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1 European Trigonometry Comes of Age (1552–1613)

The subject we know today as trigonometry has a long, complex history that weaves through several major cultures and more than two millennia. Perhaps more than any other subject in the modern mathematics curriculum, trigonometry has been shaped, has been reconfigured, and gone through metamorphoses several times. Born of needs in ancient astronomy, it has been repurposed by many scientific disciplines and worked to serve several cultural and religious perspectives. It has been a participant, active or passive, in many of humanity's most significant scientific pursuits. The tidy, polished package found in today's high school and university textbooks camouflages a tangled story that interacts with many themes in the history of science, often with implications for some of the most transformative moments in our and other cultures.

I told the first half of this story in *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry*.¹ This volume narrates the second half, but we begin with a brief summary of what went before. Trigonometry began with Greek astronomers such as Hipparchus of Rhodes, who had constructed geometric models of the motions of the sun and moon that reproduced qualitatively the phenomena he witnessed in the sky. Converting these models into tools for prediction of events like eclipses required the translation of their geometric components into numerical measures. Since these components were lines and circles, it quickly became necessary to convert the magnitudes of circular arcs into lengths of line segments and vice versa. Hence the chord function was formulated,² giving the astronomer the ability to compute the length of a chord within a circle given the magnitude of the arc that it spans. The earliest table of chords of which we are aware was constructed by Hipparchus; the earliest account of the construction of chord tables is in Claudius Ptolemy's *Almagest*. The mathematical preparation for astronomy began with these chords and grew from there. However, since the geometric arena was often the celestial sphere rather than a flat surface, plane trigonometry was only the beginning. Perhaps already from the time of

¹ [Van Brummelen 2009].

² The term “function” has a long and complicated history. Properly speaking, according to the term's modern usage, it is an anachronism to refer to functions at all before the modern period. However, there is an affinity at least between ancient numerical tables and our use of the term: ancient astronomers found the length of the chord of a given arc by inputting the numerical value of that arc into a table and treating the value obtained as an output. In this book the word “function” is used in this loose sense, unless stated otherwise.

2 Chapter 1

Hipparchus, astronomers quickly moved from the plane to the sphere, where much of the most important work was done.

The first major transformation occurred with the complicated and controversial transmission of mathematical astronomy from Greece to India. The early Indian astronomers' appropriation of the geometric models of the planets, much more than a simple transmission of knowledge (but a topic for another book), also extended to many new ways of thinking in trigonometry. The most obvious effect of the transformation of trigonometry in India is the introduction of the sine function: a slightly less intuitive quantity from a geometric point of view but a more efficient tool for astronomical computation. The versed sine followed quickly afterward. The inventions of new mathematical methods to work with these functions, such as iterative solutions to equations and higher-order interpolation within numerical tables, greatly enriched mathematical astronomy. In the fourteenth and fifteenth centuries, astronomers even employed infinitesimal arguments that we recognize today as related to calculus to derive several powerful results beneficial to astronomy, most famously the Taylor series for the sine and cosine.

The reception and naturalization of trigonometry in medieval Islam is no less complicated. In the eighth and ninth centuries Indian astronomy found its way through Persia to Baghdad. As interest grew, a translation movement brought a fresh crop of Greek texts to Islamic scholars. This produced the curious circumstance that two approaches to astronomy, both of which contained at least some trace of Greek origin, were in opposition to each other. The Greek texts gradually took precedence during the ninth and tenth centuries, but many of the Indian advances (including the sine and iterative methods) were retained. Around the end of the tenth century several advances streamlined eastern Islamic trigonometry. The tangent, invented in the process of sundial construction, became part of the trigonometric toolkit. New theorems reformulated the foundations of spherical trigonometry and delivered greater power to both astronomy and astrology. Trigonometry was also applied to new contexts, including ritual needs like determining the beginning of the month of Ramadan and the direction of prayer toward Mecca. Some of the work done on the latter problem became a standard tool in mathematical geography, bringing trigonometry down from the heavens to the earth for the first time.

From the tenth century onward, Islamic science gradually diversified according to cultural subgroups spread across its vast geographical area. The most prominent division was between eastern Islam and al-Andalus, in what is now Spain. Andalusian mathematical astronomy retained Indian and Greek influences, but after AD 1000 it developed without much conversation with the East. Rather, their knowledge spread northward into Europe, especially through the Toledan and Alfonsine Tables. Some innovations in trigonometry occurred in medieval Europe, sometimes through interactions with practical geometry

and with astronomical instruments. However, the fifteenth century saw the beginning of tremendous growth through the theoretical astronomy of people such as Giovanni Bianchini (ca. 1410–1469) and Regiomontanus (1436–1476). This period set in motion the events that we shall survey in this chapter.

It is a reflection of the richness of the history of trigonometry that after more than one and a half millennia of years of progress, in the year 1550 the word itself was still 50 years away from being coined. Indeed, triangles did not really emerge as the primitive objects of study until Regiomontanus's *De triangulis omnimodis* ("Concerning Triangles of Every Kind") became popular in the mid-sixteenth century. This volume's title, *The Doctrine of Triangles*, is taken from one of the names that was given to trigonometry in the sixteenth and seventeenth centuries.

■ What's in a Name?

By 1550, the central problem of trigonometry—determining lengths in geometric diagrams from corresponding circular arcs and vice versa—had long been solved. European astronomers had within their grasp a somewhat compact theory that allowed them to solve every problem that they needed to solve, both on the plane and on the sphere. Regiomontanus's *De triangulis omnimodis*, written in the fifteenth century but published in 1533,³ provided a unified source for the mathematical methods and most (although not quite all) of the fundamental theorems. Sine tables composed by Regiomontanus and others provided a straightforward tool for working out the practical calculations. Seemingly, there was not much left to do.

However, there was a great deal left to do. Over the next 50 years, the mathematical structure and even the basic notions of trigonometry were overhauled. New theorems were discovered, and more elegant and efficient ways of organizing the material were found. By the beginning of the seventeenth century, new ways to employ the subject, both within science and outside of it, were being devised with regularity. Even the basic functions, the fundamental building blocks of trigonometry, went through multiple reinventions. By 1613, the subject no longer looked much like Regiomontanus's *De triangulis omnimodis*.

We may begin to get a sense of the contrast by comparing basic definitions in the works of two of the dominant figures in the mid-sixteenth century, Regiomontanus and Rheticus. We start with Regiomontanus's *De triangulis omnimodis*.

³[Regiomontanus 1533]; see also the edition [Regiomontanus 1561]. *De triangulis* has been translated in [Regiomontanus (Hughes) 1967]. Finally, see [de Siebenthal 1993, chapter 5, 268–352] for an account of the mathematics in French.

Text 1.1

Regiomontanus, Defining the Basic Trigonometric Functions

(from *De triangulis omnimodis*)

Definitions:

...

An *arc* is a part of the circumference of a circle.

The straight line coterminous with the arc is usually called its *chord*.

When the arc and its chord are bisected, we call that half-chord the *right sine*⁴ of the half-arc.

Furthermore, the *complement* of any *arc* is the difference between [the arc] itself and a quadrant.

The *complement* of an *angle* is the difference between [the angle] itself and a right angle.

