We then set, when possible,

$$D_m U(m,y) = D_y \frac{\delta U}{\delta m}(m,y), \quad D^2_{mm} U(m,y,y') = D^2_{y,y'} \frac{\delta^2 U}{\delta m^2}(m,y,y'). \tag{1.16}$$

To explain the meaning of $D_m U$, let us compute the action of U onto the push-forward of a measure m by the flow an ordinary differential equation driven by a smooth vector field. For a given smooth vector field $B : \mathbb{R}^d \to \mathbb{R}^d$ and an absolutely continuous probability measure $m \in \mathcal{P}(\mathbb{R}^d)$ with a smooth density, let $(m(t))_{t\geq 0} = (\mathbb{R}^d \ni x \mapsto m(t,x))_{t \geq 0}$ be the solution to

$$\begin{cases} \dfrac{\partial m}{\partial t} + \mathrm{div}(Bm) = 0, \\ m_0 = m. \end{cases}$$

Provided that $[\partial m/\partial t](t,\cdot)$ lives in $L^2(\mathbb{R}^d)$, this expression directly gives

$$\begin{aligned} \frac{d}{dh} U(m(h))_{|h=0} &= \langle \frac{\delta U}{\delta m}, -\mathrm{div}(Bm)\rangle_{L^2(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d} D_m U(m,y) \cdot B(y)\ dm(y), \end{aligned} \tag{1.17}$$

where we used an integration by parts in the last equality.

Another way to understand these derivatives is to project the map U to the finite dimensional space $(\mathbb{R}^d)^N$ via the empirical measure: if $\boldsymbol{x} = (x_1, \dots, x_N) \in$

$(\mathbb{R}^d)^N$, let $m^N_{\boldsymbol{x}} := (1/N)\sum_{i=1}^N \delta_{x_i}$ and set $u^N(\boldsymbol{x}) = U(m^N_{\boldsymbol{x}})$. Then one checks the following relationships (see Proposition 6.1.1): for any $j \in \{1, \ldots, N\}$,

$$D_{x_j} u^N(\boldsymbol{x}) = \frac{1}{N} D_m U(m^N_{\boldsymbol{x}}, x_j), \tag{1.18}$$

$$D^2_{x_j,x_j} u^N(\boldsymbol{x}) = \frac{1}{N} D_y \left[D_m U\right](m^N_{\boldsymbol{x}}, x_j) + \frac{1}{N^2} D^2_{mm} U(m^N_{\boldsymbol{x}}, x_j, x_j) \tag{1.19}$$

while, if $j \neq k$,

$$D^2_{x_j,x_k} u^N(\boldsymbol{x}) = \frac{1}{N^2} D^2_{mm} U(m^N_{\boldsymbol{x}}, x_j, x_k). \tag{1.20}$$

1.2.3 Formal Asymptotic of the $(v^{N,i})$

Provided that (1.2) has a unique solution, each $v^{N,i}$, for $i = 1, \ldots, N$, is symmetric with respect to permutations on $\{1, \ldots, N\}\backslash\{i\}$ and, for $i \neq j$, the role played by x^i in $v^{N,i}$ is the same as the role played by x^j in $v^{N,j}$ (see Section 6.2). Therefore, it makes sense to expect, in the limit $N \to +\infty$,

$$v^{N,i}(t, \boldsymbol{x}) \simeq U(t, x_i, m^{N,i}_{\boldsymbol{x}})$$

where $U : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$. Starting from this *ansatz*, our aim is now to provide heuristic arguments explaining why U should satisfy (1.10). The sense in which the $(v^{N,i})_{i=1,\ldots,N}$ actually converge to U is stated in Theorem 2.4.8 and the proof given in Chapter 6.

The informal idea is to assume that $v^{N,i}$ is already of the form $U(t, x_i, m^{N,i}_{\boldsymbol{x}})$ and to plug this expression into the equation of the Nash equilibrium (1.2): the time derivative and the derivative with respect to x_i are understood in the usual sense, while the derivatives with respect to the other variables are computed by using the relations in the previous section.

The terms $\partial_t v^{N,i}$ and $H(x_i, D_{x_i} v^{N,i})$ easily become $\partial U/\partial t$ and $H(x, D_x U)$. We omit for a while the second-order terms and concentrate on the expression (see the second line in (1.2)):

$$\sum_{j \neq i} D_p H(x_j, D_{x_j} v^{N,j}) \cdot D_{x_j} v^{N,i} \,.$$

Note that $D_{x_j} v^{N,j}$ is just like $D_x U(t, x_j, m^{N,j}_{\boldsymbol{x}})$. In view of (1.18),

$$D_{x_j} v^{N,i} \simeq \frac{1}{N-1} D_m U(t, x_i, m^{N,i}_{\boldsymbol{x}}, x_j),$$

