# TABLE OF CONTENTS

# CHAPTER ONE
# INFINITY

## A SHORT HISTORY OF INFINITY

The symbol for infinity that one sees most often is the lazy eight curve, technically called the lemniscate. This symbol was first used in a seventeenth century treatise on conic sections.[1] It caught on quickly and was soon used to symbolize infinity or eternity in a variety of contexts. For instance, in the 1700s the infinity symbol began appearing on the Tarot card known as the Juggler or the Magus. It is an interesting coincidence that the Qabbalistic symbol associated with this particular Tarot card is the Hebrew letter ℵ, (pronounced alef), for Georg Cantor, the founder of the modern mathematical theory of the infinite, used the symbol $\aleph_0$, (pronounced alef-null), to stand for the first infinite number.

The appropriateness of the symbol $\infty$ for infinity lies in the fact that one can travel endlessly around such a curve . . . demolition derby style, if you will. Endlessness is, after all, a principal component of one's concept of infinity. Other notions associated with infinity are indefiniteness and inconceivability.

Figure 1.

Figure 2.

Infinity commonly inspires feelings of awe, futility, and fear. Who as a child did not lie in bed filled with a slowly mounting terror while sinking into the idea of a universe that goes on and on, for ever and ever? Blaise Pascal puts this feeling very well: "When I consider the small span of my life absorbed in the eternity of all time, or the small part of space which I can touch or see engulfed by the infinite immensity of spaces that I know not and that know me not, I am frightened and astonished to see myself here instead of there . . . now instead of then."[2]

It is possible to regard the history of the foundations of mathematics as a progressive enlarging of the mathematical universe to include more and more infinities. The Greek word for infinity was *apeiron,* which literally means unbounded, but can also mean infinite, indefinite, or undefined. *Apeiron* was a negative, even pejorative, word. The original chaos

out of which the world was formed was *apeiron.* An arbitrary crooked line was *apeiron.* A dirty crumpled handkerchief was *apeiron.* Thus, *apeiron* need not only mean infinitely large, but can also mean totally disordered, infinitely complex, subject to no finite determination. In Aristotle's words, ". . . being infinite is a privation, not a perfection but the absence of a limit. . . ."[3]

There was no place for the *apeiron* in the universe of Pythagoras and Plato. Pythagoras believed that any given aspect of the world could be represented by a finite arrangement of natural numbers, (where "natural number" means "whole number.") Plato believed that even his ultimate form, the Good, must be finite and definite. This was in contradistinction to almost all later metaphysicians, who assumed that the Absolute is necessarily infinite. In the next chapter I will discuss the way in which Greek mathematics was limited by this refusal to accept the *apeiron,* even in the relatively harmless guise of a real number with an infinite decimal expansion.

Aristotle recognized that there are many aspects of the world that seem to point to the actuality of the *apeiron.* For instance, it seems possible that time will go on forever; and it would seem that space is infinitely divisible, so that any line segment contains an infinity of points. In order to avoid these actual infinites that seemed to threaten the orderliness of his *a priori* finite world, Aristotle invented the notion of the *potentially infinite* as opposed to the *actually infinite.* I will describe this distinction in more detail in the next section, but for now let me characterize it as follows. Aristotle would say that the set of natural numbers is potentially infinite, since there is no largest natural number, but he would deny that the set is actually infinite, since it does not exist as one finished thing. This is a doubtful distinction, and I am inclined to agree with Cantor's opinion that ". . . in truth the potentially infinite has only a borrowed reality, insofar as a potentially infinite concept always points towards a logically prior actually infinite concept whose existence it depends on."[4]

Plotinus was the first thinker after Plato to adopt the belief that at least God, or the One, is infinite, stating of the One that, "Absolutely One, it has never known measure and stands outside of number, and so is under no limit either in regard to anything external or internal; for any such determination would bring something of the dual into it."[5]

St. Augustine, who adapted the Platonic philosophy to the Christian religion, believed not only that God was infinite, but also that God could think infinite thoughts. St. Augustine argued that, "Such as say that things infinite are past God's knowledge may just as well leap head-

long into this pit of impiety, and say that God knows not all numbers. . . . What madman would say so? . . . What are we mean wretches that dare presume to limit His knowledge?"[6]

This extremely modern position will be returned to in the last section of this chapter. Later medieval thinkers did not go as far as Augustine and, although granting the unlimitedness of God, were unwilling to grant that any of God's creatures could be infinite. In his *Summa Theologiae* St. Thomas Aquinas gives a sort of Aristotelian proof that "although God's power is unlimited, he still cannot make an absolutely unlimited thing, no more than he can make an unmade thing (for this involves contradictories being true together)."[7] The arguments are elegant, but suffer from the flaw of being circular: it is proved that the notion of an unlimited thing is contradictory by slipping in the premise that a "thing" is by its very nature limited.

Thus, with the exception of Augustine and a few others, the medieval thinkers were not prepared to deal with the infinitude of any entities other than God, be they physical, psychological, or purely abstract. The famous puzzle of how many angels can dance on the head of a pin can be viewed as a question about the relationship between the infinite Creator and the finite world. The crux of this problem is that, on the one hand, it would seem that since God is infinitely powerful, he should be able to bid an infinite number of angels to dance on the head of a pin; on the other hand, it was believed by the medieval thinkers that no actually infinite collection could ever arise in the created world.

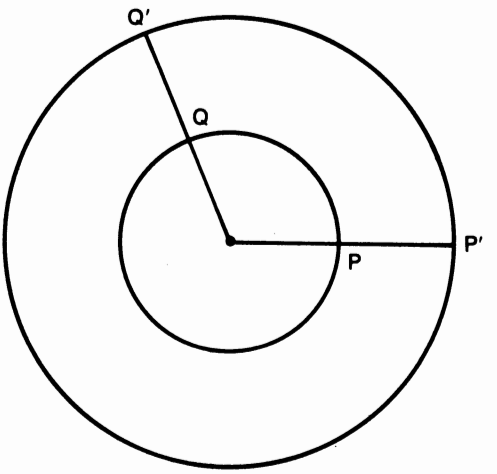


Figure 3.

Their proofs that infinity is somehow a self-contradictory notion were all flawed, but there was at least one interesting paradox involving infinity that the medieval thinkers were aware of. It would seem that any line includes infinitely many points. Since the circumference of a circle with radius two is two times as long as the circumference of a circle with radius one, then the former should include a *larger* infinity of points than the latter. But by drawing radii we can see that each point $P$ on the small circle corresponds to exactly one point $P'$ on the large circle, and each point $Q'$ on the large circle corresponds to exactly one point $Q$ on the small circle. Thus we seem to have two infinities that are simultaneously different and equal.

In the early 1600s Galileo Galilei offered a curious solution to this problem. Galileo proposed that the smaller length could be turned into the longer length by adding an infinite number of infinitely small gaps. He was well aware that such a procedure leads to various difficulties: "These difficulties are real; and they are not the only ones. But let us remember that we are dealing with infinites and indivisibles, both of which transcend our finite understanding, the former on account of their magnitude, the latter because of their smallness. In spite of this, men cannot refrain from discussing them, even though it must be done in a roundabout way."[8]

He resolved some of his difficulties by asserting that problems arise only, "when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this I think is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to another."[9] This last assertion is supported by an example that is sometimes called Galileo's paradox.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 . . . |
| ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ |
| 1 | 4 | 9 | 16 | 25 | 36 | 49 . . . |

The paradoxical situation arises because, on the one hand, it seems evident that most natural numbers are not perfect squares, so that the set of perfect squares is smaller than the set of all natural numbers; but, on the other hand, since every natural number is the square root of exactly one perfect square, it would seem that there are just as many perfect squares as natural numbers. For Galileo the upshot of this paradox was that, "we can only infer that the totality of all numbers is infinite, and that the number of squares is infinite . . . ; neither is the number of squares less than the totality of all numbers, nor the latter greater

than the former; and finally, the attributes 'equal,' 'greater,' and 'less,' are not applicable to infinite, but only to finite quantities."[10]

I have quoted Galileo at some length, because it is with him that we have the first signs of the modern attitude toward the actual infinite in mathematics. If infinite sets do not behave like finite sets, this does not mean that infinity is an inconsistent notion. It means, rather, that infinite numbers obey a different "arithmetic" from finite numbers. If using the ordinary notions of "equal" and "less than" on infinite sets leads to contradictions, this is not a sign that infinite sets cannot exist, but, rather, that these notions do not apply without modification to infinite sets. Galileo himself did not see how to carry out such a modification of these notions; *this* was to be the task of Georg Cantor, some 250 years later.

