Contents

Chapter One

An Overview of the Proof

THE PURPOSE OF this chapter is to give the main steps in the proof of Theorems A and B (stated in the introduction) that for each n the norm residue homomorphism

$$K_n^M(k)/\ell \longrightarrow H^n_{\text{ét}}(k, \mu_\ell^{\otimes n}) \tag{1.1}$$

is an isomorphism, and $H^{p,n}(X, \mu_\ell^{\otimes n}) \cong H^p_{\text{ét}}(X, \mu_\ell^{\otimes n})$ for $p \le n$. We proceed by induction on n. It turns out that in order to prove Theorems A, B, and C, we must simultaneously prove several equivalent (but more technical) assertions, H90(n) and BL(n), which are defined in 1.5 and 1.28.

1.1 FIRST REDUCTIONS

We fix a prime ℓ and a positive integer n. In this section we reduce Theorems A and B to H90(n), an assertion (defined in 1.5) about the étale cohomology of the ℓ-local motivic complex $\mathbb{Z}_{(\ell)}(n)$. We begin with a series of reductions, the first of which is a special case of the *transfer argument.*

The transfer argument 1.2. Let F be a covariant functor on the category of fields which are algebraic over some base field, taking values in $\mathbb{Z}/\ell$-modules and commuting with direct limits. We suppose that F is also contravariant for finite field extensions k'/k, and that the evident composite from $F(k)$ to itself is multiplication by $[k'{:}k]$. The contravariant maps are commonly called *transfer maps.* If $[k':k]$ is prime to ℓ, the transfer hypothesis implies that $F(k)$ injects as a summand of $F(k')$. More generally, $F(k)$ injects into $F(k')$ for any algebraic extension k' consisting of elements whose degree is prime to ℓ. Thus to prove that $F(k)=0$ it suffices to show that $F(k')=0$ for the field k'.

Both $k \mapsto K_n^M(k)/\ell$ and $k \mapsto H^n_{\text{ét}}(k, \mu_\ell^{\otimes n})$ satisfy these hypotheses, and so do the kernel and cokernel of the norm residue map (1.1), because the norm residue commutes with these transfers. Thus if the norm residue is an isomorphism for k' it is an isomorphism for k, by the transfer argument applied to the kernel and cokernel of (1.1). For this reason, we may assume that k contains all ℓ^{th} roots of unity, that k is a perfect field, and even that k has no field extensions of degree prime to ℓ.

The second reduction allows us to assume that we are working in characteristic zero, where, for example, the resolution of singularities is available.

Lemma 1.3. *If* (1.1) *is an isomorphism for all fields of characteristic* 0*, then it is an isomorphism for all fields of characteristic* $\neq \ell$*.*

Proof.[1] Let R be the ring of Witt vectors over k and K its field of fractions. By the standard transfer argument 1.2, we may assume that k is a perfect field, so that R is a discrete valuation ring. In this case, the specialization maps "sp" are defined and compatible with the norm residue maps in the sense that

$$\begin{array}{ccc} K_n^M(K)/\ell & \longrightarrow & H^n_{\text{ét}}(K,\mu_\ell^{\otimes n}) \\ \downarrow{\scriptstyle \text{sp}} & & \downarrow{\scriptstyle \text{sp}} \\ K_n^M(k)/\ell & \longrightarrow & H^n_{\text{ét}}(k,\mu_\ell^{\otimes n}) \end{array}$$

commutes (see [Wei13, III.7.3]). Both specialization maps are known to be split surjections. Since $\text{char}(K)=0$, the result follows. □

Our third reduction translates the problem into the language of motivic cohomology, as the condition H90(n) of Definition 1.5.

The (integral) motivic cohomology of a smooth variety X is written as $H^{n,i}(X,\mathbb{Z})$ or $H^n(X,\mathbb{Z}(i))$; it is defined to be the Zariski hypercohomology on X of $\mathbb{Z}(i)$; see [MVW, 3.4]. Here $\mathbb{Z}(i)$ is a cochain complex of étale sheaves which is constructed, for example, in [MVW, 3.1]. By definition, $\mathbb{Z}(i)=0$ for $i<0$ and $\mathbb{Z}(0)=\mathbb{Z}$, so $H^n(X,\mathbb{Z}(i))=0$ for $i<0$ and even $i=0$ when $n\neq 0$. There are pairings $\mathbb{Z}(i)\otimes\mathbb{Z}(j)\to\mathbb{Z}(i+j)$ making $H^*(X,\mathbb{Z}(*))$ into a bigraded ring. When k is a field, we often write $H^*(k,\mathbb{Z}(*))$ for $H^*(\operatorname{Spec} k,\mathbb{Z}(*))$.

There is a quasi-isomorphism $\mathbb{Z}(1)\xrightarrow{\simeq}\mathcal{O}^\times[-1]$; see [MVW, 4.1]. This yields an isomorphism $H^1(X,\mathbb{Z}(1))\cong\mathcal{O}_X^\times$. When $X=\operatorname{Spec}(k)$ for a field k, the Steinberg relation holds in $H^2(X,\mathbb{Z}(2))$: if $a\neq 0,1$ then $a\cup(1-a)=0$. The presentation of $K_*^M(k)$ implies that we have a morphism of graded rings $K_*^M(k)\to H^*(k,\mathbb{Z}(*))$ sending $\{a_1,\dots,a_n\}$ to $a_1\cup\cdots\cup a_n$. It is a theorem of Totaro and Nesterenko–Suslin that $K_n^M(k)\cong H^n(\operatorname{Spec} k,\mathbb{Z}(n))$ for each n; proofs are given in [NS89], [Tot92], and [MVW, Thm. 5.1].

We can of course vary the coefficients in this construction. Given any abelian group A, we may consider $H^n(X,A(i))$, where $A(i)$ denotes $A\otimes\mathbb{Z}(i)$; $H^*(X,A(*))$ is a ring if A is. Because Zariski cohomology commutes with direct limits, we have $H^n(X,\mathbb{Z}(i))\otimes\mathbb{Q}\xrightarrow{\simeq}H^n(X,\mathbb{Q}(i))$ and $H^n(X,\mathbb{Z}(i))\otimes\mathbb{Z}_{(\ell)}\xrightarrow{\simeq}H^n(X,\mathbb{Z}_{(\ell)}(i))$. Because $H^{n+1}_{\text{zar}}(\operatorname{Spec} k,\mathbb{Z}(n))=0$ [MVW, 3.6], this implies that we have

$$K_n^M(k)/\ell\cong H^n_{\text{zar}}(\operatorname{Spec} k,\mathbb{Z}/\ell(n)). \tag{1.4}$$

Since each $A(i)$ is a complex of étale sheaves, we can also speak about the étale motivic cohomology $H^*_{\text{ét}}(X,A(i))$. There is a motivic-to-étale map

1. Taken from [Voe96, 5.2].

$H^*(X, A(i)) \to H^*_{\text{ét}}(X, A(i))$; it is just the change-of-topology map $H^*_{\text{zar}} \to H^*_{\text{ét}}$. For $A = \mathbb{Z}/\ell$ we have isomorphisms $H^n_{\text{ét}}(X, \mathbb{Z}/\ell(i)) \cong H^n_{\text{ét}}(X, \mu_\ell^{\otimes i})$ for all $n, i \geq 0$; see [MVW, 10.2]. We also have $H^n_{\text{ét}}(k, \mathbb{Z}(i))_{(\ell)} = H^n_{\text{ét}}(k, \mathbb{Z}_{(\ell)}(i))$ and $H^n_{\text{ét}}(k, \mathbb{Z}(i)) \otimes \mathbb{Q} = H^n_{\text{ét}}(k, \mathbb{Q}(i))$.

The condition H90(n)

Definition 1.5. Fix n and ℓ. We say that *H90(n) holds* if $H^{n+1}_{\text{ét}}(k, \mathbb{Z}_{(\ell)}(n)) = 0$ for any field k with $1/\ell \in k$. Note that H90(0) holds as $H^1_{\text{ét}}(k, \mathbb{Z}) = 0$, and that H90(n) implicitly depends on the prime ℓ.

The name "H90(n)" comes from the observation that H90(1) is equivalent to the localization at ℓ of the classical Hilbert's Theorem 90:

$$H^2_{\text{ét}}(k, \mathbb{Z}(1)) \cong H^2_{\text{ét}}(k, \mathbb{G}_m[-1]) = H^1_{\text{ét}}(k, \mathbb{G}_m) = 0.$$

We now connect H90(n) to $K^M_n(k)$.