and the sum over j is like an integration with respect to $m_{\boldsymbol{x}}^{N,i}$. So we find, ignoring the difference between $m_{\boldsymbol{x}}^{N,i}$ and $m_{\boldsymbol{x}}^{N,j}$,

$$\begin{aligned}&\sum_{j\neq i} D_pH(x_j, D_{x_j}v^{N,j})\cdot D_{x_j}v^{N,i}\\&\qquad\simeq \int_{\mathbb{R}^d} D_pH(y, D_xU(t, m_{\boldsymbol{x}}^{N,i}, y))\cdot D_mU(t, x_i, m_{\boldsymbol{x}}^{N,i}, y)dm_{\boldsymbol{x}}^{N,i}(y).\end{aligned}$$

We now study the term $\sum_{j=1}^{N}\Delta_{x_j}v^{N,i}$ (see the first line in (1.2)). As $\Delta_{x_i}v^{N,i}\simeq\Delta_xU$, we need to analyze the quantity $\sum_{j\neq i}\Delta_{x_j}v^{N,i}$. In view of (1.19), we expect

$$\begin{aligned}\sum_{j\neq i}\Delta_{x_j}v^{N,i} &\simeq \frac{1}{N-1}\sum_{j\neq i}\mathrm{div}_y\left[D_mU\right](t, x_i, m_{\boldsymbol{x}}^{N,i}, x_j)\\&\quad+\frac{1}{(N-1)^2}\sum_{j\neq i}\mathrm{Tr}\left[D^2_{mm}U\right](t, x_i, m_{\boldsymbol{x}}^{N,i}, x_j, x_j)\\&=\int_{\mathbb{R}^d}\mathrm{div}_y\left[D_mU\right](t, x_i, m_{\boldsymbol{x}}^{N,i}, y)dm_{\boldsymbol{x}}^{N,i}(y)\\&\quad+\frac{1}{N-1}\int_{\mathbb{R}^d}\mathrm{Tr}\left[D^2_{mm}U\right](t, x_i, m_{\boldsymbol{x}}^{N,i}, y, y)dm_{\boldsymbol{x}}^{N,i}(y),\end{aligned}$$

where we can drop the last term, as it is of order $1/N$.

Let us finally discuss the limit of the term $\sum_{k,l=1}^{N}\mathrm{Tr}(D^2_{x_j,x_k}v^{N,i})$ (see the first line in (1.2)) that we rewrite

$$\Delta_{x_i}v^{N,i} + 2\sum_{k\neq i}\mathrm{Tr}\big(D_{x_i}D_{x_k}v^{N,i}\big) + \sum_{k,l\neq i}\mathrm{Tr}\big(D^2_{x_k,x_l}v^{N,i}\big). \tag{1.21}$$

The first term gives Δ_xU. Using (1.18), the second one becomes

$$\begin{aligned}2\sum_{k\neq i}\mathrm{Tr}\big(D_{x_i}D_{x_k}v^{N,i}\big) &\simeq \frac{2}{N-1}\sum_{k\neq i}\mathrm{Tr}\left[D_xD_mU\right](t, x_i, m_{\boldsymbol{x}}^{N,i}, x_k)\\&= 2\int_{\mathbb{R}^d}\mathrm{div}_x\left[D_mU\right](t, x_i, m_{\boldsymbol{x}}^{N,i}, y)dm_{\boldsymbol{x}}^{N,i}(y).\end{aligned}$$

As for the last term in (1.21), we have by (1.20):

$$\sum_{k,l\neq i} \mathrm{Tr}\big(D^2_{x_k,x_l} v^{N,i}\big) \simeq \frac{1}{(N-1)^2} \sum_{k,l\neq i} \mathrm{Tr}\left[D^2_{mm}U\right](t, x_i, m^{N,i}_{\boldsymbol{x}}, x_j, x_k)$$
$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{Tr}\left[D^2_{mm}U\right](t, x_i, m^{N,i}_{\boldsymbol{x}}, y, y')dm^{N,i}_{\boldsymbol{x}}(y)dm^{N,i}_{\boldsymbol{x}}(y').$$

