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1 The Fast Track Introduction to Calculus

Chapter Preview. *Calculus is a new way of thinking about mathematics. This chapter provides you with a working understanding of the calculus mindset, core concepts of calculus, and the sorts of problems they help solve. The focus throughout is on the* ideas *behind calculus (the big picture of calculus); the subsequent chapters discuss the* math *of calculus. After reading this chapter, you will have an intuitive understanding of calculus that will ground your subsequent studies of the subject. Ready? Let's start the adventure!*

1.1 What Is Calculus?

Here's my two-part answer to that question:

Calculus is a mindset—a dynamics mindset. Contentwise, calculus is the mathematics of infinitesimal change.

Calculus as a Way of Thinking

The mathematics that precedes calculus—often called "pre-calculus," which includes algebra and geometry—largely focuses on *static* problems: problems devoid of change. By contrast, change is central to calculus—calculus is about *dynamics*. Example:

- What's the perimeter of a square of side length 2 feet? ⟵ *Pre-calculus problem.*
- How fast is the square's perimeter *changing* if its side length is *increasing* at the constant rate of 2 feet per second? ⟵ *Calculus problem.*

This statics versus dynamics distinction between pre-calculus and calculus runs even deeper—change is the *mindset* of calculus. The subject trains you to think of a problem in terms of dynamics (versus statics). Example:

- Find the volume of a sphere of radius r. *Pre-calculus mindset:* Use $\frac{4}{3}\pi r^3$ (Figure 1.1(a)).
- Find the volume of a sphere of radius r. *Calculus mindset:* Slice the sphere into a gazillion disks of tiny thickness and then add up their volumes (Figure 1.1(b)). When the disks' thickness is made "infinitesimally small" this approach reproduces the $\frac{4}{3}\pi r^3$ formula. (We will discuss why in Chapter 5.)

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Interactive Figure

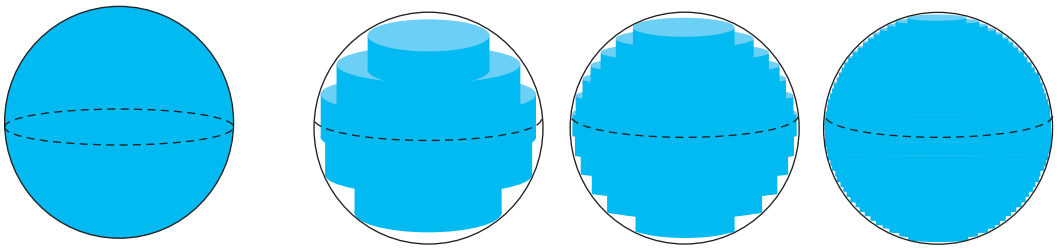

Figure 1.1: Visualizing the volume of a sphere via (a) a pre-calculus mindset and (b) a calculus mindset.

There's that mysterious word again—infinitesimal—and I've just given you a clue of what it might mean. I'll soon explain. Right now, let me pause to address a thought you might have just had: "Why the slice-and-dice approach? Why not just use the $\frac{4}{3}\pi r^3$ formula?" The answer: had I asked for the volume of some random blob in space instead, that static pre-calculus mindset wouldn't have cut it (there is no formula for the volume of a blob). The dynamics mindset of calculus, on the other hand, would have at least led us to a reasonable approximation using the same slice-and-dice approach.

That volume example illustrates the power of the dynamics mindset of calculus. It also illustrates a psychological fact: *shedding the static mindset of pre-calculus will take some time*. That was the dominant mindset in your mathematics courses prior to calculus mathematics courses, so you're accustomed to thinking that way about math. But fear not, young padawan (a *Star Wars* reference), I am here to guide you through the transition into calculus' dynamics mindset. Let's continue the adventure by returning to what I've been promising: insight into infinitesimals.

What Does "Infinitesimal Change" Mean?

The volume example earlier clued you in to what "infinitesimal" might mean. Here's a rough definition:

"Infinitesimal change" means: *as close to zero change as you can imagine, but not zero change.*

Let me illustrate this by way of Zeno of Elea (c. 490–430 BC), a Greek philosopher who devised a set of paradoxes arguing that motion is not possible. (Clearly, Zeno did not have a dynamics mindset.) One such paradox—the Dichotomy Paradox—can be stated as follows:

To travel a certain distance you must first traverse half of it.