Lemma 1.6. *For all $n > i$, $H^n_{\text{ét}}(k, \mathbb{Z}(i))$ is a torsion group, and its ℓ-torsion subgroup is $H^n_{\text{ét}}(k, \mathbb{Z}_{(\ell)}(i))$. When $1/\ell \in k$ and $n \geq i+1$ we have $H^{n+1}_{\text{ét}}(k, \mathbb{Z}_{(\ell)}(i)) \cong H^n_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}_{(\ell)}(i))$, while there is an exact sequence*

$$K^M_n(k) \otimes \mathbb{Q}/\mathbb{Z}_{(\ell)} \to H^n_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}_{(\ell)}(n)) \to H^{n+1}_{\text{ét}}(k, \mathbb{Z}_{(\ell)}(n)) \to 0.$$

Proof. We have $H^n_{\text{ét}}(k, \mathbb{Q}(i)) \cong H^n(k, \mathbb{Q}(i))$ for all n by [MVW, 14.23]. If $n > i$, $H^n(k, \mathbb{Q}(i))$ vanishes (by [MVW, 3.6]) and hence $H^n_{\text{ét}}(k, \mathbb{Z}(i))$ is a torsion group. Its ℓ-torsion subgroup is $H^n_{\text{ét}}(k, \mathbb{Z}(i))_{(\ell)} = H^n_{\text{ét}}(k, \mathbb{Z}_{(\ell)}(i))$. Set $D(i) = \mathbb{Q}/\mathbb{Z}_{(\ell)}(i)$. The étale cohomology sequence for the exact sequence $0 \to \mathbb{Z}_{(\ell)}(i) \to \mathbb{Q}(i) \to D(i) \to 0$ yields the second assertion (for $n \geq i+1$), and (taking $n = i$) yields the commutative diagram:

$$\begin{array}{ccccccc}
H^n(k, \mathbb{Z}_{(\ell)}(n)) & \to & H^n(k, \mathbb{Q}(n)) & \to & H^n(k, D(n)) & \longrightarrow & 0 \\
\downarrow & & \cong\downarrow & & \downarrow & & \\
H^n_{\text{ét}}(k, \mathbb{Z}_{(\ell)}(n)) & \to & H^n_{\text{ét}}(k, \mathbb{Q}(n)) & \to & H^n_{\text{ét}}(k, D(n)) & \xrightarrow{\text{onto}} & H^{n+1}_{\text{ét}}(k, \mathbb{Z}_{(\ell)}(n)).
\end{array}$$

The bottom right map is onto because $H^{n+1}_{\text{ét}}(k, \mathbb{Q}(i)) = 0$. Since $H^n(k, D(n)) \cong K^M_n(k) \otimes \mathbb{Q}/\mathbb{Z}_{(\ell)}$, a diagram chase yields the exact sequence. □

The example $\text{Br}(k)_{(\ell)} = H^2_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}_{(\ell)}(1)) \cong H^3_{\text{ét}}(k, \mathbb{Z}_{(\ell)}(1))$ shows that the higher étale cohomology of $\mathbb{Z}(n)$ and $\mathbb{Z}_{(\ell)}(n)$ need not vanish.

Theorem 1.7. *Fix n and ℓ. If $K^M_n(k)/\ell \xrightarrow{\simeq} H^n_{\text{ét}}(k, \mu_\ell^{\otimes n})$ holds for every field k containing $1/\ell$, then H90(n) holds.*

Of course, the weaker characteristic 0 hypothesis suffices by Lemma 1.3.

Proof. Recall that $K_n^M(k) \cong H_{\text{zar}}^n(\operatorname{Spec} k, \mathbb{Z}(n))$. The change of topologies map $H_{\text{zar}}^n \to H_{\text{ét}}^n$ yields a commutative diagram:

$$\begin{array}{ccccccccc}
K_n^M(k) & \xrightarrow{\ell} & K_n^M(k) & \longrightarrow & K_n^M(k)/\ell & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \text{Norm residue}} & & & & \\
H_{\text{ét}}^n(k, \mathbb{Z}(n)) & \xrightarrow{\ell} & H_{\text{ét}}^n(k, \mathbb{Z}(n)) & \longrightarrow & H_{\text{ét}}^n(k, \mu_\ell^{\otimes n}) & \longrightarrow & H_{\text{ét}}^{n+1}(k, \mathbb{Z}(n)) & \xrightarrow{\ell} &
\end{array}$$

The right vertical map is the Norm residue homomorphism, because the left vertical maps are multiplicative, and $H_{\text{ét}}^1(k, \mathbb{Z}(1)) = k^\times$. If the norm residue is a surjection, then $H_{\text{ét}}^{n+1}(k, \mathbb{Z}(n))$ has no ℓ-torsion. But it is a torsion group, and its ℓ-primary subgroup is $H_{\text{ét}}^{n+1}(k, \mathbb{Z}_{(\ell)}(n))$ by Lemma 1.6. As this must be zero for all k, H90(n) holds. $\square$

The converse of Theorem 1.7 is true, and will be proven in chapter 2 as Theorem 2.38 and Corollary 2.42. For reference, we state it here. Note that parts a) and b) are the conclusions of Theorems A and B (stated in the introduction).

Theorem 1.8. *Fix n and ℓ. Suppose that H90(n) holds. If k is any field containing $1/\ell$, then:*

a) the norm residue $K_n^M(k)/\ell \to H_{\text{ét}}^n(k, \mu_\ell^{\otimes n})$ is an isomorphism;
b) for every smooth X over k and all $p \le n$, the motivic-to-étale map $H^p(X, \mathbb{Z}/\ell(n)) \to H_{\text{ét}}^p(X, \mu_\ell^{\otimes n})$ is an isomorphism.

1.2 THE QUICK PROOF

With these reductions behind us, we can now present the proof that the norm residue is an isomorphism. In order to keep the exposition short, we defer definitions and proofs to later sections.

We will proceed by induction on n, assuming H90(n − 1) holds. By Theorems 1.7 and 1.8, this is equivalent to assuming that $K_{n-1}^M(k)/\ell \cong H_{\text{ét}}^{n-1}(k, \mu_\ell^{\otimes n-1})$ for all fields k containing $1/\ell$.

Definition 1.9. We say that a field k containing $1/\ell$ is *ℓ-special* if k has no finite field extensions of degree prime to ℓ. This is equivalent to the assertion that every finite extension is a composite of cyclic extensions of degree ℓ, and hence that the absolute Galois group of k is a pro-ℓ-group.

If k is a field containing $1/\ell$, any maximal prime-to-ℓ algebraic extension is ℓ-special. These extensions correspond to the Sylow ℓ-subgroups of the absolute Galois group of k.

The following theorem first appeared as [Voe03a, 5.9]; it will be proven in section 3.1 as Theorem 3.11.

Theorem 1.10. *Suppose that H90(n − 1) holds. If k is an ℓ-special field and* $K_n^M(k)/\ell = 0$, *then* $H^n_{\text{ét}}(k, \mu_\ell^{\otimes n}) = 0$ *and hence* $H^{n+1}_{\text{ét}}(k, \mathbb{Z}_{(\ell)}(n)) = 0$.

The main part of this book is devoted to proving the following deep theorem.

Theorem 1.11. *Suppose that H90(n − 1) holds. Then for every field k of characteristic 0 and every nonzero symbol* $\underline{a} = \{a_1, \dots, a_n\}$ *in* $K_n^M(k)/\ell$ *there is a smooth projective variety* $X_{\underline{a}}$ *whose function field* $K_{\underline{a}} = k(X_{\underline{a}})$ *satisfies:*

(a) $\underline{a}$ *vanishes in* $K_n^M(K_{\underline{a}})/\ell$; *and*
(b) the map $H^{n+1}_{\text{ét}}(k, \mathbb{Z}_{(\ell)}(n)) \to H^{n+1}_{\text{ét}}(K_{\underline{a}}, \mathbb{Z}_{(\ell)}(n))$ *is an injection.*

Outline of proof. (See Figure 1.1.) The varieties $X_{\underline{a}}$ we use to prove Theorem 1.11 are called *Rost varieties for* $\underline{a}$; they are defined in section 1.3 (see 1.24). Part of the definition is that any Rost variety satisfies condition (a). The proof that a Rost variety exists for every $\underline{a}$, which is due to Markus Rost, is postponed until part II of this book, and is given in chapter 11 (Theorem 11.2).

The proof that Rost varieties satisfy condition (b) of Theorem 1.11 will be given in chapter 4 (in Theorem 4.20). The proof requires the motive of the Rost variety to have a special summand called a *Rost motive*; the definition of Rost motives is given in section 4.3 (see 4.11).

The remaining difficult step in the proof of Theorem 1.11, due to Voevodsky, is to show that there is always a Rost variety for $\underline{a}$ which has a Rost motive. We give the proof of this in chapter 5, using the simplicial scheme $\mathfrak{X}$ which is defined in 1.32. The input to the proof is a cohomology class $\mu \in H^{2b+1,b}(\mathfrak{X}, \mathbb{Z})$; μ will be constructed in chapter 3, starting from $\underline{a}$; see Corollary 3.16. The class μ is used to construct a motivic cohomology operation ϕ and chapter 6 is devoted to showing that ϕ coincides with the operation βP^b $(b = (\ell^{n-1} - 1)/(\ell - 1))$; see Theorem 6.34. The proof requires facts about motivic cohomology operations which are developed in part III. □

The quick proof

Assuming Theorems 1.8, 1.10 and 1.11, we can now prove Theorems A and B of the introduction. This argument originally appeared on p. 97 of [Voe03a].

Theorem 1.12. *If H90(n − 1) holds, then H90(n) holds. By Theorem 1.8, this implies that for every field k containing* $1/\ell$:

a) the norm residue $K_n^M(k)/\ell \to H^n_{\text{ét}}(k, \mu_\ell^{\otimes n})$ *is an isomorphism;*
b) for every smooth X over k and all $p \le n$, *the motivic-to-étale map* $H^p(X, \mathbb{Z}/\ell(n)) \to H^p_{\text{ét}}(X, \mu_\ell^{\otimes n})$ *is an isomorphism.*

Since H90(1) holds, it follows by induction on n that H90(n) holds for every n. Note that Theorem A is 1.12(a) and Theorem B is 1.12(b).