Book I, Theorem 20: In every right triangle, one of whose acute vertices becomes the center of a circle and whose [hypotenuse] its radius, the side subtending this acute angle is the right sine of the arc adjacent to that [side and] opposite the given angle, and the third side of the triangle is equal to the sine of the complement of the arc.⁵

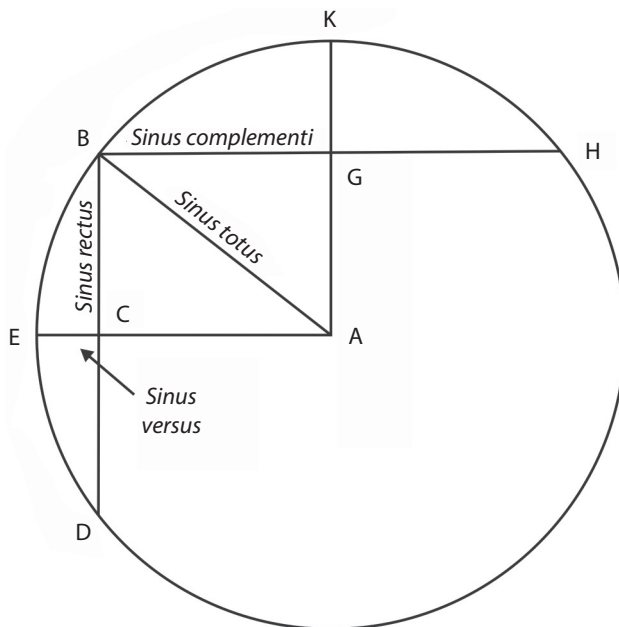


Figure 1.1

Regiomontanus's definitions of the primitive trigonometric functions.

⁴ In Latin, *sinum rectum*.

⁵ [Regiomontanus (Hughes) 1967, 31, 59].

Explanation: In right-angled triangle ABC (figure 1.1), draw a circle centered at A with radius AB . Draw AK vertically, BC parallel to AK , and BH parallel to AC ; and extend AC to E and BC to D . Several differences between Regiomontanus's structure and the modern definition are apparent. Firstly, following his predecessors, he defines the trigonometric functions as lengths of line segments in the diagrams, not as ratios. Secondly, again following convention, he relies on the ancient Greek chord function by defining the sine BC (*sinus rectus*) as half the length of the chord BD . Thirdly, he allows the radius R of the base circle to be any chosen value. In the *De triangulis* Regiomontanus at times uses $R=60,000$ but at other times uses $R=10,000,000$. Such large radii were chosen to avoid having to work with decimal fractions.

Regiomontanus calls the circle's radius $R=AB$ the *sinus totus*, a term used already in medieval Islam that represents the greatest possible sine value. The right sine of $\widehat{BE}$ is BC ; in modern terms, $\sin(\widehat{BE}) = R \sin \widehat{BE} = BC$.⁶ This is the only function used in most of the *De triangulis*. What we call the cosine is called simply the *sinus complementi*, the sine of the complement of the given arc. Near the end of the book Regiomontanus uses the versed sine, the *sinus versus EC*, the difference between the *sinus totus* and the *sinus rectus*. This function originated in India.

Just like Ptolemy's *Almagest* a millennium and a half earlier, the *De triangulis* lacks an equivalent to the tangent function. In I.28, Regiomontanus describes how to find an angle in a right triangle if the ratio between two sides is known, a simple but nontrivial process if one does not have a tangent. But Regiomontanus did not have long to wait. In his popular collection of tables for spherical astronomy, the *Tabulae directionum* ("Tables of directions"),⁷ he borrowed several tables from his predecessor Giovanni Bianchini to solve stellar coordinate conversion problems.⁸ One of these tables, repeatedly borrowed in turn by various successors, was recognized as useful in many other calculations, hence the name bestowed on it by Regiomontanus, the *tabula fecunda* ("fruitful table"). Mathematically equivalent to the tangent, it would become accepted gradually as a full-fledged trigonometric function on its own.

Regiomontanus was the most frequently quoted trigonometer of the sixteenth century, and we shall see more of his influence later in this chapter. His definitions and terms, most of them not original to him but spread by him, became the foundation of the field. One of his early adopters was Erasmus

⁶ Here and throughout, we capitalize a trigonometric function if it is used with a circle with $R \neq 1$.

⁷ See [Van Brummelen 2009, 261–263], as well as [Delambre 1819, 292–293] and [Folkerts 1977, 234–236].

⁸ [Van Brummelen 2018].

Reinhold (1511–1553), one of the best quantitative astronomers of his generation. A colleague of Georg Rheticus at the University of Wittenberg, Reinhold was one of the first to receive a copy of Copernicus's work. Reinhold is most known for his very successful astronomical *Prutenic Tables*, but more relevant to us is his posthumous 1554 *Tabularum directionum*.⁹ This collection of tables is an expansion of Regiomontanus's work of the same name and includes a tangent table (“*canon fecundus*”) greatly expanded from Regiomontanus's. This table gives values to at least seven places for every minute of arc from 0° to 89° and for every 10 seconds of arc between 89° and 90° where the values change rapidly from entry to entry.¹⁰ To give the reader a sense of calculations in typical astronomical work of the time, we provide a short passage of his commentary on Copernicus, one of many where Reinhold uses his tangent table.

Text 1.2

Reinhold, a Calculation in a Planetary Model Using Sines and Tangents (from Reinhold's commentary on Copernicus's *De revolutionibus*)

Likewise, because angle FEN is $39^\circ 37' 38''$, therefore in right [triangle] EPL the remaining angle of LEP [that is, angle ELP] is $50^\circ 22' 22''$; and when EL is 100,000, then LP is 63,779 and PE is 77,021. And now when EL is taken to be 5,943, such that it is half the eccentricity, then LP is 3,790 and EP is 4,577. And from here, their doubles are $DQ = 7,580$ and $EQ = 9,154$, when $EN \dots$ is 100,000. Therefore, the whole of these, QEN , is 109,154. And with QN taken to be 10,000,000, then QD is 694,432. And from our table, angle DNQ is $3^\circ 58' 21''$.¹¹

Explanation: (See figure 1.2.) In the figure, D is the center of the universe and E is the center of the topmost eccentric deferent circle.

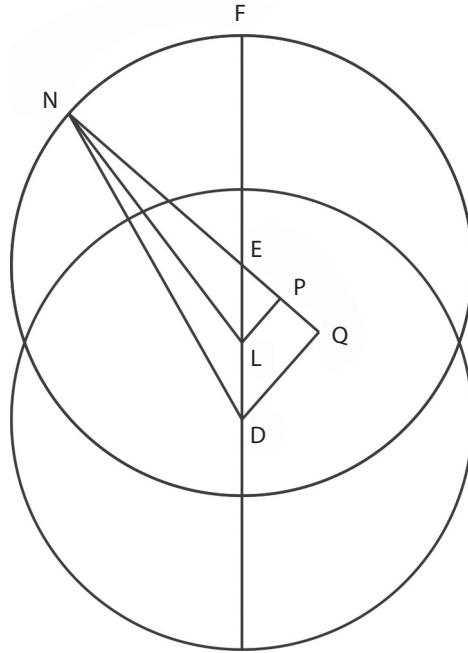
Reinhold knows that $\angle FEN = 39^\circ 37' 38''$ and wants to find $\angle QND$. Firstly, since $\angle FEN = \angle PEL$ and $\angle EPL$ is a right angle, $\angle ELP = 90^\circ - 39^\circ 37' 38'' = 50^\circ 22' 22''$. Next, in right-angled triangle EPL , Reinhold sets the hypotenuse $R = 100,000$. This allows him to use his Sine table; he finds $LP = \cos \angle ELP = R \cos \angle ELP = 63779$ and $PE = \sin \angle ELP = R \sin \angle ELP = 63779$. But EL is a known parameter with value 5,943, so LP and EP are scaled downward to 3,790 and 4,577, adjusting from the hypotenuse of 100,000 assumed by the Sine table to a hypotenuse of 5,943. Now, the astronomical model assumes that $EL = LD$, so the sides of triangle DQE are double those of $\triangle LPE$, which

⁹ [Reinhold 1554]. The “*canon fecundus*” may be found on folios 17 through 51.

¹⁰ The values in the table stray significantly away from the correct ones as the argument approaches 90° , a problem that plagued both medieval Islamic and especially early European table makers. See the account of Rheticus, Romanus, and Pitiscus in [Van Brummelen 2009, 280–282]. See also the analysis of early European tangent tables in [Pritchard, forthcoming].

