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Contents

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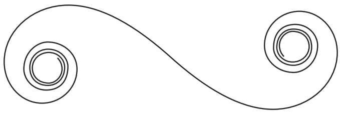
CHAPTER ONE

The Euler Spiral



Why This Curve?

It is a curve that attracts some particularly pleasing mathematics and also one that enjoys varied and surprising application. More than this, its shape is one of enduring elegance. And it is our favourite.

1.1 **An Unusual Parametrization**...

The standard practice of expressing a curve in parametric form $x = g(t)$, $y = h(t)$ brings with it variants of formulae for its common characteristics: its slope, the area under it, its arc length and its curvature. In the usual notation:

- slope: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y'}{x'}$,
- area: $\displaystyle\int_{t_1}^{t_2} yx'\,\mathrm{d}t$,
- arc length s: $\displaystyle\int_{t_1}^{t_2} \sqrt{x'^2 + y'^2}\,\mathrm{d}t$,
- curvature κ: $\dfrac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$,

where the prime indicates the derivative with respect to the parameter t. Given differentiable functions $g(t)$ and $h(t)$, the usual major concern is whether the resulting integrals can be evaluated in closed form.

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Our interest lies in what might at first appear to be a most exotic example of a parametrization:

$$x = x(t) = \int \cos f(t)\,\mathrm{d}t = \int_0^t \cos f(u)\,\mathrm{d}u,$$
$$y = y(t) = \int \sin f(t)\,\mathrm{d}t = \int_0^t \sin f(u)\,\mathrm{d}u,$$

where we choose to start the parameter at 0 and $f(u)$ is any differentiable function of u. The potentially daunting appearance of the integral actually contributes to the simplification of all but the area under the curve, with

$$\begin{aligned} x' &= x'(t) = \cos f(t) \\ y' &= y'(t) = \sin f(t) \end{aligned} \quad \text{and} \quad \begin{aligned} x'' &= -f'(t)\sin f(t) \\ y'' &= f'(t)\cos f(t) \end{aligned}$$

which lead to

- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\sin f(t)}{\cos f(t)} = \tan f(t);$
- $s = \displaystyle\int_0^t \sqrt{\cos^2 f(u) + \sin^2 f(u)}\,\mathrm{d}u = t;$
- $\kappa(t) = \dfrac{f'(t)\cos^2 f(t) + f'(t)\sin^2 f(t)}{[\cos^2 f(t) + \sin^2 f(t)]^{3/2}} = f'(t).$

We see that the parameter value, t, is precisely the arc length, s, and the curvature at the point given by parameter t is $f'(t)$; or, put conversely,

$$f(t) = \int \kappa(t)\,\mathrm{d}t = \int^t \kappa(u)\,\mathrm{d}u.$$

We can, therefore, replace the abstract parameter t with the curve's arc length, s, and rewrite the parametrization as

$$x = x(s) = \int_0^s \cos\left(\int^u \kappa(t)\,\mathrm{d}t\right)\mathrm{d}u,$$
$$y = y(s) = \int_0^s \sin\left(\int^u \kappa(t)\,\mathrm{d}t\right)\mathrm{d}u.$$

The reader may be reassured by the equations generating a straight line (the x-axis) when $\kappa(t) = 0$ and a circle ($x^2+(y-1)^2 = 1$) when $\kappa(t) = 1$. The next natural step is to take $\kappa(t) = t$, or equivalently $\kappa(s) = s$, which

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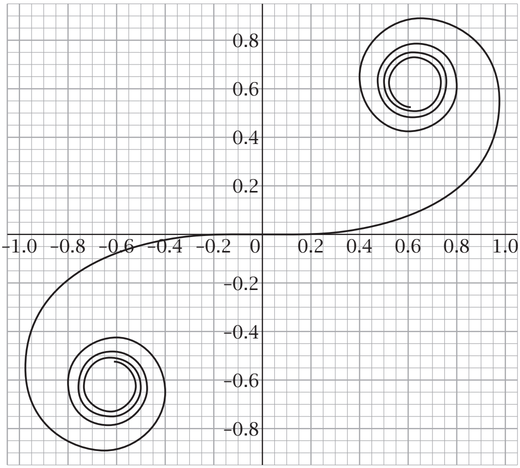