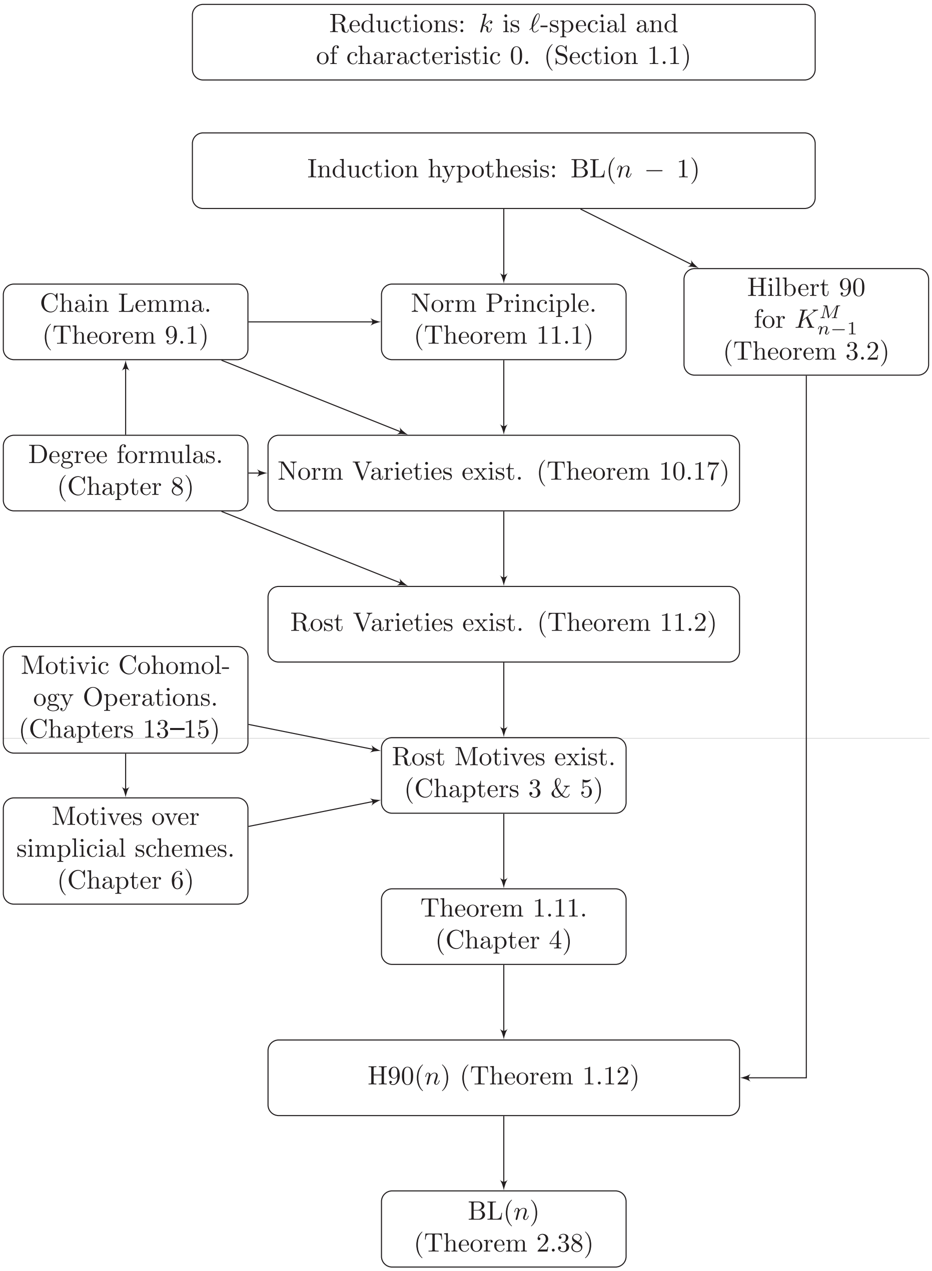

Figure 1.1: Dependency chart of Main Theorem 1.11

Proof of Theorem 1.12. Fix k, and an algebraically closed overfield Ω of infinite transcendence degree $>|k|$ over k. We first use transfinite recursion to produce an ℓ-special field k' $(k\subset k'\subset\Omega)$ such that $K_n^M(k)/\ell\to K_n^M(k')/\ell$ is zero and $H^{n+1}_{\text{ét}}(k,\mathbb{Z}_{(\ell)}(n))$ embeds into $H^{n+1}_{\text{ét}}(k',\mathbb{Z}_{(\ell)}(n))$.

Well-order the symbols in $K_n^M(k)$: $\{\underline{a}_\lambda\}_{\lambda<\kappa}$. Fix $\lambda<\kappa$; inductively, there is an intermediate field k_λ such that $\underline{a}_\mu$ vanishes in $K_n^M(k_\lambda)/\ell$ for all $\mu<\lambda$ and $H^{n+1}_{\text{ét}}(k,\mathbb{Z}_{(\ell)}(n))$ embeds into $H^{n+1}_{\text{ét}}(k_\lambda,\mathbb{Z}_{(\ell)}(n))$. If $\underline{a}_\lambda$ vanishes in $K_n^M(k_\lambda)/\ell$, set $k_{\lambda+1}=k_\lambda$. Otherwise, Theorem 1.11 states that there is a variety X_λ over k_λ whose function field $K=k_\lambda(X_\lambda)$ splits a_λ, and such that $H^{n+1}_{\text{ét}}(k_\lambda,\mathbb{Z}_{(\ell)}(n))$ embeds into $H^{n+1}_{\text{ét}}(K,\mathbb{Z}_{(\ell)}(n))$; set $k_{\lambda+1}=K$. If λ is a limit ordinal, set $k_\lambda=\cup_{\mu<\lambda}k_\mu$. Finally, let k' be a maximal prime-to-ℓ algebraic extension of k_κ. Then $H^{n+1}_{\text{ét}}(k,\mathbb{Z}_{(\ell)}(n))$ embeds into $H^{n+1}_{\text{ét}}(k_\kappa,\mathbb{Z}_{(\ell)}(n))$, which embeds in $H^{n+1}_{\text{ét}}(k',\mathbb{Z}_{(\ell)}(n))$ by the usual transfer argument 1.2. By construction, k' splits every symbol in $K_n^M(k)$.

Iterating this construction, we obtain an ascending sequence of field extensions $k^{(m)}$; let L denote the union of the $k^{(m)}$. Then L is ℓ-special and $K_n^M(L)/\ell=0$ by construction, so $H^{n+1}_{\text{ét}}(L,\mathbb{Z}_{(\ell)}(n))=0$ by Theorem 1.10. Since $H^{n+1}_{\text{ét}}(k,\mathbb{Z}_{(\ell)}(n))$ embeds into $H^{n+1}_{\text{ét}}(L,\mathbb{Z}_{(\ell)}(n))$, we have $H^{n+1}_{\text{ét}}(k,\mathbb{Z}_{(\ell)}(n))=0$. Since this holds for any k, H90(n) holds. ☐

In the remainder of this chapter, we introduce the ideas and basic tools we will use in the rest of the book.

1.3 NORM VARIETIES AND ROST VARIETIES

In this section we give the definition of norm varieties and Rost varieties; see Definitions 1.13 and 1.24. These varieties are the focus of the main theorem 1.11, and will be shown to exist in chapters 10 and 11 in part II.

We begin with the notions of a splitting variety and a norm variety for a symbol $\underline{a}\in K_n^M(k)/\ell$. Norm varieties will be the focus of chapter 10.

Definition 1.13. Let $\underline{a}$ be a symbol in $K_n^M(k)/\ell$. A field F over k is said to *split* $\underline{a}$, and be a *splitting field* for $\underline{a}$, if $\underline{a}=0$ in $K_n^M(F)/\ell$. A variety X over k is called a *splitting variety* for $\underline{a}$ if its function field splits $\underline{a}$ (i.e., if $\underline{a}$ vanishes in $K_n^M(k(X))/\ell$).

A splitting variety X is called an *ℓ-generic* splitting variety if any splitting field F has a finite extension E of degree prime to ℓ with $X(E)\neq\emptyset$.

A *norm variety* for a nonzero symbol $\underline{a}$ in $K_n^M(k)/\ell$ is a smooth projective ℓ-generic splitting variety of dimension $\ell^{n-1}-1$.

We will show in Theorem 10.17 that norm varieties always exist for all n when $\text{char}(k)=0$. When $n=1$, the 0-dimensional variety $X=\text{Spec}\ k(\sqrt[\ell]{a})$ is a norm variety for a because $K_n^M(k)/\ell=k^\times/k^{\times\ell}$. When $n=2$, Severi–Brauer varieties are norm varieties by Proposition 1.25.

Remark 1.13.1. (Specialization) Let Y be a reduced subscheme of X, not contained in the singular locus of X. If X is a splitting variety for $\underline{a}$ then so is Y. When X is a smooth splitting variety, such as a norm variety for $\underline{a}$, this implies that $\underline{a}$ is split by every field E with $X(E)\neq\emptyset$.

To see this, pick a closed nonsingular point x lying on Y. By specialization [Wei13, III.7.3], there is a map $K_n^M(k(X))\to K_n^M(k(Y))$ sending the class of $\underline{a}$ on $k(X)$ to the class of $\underline{a}$ on $k(Y)$.

Severi–Brauer varieties

Recall that the set of minimal left ideals of the matrix algebra $M_\ell(k)$ correspond to the k-points of the projective space $\mathbb{P}_k^{\ell-1}$; if I is a minimal left ideal corresponding to a line L of k^ℓ then the rows of matrices in I all lie on L.

Now fix a symbol $\underline{a}=\{a_1,a_2\}$ and a primitive ℓ^{th} root of unity in k, ζ. Let $A=A(\underline{a})$ denote the central simple algebra $k\{x,y\}/(x^\ell=a_1, y^\ell=a_2, xy=\zeta yx)$. It is well known that there is a smooth projective variety X of dimension $\ell-1$, defined over k, such that for every field F over k, $X(F)$ is the set of (nonzero) minimal ideals of $A\otimes_k F$: $X(F)\neq\emptyset$ if and only if $A\otimes_k F\cong M_\ell(F)$. The variety X is called the *Severi–Brauer* variety of A.

Here is one way to construct the Severi–Brauer variety X. If $E=k(\sqrt[\ell]{a_1})$ then $A\otimes_k E\cong M_\ell(E)$; the Galois group of E/k acts on the set of minimal ideals of $A\otimes_k E$ and hence on $\mathbb{P}_E^{\ell-1}$ and $X\times_k E$ is $\mathbb{P}_E^{\ell-1}$ with this Galois action. Now apply Galois descent. This method originated in [Ser63]; see [KMRT98].