¹¹ [Nobis/Pastori 2002, 246–247]. Translated from the Latin.

Figure 1.2
Reinhold's calculation with a planetary model using the tangent.



make $DQ = 2LP = 7,580$ and $EQ = 2EP = 9,154$. But EN is the radius of the circle, previously set to 100,000; therefore $QEN = 109,154$. Finally, consider right triangle NQD . Reinhold's *Canon fecundus* uses a radius of 10,000,000, so he sets QEN (the side adjacent to the angle we seek) equal to that value rather than 109,154. This requires him to adjust DQ 's value accordingly, from 7,580 upward to 694,432. He can now look up this value in the *Canon fecundus* (figure 1.3); we can see for ourselves that $\angle QND$ is between $3^\circ 58'$ and $3^\circ 59'$.

Clearly the tangent has come a long way from its initial role as a helper to Bianchini and Regiomontanus in solving stellar coordinate problems. Reinhold is now using his *Canon fecundus* as a general purpose tool for dealing with arbitrary right triangles.

The approach shared by Regiomontanus and Reinhold, dominant in the sixteenth century, was opposed by Georg Reticus (1514–1574). Known as the man who discovered Copernicus and convinced him to publish his heliocentric theory, Reticus hailed from the region of Rhaetia, which overlaps Austria, Switzerland, and Germany.¹² In his mid-twenties he visited Copernicus and became his student; he announced the heliocentric theory in his

¹² We have already discussed Reticus and Copernicus in [Van Brummelen 2009, 273–282]. For more on Reticus, see [Burmeister 1967–1968] and [Danielson 2006].

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Figure 1.3

A page from Reinhold's *Canon fecundus*. This section gives tangents from 0° to 4° , and cotangents from 86° to 90° . This page includes tangent values for arcs with minute values between $30'$ and $60'$; the grid on the facing page gives values for arcs with minute values between $0'$ and $30'$.

F E C U N D U S					18
	0	1	2	3	
30	87268	261859	436609	611625	30
31	90177	264770	439523	614544	31
32	93086	267681	442438	617464	32
33	95995	270592	445353	620384	33
34	98904	273503	448267	623304	34
35	101814	276414	451182	626225	35
36	104723	279325	454097	629145	36
37	107632	282237	457012	632066	37
38	110541	285148	459927	634986	38
39	113450	288059	462842	637907	39
40	116360	290970	465757	640828	40
41	119269	293882	468672	643749	41
42	122178	296794	471588	646671	42
43	125088	299705	474503	649592	43
44	127997	302617	477419	652514	44
45	130906	305528	480335	655435	45
46	133816	308439	483251	658357	46
47	136725	311351	486166	661278	47
48	139635	314262	489082	664200	48
49	142544	317174	491997	667121	49
50	145454	320085	494913	670043	50
51	148363	322997	497829	672965	51
52	151273	325909	500745	675888	52
53	154182	328821	503662	678810	53
54	157092	331733	506578	681732	54
55	160001	334645	509495	684656	55
56	162911	337558	512411	687578	56
57	165820	340470	515328	690501	57
58	168730	343382	518244	693423	58
59	171639	346295	521161	696346	59
60	174550	349207	524078	699269	60
	89	88	87	86	

Narratio prima and helped Copernicus bring his *De revolutionibus* (and separately its trigonometry under the title *De lateribus et angulis triangulorum*, “On the Sides and Angles of Triangles”)¹³ to press.

Rheticus’s accomplishments after Copernicus’s death in 1543 are primarily trigonometric, especially in the design and production of tables. His short 1551 tract *Canon doctrinae triangulorum* (“Table of the Doctrines of Triangles”),¹⁴ consisting of nothing more than a short introductory poem, 14 pages of tables, and a six-page dialogue, seems at first glance unassuming. But within its pages one finds not only tables of all six trigonometric func-

¹³ [Copernicus 1542]. For an account of the trigonometry in this treatise (which is not very original), see [Swerdlow/Neugebauer 1984, part 1, 99–104]. See also [Rosińska 1983], which argues that the sine table in this work was computed by Copernicus himself but corrected by Rheticus based on Regiomontanus’s tables.

¹⁴ [Rheticus 1551]. This treatise has an unusual history. Since it was placed on the *Index expurgatorius* (and since Rheticus’s later work, the *Opus palatinum*, rendered it obsolete), it disappeared from view after the sixteenth century. It was rediscovered by Augustus De Morgan in the mid-nineteenth century. See [De Morgan 1845], [Hunrath 1899], [Archibald 1949b], and [Archibald 1953]. [Roegel 2011d] contains a recomputation of all of its tables.

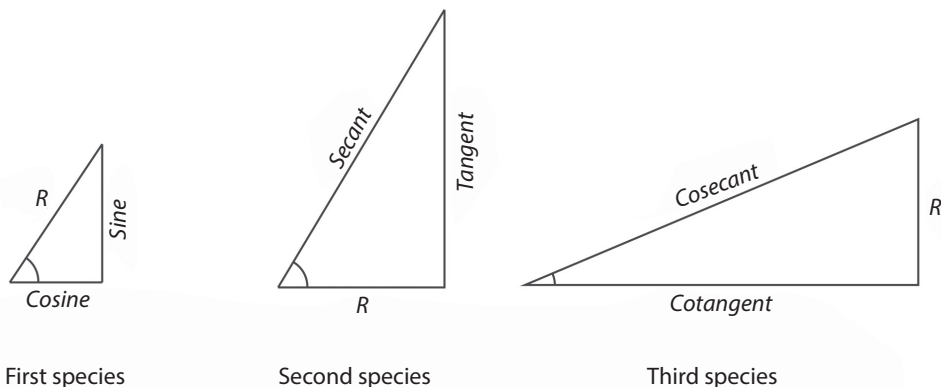


Figure 1.4

Rheticus's six trigonometric functions.

tions now considered standard (sine, cosine, secant, tangent, cosecant, and cotangent) but also a completely new and elegant set of terminology to describe them. Consider three “species” of right triangles (figure 1.4) described with respect to a given radius R .¹⁵ In the first species the hypotenuse is set equal to R ; in the second, the base; and in the third, the perpendicular. Then:

- in the first species, we have two functions, the perpendicular and the base (equivalent to Sine and Cosine respectively);
- in the second species, we have the hypotenuse and the perpendicular (Secant and Tangent); and
- in the third species, we have the hypotenuse and the base (Cosecant and Cotangent).

When Rheticus solves triangles, circles play no role. Thus, Rheticus's system not only defines all six trigonometric functions compactly but also divorces them from circular arcs: the arguments are now simply angles within the triangles, as they are today.

Rheticus found posthumous support for his design in the writings of the possibly the most well-known mathematician of the sixteenth century, François Viète (1540–1603). Viète's career was in the French civil service—not mathematics, on which he worked in his spare time. As a Huguenot during a time of unrest between Catholics and Protestants in France, his position was often hardly stable. He lived through an authorized massacre of Huguenots (which claimed the life of his older colleague Peter Ramus) and five years of

¹⁵In the *Canon doctrinae triangulorum* Rheticus sets $R = 10,000,000$; in the *Opus palatinum*, $R = 10,000,000,000$.