Figure 1.1. The Euler spiral.

leads to a curve whose curvature increases linearly with arc length, a curve whose simplest parametric equations are

$$x = x(s) = \int_0^s \cos \tfrac{1}{2}u^2 \, \mathrm{d}u,$$

$$y = y(s) = \int_0^s \sin \tfrac{1}{2}u^2 \, \mathrm{d}u.$$

Such a curve must spiral inwards since the curvature becomes greater as the curve develops, and does so to form the *Euler spiral*, shown in figure 1.1 and the curve that is the subject of this chapter.

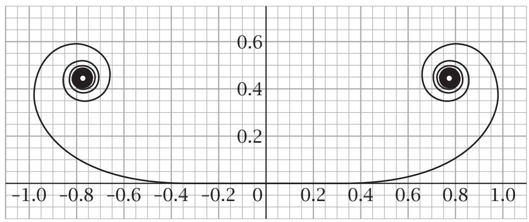

Figure 1.2. A chaise longue.

With a sympathetic curve plotter, considerable and enjoyable time can be spent experimenting with other variants of $\kappa(t)$; for example, figure 1.2 is generated by $\kappa(t) = t^2$ and figure 1.3 by $\kappa(t) = \cos t - t \sin t$ (and hence $f(t) = t \cos t$).

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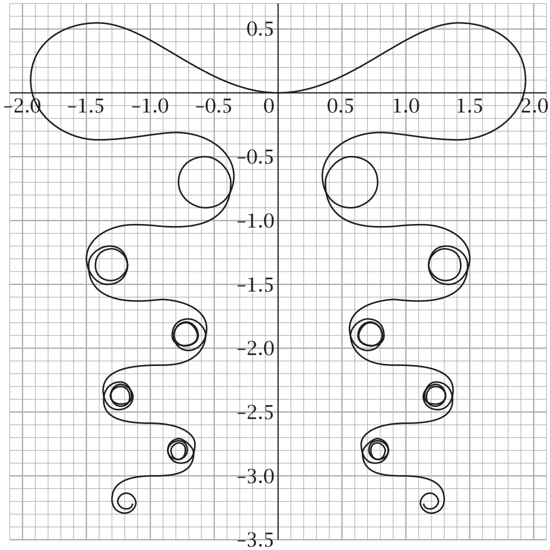

Figure 1.3. An elegant madness.

So, the curve of principal interest in this chapter appears as an example of what might at first appear to be a rather exotic parametrization involving arc length and curvature, but we will now see that such a construction is not at all strange. Quite the reverse, in fact.

1.2 ...**Yet a Natural Parametrization**

If a mathematical result is given a name, it is a clear indication of its perceived importance; if that name begins with the adjective "fundamental", that importance is magnified to the level where the result occupies a central role in the theory at hand or perhaps in mathematics more generally. Such is the case, then, for the *fundamental theorem of plane curves*, formal statements of which attract the abstraction necessary for mathematical precision, but its essence is that "curvature determines the curve". That is, if we are given a starting point in the plane and a curvature function, the curve is determined.

So, suppose that we are given a curvature function, $\kappa(s)$, parametrized in terms of arc length, s. With this, we have two quantities that are primitive, that is, ones that are not affected by external influences such as coordinate systems or frames of reference; they are termed *intrinsic* variables, and there is another too: the *tangential angle*, ψ.

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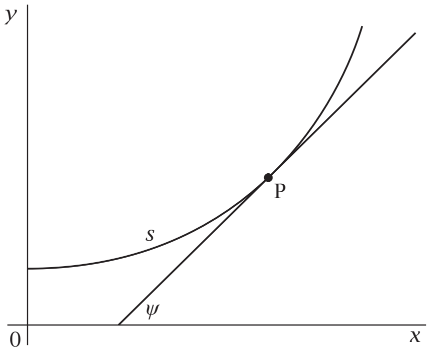

Figure 1.4. Intrinsic coordinates.