One of the reasons that Galileo felt it necessary to come to some sort of terms with the actual infinite was his desire to treat space and time as continuously varying quantities. Thus, the results of an experiment on motion can be stated in the form that $x = f(t)$, that space position is a certain function of continuously changing time. But this variable $t$ that grows continuously from, say, zero to ten is *apeiron,* both in the sense that it takes on arbitrary values, and in the sense that it takes on infinitely many values.

This view of position as a function of time introduced a problem that helped lead to the founding of the Calculus in the late 1600s. The problem was that of finding the instantaneous velocity of a moving body, whose distance $x$ from its starting point is given as a function $f(t)$ of time.

It turns out that to calculate the velocity at some instant $t_0$, one has to imagine measuring the speed over an infinitely small time interval $dt$. The speed $f'(t_0)$ at $t_0$ is given by the formula $(f(t_0 + dt) - f(t_0))/dt$, as everyone who has ever survived a first-year calculus course knows.

The quantity $dt$ is called an *infinitesimal,* and obeys many strange rules. If $dt$ is added to a regular number, then it can be ignored, treated like zero. But, on the other hand, $dt$ is regarded as being different enough from zero to be usable as the denominator of a fraction. So is $dt$ zero or not? Adding finitely many infinitesimals together just gives another infinitesimal. But adding infinitely many of them together can give either an ordinary number, or an infinitely large quantity.

Bishop Berkeley found it curious that mathematicians could swallow the Newton–Leibniz theory of infinitesimals, yet balk at the peculiarities of orthodox Christian doctrine. He wrote about this in a 1734

work, the full title of which was, *The Analyst, Or A Discourse Addressed to an Infidel Mathematician. Wherein It is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith. "First cast out the beam out of thine own Eye; and then shalt thou see clearly to cast out the mote out of thy brother's Eye."*[11]

The use of infinitely small and infinitely large numbers in calculus was soon replaced by the limit process. But it is unlikely that the Calculus could ever have developed so rapidly if mathematicians had not been willing to think in terms of actual infinities. In the past fifteen years, Abraham Robinson's non-standard analysis has produced a technique by which infinitesimals can be used without fear of contradiction. Robinson's technique involves enlarging the real numbers to the set of *hyperreal* numbers, which will be discussed in Chapter 2.

After the introduction of the limit process, calculus was able to advance for a long time without the use of any actually infinite quantities. But as mathematicians tried to get a precise *description* of the continuum or real line, it became evident that infinities in the foundations of mathematics could only be avoided at the cost of great artificiality. Mathematicians, however, still hesitated to plunge into the world of the actually infinite, where a set could be the same size as a subset, a line could have as many points as a line half as long, and endless processes were treated as finished things.

In was Georg Cantor who, in the late 1800s, finally created a theory of the actual infinite which by its apparent consistency, demolished the Aristotelian and scholastic "proofs" that no such theory could be found. Although Cantor was a thoroughgoing scholar who later wrote some very interesting philosophical defenses of the actual infinite, his point of entry was a mathematical problem having to do with the uniqueness of the representation of a function as a trigonometric series.

To give the flavor of the type of construction Cantor was working with, let us consider the construction of the Koch curve shown in Figure 4. The Koch curve is found as the limit of an infinite sequence of approximations. The first approximation is a straight line segment (stage 0). The middle third of this segment is then replaced by two pieces, each as long as the middle third, which are joined like two sides of an equilateral triangle (stage 1). At each succeeding stage, each line segment has its middle third replaced by a spike resembling an equilateral triangle.

Now, if we take infinity as something that can, in some sense, be at-

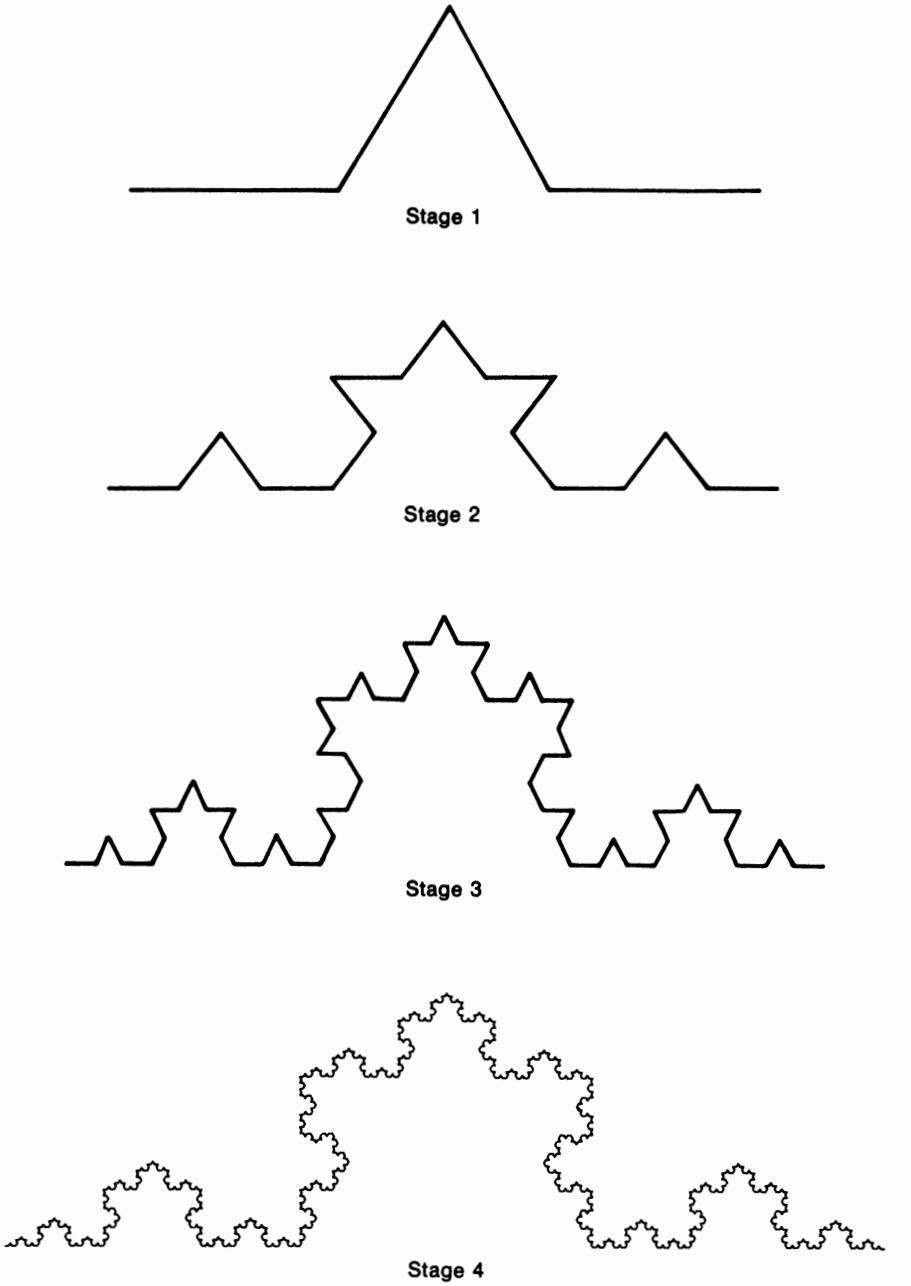


Figure 4. Adapted from Benoit Mandelbrot, *Fractals*.

tained, then we will regard the limit of this infinite process as being a curve actually existing, if not in physical space, then at least as a mathematical object. The Koch curve is discussed at length in Benoit Mandelbrot's book, *Fractals,* where he explains why there is reason to think of the Koch curve in its infinite spikiness as being a better model of a coastline than any of its finitely spiky approximations.[12]

Cantor soon obtained a number of interesting results about actually infinite sets, most notably the result that the set of points on the real line constitutes a *higher* infinity than the set of all natural numbers. That is, Cantor was able to show that infinity is not an all or nothing concept: there are degrees of infinity.