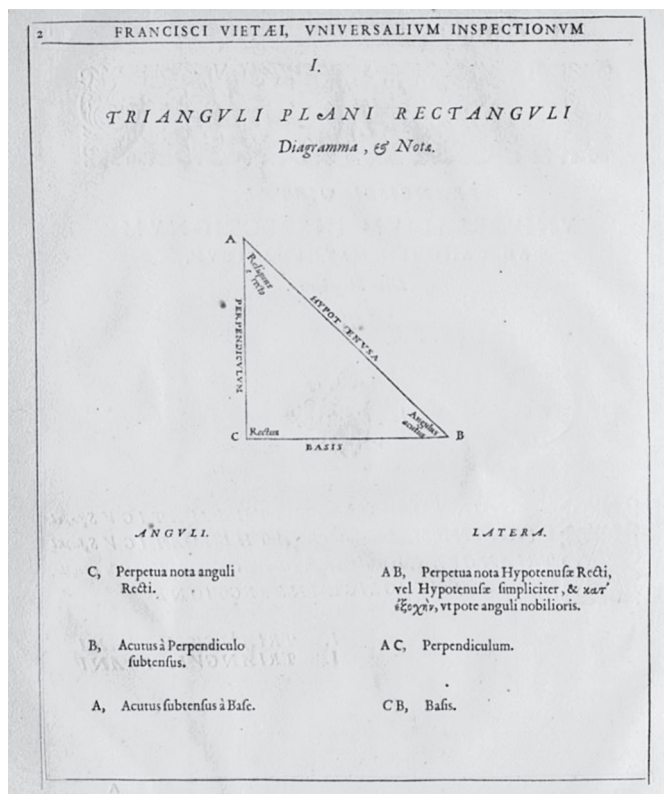


Figure 1.5

A page from Viète's *Canon mathematicus seu ad triangula* (1579), naming the sides and angles of a right triangle.

banishment from Paris, during which he worked on his mathematics. His interests were diverse, including astronomy and cryptography; but today he is recognized most for his contributions to the revolution of symbolic algebra, especially his *In artem analyticam isagoge*.¹⁶

While Viète's role in transforming algebra was fundamental, he was also deeply involved in the evolution of trigonometry. His first mathematical work, *Canon mathematicus seu ad triangula* ("Mathematical Canon, or On Trian-

¹⁶For editions and translations of Viète's mathematical treatises, see [Viète 1646; 1983] and [Viète/Girard/de Beaune 1986]; [van Egmond 1985] is a catalog of his works. None of these books contains *Canon mathematicus seu ad triangula* [Viète 1579], which occupies our attention here. See also [Ritter 1895] and [Reich/Gericke 1973]; the latter contains accounts of several of Viète's works in algebra. The secondary literature on Viète's role in the transformation of algebra is too large to be summarized here.

	<i>Hypotenusa</i>	<i>Perpendicularum</i>	<i>Basis</i>
I.	Totus	Sinus Anguli, vel Peripheriae (<i>sine</i>)	Sinus anguli Reliqui, seu Residuae peripheriae (<i>cosine</i>)
II.	Hypotenusa Faecundi Anguli, vel Peripheriae (<i>secant</i>)	Faecundus Anguli, vel Peripheriae (<i>tangent</i>)	Totus
III.	Hypotenusa Faecundi anguli Reliqui, vel Residuae peripheriae (<i>cosecant</i>)	Totus	Faecundus anguli Reliqui, vel Residuae peripheriae (<i>cotangent</i>)

Figure 1.6

Viète's nomenclature for the six trigonometric functions, taken from page 16 of *Universalium inspectionum* of his *Canon mathematicus seu ad triangula*. The Roman numerals on the left refer to Rheticus's triangle species.

gles," 1579),¹⁷ is an unusual volume—as close as it comes to being a coffee table book on trigonometry. For instance, the first page of text (figure 1.5) lays out the names of the sides and angles of a right-angled triangle with an eye to filling the page in a pleasing way. The book begins with a set of trigonometric tables designed according to the methods of Rheticus's *Canon doctrinae triangulorum*, with all six functions grouped according to the three triangle species we saw in figure 1.4. Although his names for the various functions often vary (see figure 1.6) and borrow the term *fecunda* from Regiomontanus, the structure clearly imitates that of Rheticus.¹⁸

Most of Viète's colleagues and contemporaries, however, were content to stick with the language of Regiomontanus.¹⁹ For instance, only eight years after the *De triangulis omnimodis* was published, the great German astronomical

¹⁷ See [Viète 1579], [Hunrath 1899], and [Rosenfeld 1988, 24–27]. See also [Roegel 2011g] for a recomputation of the tables.

¹⁸ See page 16 of the *Universalium inspectionum* within [Viète 1579], and [Ritter 1895, 40]. Viète applies the term *fecunda* to several quantities.

¹⁹ [Von Braunmühl 1900/1903, vol. 1, 183] suggests that Viète's unique notation here and elsewhere, brilliant as it was, may have contributed to his colleagues' lack of appetite for his trigonometric inventions. But Rheticus and Viète were not without followers; Adrianus Romanus's *Canon triangulorum* [Romanus 1609], for instance, adopts some of Viète's structure and terminology, including the terms "transsinuousae" for the secant and "prosinus" for the tangent (even though the standard terms are on the title page).

and geographical instrument maker Peter Apian (1495–1552)²⁰ had followed with his 1541 *Instrumentum sinuum seu primi mobilis*, a well-known treatise on trigonometric instruments and their use in solving various astronomical problems, which we shall consider later. Apian uses names that would have been familiar to Regiomontanus and his colleagues: the *sinus rectus primus* for the sine and the *sinus rectus secundus* for the cosine.²¹ There is no reference to Regiomontanus’s *tabula fecunda* or indeed to anything resembling a tangent function.

Apian’s traditional names for the sine and cosine are found again in the 1558 collection of works on spherical astronomy²² by Francesco Maurolico (1494–1575). A Sicilian priest, Maurolico held a variety of civil positions over the course of his life, including master of the mint, and was eventually appointed professor at the University of Messina. He was active in a wide variety of areas of mathematics and science, including optics and music; within astronomy he was especially prolific in spherical astronomy and edited several Greek works on the subject. Although he does not define the tangent and cotangent directly in his book on spherics, they do appear as *umbra versa* and *umbra recta* in Book II, Proposition 30,²³ as they often had before. These terms derive from ancient and medieval references to “shadows” in sundials, and Maurolico himself defines the *umbra versa* and *umbra recta* in this way in his astronomical treatise *De sphaera*, a work infamous for his vicious condemnation of Copernicus.²⁴ However, as we noted earlier, it was not from the *umbra versa* and *umbra recta* that the modern tangent and cotangent evolved.

We do find one innovation in Maurolico’s work on spherics. Near the end he describes a new table as follows: “In imitation of the *tabula fecunda* of Johannes Regiomontanus, we made another table which we have named *benefica*, because certain calculations become easy by means of this table.”²⁵

²⁰ For a general introduction to Apian’s mathematics see [Kaunzner 1997]; for his trigonometry see [Folkerts 1997].

²¹ See the third page of the first section of [Apian 1541], *Instrumentum hoc primi mobilis componere*.

²² [Maurolico 1558] (on which see [Moscheo 1992] on editorial issues) includes Latin editions of Theodosius’s *Spherics*, Menelaus’s *Spherics*, Autolycus’s *Spherics*, Theodosius’s *De habilitationibus*, and Euclid’s *Phenomena* as well as several small trigonometric tables (sine, *tabula fecunda*, *tabula benefica*, and declinations and ascensions) and a *Compendium mathematicae*. On Maurolico’s sources for his edition of Menelaus, see [Taha/Pinel 1997] or [Taha/Pinel 2001]. See also [Napoli 1876] for an edition of Maurolico’s *Geometricarum quaestionum*. [Rose 1975, 159–184] is a good account of Maurolico’s life and work.

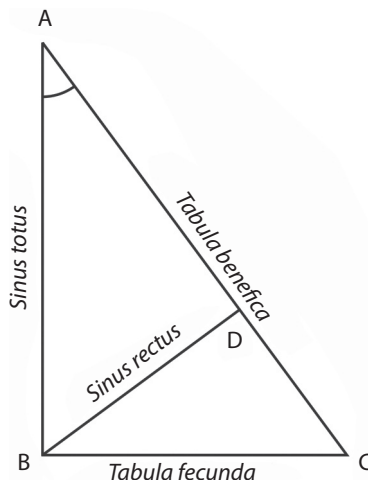
²³ [Maurolico 1558, f. 58].