Definition 1.14. If k contains a primitive ℓ^{th} root of unity, ζ, the *Severi–Brauer variety* X associated to a symbol $\underline{a}=\{a_1,a_2\}$ is defined to be the Severi–Brauer variety of $A=A(\underline{a})$. (The variety is independent of the choice of ζ.) If k does not contain a primitive ℓ^{th} root of unity, we will mean the Severi–Brauer variety for $\{a_1,a_2\}$ defined over $k(\zeta)$.

If $\zeta\in F$, there is a canonical map $K_2^M(F)/\ell \longrightarrow {}_\ell\mathrm{Br}(k)$, sending $\{a_1,a_2\}$ to its associated central simple algebra A. The Merkurjev–Suslin Theorem [MeS82] states that this is an isomorphism. Since $A\otimes_k k(X)$ is a matrix algebra by construction, the Merkurjev–Suslin Theorem implies that $k(X)$ splits $\underline{a}$. Here is a more elementary proof.

Lemma 1.15. *Every symbol $\underline{a}=\{a_1,a_2\}$ is split by its Severi–Brauer variety.*

Proof. (Merkurjev) Fix $\alpha=\sqrt[\ell]{a_1}$ and set $E=k(\alpha)$. Recall from [Wei82] (or 11.12) that the *Weil restrictions* of $\mathbb{A}^1$ along E and k are isomorphic to the affine spaces $\mathbb{A}^\ell$ and $\mathbb{A}^1$ over k, and the Weil restriction of the norm map $N_{E/k}$ is a map $N:\mathbb{A}^\ell\to\mathbb{A}^1$. Then the Severi–Brauer variety X is birationally equivalent to the subvariety of $\mathbb{A}^\ell$ defined by $N(X_0,\dots,X_{\ell-1})=a_2$.

In the function field $k(X)$, we set $x_i=X_i/X_0$ and $c=N(1,x_1,\dots,x_{\ell-1})$, so that $cX_0^\ell=a_2$. Then $k(X)=k(x_1,\dots,x_{\ell-1})(\beta)$, $\beta^\ell=a_2/c$. By construction, the element $y = 1 + \sum x_i \alpha^i$ of $E(X) = E(x_1, \dots)$ has $Ny = c = a_2/\beta^\ell$ so in $K_2^M(k(X))/\ell$ we have

$$\{a_1, a_2\} = \{a_1, a_2/\beta^\ell\} = \{a_1, Ny\} = N\{\alpha^\ell, y\} = N(0) = 0.$$

Thus the field $k(X)$ splits the symbol $\underline{a}$. □

Corollary 1.16. *The Severi–Brauer variety X of a symbol $\underline{a} = \{a_1, a_2\}$ is a norm variety for $\underline{a}$.*

Proof. Since any norm variety for $k(\zeta)$ is also a norm variety for k, and a field F splits $\underline{a}$ iff $F(\zeta)$ splits $\underline{a}$, we may assume that k contains a primitive ℓ^{th} root of unity. Thus X exists and is a smooth projective variety of dimension $\ell - 1$. By Lemma 1.15, $k(X)$ splits the symbol. Finally, suppose that a field F/k splits $\underline{a}$. Then the associated central simple algebra is trivial ($A \otimes_k F \cong M_\ell(F)$) and hence $X(F) \neq \emptyset$. □

The characteristic number $\mathbf{s_{d(X)}}$

The definition of a Rost variety also involves the notion of a ν_i-variety, which is defined using the classical characteristic number $s_d(X)$.

Let X be a smooth projective variety of dimension $d > 0$. Recall from [MS74, §16] that there is a characteristic class $s_d : K_0(X) \to CH^d(X)$ corresponding to the symmetric polynomial $\sum t_j^d$ in the Chern roots t_j of a bundle; the characteristic number is the degree of the characteristic class. We shall write $s_d(X)$ for the characteristic number of the tangent bundle T_X, i.e., $s_d(X) = \deg(s_d(T_X))$. When $d = \ell^\nu - 1$, we know that $s_d(X) \equiv 0 \pmod{\ell}$; see [MS74, 16.6 and 16-E] and [Sto68, pp. 128–29] or [Ada74, II.7].

Definition 1.17. A *ν_i-variety* over a field k is a smooth projective variety X of dimension $d = \ell^i - 1$, with $s_d(X) \not\equiv 0 \pmod{\ell^2}$.

Remark. In topology, a smooth complex variety X of dimension $d = \ell^i - 1$ for which $s_d(X) \equiv \pm\ell \pmod{\ell^2}$ is called a *Milnor manifold.* In complex cobordism theory, the bordism classes of Milnor manifolds in MU_d are among the generators of the complex cobordism ring MU_* of stably complex manifolds.

Examples 1.18. (1) It is well known that $s_d(\mathbb{P}^d) = d + 1$; see [MS74, 16.6]. Setting $d = \ell - 1$, we see that $\mathbb{P}^{\ell-1}$ (and any form of it) is a ν_1-variety. In particular, the Severi–Brauer variety of a symbol $\{a_1, a_2\}$ is a ν_1-variety, since it is a form of $\mathbb{P}^{\ell-1}$.

(2) A smooth hypersurface X of degree ℓ in $\mathbb{P}^{d+1}$ has $s_d(X) = \ell(d + 2 - \ell^d)$ by [MS74, 16-D], so if $d = \ell^i - 1$ we see that X is a ν_i-variety and $X(\mathbb{C})$ is a Milnor manifold.

(3) We will see in Proposition 10.14 that if $\mathrm{char}(k) = 0$, any norm variety for a symbol $\{a_1, \dots, a_n\}$ $(n \geq 2)$ is a ν_{n-1}-variety.

Borel–Moore homology

The Borel–Moore homology group $H^{BM}_{-1,-1}(X)$ of a scheme X is defined as $\mathrm{Hom}_{\mathbf{DM}}(\mathbb{Z}, M^c(X)(1)[1])$ if $\mathrm{char}(k)=0$ (resp., $\mathrm{Hom}_{\mathbf{DM}}(\mathbb{Z}[1/p], \mathbb{Z}[1/p]\otimes M^c(X)(1)[1])$ if $\mathrm{char}(k)=p>0$ and k is perfect); see [MVW, 16.20]. Here $M^c(X)$ is the motive of X with compact supports. $H^{BM}_{-1,-1}(X)$ is a covariant functor in X for proper maps, and contravariant for finite flat maps, because $M^c(X)$ has these properties; see [MVW, 16.13]. When X is projective, the natural map from $M(X)=\mathbb{Z}_{\mathrm{tr}}(X)$ to $M^c(X)$ is an isomorphism in **DM**, so the Borel–Moore homology group agrees with the usual motivic homology group $H_{-1,-1}(X,R)$, which is defined as $\mathrm{Hom}_{\mathbf{DM}}(R, R_{\mathrm{tr}}(X)(1)[1])$, where R is $\mathbb{Z}$ (resp., $\mathbb{Z}[1/p]$); see [MVW, 14.17].

Proposition 1.19. *Let X be a smooth variety over a perfect field k. Then $H^{BM}_{-1,-1}(X)$ is the group generated by symbols $[x,\alpha]$, where x is a closed point of X and $\alpha\in k(x)^\times$, modulo the relations*

(i) $[x,\alpha][x,\alpha']=[x,\alpha\alpha']$ and
(ii) for every point y of X such that $\dim(\overline{\{y\}})=1$, the image of the tame symbol $K_2(k(y))\to\oplus k(x)^\times$ is zero.

That is, we have an exact sequence

$$\bigoplus_y K^M_2(k(y)) \xrightarrow{\text{tame}} \bigoplus_x k(x)^\times \xrightarrow{\bigoplus[x,-]} H^{BM}_{-1,-1}(X)\to 0.$$

Proof. Let A denote the abelian group presented in the Proposition, and set $d=\dim(X)$. Note that A is uniquely p-divisible when k is a perfect field of characteristic $p>0$, because each $k(x)^\times$ is uniquely p-divisible, and the group $K^M_2(k(y))$ is also uniquely p-divisible by Lemma 1.20.

We first show that A is isomorphic to $H^{2d+1,d+1}(X,\mathbb{Z})$. To this end, consider the hypercohomology spectral sequence $E_2^{p,q}=H^p(X,\mathcal{H}^q)\Rightarrow H^{p+q,d+1}(X,\mathbb{Z})$, where $\mathcal{H}^q$ denotes the Zariski sheaf associated to the presheaf $H^{q,d+1}(-,\mathbb{Z})$. Since $H^{q,d+1}=0$ for $q>d+1$, the terms $E_2^{p,q}$ are zero unless $p\le d$ and $q\le d+1$. From this we deduce that $H^{2d+1,d+1}(X,\mathbb{Z})\cong H^d(X,\mathcal{H}^{d+1})$.

For each n, $\mathcal{H}^n$ is a homotopy invariant Zariski sheaf, by [MVW, 24.1]. Moreover, it has a canonical flasque "Gersten" resolution on each smooth X, given in [MVW, 24.11], whose c^{th} term is the coproduct of the skyscraper sheaves $H^{n-c,d+1-c}(k(z))$ for which z has codimension c in X. Taking $n=d+1$, and recalling that $K^M_n\cong H^{n,n}$ on fields, we see that the skyscraper sheaves in the $(d-1)^{st}$ and d^{th} terms take values in $K^M_2(k(y))$ and $K^M_1(k(x))$. Moreover, by [Wei13, V.9.2 and V(6.6.1)], the map $K^M_2(k(y))\to K^M_1(k(x))$ is the tame symbol if $x\in\overline{\{y\}}$, and zero otherwise. As $H^d(X,\mathcal{H}^{d+1})$ is obtained by taking global sections of the Gersten resolution and then cohomology, we see that it is isomorphic to A.