This is the angle that each of the tangents to the curve makes with a fixed direction, and it is reasonable and common to normalize matters so that $s = 0$ when $\psi = 0$ and take the direction as the positive x-axis. It is clear from figure 1.4 that $\mathrm{d}y/\mathrm{d}x = \tan\psi$. What is more – and it is a standard result easily gained – we have $\mathrm{d}\psi(s)/\mathrm{d}s = \kappa(s)$; indeed, this need not be thought of as a *result* but as a *definition* of curvature as the (normalized) rate at which the tangent to the curve turns.

Moreover, by definition,

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1 + \tan^2\psi} = \sec\psi,$$

and so $\mathrm{d}x/\mathrm{d}s = \cos\psi$, and since

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}s} \times \frac{\mathrm{d}s}{\mathrm{d}x} = \tan\psi,$$

it must be that $\mathrm{d}y/\mathrm{d}s = \sin\psi$.

We now have the necessary apparatus in place to achieve our goal of deriving the equation of a curve from a knowledge of its curvature.

Let the curve have a parametrization in terms of arc length as

$$x = g(s)$$
$$y = h(s)$$

and write

$$\frac{\mathrm{d}x}{\mathrm{d}s} = g'(s) \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}s} = h'(s).$$

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We have, then, the following set of differential equations defining the curve in terms of intrinsic coordinates:

$$\kappa(s) = \frac{\mathrm{d}\psi(s)}{\mathrm{d}s},$$
$$g'(s) = \cos\psi(s),$$
$$h'(s) = \sin\psi(s).$$

So, given $\kappa(s)$ we first find $\psi(s)$ and from this the parametric equations of the curve.

We have already danced between definite and indefinite integrals and we will continue to do so here, again using the variable as a limit and remembering any arbitrary constant. In these terms, the first equation has general solution

$$\psi(s) = \int \kappa(s)\,\mathrm{d}s = \int_0^s \kappa(t)\,\mathrm{d}t + \psi_0.$$

This means that

$$g'(s) = \cos\left(\int_0^s \kappa(t)\,\mathrm{d}t + \psi_0\right) \quad \text{and} \quad h'(s) = \sin\left(\int_0^s \kappa(t)\,\mathrm{d}t + \psi_0\right),$$

and these have the general solution

$$g(s) = \int \cos\left(\int_0^s \kappa(t)\,\mathrm{d}t + \psi_0\right)\mathrm{d}s = \int_0^s \cos\left(\int_0^u \kappa(t)\,\mathrm{d}t + \psi_0\right)\mathrm{d}u + x_0,$$

$$h(s) = \int \sin\left(\int_0^s \kappa(t)\,\mathrm{d}t + \psi_0\right)\mathrm{d}s = \int_0^s \sin\left(\int_0^u \kappa(t)\,\mathrm{d}t + \psi_0\right)\mathrm{d}u + y_0.$$

We now make explicit the influence of those arbitrary constants of integration, using the standard, elementary trigonometric identities

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi,$$
$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi,$$

to obtain

$$\begin{aligned} g(s) &= \int_0^s \cos\left(\int_0^u \kappa(t)\,\mathrm{d}t + \psi_0\right)\mathrm{d}u + x_0 \\ &= \int_0^s \cos\left(\int_0^u \kappa(t)\,\mathrm{d}t\right)\cos\psi_0 - \sin\left(\int_0^u \kappa(t)\,\mathrm{d}t\right)\sin\psi_0\,\mathrm{d}u + x_0 \\ &= \cos\psi_0 \int_0^s \cos\left(\int_0^u \kappa(t)\,\mathrm{d}t\right)\mathrm{d}u \\ &\qquad - \sin\psi_0 \int_0^s \sin\left(\int_0^u \kappa(t)\,\mathrm{d}t\right)\mathrm{d}u + x_0, \end{aligned}$$