²⁴ *De sphaera* is the first of a number of short treatises in *Opuscula mathematica*, [Maurolico 1575]; the definitions of *umbra versa* and *umbra recta* may be found on page 13. For Maurolico’s attack on Copernicus, see [Rosen 1957].

²⁵ [Maurolico 1558, f. 60], *Demonstratio tabulae beneficae*.

Figure 1.7

Maurolico's trigonometric functions.



Maurolico's new table introduces what today we call the secant.²⁶ A short table at the end of the book²⁷ gives secant values, with $R = 100,000$, for integer arguments from 1° to 89° . Rheticus, of course, had already published tables of all six trigonometric functions seven years earlier in his *Canon doctrinae triangulorum*. But he had used his own unique terms and definitions, which make no appearance in Maurolico's work.²⁸ Instead, consider Maurolico's figure 1.7: within right triangle ABC , segment BD is perpendicular to AC . Set AB equal to $R = 100,000$. Then, given $\angle A$ at the top of the diagram, we may find BD from a table of sines, BC from the *tabula fecunda*, and AC from the *tabula benefica*.

It took another quarter century for the *tabula fecunda* and *tabula benefica* to take on their modern names of tangent and secant in Danish scholar Thomas Fincke's (1561–1656) *Geometriae rotundi* ("Geometry of Circles and Spheres").²⁹ Still a 22-year-old student in 1583 at its publication, Fincke switched to the study of medicine that same year. Over the course of his very long career, he held professorial positions in medicine, rhetoric, and mathematics and held a number of senior administrative posts (including rector and

²⁶ Copernicus composed a table of secants by hand, but it was never published. See [Glowatzki/Göttsche 1990, 190–192]. For an analysis of Maurolico's table, see [Van Brummelen/Byrne, forthcoming].

²⁷ Folio 66. As we shall see later, a controversy arose over whether Maurolico's table owed an unpaid debt to Rheticus.

²⁸ Here we differ from von Braunmühl's opinion that Maurolico was following Rheticus; see [von Braunmühl 1900/1903, I, 150–151].

²⁹ [Fincke 1583]. De Morgan first makes this identification in [De Morgan 1846]. See [Schönbeck 2004] for a detailed account of Fincke's life and a summary of the *Geometriae rotundi*.

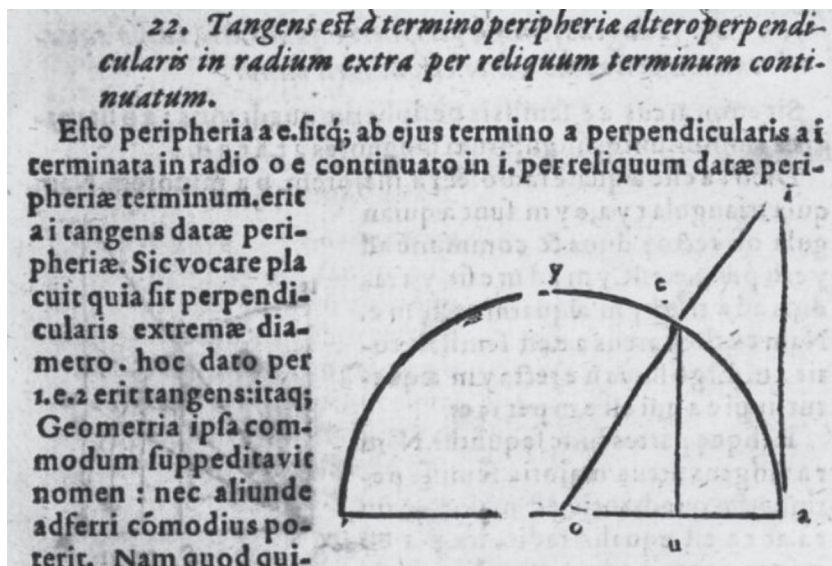


Figure 1.8
 Defining the tangent in Fincke's *Geometria rotundi*.

dean of the medical school for over half a century) at the University of Copenhagen. But the *Geometriae rotundi* remains his most enduring legacy. Inspired by Peter Ramus's 1569 *Geometria*, in a way the book is a step back to an older time, with its emphasis on the ancient spherical Menelaus's theorem.³⁰ However, it was found to be extremely clear and readable, and it was spoken of highly for several decades.

One of the *Geometriae rotundi*'s most lasting contributions was its creative use of language to simplify the presentation. Among his innovations were the inventions of the names "*tangens*" and "*secans*" for the tangent and secant functions respectively. In Proposition V.22 (figure 1.8), Fincke takes a semicircle of given radius, draws a vertical tangent from its rightmost point, and extends a diagonal at a given angle from center *O* until it touches the tangent line at *I*. Then the length of *AI*, naturally, is the "tangent" of that angle. A few propositions later (V.27), Fincke calls *OI* the secant since it crosses the circle's edge.³¹

The new names were instantly popular among Fincke's colleagues; they are found already three years later in Christoph Clavius's 1586 edition of The-

³⁰ See [Van Brummelen 2009, 56–61].

³¹ [Fincke 1583, 73–74, 76].

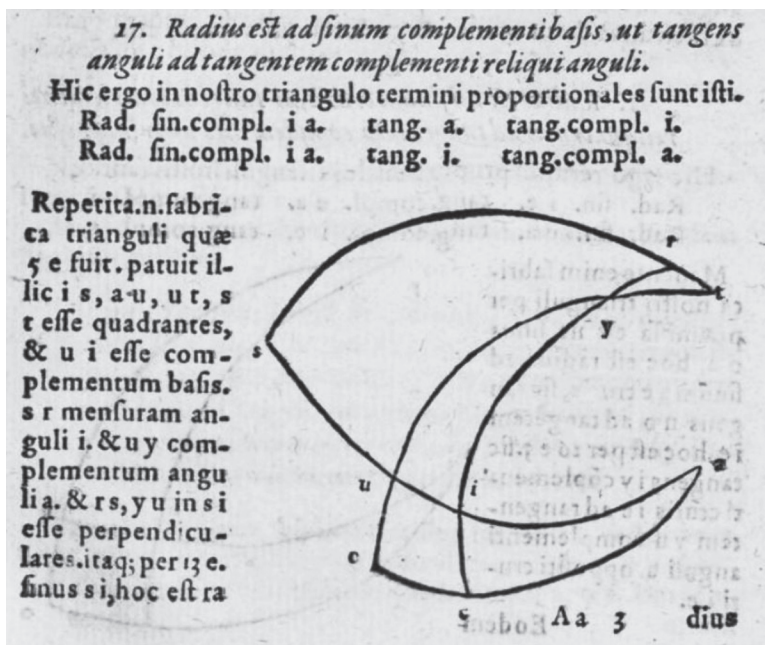


Figure 1.9

Fincke's expression of the relation $\cos c = \cot A \cot B$ for a right-angled spherical triangle, *Geometria rotundi* XIV.17. Book XIV contains the earliest appearances of the abbreviations "sin," "tang," and "sec"; the first two can be seen here.

odosius's *Spherics*³² as well as in Antonio Magini's 1592 *De planis triangulis* (which also contains the terms *tangens secunda* and *secans secunda* for cotangent and cosecant, paralleling the earlier usages of *sinus primus* and *sinus secundus* for sine and cosine), among other works.³³ The abbreviations for the words varied from one author to the next; well into the seventeenth century they had not yet become standardized. François Viète himself objected to the new terms, arguing that they could be too easily confused with other ways that the terms are used in geometry.³⁴ But in this case at least, Viète's opinion did not transform into practice.