This fact runs counter to the naive concept of infinity: there is only one infinity, and this infinity is unattainable and not quite real. Cantor keeps this naive infinity, which he calls the Absolute Infinite, but he allows for many intermediate levels between the finite and the Absolute Infinite. These intermediate stages correspond to his *transfinite numbers* . . . numbers that are infinite, but none the less conceivable.

In the next section we will discuss the possibility of finding *physically* existing transfinite sets. We will then look for ways in which such actual infinities might exist *mentally.* Finally we will discuss the Absolute, or metaphysical, infinite.

This threefold division is due to Cantor, who, in the following passage, distinguishes between the Absolute Infinite, the physical infinities, and the mathematical infinities:

> The actual infinite arises in three contexts: *first* when it is realized in the most complete form, in a fully independent other-worldly being, *in Deo,* where I call it the Absolute Infinite or simply Absolute; *second* when it occurs in the contingent, created world; *third* when the mind grasps it *in abstracto* as a mathematical magnitude, number, or order type. I wish to make a sharp contrast between the Absolute and what I call the Transfinite, that is, the actual infinities of the last two sorts, which are clearly limited, subject to further increase, and thus related to the finite.[13]

## PHYSICAL INFINITIES[14]

There are three ways in which our world appears to be unbounded and thus, perhaps, infinite. It seems that time cannot end. It seems that space cannot end. And it seems that any interval of space or time can be divided and subdivided endlessly. We will consider these three apparent physical infinites in three subsections.

## TEMPORAL INFINITIES

Suppose that the human race was never going to die out—that any given generation would be followed by another generation. Would we not then have to admit that the number of generations of man is actually infinite?

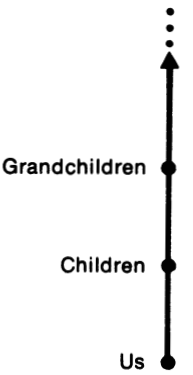


Figure 5.

Aristotle argued against this conclusion, asserting that in this situation the number of generations of man would be but potentially infinite; that is, infinite only in the sense of being inexhaustible. He maintained that at any given time there would only have been some finite number of generations, and that it was not permissible to take the entire future as a single whole containing an actual infinitude of generations.

It is my opinion that this sort of distinction rests on a view of time that has been fairly well discredited by modern relativistic physics. In order to agree with Aristotle that, although there will never be a last generation, there is no infinite set of all the generations, we must believe that the future does not exist as a stable, definite thing. For if we have the future existing in a fixed way, then we have all of the infinitely many future generations existing "at once."

But one of the chief consequences of Einstein's Special Theory of Relativity is that it is space-time that is fundamental, not isolated space which evolves as time passes. I will not argue this point in detail here, but let me repeat that on the basis of modern physical theory we have every reason to think of the passage of time as an illusion. Past, present, and future all exist together in space-time.

So the question of the infinitude of time is not one that is to be dodged by denying that time can be treated as a fixed dimension such as

space. The question still remains: is time infinite? If we take the entire space-time of our universe, is the time dimension infinitely extended or not?

Fifty, or even twenty, years ago it would have been natural to assert that our universe has no beginning or end and that time is thus infinite in both directions. But recently it has become an established fact that the universe does have a beginning in time known as the Big Bang. The Big Bang took place approximately 15 billion years ago. At that time our universe was the size of a point, and it has been expanding ever since. What happened before the Big Bang? It is at least possible to answer, "Nothing." The apparent paradox of having a *first* instant in time is sometimes avoided by saying that the Big Bang did not occur *in* time . . . that time is open, rather than closed, in the past.

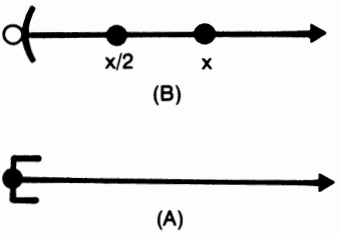


Figure 6A (bottom) and Figure 6B (top).

This is a subtle distinction, but a useful one. If we think of time as being all the points greater than or equal to zero, then there is a first instant: zero. But if we think of time as being all the points strictly greater than zero, then there is no first instant. For any instant $t$ greater than zero, one has an earlier instant $t/2$ that is also greater than zero.

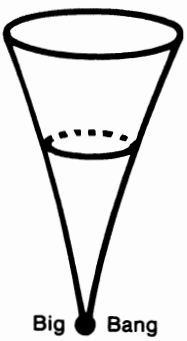


Figure 7.

But in any case, if we think of time as not existing before the Big Bang, then there are certainly not an infinite number of years in our past. And what about the future? There is no real consensus on this. Many cosmologists feel that our universe will eventually stop expanding and collapse to form a single huge black hole called the Big Stop or the Gnab Gib; others feel that the expansion of the universe will continue indefinitely.

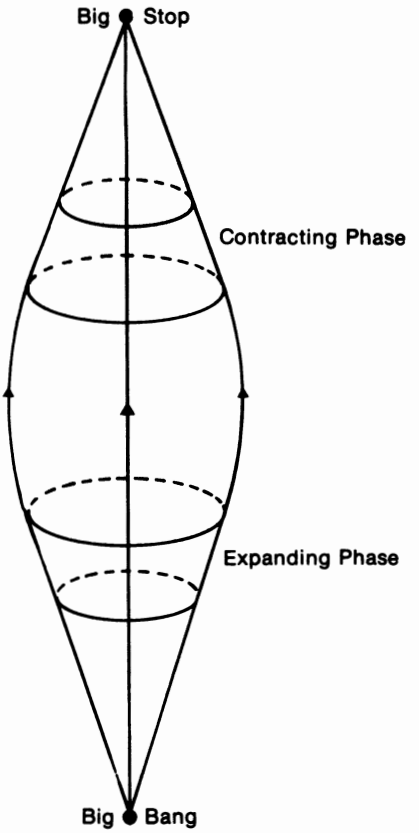


Figure 8.

If the universe really does start as a point and eventually contract back to a point, is it really reasonable to say that there is no time except for the interval between these points? What comes before the beginning and after the end?

One response is to view the universe as an oscillating system, which

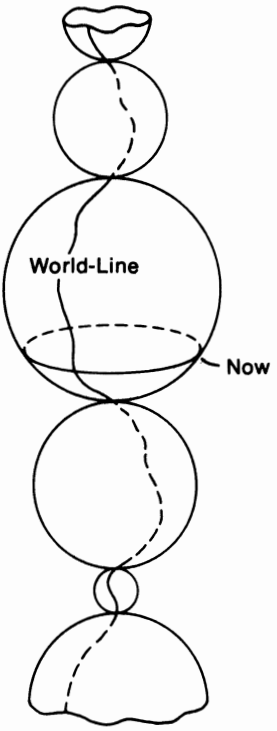


Figure 9.

repeatedly goes through expansions and contractions. This would reintroduce an infinite time, which could, however, be avoided.

The way in which one would avoid infinite time in an endlessly oscillating universe would be to adopt a belief in what used to be called "the eternal return." This is the belief that every so often the universe must repeat itself. The idea is that a finite universe must return to the same state every so often, and that once the same state has arisen, the future evolution of the universe will be the same as the one already undergone. The doctrine of eternal recurrence amounts to the assumption that time is a vast circle. An oscillating universe with circular time is pictured in Figure 10.

There is a simpler model of an oscillating universe with circular time, which can be called *toroidal space-time.* In toroidal space-time we have an oscillating universe that repeats itself after every cycle. Such a model is obtainable by identifying the two points, "Big Bang" and "Big Stop," in Figure 11.

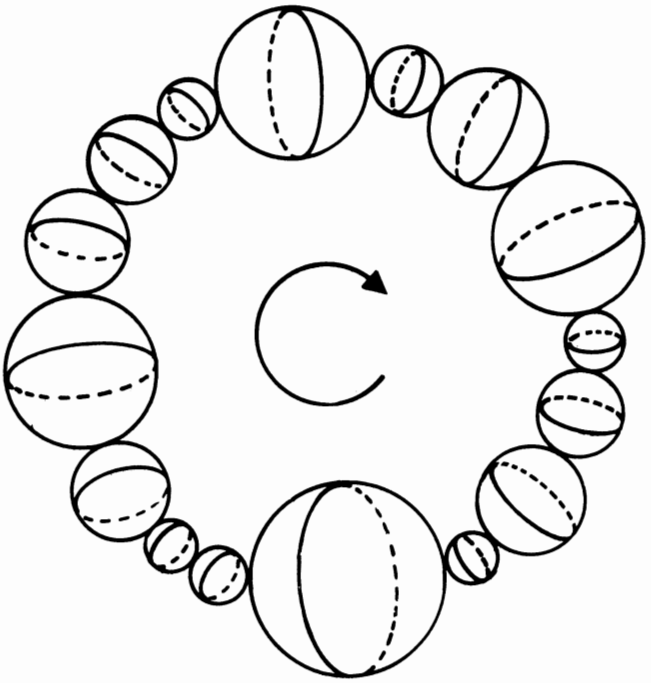

Figure 10.