Now suppose that $\mathrm{char}(k)=0$. Using motivic duality with $d=\dim(X)$ (see [MVW, 16.24] or [FV00, 7.1]), the proof is finished by the duality calculation:

$$\begin{aligned} H^{BM}_{-1,-1}(X,\mathbb{Z}) &= \mathrm{Hom}(\mathbb{Z}, M^c(X)(1)[1]) \\ &= \mathrm{Hom}(\mathbb{Z}(d)[2d], M^c(X)(d+1)[2d+1]) \qquad (1.19a) \\ &= \mathrm{Hom}_{\mathbf{DM}}(M(X), \mathbb{Z}(d+1)[2d+1]) = H^{2d+1,d+1}(X). \end{aligned}$$

Now suppose that $\mathrm{char}(k)>0$. Since $H^{2d+1,d+1}(X,\mathbb{Z})\cong A$ is uniquely divisible, the duality calculation (1.19a) goes through with $\mathbb{Z}$ replaced by $\mathbb{Z}[1/p]$, using the characteristic p version of motivic duality (see [Kel13, 5.5.14]). □

Lemma 1.20. *(Bloch–Kato–Gabber) If F is a field of transcendence degree 1 over a perfect field k of characteristic p, $K_2^M(F)$ is uniquely p-divisible.*

Proof. For any field F of characteristic p, the group $K_2(F)$ has no p-torsion (see [Wei13, III.6.7]), and the $d\log$ map $K_2(F)/p\to\Omega^2_F$ is an injection with image $\nu(2)$; see [Wei13, III.7.7.2]. Since k is perfect, $\Omega^1_k=0$ and Ω^1_F is 1–dimensional, so $\Omega^2_F=0$ and hence $K_2(F)/p=0$. □

The motivic homology functor $H^{BM}_{-1,-1}(X)$ has other names in the literature. It is isomorphic to the K-cohomology groups $H^d(X,\mathcal{K}_{d+1})$ [Qui73] and $H^d(X,\mathcal{K}^M_{d+1})$, where $d=\dim(X)$, and to Rost's Chow group with coefficients $A_0(X,\mathcal{K}_1)$ [Ros96]. Since we will only be concerned with smooth projective varieties X and integral coefficients, we will omit the superscript "BM" and the coefficients and just write $H_{-1,-1}(X)$.

Examples 1.21. (i) $H_{-1,-1}(\mathrm{Spec}\,E)=E^\times$ for every field E over k. This is immediate from the presentation in 1.19.
(ii) If E is a finite extension of k, the proper pushforward from $E^\times=H_{-1,-1}(\mathrm{Spec}\,E)$ to $k^\times=H_{-1,-1}(\mathrm{Spec}\,k)$ is just the norm map $N_{E/k}$.
(iii) For any proper variety X over k, the pushforward map

$$N_{X/k}: H_{-1,-1}(X)\to H_{-1,-1}(\mathrm{Spec}\,k)=k^\times$$

is induced by the composites $\mathrm{Spec}\,k(x)\to X\to\mathrm{Spec}\,k$, $x\in X$. By (ii), we see that $N_{X/k}$ sends $[x,\alpha]$ to the norm $N_{k(x)/k}(\alpha)$.

Definition 1.22. When X is proper, the projections $X\times X\to X$ are proper and we may define the reduced group $\overline{H}_{-1,-1}(X)$ to be the coequalizer of $H_{-1,-1}(X\times X)\rightrightarrows H_{-1,-1}(X)$, i.e., the quotient of $H_{-1,-1}(X)$ by the difference of the two projections.

Example 1.23. When $E=k(\sqrt[\ell]{a})$ is a cyclic field extension of k, with Galois group generated by σ, then $\overline{H}_{-1,-1}(\mathrm{Spec}\,E)$ is the cokernel of $E^\times\xrightarrow{1-\sigma}E^\times$, and Hilbert's Theorem 90 induces an exact sequence

$$0\to\overline{H}_{-1,-1}(\mathrm{Spec}\,E)\xrightarrow{N_{E/k}}k^\times\xrightarrow{a\cup}\mathrm{Br}(E/k)\to 0.$$

Note that $\mathrm{Br}(E/k)$ is a subgroup of $K_2^M(k)/\ell$ when $\mu_\ell \subset k^\times$. We will generalize this in Proposition 7.7, using $K_{n+1}^M(k)/\ell$.

Rost varieties

Definition 1.24. A *Rost variety* for a sequence $\underline{a}=(a_1,\dots,a_n)$ of units of k is a ν_{n-1}-variety X satisfying:

(a) X is a splitting variety for $\underline{a}$, i.e., $\underline{a}$ vanishes in $K_n^M(k(X))/\ell$;
(b) for each integer i, $1\le i<n$, there is a ν_i-variety mapping to X;
(c) the map $N:\overline{H}_{-1,-1}(X)\to k^\times$ is an injection.

When $n=1$, $\mathrm{Spec}(k(\sqrt[\ell]{a}))$ is a Rost variety for a. When $n=2$, Proposition 1.25 shows that Severi–Brauer varieties of dimension $\ell-1$ are Rost varieties. In chapter 11 we will show that Rost varieties exist over ℓ-special fields for all n, ℓ and $\underline{a}$, at least when $\mathrm{char}(k)=0$. More specifically, Theorem 11.2 shows that norm varieties for $\underline{a}$ are Rost varieties for $\underline{a}$.

Proposition 1.25. *The Severi–Brauer variety X of a symbol $\underline{a}=\{a_1,a_2\}$ is a Rost variety for $\underline{a}$.*

Proof. By Lemma 1.15, X splits $\underline{a}$; by Example 1.18(1), X is a ν_1-variety. Finally, Quillen proved that $H_{-1,-1}(X)=H^1(X,\mathcal{K}_2)$ is isomorphic to $K_1(A)$, and it is classical that $K_1(A)$ is the image of $A^\times\to k^\times$; see [Wan50, p. 327]. □

1.4 THE BEILINSON–LICHTENBAUM CONDITIONS

Our approach to Theorems A and B (for n) will use their equivalence with a more general condition, which we call the *Beilinson–Lichtenbaum* condition BL(n). In this section, we define BL(n) (in 1.28); in section 2.1 we show that it implies the corresponding condition BL(p) for all $p<n$.

Consider the morphism of sites $\pi:(\mathbf{Sm}/k)_{\text{ét}}\to(\mathbf{Sm}/k)_{\text{zar}}$, where π_* is restriction and π^* sends a Zariski sheaf $\mathcal{F}$ to its associated étale sheaf $\mathcal{F}_{\text{ét}}$. The total direct image $\mathbf{R}\pi_*$ sends an étale sheaf (or complex of sheaves) $\mathcal{F}$ to a Zariski complex such that $H^*_{\text{zar}}(X,\mathbf{R}\pi_*\mathcal{F})=H^*_{\text{ét}}(X,\mathcal{F})$. In particular, the Zariski cohomology of $\mathbf{R}\pi_*\mu_\ell^{\otimes n}$ agrees with the étale cohomology of $\mu_\ell^{\otimes n}$.

Recall [Wei94, 1.2.7] that the good truncation $\tau^{\le n}\mathcal{C}$ of a cochain complex $\mathcal{C}$ is the universal subcomplex which has the same cohomology as $\mathcal{C}$ in degrees $\le n$ but is acyclic in higher degrees. Applying this to $\mathbf{R}\pi_*\mathcal{F}$ leads to the following useful complexes.

Definition 1.26. The cochain complexes of Zariski sheaves $L(n)$ and $L/\ell^\nu(n)$ are defined to be

$$L(n)=\tau^{\le n}\mathbf{R}\pi_*[\mathbb{Z}_{(\ell)}(n)] \quad\text{and}\quad L/\ell^\nu(n)=\tau^{\le n}\mathbf{R}\pi_*[\mathbb{Z}/\ell^\nu(n)].$$

We know by [MVW, 10.3] that for each n (and all ν) there is a quasi-isomorphism of complexes of étale sheaves $\mu_{\ell^\nu}^{\otimes n} \xrightarrow{\sim} \mathbb{Z}/\ell^\nu(n)$. When X is a Zariski local scheme this implies that $H^n(X, L(n))$ is: $H^n_{\text{ét}}(X, \mathbb{Z}_{(\ell)}(n))$ for $p \le n$ and zero for $p > n$; while $H^n(X, L/\ell^\nu(n))$ is: $H^p_{\text{ét}}(X, \mu_{\ell^\nu}^{\otimes n})$ for $p \le n$ and zero for $p > n$.

Now $\mathbb{Z}_{(\ell)}(n)$ and the $\mathbb{Z}/\ell^\nu(n)$ are étale sheaves with transfers, so their canonical flasque resolutions $E^\bullet$ are complexes of étale sheaves with transfers by [MVW, 6.20]. The restriction $\pi_* E^\bullet$ to the Zariski site inherits the transfer structure, so the truncations $L(n)$ and $L/\ell^\nu(n)$ are complexes of Zariski sheaves with transfers.

The adjunction $1 \to \mathbf{R}\pi_*\pi^*$ gives a natural map of Zariski complexes $\mathbb{Z}/\ell^\nu(n) \to \mathbf{R}\pi_*[\mathbb{Z}/\ell^\nu(n)]$. Since the complexes $\mathbb{Z}(n)$ and $\mathbb{Z}/\ell^\nu(n)$ are zero above degree n by construction ([MVW, 3.1]), we may apply $\tau^{\le n}$ to obtain morphisms of sheaves on $\mathbf{Sm}/k$:

$$\mathbb{Z}_{(\ell)}(n) \xrightarrow{\tilde{\alpha}_n} L(n), \quad \mathbb{Z}/\ell^\nu(n) \xrightarrow{\alpha_n} L/\ell^\nu(n). \tag{1.27}$$

Definition 1.28. We will say that *BL(n) holds* if the map $\mathbb{Z}/\ell(n) \xrightarrow{\alpha_n} L/\ell(n)$ is a quasi-isomorphism for any field k containing $1/\ell$. This is equivalent to the seemingly stronger but analogous assertion with coefficients $\mathbb{Z}/\ell^\nu$; see 1.29(a).