³² [Clavius 1586]. In addition to Theodosius's *Spherics*, the book contains tables of tangents and secants (in which the name *benefica* also appears) and trigonometric treatises by Clavius himself.

³³ [Magini 1592]. [Cajori 1928–1929, vol. 2, 150–151] also refers to the use of these terms by Brahe, Lansberg, Blundeville, and Pitiscus.

³⁴ [Viète 1593, the third folio numbered 38] ("Immo vero artem confundunt, cum his vocibus necessae habeat uti Geometra abs relatione"); see also [Cajori 1928–1929, vol. 2, 150].

Book XIV, concluding Fincke’s *Geometriae rotundi* with some spherical trigonometric results, contains a significant notational development. Perhaps due to the length of text that would otherwise be required to state these theorems, Fincke abbreviates the trigonometric functions in ways that we recognize today. Here we find for the first time “sin.” for sine; “tan.” and “tang.” for tangent; “sec.” for secant; and “sin. comp.” or “sin. compl.” for cosine (and similarly for cotangent and cosecant). In figure 1.9, for instance, we see Fincke’s expression of the relation “ R is to $\text{Cos } ia$ as $\text{Tan } a$ is to $\text{Cot } i$ ” in the right-angled spherical triangle at the bottom of the diagram, equivalent to our $\cos c = \cot A \cot B$.

■ Trigonometric Tables Evolving

Until machines took over the world of computation, numerical tables were how trigonometry was used in the sciences, surveying, and navigation. Hipparchus’s invention of the trigonometric table to convert geometric statements into quantitative results was to extend far beyond his predictions of eclipses. In turn, the need for easily computed, yet accurate tables was the motive behind many of the theorems that are now taught in school. The basic formulas of plane trigonometry—for instance, the sine and cosine sum and difference laws and the half-angle formulas—were invented to simplify computations of tables.³⁵ And as we just saw, the tangent and the secant functions were introduced in Europe not as functions but as tables (the *tabula fecunda* and *tabula benefica*).

The late sixteenth century saw a spectacular rise in the production of trigonometric tables in terms of both the industry required to generate them and the quality of the results.³⁶ Almost every author participated in the table-making process (see figure 1.10); composing a table was a major part of what it meant to be a practitioner of the doctrine of triangles. Dealing with fractional quantities outside of the astronomers’ traditional sexagesimal (base 60) arithmetic was not in the standard toolbox until late in the sixteenth century; table makers usually got around this problem by using a base circle radius equal to some large power of ten.³⁷ Then, they could represent Sines, Cosines, and so on as large whole numbers.

³⁵ See [Van Brummelen 2009, 41–46, 70–77] for descriptions of trigonometric tables in ancient Greece and in multiple places elsewhere in the book for discussions of tables in medieval cultures.

³⁶ See [Glawatzki/Göttsche 1990] for a study of Regiomontanus’s trigonometric tables and those of his successors.

³⁷ At least one astronomer of the fifteenth century (Giovanni Bianchini) took some early steps toward decimal fractional notation, including the invention of the decimal point, which we shall describe shortly.

Author	Work	sin	tan	sec	R	Step size	Worst case error
Regiomontanus	<i>Tabulae directionum</i> (1490)	✓	✓		60,000 (sine) 100,000 (tangent)	1' (sine) 1° (tangent)	4 th of 7 decimal places
Apian	<i>Introductio geographica</i> (1541)	✓			100,000	1'	
Regiomontanus	<i>Tractatus Georgii Peurbachii...</i> (1541)	✓			6,000,000; 10,000,000	1'	
Copernicus	<i>De lateribus triangulorum</i> (1542)	✓			10,000,000	1'	
Rheticus	<i>Canon doctrinae triangulorum</i> (1551)	✓	✓	✓	10,000,000	10'	5 th of 10 decimal places
Reinhold	<i>Tabularum directionum</i> (1554)		✓		10,000,000	1' (10" after 89°)	4 th of 12 decimal places (for 89°59':5 th of 11 places)
Maurolico	<i>Theodosii sphaericorum</i> (1558)	✓	✓	✓	100,000	1°	7 th of 7 decimal places (for 89°59':5 th of 9 places)
Viète	<i>Canon mathematicus seu ad triangula</i> (1579)	✓	✓	✓	100,000,000	1'	9 th of 9 decimal places
Bressieu	<i>Metrices astronomicae</i> (1581)	✓	✓	✓	60 (three sexagesimal places)	1°	3 rd of 4 sexagesimal places
Fincke	<i>Geometriae rotundi</i> (1583)	✓	✓	✓	10,000,000	1'	5 th of 11 decimal places
Rheticus/Otho	<i>Opus palatinum</i> (1596)	✓	✓	✓	10,000,000,000	10"	7 th of 15 decimal places (for 89°59':9 th of 14 places)
Pitiscus	<i>Trigonometriae</i> (1600)	✓	✓	✓	100,000	1'	5 th of 9 decimal places
Van Roomen	<i>Canon triangulorum sphaericorum</i> (1607)	✓	✓	✓	1,000,000,000	10'	6 th of 12 decimal places
Pitiscus (Rheticus)	<i>Thesaurus mathematicus</i> (1613)	✓			1,000,000,000,000,000	10"	

Figure 1.10

Trigonometric tables from Regiomontanus to the eve of logarithms.

A quick examination of figure 1.10 reveals several noteworthy facts. Firstly, it took almost no time for the tangent and the secant functions, under various names, to be accepted and tabulated along with the sine.³⁸ Secondly, the increments between the arguments became smaller and smaller, achieving more accuracy at the cost of increased labor; the standard increment soon became 1' or even smaller. Finally, often unaware of it, all authors struggled with the entries of a trigonometric table that are most difficult to compute accurately: namely, values for the tangent and secant where the argument approaches 90°. These values were often calculated by dividing by a very small quantity such as the cosine of an angle near 90°.³⁹ Small rounding errors in

³⁸ In spherical trigonometry the function $\arcsin(\sin x \sin y)$ had currency through the sixteenth century and was often tabulated; see [Van Brummelen 2009, 263] on Regiomontanus's table and [Głowatski/Göttsche 1990, 197–207] for a summary.

³⁹ See [Pritchard, forthcoming].

the cosine values were thus magnified and became much larger errors in the tangent and secant values.⁴⁰

Several sixteenth-century European authors discussed their methods for computing sines.⁴¹ Usually their methods did not go much beyond what one finds already in the chord table in Ptolemy's *Almagest* along with those developed in early Islam and transmitted to Europe through al-Andalus. A typical early sixteenth-century text is Regiomontanus's *Compositio tabularum sinuum rectorum*, published 65 years after his death in 1541.⁴² Regiomontanus begins this work simply by stating that one can find the Sine of the complement of an arc whose Sine is known, using the Pythagorean Theorem:

$$\sin(90^\circ - \theta) = \sqrt{R^2 - \sin^2 \theta}. \quad (1.1)$$

He then determines the Sines of the *kardajas*, namely, the multiples of 15° , which can be obtained from the Sines of 30° , 45° , and 60° , a simple geometric argument deriving the Sine of 15° , and (1.1).⁴³ This results in a small table of sines, listed in the order of their computation rather than in increasing order, with $R = 600,000,000$:

<i>Arcus</i>	<i>Sinus</i>
90	600000000
30	300000000
60	519615242
45	424264069
15	155291427
75	579555496

⁴⁰ We have already discussed this problem with respect to Rheticus's tables in the *Opus palatinum*, their identification by Adriaan van Roomen, and the repairs to the table made by Pitiscus; see [Van Brummelen 2009, 280–282]. For the secant function, the alternative method $\sec^2 \theta = 1 + \tan^2 \theta$ was much less prone to error (assuming one has an accurate tangent table) and used occasionally; see [Van Brummelen/Byrne, forthcoming].