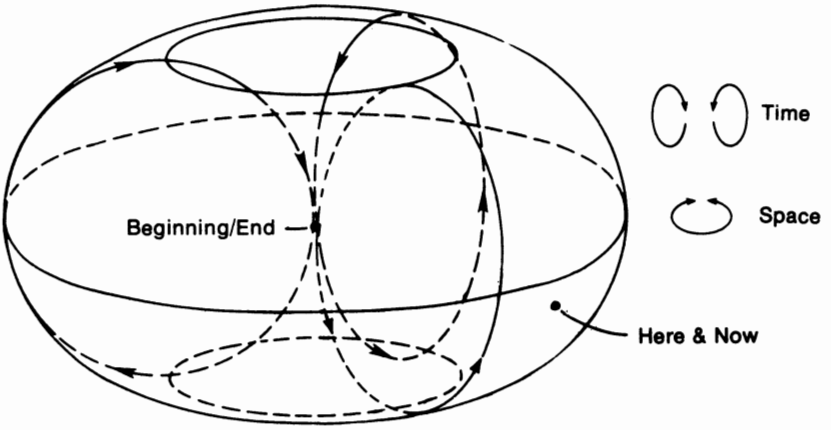


Figure 11. From R. v.B. Rucker, *Geometry, Relativity, and the Fourth Dimension.*

Note, however, that if the universe really *expands* forever, then it cannot ever repeat itself, as the average distance between galaxies is a continually increasing quantity that never returns to the same value.

## SPATIAL INFINITIES

We now turn to a consideration of the possibility of spatial infinites. The potential versus actual infinity distinction is sometimes used to try to scotch this question at the outset. Immanuel Kant, for instance, argues that the world cannot be an infinite whole of coexisting things because "in order therefore to conceive the world, which fills all space, as a whole, the successive synthesis of the parts of an infinite world would have to be looked upon as completed; that is, an infinite time would have to be looked upon as elapsed, during the enumeration of all coexisting things."[15]

Kant's point is that space is in some sense not already really there—that things exist together in space only when a mind perceives them to do so. If we accept this, then it is true that an infinite space is something that no finite mind can know of after any finite amount of time. But one feels that the world does exist as a whole, in advance of any efforts on our part to see it as a unity. And if we take all of space-time, it certainly does not seem to be meaningless to ask whether the spatial extent of space-time is infinite or not.

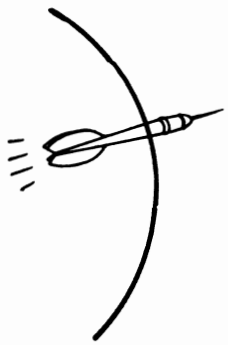

Figure 12A. Dart goes beyond "boundary."

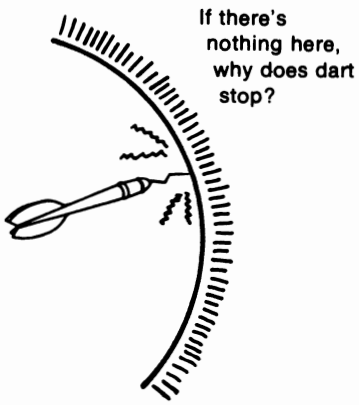


Figure 12B. Dart stops at boundary.

In *De Rerum Natura,* Lucretius first gave the classic argument for the unboundedness of space: "Suppose for a moment that the whole of

space were bounded and that someone made his way to its uttermost boundary and threw a flying dart."[16] It seems that either the dart must go past the boundary, in which case it is no boundary of space; or the dart must stop, in which case there is something just beyond the boundary that stops it, which again means that the purported boundary is not really the end of the universe.

So great was their revulsion against the *apeiron* that Parmenides, Plato, and Aristotle all held that the space of our universe is bounded and finite, having the form of a vast sphere. When faced with the question of what lies outside this sphere, Aristotle maintained that "what is limited, is not limited in reference to something that surrounds it."[17]

In modern times we have actually developed a way to make Aristotle's claim a bit more reasonable. As Lucretius realized, the weak point in the claim that space is a finite sphere is that such a space has a definite boundary. But there is a way to construct a three-dimensional space which is finite and which does *not* have boundary points: simply take the hypersurface of a hypersphere. Such a space is endless but not infinite.

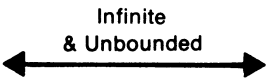


Figure 13A.

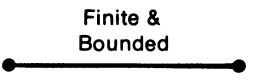


Figure 13B.

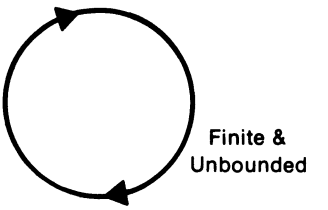


Figure 13C.

To understand how something can be endless but not infinite, think of a circle. A fly can walk around and around the rim of a glass without ever coming to a barrier or stopping point, but none the less he will soon retrace his steps.

Again, the surface of the Earth is a two-dimensional manifold which is finite but unbounded (unbounded in the sense of having no edges). You can travel and travel on the Earth's surface without ever coming to

any truly impassible barrier . . . but if you continue long enough, you will begin to recross your steps.

The reason that the two-dimensional surface of the Earth is finite but unbounded is that it is bent, in three-dimensional space, into the shape of a sphere. In the same way, it is possible to imagine the three-dimensional space of our universe as being bent, in some four-dimensional space, into the shape of a hypersphere. It was Bernhard Riemann who first realized this possibility in 1854. There is, however, a traditional belief that anticipates the hypersphere. This tradition, described in the essay, "The Fearful Sphere of Pascal," by Jorge Luis Borges, is summarized by the saying (attributed to the legendary magician Hermes Trismegistus) that "God is an intelligible sphere, whose center is everywhere and whose circumference is nowhere."[18] If the universe is indeed a hypersphere, then it would be quite accurate to regard it as a sphere whose center is everywhere and whose circumference is nowhere.

To see why this is so, consider the fact that if space is hyperspherical, then one can cover all of space by starting at any point and letting a sphere expand outwards from that point. The curious thing is that if one lets a sphere expand in hyperspherical space, there comes a time when the circumference of the sphere turns into a point and disappears. This fact can be grasped by considering the analogous situation of the sequence of circular latitude lines on the spherical surface of the earth.[19] This line of thought appears in Dante's *Paradisio* (1300).[20]

Aristotle had believed that the world was a series of nine spheres centered around the Earth. The last of these crystalline spheres was called the *Primum Mobile* and lay beyond the sphere upon which were fastened all of the stars (other than the sun, which was attached to the fourth sphere). In the *Paradisio,* Dante is led out through space by Beatrice. He passes through each of the nine spheres of the world: Moon, Mercury, Venus, Sun, Mars, Jupiter, Saturn, Fixed Stars, Primum Mobile. Beyond these nine spheres lie nine spheres of angels, corresponding to the nine spheres of the world. Beyond the nine spheres of angels lies a *point* called the Empyrean, which is the abode of God.