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$$\begin{aligned}
h(s) &= \int_0^s \sin\left(\int_0^u \kappa(t)\,\mathrm{d}t + \psi_0\right)\mathrm{d}u + y_0 \\
&= \int_0^s \sin\left(\int_0^u \kappa(t)\,\mathrm{d}t\right)\cos\psi_0 + \cos\left(\int_0^u \kappa(t)\,\mathrm{d}t\right)\sin\psi_0\,\mathrm{d}u + y_0 \\
&= \cos\psi_0 \int_0^s \sin\left(\int_0^u \kappa(t)\,\mathrm{d}t\right)\mathrm{d}u \\
&\qquad + \sin\psi_0 \int_0^s \cos\left(\int_0^u \kappa(t)\,\mathrm{d}t\right)\mathrm{d}u + y_0,
\end{aligned}$$

which we can conveniently rewrite in matrix form as

$$\begin{pmatrix} g(s) \\ h(s) \end{pmatrix} = \begin{pmatrix} \cos\psi_0 & -\sin\psi_0 \\ \sin\psi_0 & \cos\psi_0 \end{pmatrix} \begin{pmatrix} \int_0^s \cos\left(\int_0^u \kappa(t)\,\mathrm{d}t\right)\mathrm{d}u \\ \int_0^s \sin\left(\int_0^u \kappa(t)\,\mathrm{d}t\right)\mathrm{d}u \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

The pre-multiplying matrix represents an anticlockwise rotation about the origin by angle ψ_0 and the final vector a translation. We have shown that, up to these two isometries, our curve is uniquely determined by the parametric equations

$$\begin{aligned}
x = g(s) &= \int_0^s \cos\left(\int_0^u \kappa(t)\,\mathrm{d}t\right)\mathrm{d}u, \\
y = h(s) &= \int_0^s \sin\left(\int_0^u \kappa(t)\,\mathrm{d}t\right)\mathrm{d}u,
\end{aligned}$$

with arc length as parameter. These equations are the embodiment of the fundamental theorem of plane curves and confirm that curvature does indeed determine the curve. With these thoughts, we have the Euler spiral as the curve naturally defined by the simplest non-trivial relationship possible between its curvature and arc length: $\kappa = s$. Yet, this is not the manner in which the curve came into existence – and Euler was not the first to consider it.

1.3 A Challenge

With two Jakobs, three Johanns, two Daniels and five Nikolauses, along with linguistic variants of their names, it is extremely easy to confuse one Bernoulli family member with another. The mathematical dynasty that spread over a century and comprised nine sons and sons-of-sons was singularly remarkable for the contribution it made to the various areas of mathematics and associated studies, but it is to James I (or Jakob I or Jacques I), by age the dynasty's mathematical patriarch,

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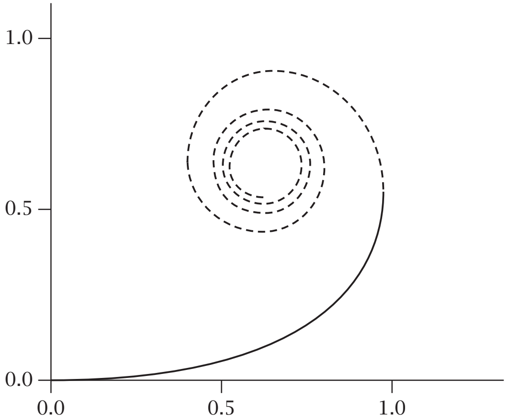

Figure 1.5. Bernoulli's curve.

whom we must look for the first appearance of our curve – although not the first real appreciation of it.

A seminal 1694 publication (Bernoulli 1694) formed the culmination of James Bernoulli's study of what we would now term the *cantilever problem*: a thin horizontal beam of negligible mass, fixed at one end and loaded at the other assumes a curved shape – but what shape? The answer he gave is what we now call the *elastica*: a curve that has its own considerable importance in various areas and that is closely related to our spiral. At the end of his work he had, characteristically, posed a number of challenges for the readers' consideration, one of which could be thought of as the converse to the one he had already answered:

> To find the curvature a lamina must have in order to be straightened out horizontally by a weight at one end.