Beilinson and Lichtenbaum had conjectured that BL(n) holds for all n, whence the name; see [Lic84, §3] and [Beĭ87, 5.10.D].

Lemma 1.29. *If BL(n) holds then:*

(a) $\alpha_n : \mathbb{Z}/\ell^\nu(n) \xrightarrow{\sim} \tau^{\le n}\mathbf{R}\pi_*\mu_{\ell^\nu}^{\otimes n}$ *is a quasi-isomorphism for all* $\nu \ge 1$*;*
(b) $\mathbb{Q}/\mathbb{Z}_{(\ell)}(n) \xrightarrow{\sim} \tau^{\le n}\mathbf{R}\pi_*[\mathbb{Q}/\mathbb{Z}_{(\ell)}(n)]$ *is a quasi-isomorphism;*
(c) $\tilde{\alpha}_n : \mathbb{Z}_{(\ell)}(n) \xrightarrow{\sim} L(n) = \tau^{\le n}\mathbf{R}\pi_*[\mathbb{Z}_{(\ell)}(n)]$ *is also a quasi-isomorphism;*
(d) $K_n^M(k)_{(\ell)} \to H^n_{\text{ét}}(k, \mathbb{Z}_{(\ell)}(n))$ *is an isomorphism for all* k *containing* $1/\ell$*.*

Proof. The statement for $\mathbb{Z}/\ell^\nu$ coefficients follows by induction on ν using the morphism of distinguished triangles:

$$\begin{array}{ccccccccc} \mathbb{Z}/\ell(n)[-1] & \to & \mathbb{Z}/\ell^{\nu-1}(n) & \to & \mathbb{Z}/\ell^\nu(n) & \to & \mathbb{Z}/\ell(n) & \to & \mathbb{Z}/\ell^{\nu-1}(n)[1] \\ \downarrow\cong & & \downarrow\cong & & \alpha_n\downarrow & & \downarrow\cong & & \downarrow\cong \\ L/\ell(n)[-1] & \to & L/\ell^{\nu-1}(n) & \to & L/\ell^\nu(n) & \to & L/\ell(n) & \to & L/\ell^{\nu-1}(n)[1]. \end{array}$$

Taking the direct limit over ν in part (a) yields part (b).

Since $\tilde{\alpha}_n$ is also an isomorphism for $\mathbb{Q}$ coefficients by [MVW, 14.23], the coefficient sequence for $0 \to \mathbb{Z}_{(\ell)}(n) \to \mathbb{Q}(n) \to \mathbb{Q}/\mathbb{Z}_{(\ell)}(n) \to 0$ shows that $\mathbb{Z}_{(\ell)}(n) \xrightarrow{\sim} L(n)$ is also a quasi-isomorphism. Part (d) is immediate from (c) and $K_n^M(k)_{(\ell)} \cong H^n_{\text{zar}}(k, \mathbb{Z}(n))_{(\ell)} = H^n_{\text{zar}}(k, \mathbb{Z}_{(\ell)}(n))$. □

The main result in chapter 2 is that BL(n) is equivalent to H90(n) and hence Theorem A, that $K_n^M(k)/\ell \cong H^n_{\text{ét}}(k,\mu_\ell^{\otimes n})$. The fact that H90(n) implies BL(n) is proven in Theorem 2.38. Here is the easier converse, that BL(n) implies H90(n).

Lemma 1.30. *If BL(n) holds then H90(n) holds.*
In addition, if BL(n) holds then for any field k containing $1/\ell$:

(a) $K_n^M(k)/\ell \cong H^n(k,\mathbb{Z}/\ell(n)) \cong H^n_{\text{ét}}(k,\mu_\ell^{\otimes n})$.
(b) For all $p \le n$, $H^p(k,\mathbb{Z}/\ell(n)) \cong H^p_{\text{ét}}(k,\mu_\ell^{\otimes n})$.

Proof. Applying $H^p(k,-)$ to α_n yields (b). Setting $p=n$ in (b) proves (a), because $K_n^M(k)/\ell \cong H^n(k,\mathbb{Z}/\ell(n))$. By Theorem 1.7, (a) implies H90(n). □

Corollary 1.31.[2] *If BL(n) holds then for every smooth simplicial scheme $X_\bullet$ the map $H^{p,n}(X_\bullet,\mathbb{Z}/\ell) \to H^p_{\text{ét}}(X_\bullet,\mu_\ell^{\otimes n})$ is an isomorphism for all $p \le n$. It is an injection when $p=n+1$.*

Proof. First, suppose that X is a smooth scheme. A comparison of the hypercohomology spectral sequences $H^p(X,\mathcal{H}^q) \Rightarrow H^{p+q}(X)$ for coefficient complexes $L/\ell(n)$ and $R\pi_*[\mathbb{Z}/\ell(n)]$ shows that $\alpha_n : H^{p,n}(X,\mathbb{Z}/\ell) \to H^p_{\text{ét}}(X,\mu_\ell^{\otimes n})$ is an isomorphism for $p \le n$ and an injection for $p=n+1$.

For $X_\bullet$, the assertion follows from a comparison of the spectral sequences $E_2^{p,q} = H^q(X_p) \Rightarrow H^{p+q}(X_\bullet)$ for the Zariski and étale topologies, and the result for each smooth scheme X_p. □

1.5 SIMPLICIAL SCHEMES

In this section, we construct a certain simplicial scheme $\mathfrak{X}$ which will play a crucial role in our constructions, and introduce some features of its cohomology.

It is well known that the hypercohomology of a simplicial scheme $X_\bullet$ agrees with the group of morphisms in the derived category of sheaves of abelian groups, from the representable simplicial sheaf $\mathbb{Z}[X_\bullet]$ (regarded as a complex of sheaves via the Dold–Kan correspondence) to the coefficient sheaf complex. Applying this to the coefficient complex $A(q)$, we obtain the original definition of the motivic cohomology of $X_\bullet$: $H^{p,q}(X_\bullet,A) = H^p_{\text{zar}}(X_\bullet,A(q))$; see [MVW, 3.4].

For our purposes, it is more useful to work in the triangulated category **DM**, which is a quotient of the derived category of Nisnevich sheaves with transfers, or its triangulated subcategory $\mathbf{DM}^{\text{eff}}_{\text{nis}}$, where we have

$$H^{p,q}(X_\bullet,A) \cong \operatorname{Hom}_{\mathbf{DM}^{\text{eff}}_{\text{nis}}}(\mathbb{Z}_{\text{tr}}(X_\bullet),A(q)[p]) = \operatorname{Hom}_{\mathbf{DM}}(\mathbb{Z}_{\text{tr}}(X_\bullet),A(q)[p]).$$

2. Taken from [Voe03a, 6.9]. It is needed for Lemma 3.13.

See [MVW, 14.17]. Similarly, the étale motivic cohomology $H^*_{\text{ét}}(X_\bullet, A(q))$ is the étale hypercohomology of the étale sheaf $A(q)_{\text{ét}}$ underlying $A(q)$, and agrees with $\text{Hom}_{\mathbf{DM}^-_{\text{ét}}}(\mathbb{Z}_{\text{tr}}(X_\bullet), A(q)[p])$; see [MVW, 10.1, 10.7].

We begin with a simplicial set construction. Associated to any nonempty set S there is a contractible simplicial set $\check{C}(S): n \mapsto S^{n+1}$; the face maps are projections (omit a term) and the degeneracy maps are diagonal maps (duplicate a term). In fact, $\check{C}(S)$ is the 0-coskeleton of S; see Lemma 12.6. More generally, for any set T, the projection $T \times \check{C}(S) \to T$ is a homotopy equivalence; it is known as the canonical cotriple resolution of T associated to the cotriple $\perp(T) = T \times S$; see [Wei94, 8.6.8].

Definition 1.32. Let X be a (nonempty) smooth scheme over k. We write $\mathfrak{X} = \check{C}(X)$ for the simplicial scheme $\mathfrak{X}_n = X^{n+1}$, whose face maps are given by projection:

$$X \leftleftarrows X \times X \,\overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}}\, X^3 \,\overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}}\, X^4 \;\cdots\;.$$

That is, $\mathfrak{X}$ is the 0-coskeleton of X.

We may regard $\mathfrak{X}$ and $\mathfrak{X} \times Y$ as simplicial representable presheaves on $\mathbf{Sm}/k$; for any smooth U, $\mathfrak{X}(U) = \check{C}(X(U))$. Thus if $X(Y) = \text{Hom}(Y, X) \neq \emptyset$ then the projection $(\mathfrak{X} \times Y)(U) \to Y(U)$ is a homotopy equivalence for all U by the cotriple remarks above. In particular, $\mathfrak{X}(k)$ is either contractible or $\emptyset$, according to whether or not X has a k-rational point.

Remark 1.32.1. A map of simplicial presheaves is called a *global weak equivalence* if its evaluation on each U is a weak equivalence of simplicial sets. It follows that $\mathfrak{X} \to \text{Spec}(k)$ is a global weak equivalence if and only if X has a k-rational point, and more generally that the projection $\mathfrak{X} \times Y \to Y$ is a global weak equivalence if and only if $\text{Hom}(Y, X) \neq \emptyset$.

We will frequently use the following standard fact. We let R denote $\mathbb{Z}$ if $\text{char}(k) = 0$, and $\mathbb{Z}[1/\text{char}(k)]$ if k is a perfect field of positive characteristic.