The puzzling thing about Dante's cosmos as it is drawn in Figure 14 is that here the Empyrean appears not to be a point, but rather to be all of space (except for the interior of the last sphere of angels). But this can be remedied if we take space to be hyperspherical! In Figure 15 I have drawn the model we obtain if we take the diagram on the last page and curve it up into a sphere with a point-sized Empyrean. In the same way, the three-dimensional model depicted by the first picture can be turned

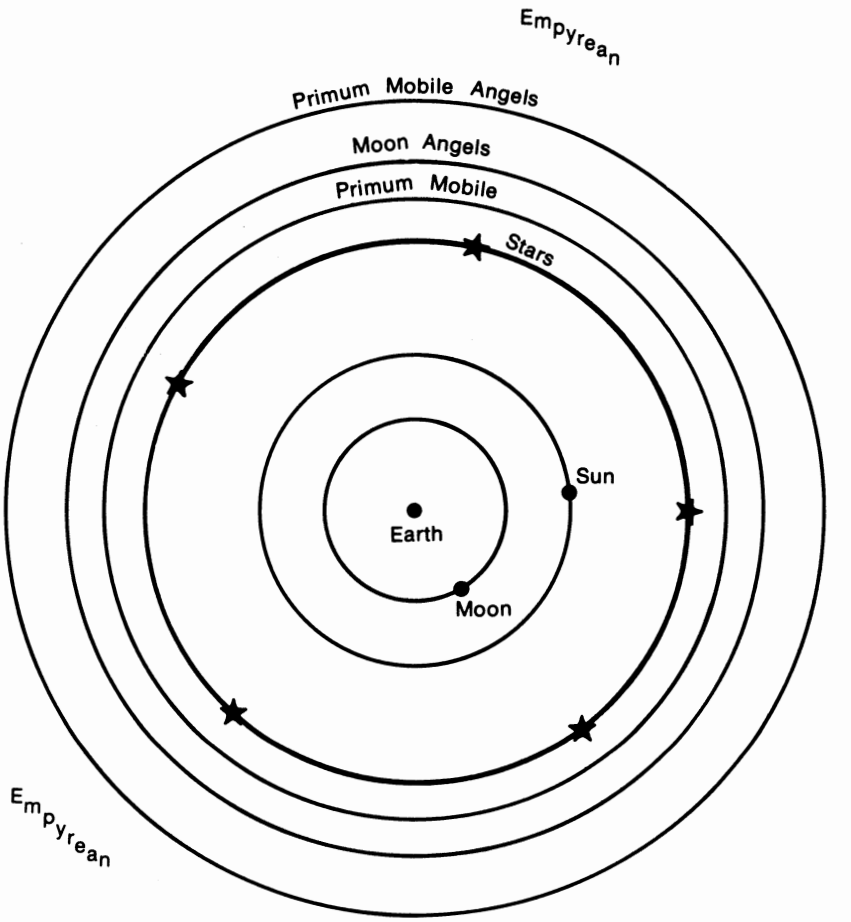


Figure 14.

into the finite unbounded space of the second picture if we bend our three-dimensional space in such a way that all of the space outside our last angelic sphere is compressed to a point.[21] Figure 16 is Doré's engraving of the Empyrean surrounded by its spheres of angels.

This whole notion of hyperspherical space was not consciously developed until the mid-nineteenth century. In the Middle Ages there was a general and uncritical acceptance of Aristotle's view of the universe–without Dante's angelic spheres.

Lucretius, of course, had insisted that space is infinite, and there were

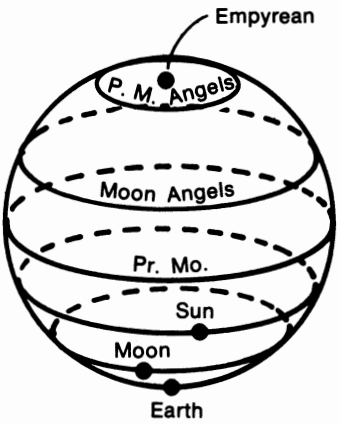


Figure 15.

many other thinkers, such as Nicolas of Cusa and Giordano Bruno, who believed in the infinitude of space. Some kept to the Aristotelian world system, but suggested that there were many such setups drifting around; others opted for a looser setup under which stars and planets are more or less randomly mixed together in infinite space.

Bruno strongly advocated such viewpoints in his writings, especially his dialogue of 1584, "On the Infinite Universe and Worlds."[22] Bruno travelled freely around Europe during his lifetime, teaching his doctrine of the infinite universe at many centers of learning. In 1591, a wealthy Venetian persuaded Bruno to come from Frankfurt to teach him "the art of memory and invention." Shortly after Bruno arrived, the trap was sprung. His host had been working closely with the ecclesiastical authorities, who considered Bruno a leading heretic or heresiarch. Bruno was turned over to the Inquisition. For nine years Bruno was interrogated, tortured, and tried, but he would not give up his beliefs; early in 1600 he was burned at the stake in the Roman *Piazza* Campo di Fiori. Bruno's example caused Galileo to express himself a good deal more cautiously on scientific questions in which the Church had an interest.

Whether or not our space is actually infinite is a question that could conceivably be resolved in the next few decades. Assuming that Einstein's theory of gravitation is correct, there are basically two types of universe: i) a hyperspherical (closed and unbounded) space that expands and then contracts back to a point; ii) an infinite space that expands forever. It is my guess that case i) will come to be most widely accepted, if only because the notion of an actually infinite space extend-

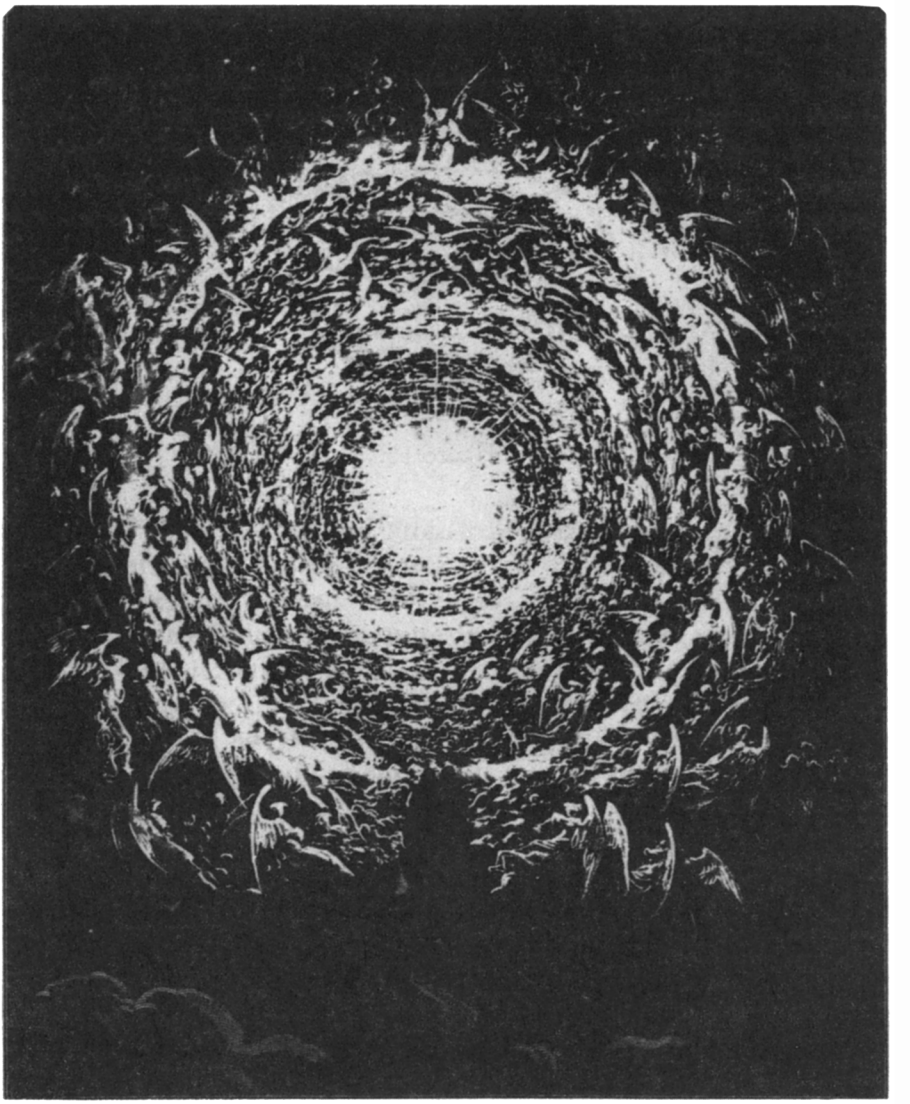

Figure 16. From Gustav Doré's *Divine Comedy* (Dover).

ing out in every direction is so unsettling. The fate of the universe in case i) is certainly more interesting, since such a universe collapses back to an infinitely dense space-time singularity that may serve as the seed for a whole new universe. In case ii), on the other hand, we simply have cooling and dying suns drifting further and further apart in an utterly