Figure 1.2 illustrates this. Here Zeno is trying to walk a distance of 2 feet. But because of Zeno's mindset, with his first step he walks only half the distance: 1 foot

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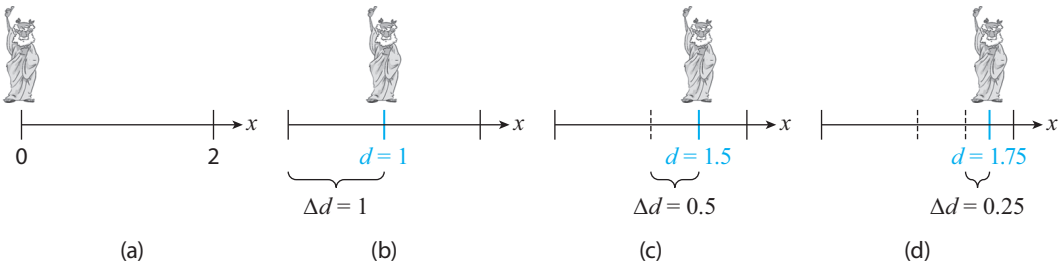

Figure 1.2: Zeno trying to walk a distance of 2 feet by traversing half the remaining distance with each step.

(Figure 1.2(b)). He then walks half of the remaining distance in his second step: 0.5 foot (Figure 1.2(c)). Table 1.1 keeps track of the total distance d, and the change in distance Δd, after each of Zeno's steps.

Table 1.1: The distance d and change in distance Δd after each of Zeno's steps.

Δd	1	0.5	0.25	0.125	0.0625	0.03125	0.015625	0.0078125	⋯
d	1	1.5	1.75	1.875	1.9375	1.96875	1.984375	1.9921875	⋯

Each change Δd in Zeno's distance is half the previous one. So as Zeno continues his walk, Δd gets closer to zero but never becomes zero (because each Δd is always half of a positive number). If we checked back in with Zeno after he's taken an infinite amount of steps, the change Δd resulting from his next step would be . . . drum roll please . . . an *infinitesimal change*—as close to zero as you can imagine but not equal to zero.

This example, in addition to illustrating what an infinitesimal change is, also does two more things. First, it illustrates the dynamics mindset of calculus. We discussed Zeno *walking*; we thought about the *change* in the distance he traveled; we visualized the situation with a figure and a table that each conveyed *movement*. (Calculus is full of action verbs!) Second, the example challenges us. Clearly, one *can* walk 2 feet. But as Table 1.1 suggests, that doesn't happen during Zeno's walk—he *approaches* the 2-foot mark with each step yet never *arrives*. How do we describe this fact with an equation? (That's the challenge.) No pre-calculus equation will do. We need a new concept that quantifies our very *dynamic* conclusion. That new concept is the mathematical foundation of calculus: limits.

1.2 Limits: The Foundation of Calculus

Let's return to Table 1.1. One thing you may have already noticed: Δd and d are related. Specifically:

$$\Delta d + d = 2, \qquad \text{or equivalently,} \qquad d = 2 - \Delta d. \tag{1.1}$$

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This equation relates each Δd value to its corresponding d value in Table 1.1. Great. But it is not the equation we are looking for, because it doesn't encode the dynamics inherent in the table. The table clearly shows that the distance d traveled by Zeno approaches 2 as Δd approaches zero. We can shorten this to

$$d \to 2 \text{ as } \Delta d \to 0.$$

(We are using "$\to$" here as a stand-in for "approaches.") The table also reiterates what we already know: were we to let Zeno continue his walk forever, he would be closer to the 2-foot mark than anyone could measure; in calculus we say: "infinitesimally close to 2." To express this notion, we write

$$\lim_{\Delta d \to 0} d = 2, \tag{1.2}$$

read "the limit of d as $\Delta d \to 0$ (but is never equal to zero) is 2."

Equation (1.2) is the equation we've been looking for. It expresses the intuitive idea that the 2-foot mark is the *limiting* value of the distance d Zeno's traversing. (This explains the "lim" notation in (1.2).) Equation (1.2), therefore, is a statement about the dynamics of Zeno's walk, in contrast to (1.1), which is a statement about the static snapshots of each step Zeno takes. Moreover, the Equation (1.2) reminds us that d is always *approaching* 2 yet never *arrives* at 2. The same idea holds for Δd: it is always *approaching* 0 yet never *arrives* at 0. Said more succinctly:

Limits approach indefinitely (and thus never arrive).

We will learn much more about limits in Chapter 2 (including that (1.2) is actually a "right-hand limit"). But the Zeno example is sufficient to give you a sense of what the calculus concept of limit is and how it arises. It also illustrates this section's title—the limit concept is the foundation on which the entire mansion of calculus is built. Figure 1.3 illustrates the process of building a new calculus concept that we will use over and over again throughout the book: *start with a finite change Δx in a quantity Y that depends on x, then shrink Δx to zero without letting it equal zero (i.e., take the limit as $\Delta x \to 0$) to arrive at a calculus result.* Working through this process—like we just did with the Zeno example, and like you can now recognize in Figure 1.1—is part of what *doing* calculus is all about. This is what I meant earlier when I said that calculus is the mathematics of infinitesimal change—contentwise, calculus is the collection of what results when we apply the workflow in Figure 1.3 to various quantities Y of real-world and mathematical interest.