Lemma 1.33. *For all smooth Y and $p>q$,* $\text{Hom}_{\mathbf{DM}}(R, R_{\text{tr}}(Y)(q)[p]) = 0$.

Proof. By definition [MVW, 3.1], $R_{\text{tr}}(Y)(q)[q]$ is a chain complex $C_*(Y \times \mathbb{G}_m^{\wedge q})$ of sheaves which is zero in positive cohomological degrees. By [MVW, 14.16],

$$\text{Hom}(R, R_{\text{tr}}(Y)(q)[p]) \cong H^{p-q}_{\text{zar}}(k, R_{\text{tr}}(Y)(q)[q]) = H^{p-q} R_{\text{tr}}(Y)(q)[q](k). \qquad \square$$

Lemma 1.34. *For every smooth X, $H_{-1,-1}(\mathfrak{X}) \cong \overline{H}_{-1,-1}(X)$.*

Proof. For all p and $n > 1$, Lemma 1.33 yields $\text{Hom}_{\mathbf{DM}}(R, R_{\text{tr}} X^p(1)[n]) = 0$. Therefore every row below $q = -1$ in the spectral sequence

$$E^1_{pq} = \text{Hom}(R[q], R_{\text{tr}} X^{p+1}(1)) \Rightarrow \text{Hom}(R, R_{\text{tr}}\mathfrak{X}(1)[p-q]) = H_{q-p,-1}(\mathfrak{X})$$

is zero. The homology at $(p,q)=(0,-1)$ yields the exact sequence

$$0 \longleftarrow H_{-1,-1}(\mathfrak{X}) \longleftarrow H_{-1,-1}(X) \longleftarrow H_{-1,-1}(X\times X).$$

Since $\overline{H}_{-1,-1}(X)$ is the cokernel of the right map, the result follows. ☐

Lemma 1.35.[3] *For every smooth X, $H^{0,0}(\mathfrak{X},R)=R$ and $H^{p,0}(\mathfrak{X},R)=0$ for $p>0$; $H^{0,1}(\mathfrak{X},\mathbb{Z})=H^{2,1}(\mathfrak{X},\mathbb{Z})=0$ and $H^{1,1}(\mathfrak{X};\mathbb{Z})\cong H^{1,1}(\operatorname{Spec}k;\mathbb{Z})\cong k^\times$.*

Proof. The spectral sequence $E_1^{p,q}=H^q(X^{p+1};R)\Rightarrow H^{p+q,0}(\mathfrak{X};R)$ degenerates at E_2 for X smooth, being zero for $q>0$, and the R-module cochain complex of the contractible simplicial set $\check{C}(\pi_0(X))$ for $q=0$.

The spectral sequence $E_1^{p,q}=H^q(X^{p+1};\mathbb{Z}(1))\Rightarrow H^{p+q,1}(\mathfrak{X};\mathbb{Z})$ degenerates at E_2, all rows vanishing except for $q=1$ and $q=2$, because $\mathbb{Z}(1)\cong\mathcal{O}^\times[-1]$; see [MVW, 4.2]. We compare this with the spectral sequence converging to $H^{p+q}_{\text{ét}}(\mathfrak{X};\mathbb{G}_m)$; $H^q_{\text{zar}}(Y,\mathcal{O}^\times)\to H^q_{\text{ét}}(Y,\mathbb{G}_m)$ is an isomorphism for $q=0,1$ (and an injection for $q=2$). Hence we have $H^{q,1}(\mathfrak{X})=H^{q,1}_{\text{ét}}(\mathfrak{X})$ for $q\le 2$, and $H^{q,1}_{\text{ét}}(\mathfrak{X})\cong H^{q,1}_{\text{ét}}(k)=H^{q-1}_{\text{ét}}(k,\mathbb{G}_m)$ by Lemma 1.37. ☐

Recall that if $f:X_\bullet\to Y_\bullet$ is a morphism of simplicial objects in any category with coproducts and a final object, the cone of f is also a simplicial object. It is defined in [Del74, 6.3.1].

Definition 1.36. The suspension $\Sigma X_\bullet$ of a simplicial scheme $X_\bullet$ is the cone of $(X_\bullet)_+\to\operatorname{Spec}(k)_+$. The reduced suspension $\widetilde{\Sigma}X_\bullet$ of any simplicial scheme $X_\bullet$ is the pointed pair $(\Sigma X_\bullet,\text{point})$, where "point" is the image of $\operatorname{Spec}(k)$ in $\Sigma X_\bullet$.

If $X_\bullet$ is pointed then $H^{p,q}(\widetilde{\Sigma}X_\bullet)=\widetilde{H}^{p,q}(\Sigma X_\bullet)$, but this makes little sense when $X_\bullet$ has no k-points. The pointed pair is chosen to avoid this problem. By construction there is a long exact sequence on cohomology:

$$\cdots\to H^{p-1,q}(X_\bullet)\to H^{p,q}(\widetilde{\Sigma}X_\bullet)\to H^{p,q}(\operatorname{Spec}k)\to H^{p,q}(X_\bullet)\to\cdots.$$

In particular, if $X_\bullet$ is pointed then we have the suspension isomorphism σ_s: $\widetilde{H}^{p-1,q}(X_\bullet)\to H^{p,q}(\widetilde{\Sigma}X_\bullet)$. If $p>q$ then $H^{p,q}(X_\bullet)\xrightarrow{\simeq}H^{p+1,q}(\widetilde{\Sigma}X_\bullet)$, because in this range $H^{p,q}(\operatorname{Spec}k)=0$.

Lemma 1.37.[4] *If X has a point x with $[k(x):k]=e$ then for each (p,q) the group $H^p(\widetilde{\Sigma}\mathfrak{X},\mathbb{Z}(q))$ has exponent e. Hence the kernel and cokernel of each $H^p(k,\mathbb{Z}(q))\to H^p(\mathfrak{X},\mathbb{Z}(q))$ has exponent e.*

The maps $H^{p,q}_{\text{ét}}(k,\mathbb{Z})\xrightarrow{\sim}H^{p,q}_{\text{ét}}(\mathfrak{X},\mathbb{Z})$ are isomorphisms for all (p,q). Therefore $H^{*,*}_{\text{ét}}(\widetilde{\Sigma}\mathfrak{X},\mathbb{Z})=0$ and $H^{*,*}_{\text{ét}}(\widetilde{\Sigma}\mathfrak{X},\mathbb{Z}/\ell)=0$.

---

3. $H^{0,0}(\mathfrak{X})$ and $H^{0,1}(\mathfrak{X})$ are used in 4.5 and 4.15.
4. Based on Lemmas 9.3 and 7.3 of [Voe03a], respectively.

Proof. Set $F(Y) = H^p(\widetilde{\Sigma}\mathfrak{X} \times Y, \mathbb{Z}(q))$; this is a presheaf with transfers which vanishes on $\mathrm{Spec}(k(x))$. As with any presheaf with transfers, the composition $F(k) \to F(k(x)) \to F(k)$ is multiplication by e. It follows that $e \cdot F(k) = 0$.

Now any nonempty X has a point x with $k(x)/k$ étale, and $\mathfrak{X}_x = \mathfrak{X} \times \mathrm{Spec}\, k(x)$ is an étale cover of $\mathfrak{X}$. Since the map from $\mathfrak{X}_x$ to the étale cover x of $\mathrm{Spec}(k)$ is a global weak equivalence, the second assertion follows from a comparison of the descent spectral sequences for the covers of $\mathfrak{X}$ and $\mathrm{Spec}\, k$. □

As in topology, the integral Bockstein $\tilde{\beta}: H^{p,q}(Y, \mathbb{Z}/\ell) \to H^{p+1,q}(Y, \mathbb{Z})$ is the boundary map in the cohomology sequence for the coefficient sequence $0 \to \mathbb{Z}(q) \xrightarrow{\ell} \mathbb{Z}(q) \to \mathbb{Z}/\ell(q) \to 0$; the usual Bockstein $\beta: H^{p,q}(Y, \mathbb{Z}/\ell) \to H^{p+1,q}(Y, \mathbb{Z}/\ell)$ is the boundary map for $0 \to \mathbb{Z}(q) \xrightarrow{\ell} \mathbb{Z}/\ell^2(q) \to \mathbb{Z}/\ell(q) \to 0$. Both are natural in Y; see 1.42(3) and section 13.1 for more information.

Corollary 1.38. *Suppose that X has a point of degree ℓ. Then the motivic cohomology groups $H^{*,*}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z})$ have exponent ℓ, and we have exact sequences:*

$$0 \to H^{p,q}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}) \to H^{p,q}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}/\ell) \xrightarrow{\tilde{\beta}} H^{p+1,q}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}) \to 0,$$
$$H^{p-1,q}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}/\ell) \xrightarrow{\beta} H^{p,q}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}/\ell) \xrightarrow{\beta} H^{p+1,q}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}/\ell).$$

Corollary 1.39. *If $BL(n-1)$ holds and X is smooth then $H^{p,n-1}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}/\ell) = 0$ for all $p \le n$, and $H^{p,q}(\mathrm{Spec}\, k, \mathbb{Z}/\ell) \xrightarrow{\cong} H^{p,q}(\mathfrak{X}, \mathbb{Z}/\ell)$ for all $p \le q < n$.*

Proof. As $H^{p,n-1}_{\text{ét}}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}/\ell) = 0$ by Lemma 1.37, the first assertion follows from 1.31. The second assertion follows from the cohomology sequence in Definition 1.36, and Lemma 1.30. □

Example 1.40. Assume that $\mathrm{BL}(n-1)$ holds, and that X has a point of degree ℓ. Then $H^{n,n-1}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}/\ell) = 0$ by 1.39. From the first sequence in 1.38, and naturality of $\tilde{\beta}$, we see that $H^{n+1,n-1}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}) = 0$ and hence the integral Bockstein

$$H^{n+1,n-1}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}/\ell) \xrightarrow{\tilde{\beta}} H^{n+2,n-1}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z})$$

is injective. It follows that the integral Bockstein $\tilde{\beta}: H^{n,n-1}(\mathfrak{X}, \mathbb{Z}/\ell) \to H^{n+1,n-1}(\mathfrak{X}, \mathbb{Z})$ is an injection because, as noted in 1.36, $H^{n,n-1}(\mathfrak{X}, \mathbb{Z}/\ell) \cong H^{n+1,n-1}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}/\ell)$ and $H^{n+1,n-1}(\mathfrak{X}, \mathbb{Z}) \cong H^{n+2,n-1}(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z})$.