The beam remains fixed at one end but now assumes a curved, upward shape that transforms to a horizontal line by the action of a weight placed at its other extremity. In a note dated that same year, Bernoulli showed that he had solved his own problem and gave the intrinsic equation of such a curve as $a^2 = sR$, where a is constant and R is the curve's radius of curvature, defined as the reciprocal of its curvature, κ; its nature is, then, that its curvature is proportional to its arc length. His argument is terse and more of a statement than a proof, and in consequence it is obscure – a view echoed in 1744 by his nephew, Nicholas I, when, in editing his uncle's work for publication,

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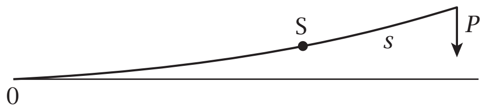

Figure 1.6. Euler's argument.

he commented "I have not found this identity established". Referring to figure 1.5, Bernoulli's curve is the solid line, drawn until it becomes vertical. Beyond this, he would have had no interest, and it would be left to the inimitable Leonhard Euler to provide the convincing arguments which extended it to the dotted spiral.

It was in that same year of 1744 that Euler published a work (see Euler 1744, (E65)) that, even by his lofty standards, is extraordinary in its compass. There are two appendices, the first of which, *Additamentum 1*, deals with elastic curves, and in sections 51 and 52 he considers this problem. The mixture of pure mathematics and physics that Euler uses is a clearer version of what was, apparently, Bernoulli's argument.

Referring to figure 1.6, suppose that the curvature of the beam at a point S on it before and after the application of a bending force is $\kappa_1(s)$ and $\kappa_2(s)$ respectively. Also suppose that S is distant s along the beam from the point of application of the force. In modern terms, the force's influence is measured by the equation $M = \kappa EI$, where M, E and I are the moment of the force, Young's modulus and the second moment of area of the beam (about its neutral axis), respectively. With $M = Ps$ we measure the contribution to the curvature of the beam as

$$\kappa = \frac{Ps}{EI} = \frac{s}{a^2},$$

as Euler did. This means that $\kappa_2(s) = \kappa_1(s) - \kappa$, and, since the final shape of the beam is required to be a straight line, $\kappa_2(s) = 0$, and so

$$0 = \kappa_1(s) - \frac{s}{a^2} \quad \text{and} \quad \kappa_1(s) = \frac{1}{r} = \frac{s}{a^2}.$$

From this intrinsic equation Euler somewhat rapidly extracted the curve's parametric equations[1]

$$x = \int \cos \frac{s^2}{2a^2}\,\mathrm{d}s = \int_0^s \cos \frac{u^2}{2a^2}\,\mathrm{d}u$$

$$y = \int \sin \frac{s^2}{2a^2}\,\mathrm{d}s = \int_0^s \sin \frac{u^2}{2a^2}\,\mathrm{d}u$$

[1] Actually, we have interchanged sin and cos to be consistent with modern usage.

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with his largely implied argument being

$$\int \frac{1}{r}\,\mathrm{d}s = \int \frac{s}{a^2}\,\mathrm{d}s \quad \Rightarrow \quad \int \frac{\mathrm{d}\psi}{\mathrm{d}s}\,\mathrm{d}s = \frac{s^2}{2a^2} \quad \Rightarrow \quad \psi = \frac{s^2}{2a^2}.$$

With

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\psi \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\psi,$$

the solution follows.

The parametric equations allow for the extension of the curve beyond our assumed bound: to the vertical and, beyond that, to the infinite spiral.

Euler acknowledged that neither integral can be evaluated in closed terms and he used series expansions and term-by-term integration to yield the still-useful infinite series forms:

$$x = s - \frac{s^5}{2! \times 5} + \frac{s^9}{4! \times 9} - \frac{s^{13}}{6! \times 13} + \cdots,$$
$$y = \frac{s^3}{1! \times 3} - \frac{s^7}{3! \times 7} + \frac{s^{11}}{5! \times 11} - \frac{s^{15}}{7! \times 15} + \cdots.$$

He also made the following observation.