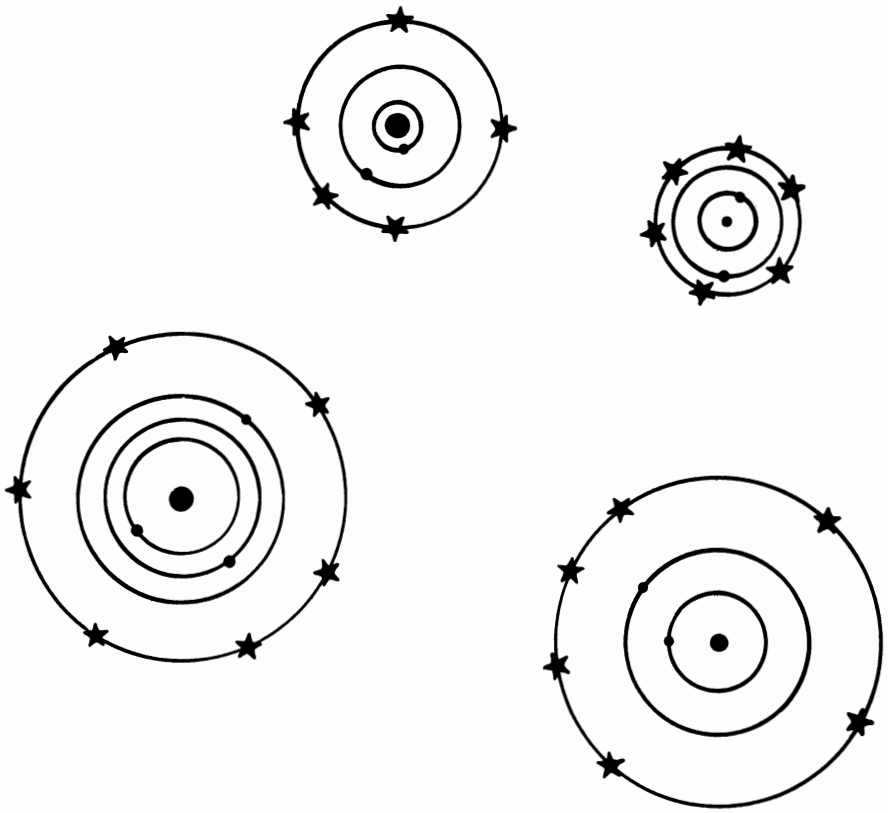

Figure 17.

empty black immensity . . . and in the end there are only ashes and cinders in an absolute and eternal night.

Even though I am basically pro-infinity, my emotions lie with the hyperspherical space. But is there any way of finding a spatial infinity here? Well, what about that four-dimensional space in which our hyperspherical universe is floating? Many would dismiss this space as a mere mathematical fiction . . . as a colorful way of expressing the finite, but unbounded, nature of our universe. This widely held position is really a more sophisticated version of Aristotle's claim that what is limited need not be limited with reference to something outside itself.

But what if one chooses to believe that the four-dimensional space in which our universe curves is real? We might imagine a higher 4-D (four-dimensional) world called, let us say, a duoverse. The duoverse would be 4-D space in which a number of hyperspheres were floating.

The hypersurface of each of the hyperspheres would be a finite, unbounded 3-D universe.

Thus, a duoverse would contain a number of 3-D universes, but no inhabitant of any one of these universes could reach any one of the others, unless he could somehow travel through 4-D space. By lowering all the dimensions by one, one can see that this situation is analogous to a universe that is a 3-D space in which a number of spheres are floating. The surface of each sphere or planet is a finite, unbounded 2-D space; and no one can get from one planet surface to another planet surface without travelling through 3-D space.

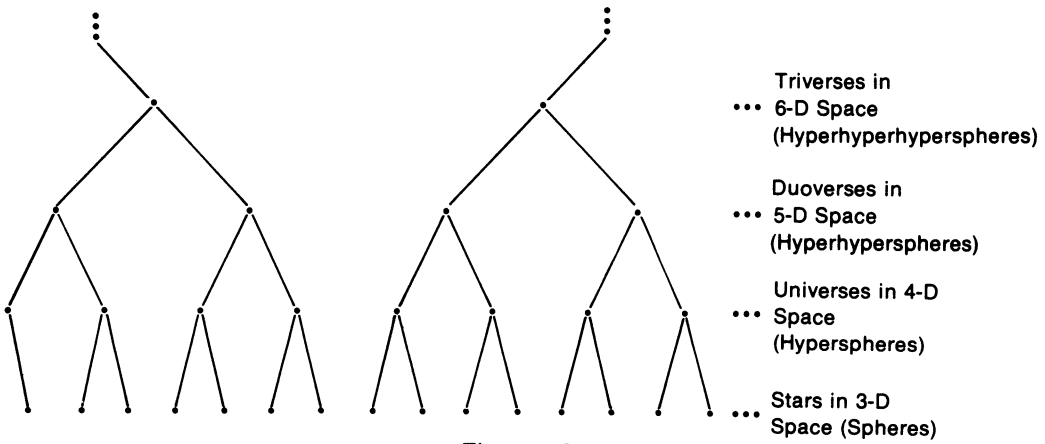


Figure 18.

Following the Hermetic principle, "As above, so below," one is tempted to believe that the duoverse we are in is actually a finite and unbounded 4-D space (the 4-D surface of a 5-D sphere in 5-D space), and that there are a number of such duoverses drifting about in a 5-D triverse. This could be continued indefinitely. One is reminded of those Eastern descriptions of the world as a disk resting on the backs of elephants, who stand upon a turtle, who stands upon a turtle, who stands upon a turtle, who stands upon a turtle, etc.

Note that in that particular sort of cosmos there is only one universe, one duoverse, one triverse, and so on. But in the kind of infinitely regressing cosmos that I have drawn in Figure 18, we have infinitely many objects at each level. Note also that to get from star A to star B one would have to move through 5-D space to get to a different duoverse. It is a curious feature of such a cosmos that, although there are an infinite

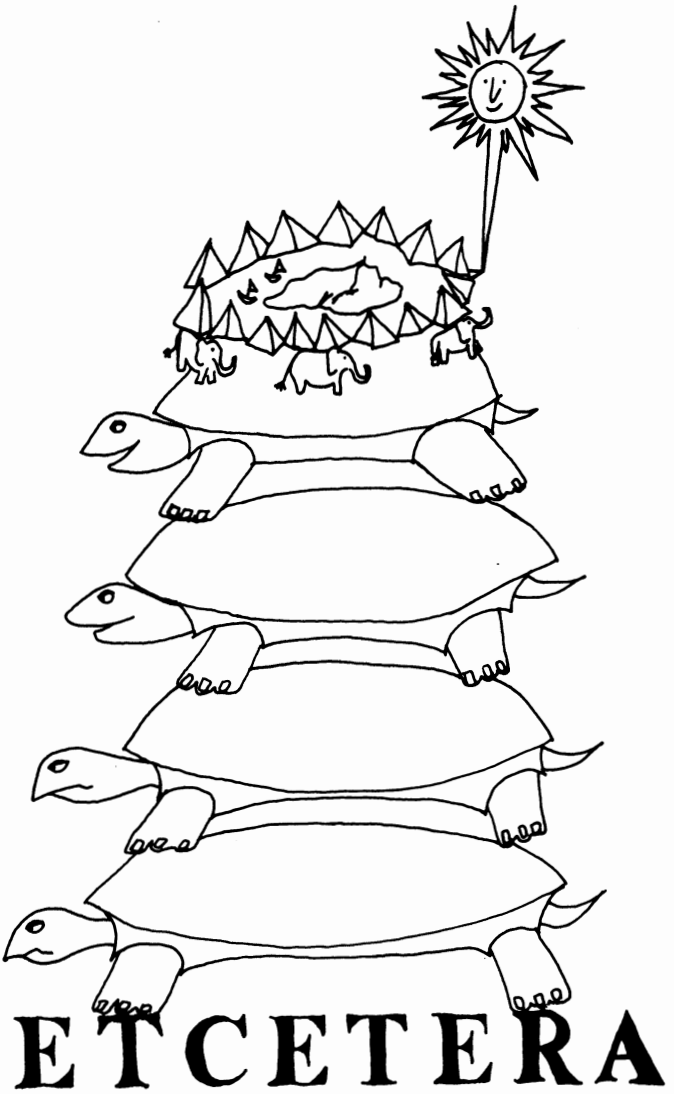


Figure 19.

number of stars, no one $n$-dimensional space has more than a finite number of them.