1.6 MOTIVIC COHOMOLOGY OPERATIONS

Cohomology operations are another fundamental tool we shall need, both in section 3.4 (to construct the element μ of Corollary 3.16), and in chapter 5 (to show that Rost motives exist). We refer the reader to chapter 13 for more discussion.

Recall that for each coefficient group A, and all $p, q \geq 0$, the motivic cohomology groups $H^{p,q}(-, A) = H^p(-, A(q))$ are contravariant functors from the category $\Delta^{\text{op}}\mathbf{Sm}/k$ of smooth simplicial schemes over k to abelian groups. For each set of integers n, i, p, q and every two groups A and B, a *cohomology operation* ϕ from $H^{n,i}(-, A)$ to $H^{p,q}(-, B)$ is just a natural transformation. The *bidegree* of ϕ is $(p - n, q - i)$.

There is a *twist isomorphism* $\sigma_t : H^{n,i}(X, A) \xrightarrow{\sim} H^{n+1,i+1}(X_+ \wedge \mathbb{G}_m, A)$ of bidegree $(1, 1)$ in motivic cohomology; see [Voe03c, 2.4] or [MVW, 16.25].

Definition 1.41. A family of operations $\phi_{(n,i)} : H^{n,i}(-, A) \to H^{n+p,i+q}(-, B)$ with a fixed bidegree (p, q) is said to be *bi-stable* if it commutes with the suspension and twist isomorphisms, σ_s and σ_t.

Examples 1.42. There are several kinds of bi-stable operations.

1. Any homomorphism $A \to B$ induces a bi-stable operation of bidegree $(0, 0)$, the change of coefficients map $H^{*,*}(-, A) \to H^{*,*}(-, B)$.
2. If R is a ring and A is an R-module then multiplication by $\lambda \in H^{p,q}(k, R)$ is a bi-stable operation of bidegree (p, q) from $H^{*,*}(-, A)$ to itself.
3. The *integral Bockstein* $\tilde{\beta}$: $H^{n,i}(X, \mathbb{Z}/\ell) \to H^{n+1,i}(X, \mathbb{Z})$ and its reduction modulo ℓ, the usual *Bockstein* β: $H^{n,i}(X, \mathbb{Z}/\ell) \to H^{n+1,i}(X, \mathbb{Z}/\ell)$ are both bi-stable operations. They are the boundary maps in the long exact cohomology sequence associated to the coefficient sequences

$$0 \to \mathbb{Z}(q) \xrightarrow{\ell} \mathbb{Z}(q) \to \mathbb{Z}/\ell(q) \to 0, \quad \text{and}$$
$$0 \to \mathbb{Z}/\ell(q) \xrightarrow{\ell} \mathbb{Z}/\ell^2(q) \to \mathbb{Z}/\ell(q) \to 0.$$

4. In [Voe03c, p. 33], Voevodsky constructed the *reduced power* operations

$$P^i: \ H^{p,q}(X, \mathbb{Z}/\ell) \to H^{p+2i(\ell-1),q+i(\ell-1)}(X, \mathbb{Z}/\ell)$$

and proved that they are bi-stable. If $\ell = 2$ it is traditional to write Sq^{2i} for P^i and Sq^{2i+1} for βP^i.

We may compose bi-stable operations if the coefficient groups match: $\phi' \circ \phi$ is a bi-stable operation whose bidegree is bidegree(ϕ') + bidegree(ϕ). It follows that the stable cohomology operations with $A = B = R$ form a bigraded ring, and that $H^{*,*}(k, R)$ is a subring.

Definition 1.43. (Milnor operations). There is a family of motivic operations Q_i on $H^{*,*}(X, \mathbb{Z}/\ell)$ constructed in [Voe03c, §13], called the *Milnor operations.* The bidegree of Q_i is $(2\ell^i - 1, \ell^i - 1)$, Q_0 is the Bockstein β, Q_1 is $P^1\beta - \beta P^1$, and the other Q_i are defined inductively.

If $\ell > 2$ the inductive formula is $Q_{i+1} = [P^{\ell^i}, Q_i]$. If $\ell = 2$ the inductive formula is $Q_i = [\beta, P^{\mathbf{r}_i}]$; this differs from $[P^{2^{i-1}}, Q_i]$ by correction terms involving $[-1] \in k^\times/k^{\times 2} = H^{1,1}(k, \mathbb{Z}/2)$. See section 13.4 in part III.

We list a few properties of these operations here, referring the reader to section 13.4 for a fuller discussion. The Q_i satisfy $Q_i^2 = 0$ and $Q_i Q_j = -Q_j Q_i$, are $K_*^M(k)$-linear and generate an exterior algebra under composition.

The following theorem concerns the vanishing of a motivic analogue of the classical Margolis homology; see section 13.6 in part III. It was established for $i = 0$ in 1.38, and will be proven for all i in Theorem 13.24. This exact sequence will be used in Propositions 3.15 and 3.17 to show that the Q_i are injections in an appropriate range.

Theorem 1.44. *If X is a Rost variety for $(a_1, ..., a_n)$, the following sequence is exact for all $i < n$ and all (p, q).*

$$H^{p-2\ell^i+1, q-\ell^i+1}\left(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}/\ell\right) \xrightarrow{Q_i} H^{p,q}\left(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}/\ell\right) \xrightarrow{Q_i} H^{p+2\ell^i-1, q+\ell^i-1}\left(\widetilde{\Sigma}\mathfrak{X}, \mathbb{Z}/\ell\right)$$

Remark. In Theorem 1.44, it suffices that for each $i < n$ there is a ν_i-variety X_i and a map $X_i \to X$. This is the formulation given in Theorem 13.24.

1.7 HISTORICAL NOTES

As mentioned in the introduction, the question of whether the norm residue is always an isomorphism was first raised by Milnor in his 1970 paper [Mil70] defining what we now call "Milnor K-theory." For local and global fields, Tate had already checked that it was true for $n = 2$ (i.e., for K_2) and all primes ℓ (published in [Tat76]), and Milnor checked in his paper that it was true for all $n > 2$ (where the groups have exponent 2). Kato verified that the norm residue was an isomorphism for fields arising in higher class field theory, and stated the question as a conjecture in [Kat80]. Bloch also asked about it in [Blo80, p. 5.12].

Originally, *norm residue homomorphism* referred to the symbol $(a, b)_k$ of a central simple algebra in the group $\mu_\ell(k)$ of a local field, arising in Hilbert's 9^{th} Problem. Later it was realized that the symbol should take values in the Brauer group, or more precisely $\mu_\ell \otimes \mathrm{Br}_\ell(k)$, and that this map factored through $K_2(k)/\ell$; see [Mil71, 15.5]. The use of this term for the map from $K_*^M(F)/m$ to $H^2_{\text{ét}}(F, \mu_\ell^{\otimes 2})$ seems to have originated in Suslin's 1986 ICM talk [Sus87, 4.2].

The question was completely settled for $n = 2$ by Merkurjev and Suslin in the 1982 paper [MeS82]. Their key geometric idea was the use of Severi–Brauer varieties, which we now recognize as the Rost varieties for $n = 2$. The case $n = 3$ for $\ell = 2$ was settled independently by Rost and Merkurjev–Suslin in the late 1980s. In 1990, Rost studied Pfister quadrics (Rost varieties for $\ell = 2$) and constructed what we now call its Rost motive; see [Ros90] and [Voe03a, 4.3].

In 1994, Suslin and Voevodsky noticed that this conjecture about the norm residue being an isomorphism would imply a circle of conjectures due to Beilinson [Beĭ87] and Lichtenbaum [Lic84] regarding the (then hypothetical) complexes of sheaves $\mathbb{Z}(n)$; the preprint was posted in 1995 and an expanded version was eventually published in [SV00a]. This is the basis of our chapter 2.

In 1996, Voevodsky announced the proof of Milnor's conjecture for $\ell = 2$, using work of Rost on the motive of a Pfister quadric. The 1996 preprint [Voe96] was expanded into [Voe03a] and [Voe03c], which appeared in 2003.

In 1998, Voevodsky announced the proof of the Bloch–Kato conjecture, i.e., Milnor's conjecture for $\ell > 2$, assuming the existence of what we call Rost varieties (1.24). Details of this program appeared in the 2003 preprint [Voe03b], and the complete proof was published in 2011 [Voe11].

Later in 1998, Rost announced the construction of norm varieties; the construction was released in the preprints [Ros98a] and [Ros98b], but did not contain the full proof that his norm varieties were "Rost varieties," i.e., satisfied the properties (1.24) required by Voevodsky's program. Most of those details appeared in [SJ06]; Rost's informal notes [Ros06] provided other details, and the final details were published in [HW09].

The combination of Rost's construction and Voevodsky's work combines to verify not only the Bloch–Kato conjecture (proving Theorem A) but also proving Theorems B and C, which are stated in the overview of this book.

The material in section 1.5 is taken from the appendix of [Voe03a].

Index

A boldface page number indicates that the term is defined on that page.