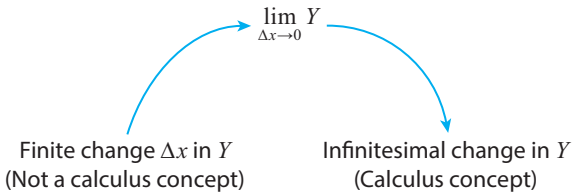

Figure 1.3: The calculus workflow.

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Three such quantities drove the historical development of calculus: instantaneous speed, the slope of the tangent line, and the area under a curve. In the next section we'll preview how the calculus workflow in Figure 1.3 solved all of these problems. (We'll fill in the details in Chapters 3–5.)

1.3 The Three Difficult Problems That Led to the Invention of Calculus

Calculus developed out of a need to solve three Big Problems (refer to Figure 1.4):[1]

1. *The instantaneous speed problem:* Calculate the speed of a falling object at a particular instant during its fall. (See Figure 1.4(a).)
2. *The tangent line problem:* Given a curve and a point P on it, calculate the slope of the line "tangent" to the curve at P. (See Figure 1.4(b).)
3. *The area under the curve problem:* Calculate the area under the graph of a function and bounded by two x-values. (See Figure 1.4(c).)

Figure 1.4 already gives you a sense of why these problems were so difficult to solve—the standard approach suggested by the problem itself just doesn't work. For example, you've been taught you need *two* points to calculate the slope of a line. The tangent line problem asks you to calculate the slope of a line using just *one* point (point P in Figure 1.4(b)). Similarly, we think of speed as "change in distance divided by change in time." How, then, can one possibly calculate the speed at an *instant*, for which there is no change in time? These are examples of the sorts of roadblocks that stood in the way of solving the three Big Problems.

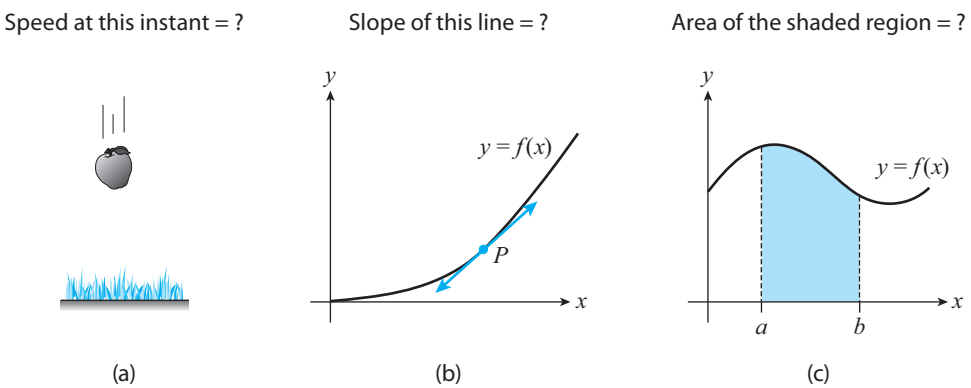

Figure 1.4: The three problems that drove the development of calculus.

[1] These may not seem like important problems. But their resolution revolutionized science, enabling the understanding of phenomena as diverse as gravity, the spread of infectious diseases, and the dynamics of the world economy.