The question we are concerned with here is whether or not space is infinitely large. There seem to be three options: i) There is some level $n$ for which $n$-dimensional space is real and infinitely extended. The situa-

tion where our three-dimensional space is infinitely large falls under this case. ii) There is some $n$ such that there is only one $n$-dimensional space. This space is to be finite and unbounded, and there is to be no reality to $n + 1$ dimensional space. The situation where our three-dimensional space is finite and unbounded, and the reality of four-dimensional space denied, falls under this case. iii) There are real spaces of every dimension, and each of these spaces is finite and unbounded. In this case we either have an infinite number of universes, duoverses, etc., or we reach a level after which there is only one $n$-verse for each $n$.

So is space infinite? It seems that we can insist that at some dimensional level it is infinite; adopt the Aristotelian stance that space is finite at some level beyond which nothing lies; or accept the view that there is an infinite sequence of dimensional levels. In this last case we already have a qualitative infinity in the dimensionality of space, and we may or may not have a quantitative infinity in terms, say, of the total volume of all the 3-D spaces involved.

## INFINITIES IN THE SMALL

In this subsection I will discuss the existence of the infinity in the *small,* as opposed to the infinity in the *large,* which has just been discussed. Since a point has no length, no finite number of points could ever constitute a line segment, which *does* have length. So it seems evident that every line segment, or, for that matter, every continuous plane segment or region of space, must consist of an infinite number of points. By the same token, any interval of time should consist of an infinite number of instants; and any continuous region of space-time would consist of an infinite number of events (*event* being the technical term for a space-time location, i.e., point at an instant).

It is undeniable that a continuous region of *mathematical* space has an infinite number of *mathematical* points. Right now, however, we are concerned with *physical* space. We should not be too hasty in assuming that every property of the abstract mathematical space we use to organize our experiences is an actual property of the concrete physical space we live in. But what *is* "the space we live in"? If it is not the space of mathematical physics, is it the space of material objects? Is it the space of our perceptions? In terms of material objects or of perceptions, points do not really exist; for any material or perceptual phenomenon is spread over a certain finite region of space-time. So when we look for the infinity in the small in matter, we do not ask whether matter consists

of an infinity of (unobservable) mass-points, but, rather, whether matter is infinitely divisible.

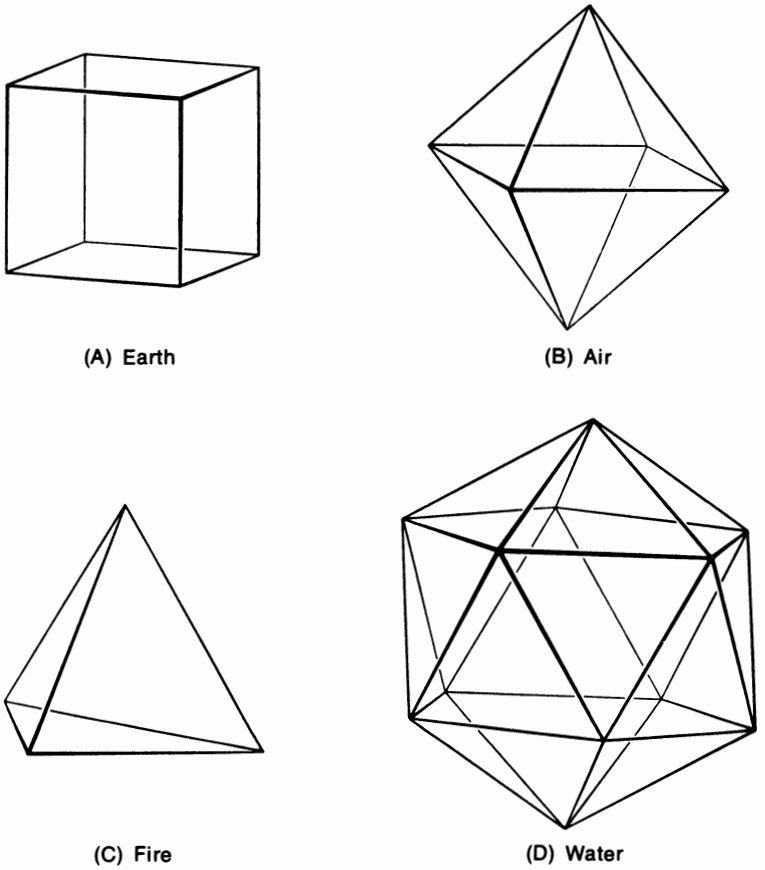


Figure 20. (A–D). From D. Hilbert and H. Cohn-Vossen, *Geometry and the Imagination.*

A commitment to avoiding the formless made it natural for Greek atomists such as Democritus to adopt a theory of matter under which the seemingly irregular bodies of the world are in fact collections of indivisible, perfectly formed atoms. (The four kinds of atoms were shaped, according to Plato, like four of the regular polyhedra. There is one other polyhedron, the twelve-sided dodecahedron, and this was thought somehow to represent the Universe with its twelve signs of the zodiac.) For the atomists, it was as if the world were an immense Lego

set, with four kinds of blocks. The diverse substances of the world–oil, wood, stone, metal, flesh, wine, and so on–were regarded as being mixtures of the four elemental substances: Earth, Air, Fire, and Water. Thus, gold was regarded by Plato as being a very dense sort of Water, and copper was viewed as gold with a small amount of Earth mixed in.

The alchemists and early chemists adopted a similar system, only the number of elemental substances became vastly enlarged to include all homogeneous substances, such as the various ores, salts, and essences. The fundamental unit here was the molecule.

A new stage in man's conception of matter came when it was discovered that if an electric current is passed through water, it can be decomposed into hydrogen and oxygen. Eventually, the vast diversity of existing molecules was brought under control by regarding molecules as collections of atoms. Soon some ninety different types of atoms or chemical elements were known. A new simplification occurred when it was discovered, by bombarding a sheet of foil with alpha rays, that an atom consists of a positive nucleus surrounded by electrons. Shortly after this the neutron was discovered, and the physical properties of the various atoms were accounted for by regarding them as collections of protons, neutrons, and electrons.

Over the last half century it has been learned, by using particle accelerators, that there are actually many types of "elementary particles" other than the neutron, electron, and proton. The situation in high-energy physics today is as follows. A few particles–electrons, neutrinos, and muons–seem to be absolutely indivisible. These particles are called *leptons*. All others–protons, neutrons, mesons, lambdas, etc.–can be broken up into smaller units, which then reassemble to form more particles.

The historical pattern in the investigation of matter has been the explanation of diverse substances as combinations of a few simpler substances. Diversity of form replaces diversity of substance. So it is no surprise that it has been proposed that the great variety of divisible particles that exist can be accounted for by assuming that these particles are all built up out of *quarks*.

A second element in the historical pattern is that as more powerful tools of investigation are used, it becomes evident that there are more types of new building blocks than had been suspected initially. This is the phase that high-energy physics is currently moving into. First there were three kinds of quark: up, down, and strange. Now, the charmed quark has been admitted, and there are two new possible quarks: the

top quark and the bottom quark. It seems likely that the many diverse types of quark will eventually be accounted for by assuming that each quark is a combination of a few, let us say, *darks* . . . and that there are only a very small number of possible kinds of dark. The cycle will then repeat, with more and more different sorts of dark being indirectly observed, the new diversity being accounted for by viewing each dark as a collection of a few smaller particles of which there are a limited variety, this limited variety beginning to proliferate, and so on.

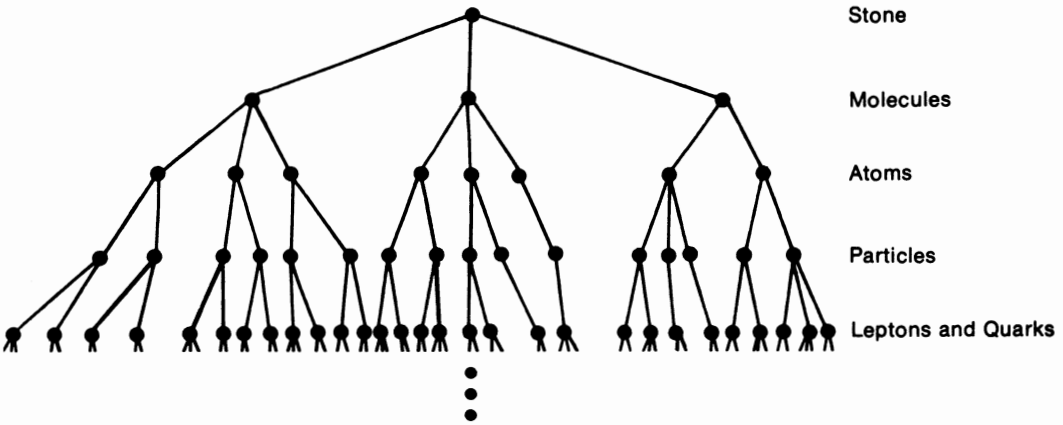


Figure 21.