> Now, from the fact that the radius of curvature continually decreases the greater the arc taken, it is manifest that the curve cannot become infinite, even if the arc is taken infinite. Therefore the curve will belong to the class of spirals, in such a way that after an infinite number of windings it will roll up a definite point as a centre, which point seems very difficult to find from this construction. Analysis therefore must be considered to gain no slight advantage if anyone should discover a method by the aid of which at least an approximate value can be assigned to the integrals in the case where s is taken as infinite. This seems not an unworthy problem upon which mathematicians may exercise their powers.

That is, the curve spirals towards its two limit points

$$x = \pm \int_0^\infty \sin \frac{u^2}{2a^2}\,\mathrm{d}u,$$
$$y = \pm \int_0^\infty \cos \frac{u^2}{2a^2}\,\mathrm{d}u.$$

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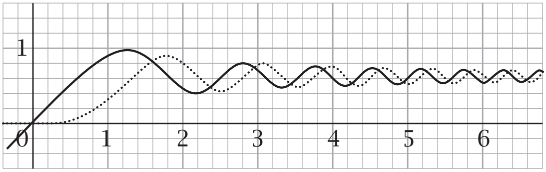

Figure 1.7. The limit point.

Figure 1.7 is the plot (for an arbitrary value of $a = 0.7$)

$$x(t) = \int_0^t \sin \frac{u^2}{2a^2}\,du \quad \text{(dotted)},$$
$$y(t) = \int_0^t \cos \frac{u^2}{2a^2}\,du \quad \text{(full)},$$

for $t \geqslant 0$, and it leads the eye to a limit, the exact value of which would be very nice to know.

There appears to be no record of any contribution from other mathematicians, but 38 years later, in 1781, Euler published his own answer to his own question:[2]

$$x = \pm \frac{a}{\sqrt{2}}\sqrt{\frac{\pi}{2}},$$
$$y = \pm \frac{a}{\sqrt{2}}\sqrt{\frac{\pi}{2}}.$$

His approach exhibits typical Eulerian masterful symbolic manipulation, with a mixture of integrals, complex exponentials and the Gamma function (which he had already brought into existence in 1729): a method that he described as *singular*. This singular method yielded various other results and, most particularly,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Bernoulli's original question had been answered and the solution curve extensively investigated, with Euler's contribution comprising a tiny part of his vast output and seemingly lost within it.

1.4 One Curve, Many Names

And so the curve slumbered. That is, until 1814, when the French physicist Augustin Fresnel deduced an expression for the intensity of the

[2] On the values of integrals extended from the variable term $x = 0$ to $x = \infty$ (E675).

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illumination at any point of a diffraction pattern, which has the form (under certain simplifying assumptions)

$$I_v = \left[\int_0^v \cos \tfrac{1}{2}\pi t^2 \, \mathrm{d}t\right]^2 + \left[\int_0^v \sin \tfrac{1}{2}\pi t^2 \, \mathrm{d}t\right]^2.$$

So, if not the spiral itself, its components resurfaced, and in a note to the French Academy of Science in 1818 Fresnel produced a table of values of the two integrals for values of v, differing by 0.1, from $v = 0.1$ to $v = 5.1$, later extended to $v = 5.5$, all to four decimal places. More tables followed, of course, but these two defining components of the Euler spiral are nearly universally known as *Fresnel integrals*. The spiral itself reemerged in 1874, when the French scientist (Marie) Alfred Cornu,[3] following Fresnel, plotted the curve and identified its use as a computational device for problems involving diffraction. The influence of this most eminent scientist's involvement caused his name to be attached to the curve – the following comes from his 1902 funeral oration by his former student, Henri Poincaré:

> Today, to predict the effect of an arbitrary screen on a beam of light, everyone makes use of the spiral of Cornu.

And from one of his obituaries (Ames 1902):

> and the method of studying problems of diffraction by the use of Cornu Spirals is familiar to everyone.

