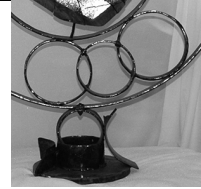


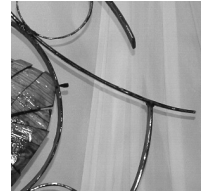
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## Chapter One

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### Critical Embedded Software

#### *Control Software Development and V&V*

CYBER-PHYSICAL SYSTEMS (CPS) is a kind of buzzword capturing the set of physical devices controlled by an onboard computer, an embedded system. Critical embedded systems are a subset of these for which failure is not acceptable. Typically this covers transportation systems such as cars, aircraft, railway systems, space systems, or even medical devices, all of them either for the expected harmfulness for people, or for the huge cost associated with their failure.

A large part of these systems are controllers. They are built as a large running loop which reads sensor values, computes a feedback, and applies it to the controlled system through actuators. For most systems, at least in the aerospace industry, the time schedule for controllers is so tight that these systems have to be “real time.” The way these systems have been designed requires the execution of the loop body to be performed within some time to maintain the system in a reasonable state. In the civil aircraft industry, the controller itself is rather complex, but is built as a composition of simpler controllers. Furthermore, the global system accounts for potential failures of components: sensors, networks, computers, actuators, etc., and adapts the control to these discrepancies.

The increase of computer use in those systems has led to huge benefits but also an exponential growth in complexity. Computer based systems compared to analog circuits enable more efficient behaviors, as well as size and weight reductions. For example, aircraft manufacturers are building control laws for their aircraft that maintain them at the limit of instability, allowing more fuel efficient behavior;<sup>1</sup> Rockwell Collins implemented a controller for a fighter aircraft able to recover controllability when the aircraft loses, in flight, from 60 to 80% of one of its wings;<sup>2</sup> United Technology has been able to replace huge and heavy

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<sup>1</sup>In an A380, fuel is transferred between tanks to move the center of gravity to the aft (backward). This degrades natural stability but reduces the need for lift surfaces and therefore improves fuel efficiency by minimizing total weight and drag. See the book *Airbus A380: Superjumbo of the 21st Century* by Noris and Wagner [58].

<sup>2</sup>Search for Damage Tolerance Flight Test video, e.g., at [https://www.youtube.com/watch?v=PTMpq\\_8SSCI](https://www.youtube.com/watch?v=PTMpq_8SSCI)

power electric systems with their electronic counterpart, with a huge reduction in size and weight.<sup>3</sup>

The drawback of this massive introduction of computers to control systems is the lack of predictability for both computer and software. While the industry has been accustomed to having access to the precise characteristic of its components, e.g., a failure rate for a physical device running in some specific conditions, these figures are hardly computable for software, because of the intrinsic complexity of computer programs.

Still, all of us are now used to accepting software licenses where the software vendor assumes nothing related to the use of the software and its possible impact. These kinds of licenses would be unacceptable for any other industry.

To conclude with this brief motivation, the aerospace industry, and more generally critical embedded systems industries, is are now facing a huge increase in the software size in their systems. This is motivated first by system complexity increases because of safety or performance objectives, but also by the need to integrate even more advanced algorithms to sustain autonomy and energy efficiency.

**Guaranteeing the good behavior of those systems is essential to enable their use.**

Until now, classical means to guarantee good behavior were mainly relying on tests. In the aerospace industry the development process is strictly constrained by norms such as the DO-178C [104] specifying how to design software and perform its verification and validation (V&V). This document shapes the V&V activities and requires the verification to be specification-driven. For each requirement expressed in the design phases, a set of tests has to be produced to argue that the requirement is satisfied. However, because of the increase in complexity of the current and future systems, these test-based verifications are reaching their limit. As a result the cost of V&V for systems has exploded and the later a bug is found, the more expensive it is to solve.<sup>4</sup>

Last, these certification documents such as DO-178C have been recently updated, accounting for the recent applicability of formal methods to argue about the verification of a requirement. Despite their possible lack of results in a general setting, these techniques, in cases of success, provide an exhaustive result, i.e., they guarantee that the property considered is valid for all uses, including systems admitting infinite behaviors.