Collecting the above relations, we expect that the Nash system

$$\begin{cases} -\dfrac{\partial v^{N,i}}{\partial t} - \displaystyle\sum_{j=1}^N \Delta_{x_j} v^{N,i} - \beta \sum_{k,l=1}^N \mathrm{Tr}\big(D^2_{x_k,x_l} v^{N,i}\big) + H(x_i, D_{x_i} v^{N,i}) \\ \qquad\qquad + \displaystyle\sum_{j\neq i} D_p H(x_j, D_{x_j} v^{N,j}) \cdot D_{x_j} v^{N,i} = F(x_i, m^{N,i}_{\boldsymbol{x}}), \\ v^{N,i}(T, \boldsymbol{x}) = G(x_i, m^{N,i}_{\boldsymbol{x}}), \end{cases}$$

has for limit

$$\begin{cases} -\dfrac{\partial U}{\partial t} - \Delta_x U - \displaystyle\int_{\mathbb{R}^d} \mathrm{div}_y\left[D_m U\right] dm(y) + H(x, D_x U) \\ \quad -\beta\Big(\Delta_x U + 2\displaystyle\int_{\mathbb{R}^d} \mathrm{div}_x\left[D_m U\right] dm(y) + \int_{\mathbb{R}^d} \mathrm{div}_y\left[D_m U\right] dm(y) \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^{2d}} \mathrm{Tr}\left[D^2_{mm}U\right] dm^{\otimes 2}(y,y')\Big) \\ \quad + \displaystyle\int_{\mathbb{R}^d} D_m U \cdot D_p H\big(y, D_x U(\cdot, y, \cdot)\big) dm(y) = F(x,m) \\ U(T,x,m) = G(x,m). \end{cases}$$

This is the master equation. Note that there are only two genuine approximations in the foregoing computation. One is where we dropped the term of order $1/N$ in the computation of the sum $\sum_{j\neq i} \Delta_{x_j} v^{N,i}$. The other one was at the very beginning, when we replaced $D_x U(t, x_j, m^{N,j}_{\boldsymbol{x}})$ by $D_x U(t, x_j, m^{N,i}_{\boldsymbol{x}})$. This is again of order $1/N$.

1.2.4 The Master Equation and the MFG System

We complete this informal discussion by explaining the relationship between the master equation and the MFG system. This relation plays a central role in the text. It is indeed the cornerstone for constructing a solution to the master equation via a method of (infinite dimensional) characteristics.

We proceed as follows. Assuming that *the value function* of the MFG system is regular—while it is part of the challenge to prove that it is indeed smooth—we show that it solves the master equation.

We start with the first-order case, i.e., $\beta = 0$, as it is substantially easier. For any $(t_0, m_{(0)}) \in [0, T] \times \mathcal{P}(\mathbb{R}^d)$, let us define the value function $U(t_0, \cdot, m_{(0)})$ as

$$U(t_0, x, m_{(0)}) := u(t_0, x) \qquad \forall x \in \mathbb{R}^d,$$

where (u, m) is a solution of the MFG system (1.7) with the initial condition $m(t_0) = m_{(0)}$ at time t_0. We claim that U is a solution of the master equation (1.10) with $\beta = 0$. As indicated, we check the claim assuming that U is smooth, although the main difficulty comes from the fact that this has to be proved. We note that, by its very definition, U must satisfy

$$U(t, x, m(t)) = u(t, x) \qquad \forall (t, x) \in [t_0, T] \times \mathbb{R}^d.$$

Using the equation satisfied by m (and provided that $\partial_t m$ can be regarded as an $L^2(\mathbb{R}^d)$ valued function), the time derivative of the left-hand side at t_0 is given by

$$\begin{aligned} \partial_t u(t_0, x) &= \partial_t U + \Big\langle \frac{\delta U}{\delta m}, \partial_t m \Big\rangle_{L^2(\mathbb{R}^d)} \\ &= \partial_t U + \Big\langle \frac{\delta U}{\delta m}, \Delta m + \operatorname{div}\big(m D_p H(\cdot, D_x U)\big) \Big\rangle_{L^2(\mathbb{R}^d)} \\ &= \partial_t U \\ &\quad + \int_{\mathbb{R}^d} \Big(\operatorname{div}_y [D_m U] - D_m U \cdot D_p H(y, D_x U(\cdot, y, \cdot)) \Big) dm_{(0)}(y), \end{aligned} \tag{1.22}$$

where the function U and its derivatives are evaluated at time t_0 and at the measure argument $m_{(0)}$; with the exception of the last term in the right-hand side, they are evaluated at point x in space; the auxiliary variable in $D_m U$ is always equal to y. Recalling the equation satisfied by u, we also have

$$\begin{aligned} \partial_t u(t_0, x) &= -\Delta u(t_0, x) + H\big(x, D_x u(t_0, x)\big) - F(x, m_{(0)}) \\ &= -\Delta_x U + H(x, D_x U) - F(x, m_{(0)}). \end{aligned}$$

This shows that

$$\begin{aligned} &\partial_t U + \int_{\mathbb{R}^d} \Big(\operatorname{div}_y [D_m U] - D_m U \cdot D_p H\big(y, D_x U(\cdot, y, \cdot)\big) \Big) dm_{(0)}(y) \\ &\quad = -\Delta_x U + H(x, D_x U) - F(x, m_{(0)}). \end{aligned}$$

Rearranging the terms, we deduce that U satisfies the master equation (1.10) with $\beta = 0$ at $(t_0, \cdot, m_{(0)})$.