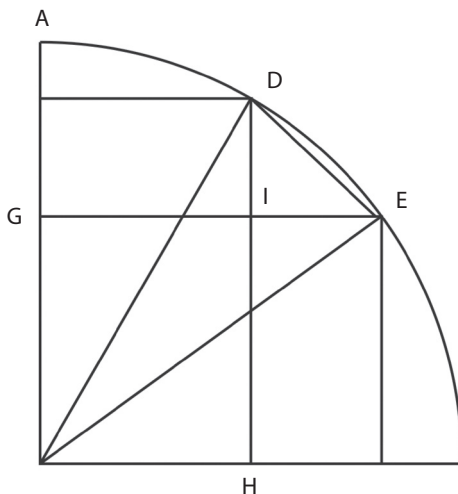
⁴¹ Occasionally they also discussed the computation of tangents and secants but usually only briefly and simply.

⁴² Published as an appendix to [Peurbach 1541]; [Glowatzki/Göttsche 1990, 11–24] contains a reproduction of the manuscript and a translation to German. This is not the earliest sixteenth-century publication describing the calculation of a sine table; Peter Apian's *Introductio geographica* (1533) contains both a sine table (reprinted a year later in his *Instrumentum sinuum seu primi mobilis*) and a description. See [Folkerts 1997, 225–226] for a brief account. The *Instrumentum sinuum seu primi mobilis* also contains a small table of arc sines, the earliest such table of which I am aware with clearly trigonometric intent. An early description of the construction of a sine table, using similar methods and almost contemporaneous with Regiomontanus, may be found in Oronce Fine's 1542 *De sinibus*; see [Ross 1977].

⁴³ The *kardajas*, from the Persian for “sections,” are found in medieval India, Islam, and Europe. For a modern account of this and the following proposition, see [Zeller 1944, 33–34].

Figure 1.11

Regiomontanus's calculation of the length of a side of a regular 15-gon.



Proposition 3 gives

$$\frac{\frac{1}{2}R}{\sin \theta} = \frac{\sin \theta}{\text{Vers } 2\theta}, \quad (1.2)$$

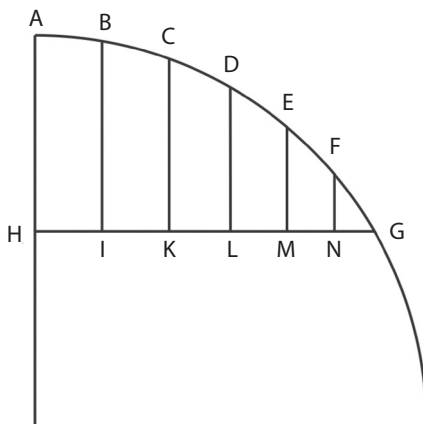
an equivalent to the Sine half-angle formula. This gives Regiomontanus all the tools he needs to find the Sines of all the multiples of $3^\circ 45'$, which he promptly does, in a table similar in form to the above. Proposition 4 uses constructions of the regular pentagon and decagon inscribed in a circle, as Ptolemy and many others had done, to determine the values of a couple of more difficult Sines.⁴⁴ For instance, since the side of the inscribed pentagon is equal to the chord of a 72° arc, half of the side of the pentagon is the Sine of 36° . Once these values are known, proposition 5 allows Regiomontanus to find the length of a side of a regular 15-gon inscribed in a circle, as follows (figure 1.11): in a quadrant of radius R , let $AD = 30^\circ$ and $AE = 54^\circ$. Then $EI = EG - IG = \sin 54^\circ - \sin 30^\circ = \sin(90^\circ - 36^\circ) - \sin 30^\circ$ and $ID = DH - HI = \sin 60^\circ - \sin 36^\circ$, so we can calculate a value for $ED = \sqrt{EI^2 + ID^2}$. But $\widehat{ED} = 24^\circ$ is one side of the regular 15-gon, so $\frac{1}{2}ED = \sin 12^\circ$. Now that we have a value for $\sin 12^\circ$, apply the half-angle formula four times to get $\sin 45'$. Once Regiomontanus has this value under his belt, he needs only time, patience, and the Sine sum and difference laws to find the Sines of all multiples of $45'$.

But all of this work is only a precursor to the most challenging problem in the calculation of Sine tables, namely that of finding the Sine values of multiples not of $45'$ but of 1° (or $1'$). The only Sines that can be found using

⁴⁴ See [Van Brummelen 2009, 72–74].

Figure 1.12

Regiomontanus's method to calculate $\sin 1^\circ$.



geometry alone are those whose arcs can be written in the form $3m/2^n$ for whole numbers m and n . To go beyond this set, mathematicians since Ptolemy had had to find a way somehow to break the bounds of the methods available to geometry. Regiomontanus proceeds as follows. Within the quadrant, cut six arcs of $1/4^\circ$ each: $AB, BC, \dots, FG$ (figure 1.12); and drop a perpendicular from G onto AH . Then drop perpendiculars from B, C, D, E , and F to HG . $HI, HK, \dots$ are then the Sines of the successive multiples of $1/4^\circ$, up to $HG = \sin 1\frac{1}{2}^\circ$. By a lemma (omitted here, although one can see it is true by inspection), Regiomontanus knows that $HI > IK > \dots > NG$. Since he already knows from his table calculations that $HL = \sin \frac{3^\circ}{4} = 7,853,773$ (in a circle of radius 600,000,000), he determines

$$\begin{aligned} \sin 1^\circ = HM = HL + LM &< \frac{4}{3} HL = \frac{4}{3} \sin \frac{3^\circ}{4} = \frac{4}{3} (7853773) \\ &= 10,471,697. \end{aligned} \quad (1.3)$$

Similarly, knowing also that $HG = \sin 1\frac{1}{2}^\circ = 15,706,169$, he finds

$$\begin{aligned} \sin 1^\circ = HM = HL + LM &> HL + \frac{1}{3} LG = \\ \sin \frac{3^\circ}{4} + \frac{1}{3} \left(\sin 1\frac{1}{2}^\circ - \sin \frac{3^\circ}{4} \right) &= 10,471,238. \end{aligned} \quad (1.4)$$

The result is a narrow interval containing $\sin 1^\circ$:

$$10,471,238 < \sin 1^\circ < 10,471,697. \quad (1.5)$$

From here Regiomontanus uses his half-angle formula to obtain⁴⁵

$$5,235,818 < \sin \frac{1^\circ}{2} < 5,236,044. \quad (1.6)$$

Since he wishes to compute a sine table with $R=6,000,000$ rather than 600,000,000, Regiomontanus divides by 100, leaving

$$52,358 < \sin \frac{1^\circ}{2} < 52,360, \quad (1.7)$$

from which he concludes that $\sin \frac{1^\circ}{2} = 52,359$. Armed with this approximation, the half-angle formula, the Sine sum and difference laws, and a lot of patience, he is able to fill in the Sines of all the multiples of $\frac{1}{4}^\circ$.⁴⁶

This technique is an enhancement on the approach used by Ptolemy in the *Almagest*, but it is essentially the same idea. Various eastern Arabic enhancements of Ptolemy's procedure from the tenth and eleventh centuries had generated similar results.⁴⁷ Curiously, only a few decades before Regiomontanus wrote this treatise but far to the East in Samarqand, Jāmshīd al-Kāshī had overturned the rules of this problem by introducing algebra and an iterative procedure that allows the determination of $\sin 1^\circ$ to as many places as one has the patience to calculate. However, his solution was not to find its way to Europe.⁴⁸ Even more curiously and much closer to Regiomontanus's home, his older colleague Giovanni Bianchini had done something similar, also with a method capable of generating arbitrary levels of precision, and we know that Regiomontanus became aware of it at some point.⁴⁹ However, there is no trace of anything new on this topic in this work.