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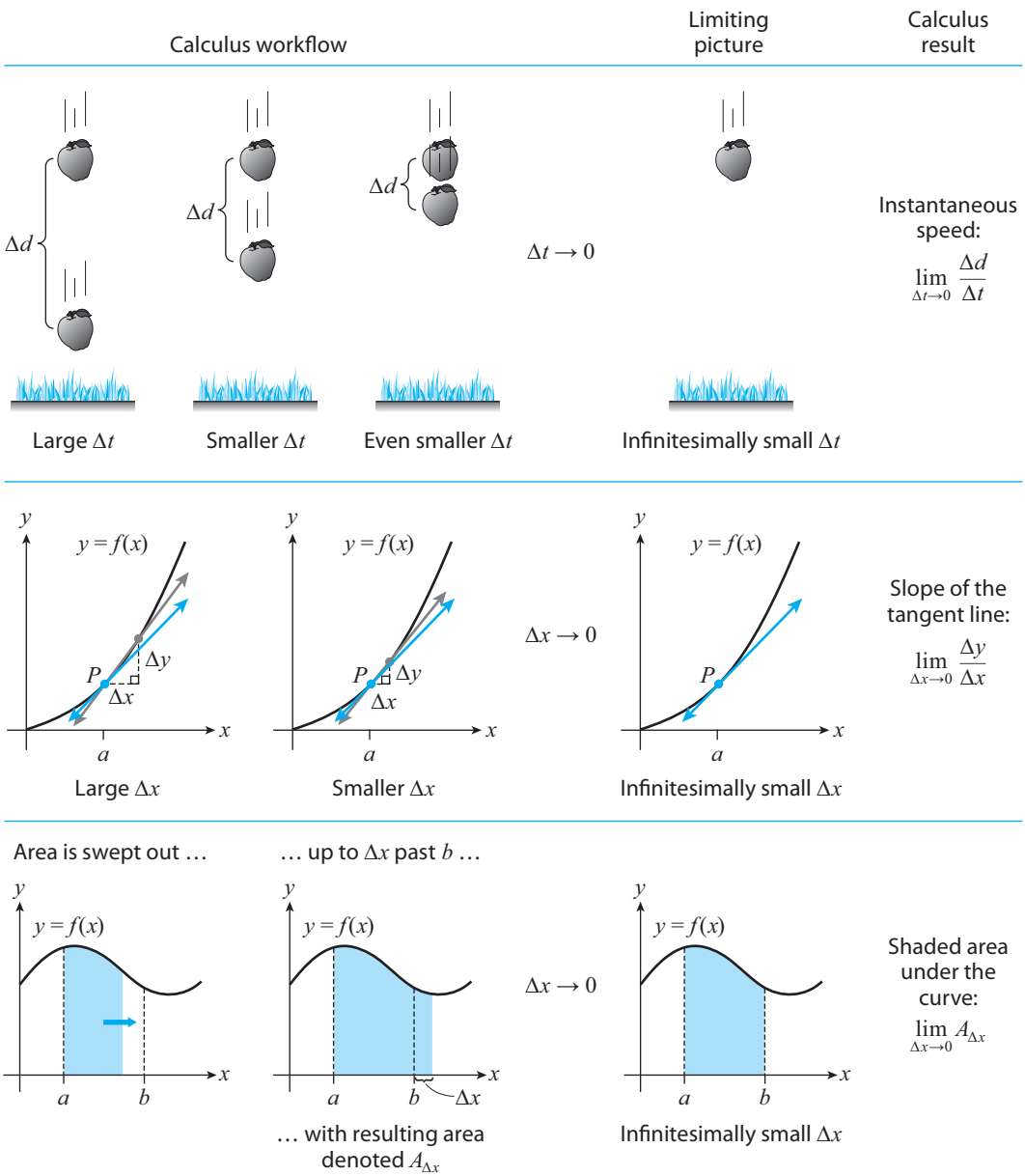

Figure 1.5: The calculus workflow (from Figure 1.3) applied to the three Big Problems.

But recall my first characterization of calculus: calculus is a dynamics mindset. Nothing about Figure 1.4 says "dynamics." Every image is a static snapshot of something (e.g., an area). So let's *calculus* the figure. (Yep, I'm encouraging you to think of calculus as a verb.)

Figure 1.5 illustrates the application of the calculus workflow (from Figure 1.3) to each Big Problem. Each row uses a dynamics mindset to recast the problem as

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the limit of a sequence of similar quantities (e.g., slopes) involving *finite* changes. Specifically:

- *Row #1:* The instantaneous speed of the falling apple is realized as the limit of its **average speeds** $\frac{\Delta d}{\Delta t}$ (ratios of changes in distance to changes in time) as $\Delta t \to 0$.
- *Row #2:* The slope of the tangent line is realized as the limit of the slopes of the **secant lines** $\frac{\Delta y}{\Delta x}$ (the gray lines in the figure) as $\Delta x \to 0$.
- *Row #3:* The area under the curve is realized as the limit as $\Delta x \to 0$ of the area swept out from $x = a$ up to Δx past b.

The limit obtained in the second row of the figure is called the **derivative** of $f(x)$ at $x = a$, the x-value of point P. The limit obtained in the third row of the figure is called the **definite integral** of $f(x)$ between $x = a$ and $x = b$. Derivatives and integrals round out the three most important concepts in calculus (limits are the third). We will discuss derivatives in Chapters 3 and 4 and integrals in Chapter 5, where we'll also fill in the mathematical details associated with the three limits in Figure 1.5.

This completes my big picture overview of calculus. Looking back now at Figures 1.1, 1.2, and 1.5, I hope I've convinced you of the power of the calculus mindset and the calculus workflow. We will employ both throughout the book. And because the notion of a limit is at the core of the workflow, I'll spend the next chapter teaching you all about limits—their precise definition, the various types of limits, and the myriad techniques to calculate them. See you in the next chapter.

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