For the second-order master equation ($\beta > 0$) the same principle applies except that, now, the MFG system becomes stochastic. Let $(t_0, m_{(0)}) \in [0, T] \times$

$\mathcal{P}(\mathbb{R}^d)$ and (u_t, m_t, v_t) be a solution of the MFG system with common noise (1.11). We set as before

$$U(t_0, x, m_{(0)}) := u_{t_0}(x) \qquad \forall x \in \mathbb{R}^d,$$

and notice that

$$U(t, x, m_t) = u_t(x) \qquad \forall (t, x) \in [t_0, T] \times \mathbb{R}^d.$$

Assuming that U is smooth enough, we have, by Itô's formula for Banach-valued processes and by the equation satisfied by m:

$$\begin{aligned} d_t u_t(x) = \Big\{ \partial_t U &+ \Big\langle \frac{\delta U}{\delta m}, (1+\beta)\Delta m_t + \operatorname{div}\big(m_t D_p H(\cdot, D_x U)\big) \Big\rangle_{L^2(\mathbb{R}^d)} \\ &+ \beta \sum_{i=1}^d \Big\langle \frac{\delta^2 U}{\delta m^2} D_{x_i} m_t, D_{x_i} m_t \Big\rangle_{L^2(\mathbb{R}^d)} \Big\} dt \\ &- \sqrt{2\beta} \sum_{i=1}^d \Big\langle \frac{\delta U}{\delta m}, D_{x_i} m_t \Big\rangle_{L^2(\mathbb{R}^d)} dW_t^i, \end{aligned} \tag{1.23}$$

where, as before, the function U and its derivatives are evaluated at time t and at the measure argument m_t; with the exception of $D_x U$ in the right-hand side, they are evaluated at point x in space.

In comparison with the first-order formula (1.22), equation (1.23) involves two additional terms: The stochastic term on the third line derives directly from the Brownian part in the forward part of (1.11) while the second-order term on the second line is reminiscent of the second-order term that appears in the standard Itô calculus. We provide a rigorous proof of (1.23) in Section 5.

Using (1.16), we obtain

$$\begin{aligned} &d_t u_t(x) \\ &= \Big\{ \partial_t U \\ &\quad + \int_{\mathbb{R}^d} \Big((1+\beta)\operatorname{div}_y [D_m U] - D_m U \cdot D_p H(\cdot, D_x U(\cdot, y, \cdot) \Big) dm_t(y) \\ &\quad + \beta \int_{\mathbb{R}^d \times \mathbb{R}^d} \operatorname{Tr}\big[D^2_{mm} U\big] \, dm_t^{\otimes 2}(y, y') \Big\} dt \\ &\quad + \Big(\int_{\mathbb{R}^d} D_m U \, dm_t(y) \Big) \cdot \sqrt{2\beta} \, dW_t. \end{aligned} \tag{1.24}$$

On the other hand, by the equation satisfied by u, we have

$$\begin{aligned} d_t u_t(x) &= \big\{-(1+\beta)\Delta u_t + H(x, Du_t) - F(x, m_t) - \sqrt{2\beta}\operatorname{div}(v_t)\big\}dt \\ &\quad + v_t \cdot dW_t \\ &= \big\{-(1+\beta)\Delta_x U + H(x, D_x U) - F(x, m_t) - \sqrt{2\beta}\operatorname{div}(v_t)\big\}dt \\ &\quad + v_t \cdot dW_t, \end{aligned} \tag{1.25}$$

where, on the right-hand side, u_t and v_t and their derivatives are evaluated at point x.

Identifying the absolutely continuous part and the martingale part, we find

$$\begin{aligned} &\partial_t U + \int_{\mathbb{R}^d} \Big((1+\beta)\operatorname{div}_y [D_m U] - D_m U \cdot D_p H(\cdot, D_x U(\cdot, y, \cdot))\Big) dm_t(y) \\ &\quad + \beta \int_{\mathbb{R}^d \times \mathbb{R}^d} \operatorname{Tr}\big[D^2_{mm} U\big] dm_t^{\otimes 2}(y, y') \\ &\quad = -(1+\beta)\Delta_x U + H(x, D_x U) - F(x, m_t) - \sqrt{2\beta}\operatorname{div}(v_t) \end{aligned} \tag{1.26}$$

and

$$\sqrt{2\beta} \int_{\mathbb{R}^d} D_m U dm_t(y) = v_t.$$

Inserting the latter identity in the former one, we derive the master equation. Note that, compared with the first-order setting (i.e., $\beta = 0$), one faces here the additional issue that, so far, there has not been any solvability result for (1.9) and that the regularity of the map U—which is defined through (1.9)—is much more involved to investigate than in the first-order case.

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