If this sort of development can indeed continue indefinitely, then we are left with the fact that a stone is a collection of collections of collections of. . . . The stone thus consists of an infinite number of particles, no one of which is indivisible. There is, finally, no matter—only form. For a stone is mostly empty space with a few molecules in it, a molecule is a cloud of atoms, an atom is a few electrons circling a tiny nucleus. . . . What if any seemingly solid bit of matter proves on closer inspection to be a cloud of smaller bits of matter, which are in turn clouds, and so on? Note that the branching matter tree that I began to draw for the stone has only a finite number of forks or nodes at each level, but that since there are infinitely many levels, there are in all an infinite number of nodes or component particles.

There are various objections to this sort of physical infinity. One is the Aristotelian argument that unless one is actually smashing the stone down to the quark level, the quarks are only potentially (as opposed to actually) there. The point would be that the stone may be indefinitely

divisible, but that since no one will ever carry out infinitely many divisions, there are not really infinite numbers of particles in the stone right now.

There is a more practical objection as well. This is that no quark has ever been observed in isolation; the existence of quarks is deduced only indirectly as a way of explaining the symmetries of structure that occur in tables of the elementary particles. This argument is not very strong, however. For one thing, a great number of the things we believe in can be observed only indirectly; and, more practically, if we can continue to increase the energy of our measuring tools, there is no reason to think that quarks cannot be more convincingly detected.

A more fundamental objection to the whole idea of particles, subparticles, etc., is that the underlying reality of the world may be field-like, rather than particle-like. By splitting particles indefinitely we arrived at the conclusion that there is only form, and no content; many physicists prefer to start with this viewpoint. For these physicists, the various features of the world are to be explained in terms of the geometry of

Figure 22.

space-time. To get a feeling for this viewpoint, one should look carefully at the surface of a river or small brook. There are circular ripples, flow bulges, whirlpools and eddies, bubbles that form, drops that fly up and fall back, waves that crest into foam. The *geometrodynamic* worldview regards space-time as a substance like the surface of a brook; the

various fields and particles that seem to exist are explained as features of the flow.

Does the space-time of geometrodynamics allow an infinity in the small? There is really no answer to this question at present. According to one viewpoint there should be a sort of graininess to space-time, and the grain size would represent a sort of indivisible atom; a different viewpoint suggests that space-time should be as infinitely continuous as mathematical space.

What if there really is nothing smaller than electrons and quarks? Is there then any hope of an infinity in the small? One can argue that a given electron can have infinitely many locations along a given meter stick, so that our space really does have infinitely many points. It is sometimes asserted that the uncertainty principle of quantum mechanics nullifies this argument, but this is not the case.

Quantum mechanics puts no upper limit on the precision with which one can, in principle, determine the position of an electron. It is just that the more precisely the electron's position is known, the less precisely are its speed and direction of motion known. Infinite precision is basically a nonphysical notion, but any desired finite degree of precision is, in principle, obtainable. The precision with which something can be measured is thus a good example of something that is potentially infinite, but never actually infinite.

But this still gives us an actual infinity in the world. For if our electron is located somewhere between zero and one, then each member of the following infinite collection is a possible outcome of a possible measurement:

$$.2 \pm .1,\ .23 \pm .01,\ .235 \pm .001,\ .2356 \pm .0001,\ \ldots,\ .235608947 \pm .000000001,\ \ldots$$

Although infinite precision is impossible, an electron can be found to occupy any of the infinitely many points between zero and one whose distance from zero is a terminating decimal.

There are, however, some modern physical speculations that regard "space" and "time" as being abstractions which apply to our size level, but which become utterly meaningless out past the thirtieth decimal place. What would be there instead? Our old friend the *apeiron*. But even if we cannot really speak of infinitely many space locations, we might hope to find infinitely many sorts of particle.

It is sometimes thought that quantum mechanics *proves* that there is a smallest size of particle that could exist. This is not true. Quantum me-

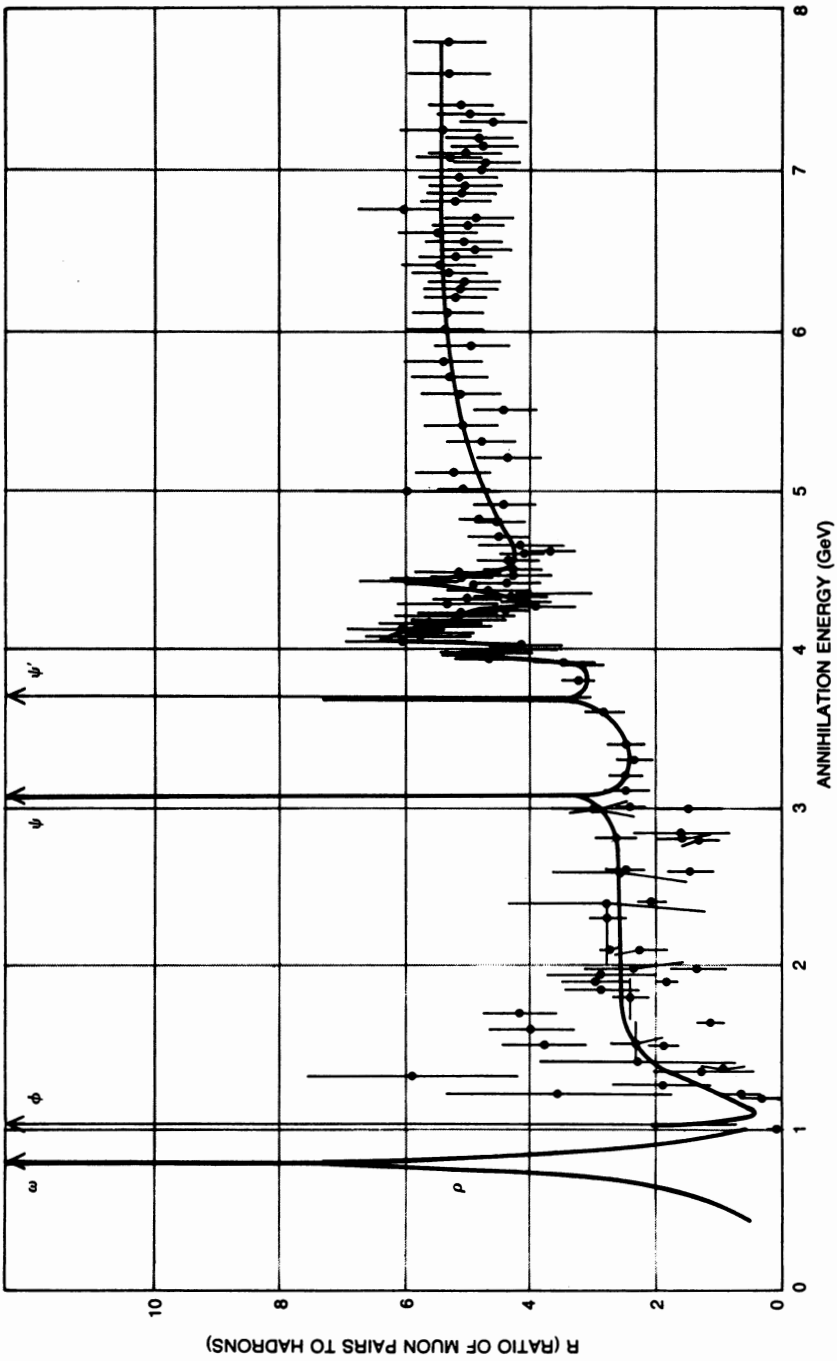


Figure 23. From *Scientific American*, Oct. 77, p. 65.

*(continued...)*

# INDEX