The *Cornu spiral* is a term still in common use. From the late nineteenth through to the twentieth century, the curve's further mathematical properties were investigated by the Italian mathematician Ernesto Cesàro, to whom the curve resembled the shape that a length of thread assumes as it is wrapped around a spindle. In 1886 and from this image, Cesàro coined *clothoid(e)* for the spiral's name, after *Clotho*, the spinner among the *Three Fates*, who spins the thread of human life by winding it around a spindle. This Italian romance is balanced by Cesàro's geometric findings regarding the curve, which were many but whose nature is somewhat remote from modern mathematics. As an indication, we provide the statements of two of them below, leaving the reader to make use of the internet to identify such terms as are necessary:

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[3] It is a nice coincidence that the French word *cornu* translates to *horned*, as with an ibex, etc.

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> When a clothoide rolls on a straight line, the locus of the centre of curvature corresponding to the point of contact is an equilateral hyperbola asymptotic to the line considered.
>
> The clothoide is the only curve enjoying the property that the centre of gravity of any arc is a centre of similitude of the circles osculating the extremities of the arc.

Again, the name *clothoid* remains in common use.

Meanwhile, trains moved faster. Modern model train sets can be equipped with a plethora of differently designed track pieces, but those that offer a change in direction are shaped as arcs of circles of various lengths and with various radii. These do not, as any serious model railway enthusiast will know, reflect the real world. A train travelling at a constant speed v along a straight section of track would encounter an instant change in its acceleration, from 0 to the centripetal acceleration of $v^2/r = \kappa v^2$, as it began to negotiate the change of direction afforded by a circular arc of radius r and, therefore, of curvature $\kappa = 1/r$. The resulting unpleasant experience, for the train and its passengers, is technically known as a *jerk* and is to be avoided. From the early days of the railway, this avoidance took the form of various *transition curves* of several types replacing the circular arc, yet best of all is one where the curvature, and so acceleration, increases linearly from 0 along the track – which is precisely the nature of the Euler spiral. It would appear to be in 1881 that the spiral was put to such use (Talbot 1890–91):

> The transition spiral was probably first used on the Pan Handle Railroad in 1881, by Mr. Elliot Holbrook. The principal part of the treatment here given was made before the writer's attention was called to Mr. Holbrook's use of the curve, and it is believed that most of the formulas and methods appear here for the first time.

A visualization of things is provided by figure 1.8. Part (a) is the plan view of a track initially formed as two straights joined by two semicircular arcs, which has been amended at the right end with the replacement of the semicircular arc with two parts of an Euler spiral (shown dotted). Part (b) shows the train's acceleration as it first moves with constant speed along the top straight, then negotiates the Euler spirals, then returns along the bottom straight and finally completes the circuit along the semicircular arc: the jerk has been replaced by a linearly increasing acceleration, which will be slightly greater, but much preferable.

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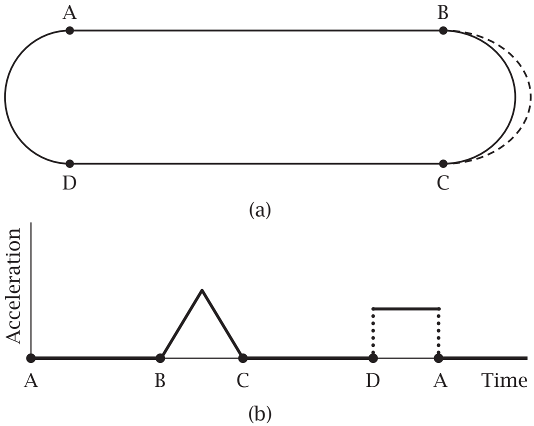

Figure 1.8. A transition curve in action.

Surprisingly, there appears to be no *Holbrook spiral*, although there is a *Glover spiral*, after one James Glover who trumpeted the curve's usefulness as a railway transition curve in 1900. In this usage, Americans may well know it as the *AREMA spiral*, and it remains in use to this day in any situation in which straight lines and curves need smooth joining – including road transitions and the vertiginous curves on roller coasters.

Whatever the context, whatever the theory or practice associated with it, and whatever name it is known by, the curve came into existence as a well-defined, well-understood entity through the omnipresent involvement of Leonhard Euler: it is surely the *Euler spiral*.

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Index

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