All the works presented in this book are motivated by this context. We present formal methods sustaining the verification of controller properties at multiple stages of their development. The goal is to define new means of verification, specific to controller analysis.

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<sup>3</sup>E.g., Active EMI filtering for inverters used at Pratt and Whitney, Patent US20140043871.

<sup>4</sup>USA NIST released in 2002 an interesting survey, “The Economic Impacts of Inadequate Infrastructure for Software Testing,” detailing the various costs of verification and bugs. Chapter 6 is focused on the transportation industry.

## CURRENT LIMITS & OBJECTIVES

The objectives of the presented works are restricted to the definition of formal methods-based analyses to support the verification of controller programs.

More specifically we can identify the following limits in the current state of the art:

**Need to compute invariants of dynamical systems** New advances in formal methods are often not specialized for a particular kind of program. They rather try to handle a large set of programming language constructs and deal with scalability issues. In specific cases, such as the application of static analysis to Airbus programs [54], dedicated analyses, like the second-order filter abstraction [47], have been defined. But the definition of these domains is tailored to the program for which they are defined.

**Lack of means to compute nonlinear invariants** As we will see in this book, the simplest properties of controllers are often based on at least quadratic properties. Again, because of efficiency and scalability, most analyses are bound to linear properties. We claim that more expressive yet more costly analyses are required in specific settings such as the analysis of control software. The scalability issues have to be addressed by carefully identifying the local part of the program on which to apply these more costly analyses.

**Expressivity of static analysis properties** Formal methods applied at model or code level are hardly used to express or analyze system-level properties. In practice, static analysis is mainly bound to numerical invariants while deductive methods or model-checking can manipulate more expressive first-order logic formulas. However, computer scientists are usually not aware of the system-level properties satisfied or to be satisfied by the control program they are analyzing. An important research topic is therefore the use of these formalisms (first-order logic and numerical invariants) to express and analyze system-level properties.

**Scope of current analyses** In the current state of the practice, concerns are split and analyzed locally. For example the control-level properties such as stability are usually analyzed by linearizing the plant and the controller description. At the code level this can be compared to the analysis of a simplified program without if-then-else or nonlinear computations. Similarly, the complete fault-tolerant architecture, which is part of the implemented embedded program, is abstracted away when analyzing system-level properties. A last example of such—potentially unsound—simplifications, is the assumption of a real semantics when performing analyses, while the actual implementation will be executed with floating-point semantics and the associated errors. The vision supported by the book is that more integrated analyses should address the study of the global system.

The proposal is mainly developed in two complementary directions:

- nonlinear invariant synthesis mainly based on the use of convex optimization techniques;
- consideration of system-level properties on discrete representation, at code level, with floating-point semantics.

This book is structured in four parts:

Part I introduces formal methods and controller design. It is intended to be readable both by a control scientist unaware of formal methods, and by a computer scientist unaware of controller design. References are provided for more scholastic presentations.

Part II focuses on invariant synthesis for discrete dynamical systems, assuming a real semantics. All techniques are based on the computation of an inductive invariant as the resolution of a convex optimization problem.

Part III revisits basic control-level properties as numerical invariants. These properties are typically expressed on the so-called *closed-loop representation*. In these chapters we assume that the system description is provided as a discrete dynamical system, without considering its continuous representation with ordinary differential equations (ODEs).

Part IV extends the previous contributions by considering floating-point computations. A first part considers that the program analyzed is executed with floating-point semantics and searches for an inductive invariant considering the numerical errors produced. A second part ensures that the use of convex optimization, a numerical technique, does not suffer from similar floating-point errors.

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