The divide over terminology that we saw in the previous section was about to make a reappearance in the context of tables. Rheticus's new structure and his tables for all six trigonometric functions appeared only a decade after the publication of Regiomontanus's book, in the 1551 *Canon doctrinae triangulorum*. While this latter work eventually became very difficult to find, clearly the word about it spread through the mathematical community; his name is mentioned frequently in the late sixteenth century in conjunction with the new trigonometric functions well before his massive *Opus palatinum*,

⁴⁵ These two values are in error in the last two places, but this is about to become irrelevant.

⁴⁶ Regiomontanus goes on to describe how to enhance the process to work one's way down to $\sin 1'$, which would allow him to build a table with an increment of $1'$, but he does not provide the calculations.

⁴⁷ See [Van Brummelen 2009, 140–145].

⁴⁸ See [Van Brummelen 2009, 146–149].

⁴⁹ See [Gerl 1989, 265–268]. A marginal note by Regiomontanus in the margin of the manuscript Cracow BJ 558 (f.22v) states that Bianchini's method is superior to Ptolemy's.

a full treatment of his trigonometry with gigantic tables, was published in 1596. In fact, although Maurolico published his table of secants under a different name (*tabula benefica*) imitating the style of Regiomontanus in 1558, Thomas Fincke asserted in his 1583 *Geometriae rotundi* that Maurolico had simply taken over Rheticus's secant table. Magini, in his 1592 *De planis triangulis*, defended Maurolico, arguing that he had worked independently of Rheticus.

The question may be resolved by a closer inspection of Maurolico's table, which gives the secant for $R = 100,000$ and for every degree up to 89° .⁵⁰ Since the secant grows without bound as the argument approaches 90° , the last few values in any secant table are difficult to compute and are highly sensitive to rounding errors. For instance, the correct value of $\text{Sec } 89^\circ$ is 5,729,869. Maurolico's value is 5,729,868 while Rheticus's is 5,729,838.⁵¹ Another example: immediately below Maurolico's table, he gives a few values of $\text{Sec } \theta$ for arguments greater than 89° , one of which ($89^\circ 30'$) has the same argument as an entry in Rheticus's table. The correct value of $\text{Sec } 89^\circ 30'$ is 11,459,301; Maurolico's is 11,459,309; Rheticus's value is 11,459,348. In both cases (and in others) Maurolico's value is much more accurate than Rheticus's. Therefore, he did not appropriate Rheticus's table.⁵²

François Viète dealt with the problem of finding Sine values for arguments where geometry alone does not suffice, both early and late in his career. In his 1579 *Canon mathematicus seu ad triangula*, he determines $\sin 1'$ as follows.⁵³ Beginning with $\sin 30^\circ = 0.5$, he applies the sine half-angle formula (in the form $\sin^2(\theta/2) = \frac{1}{2} \text{vers } \theta$) 11 times in a row. In his last two iterations he finds

$$\begin{aligned}\sin^2\left(\frac{450'}{256}\right) &= 0.000000261455205834 \text{ and} \\ \sin^2\left(\frac{225'}{256}\right) &= 0.000000065363805733.\end{aligned}\tag{1.8}$$

⁵⁰ [Von Braunmühl 1900/1903, vol. 1, 151–152] reports on the controversy and mentions a table of secants by Maurolico with arguments up to 45° ; this table is mentioned by several later writers, apparently taking their information from von Braunmühl. The manuscript in fact does contain a secant table as described by von Braunmühl but in two columns, the first of which ends at 45° . Perhaps von Braunmühl did not notice the second column and thus did not have the opportunity to compare the values in the two secant tables for arguments near 90° .

⁵¹ This entry cannot be a typographical error since Rheticus's interpolation column confirms this value. Since Rheticus's value for R is larger, it contains two more decimal places, suppressed here; likewise for the entry for $\text{sec } 89^\circ 30'$.

⁵² For a full analysis and the background to the controversy, see [Van Brummelen/Byrne, forthcoming].

⁵³ See [Viète 1579, 62–67]. For the reader's ease, we have converted Viète's calculations to a base circle of $R = 1$.

From these values Viète derives two estimates for $\sin 1'$ as follows:

$$\sin 1' > \sqrt{\left(\frac{256}{450}\right)^2 \cdot 0.000000261455205834} = 0.0002908881959 \quad (1.9)$$

and

$$\sin 1' < \sqrt{\left(\frac{256}{225}\right)^2 \cdot 0.000000065363805733} = 0.0002908882056. \quad (1.10)$$

The former comes from the assertion that $\frac{\sin(\frac{450'}{256})}{\sin 1'} < \frac{450/256}{1}$; the latter comes from $\frac{\sin 1'}{\sin(\frac{225'}{256})} < \frac{1}{225/256}$. As impressive as these calculations are, this inequality—the heart of Viète’s method—goes all the way back to Ptolemy’s *Almagest*. Now, since $\frac{225'}{256}$ is closer to $1'$ than $\frac{450'}{256}$ is, Viète proposes (but does not carry out in the text) that the final value for $\sin 1'$ should be a weighted average favoring (1.10) over (1.9). This would result in $\sin 1' \approx 0.0002908882042$, a value that is completely accurate except for the last decimal place. Decades later, Viète would invent (but not carry out) a method that applies algebra to the problem in the spirit of al-Kāshī; we shall examine it later in this chapter.

Also in the *Canon mathematicus*, we find a very large and rather odd table, the *Canonion triangulorum laterum rationalium*.⁵⁴ Within it, Viète provides 45 pages of over 1,400 Pythagorean triples, scaled so that one of the three sides of the triangle is exactly equal to 100,000. These triples are ordered sequentially so that they can be used as a trigonometric table. Their values can be quite complicated. For instance, the first entry is

$$\frac{19,988,480,000}{49,942,416,589} \text{ and } 99,999 \frac{49,942,376,589}{49,942,416,589};$$

and in fact, the square root of the sum of the squares of these two numbers is precisely 100,000. Viète himself states at the end of the *Canon mathematicus* that the *Canonion* “is of very little use.”⁵⁵ One wonders, then, why he put so much effort into it. Perhaps he was concerned about issues of roundoff error in conventional tables, or he wished not to stray from the realm of pure geometry into approximation, or he thought of this work more as number theory

⁵⁴ [Viète 1579], pages numbered separately as pp. 1–45. See also [Tanner 1977] for offshoots of this work by Torporley and Harriot, [Hutton 1811b, 5–6], [Zeller 1944, 73–74], and [Roegel 2011h] for a reconstruction of Viète’s table.

⁵⁵ [Viète 1579, 75].

than as support for astronomy. We shall encounter this “rational trigonometry” again in chapter 5.

Before we move on, it is also worth mentioning an unusual small treatise by Nicolaus Raymarus Ursus (1551–1600), a German astronomer known primarily for his rivalry with Tycho Brahe over priority to the geoheliocentric system for the motions of the planets. The work in which he propounded this model, his 1588 *Fundamentum astronomicum*,⁵⁶ also contains some computational mathematics, including discussions of the computation of sine tables. Here he refers, not entirely clearly, to a method developed by his teacher Joost Bürgi involving finite differences, which we shall discuss later.⁵⁷ The method Ursus describes for finding $\sin 1'$ is similar to those we have seen before. However, once he has it, he uses the identity

$$2 \sin(A - x) \cos x - \sin A = \sin(A - 2x) \quad (1.11)$$

cleverly to fill in the remaining entries: starting with $A = 90^\circ$ and $x = 1'$ and the knowledge of $\sin 90^\circ$ and $\sin 89^\circ 59'$, he uses it to calculate $\sin 89^\circ 58'$; and by decreasing A again and again by one minute, he is able to calculate the sines of $89^\circ 57'$, $89^\circ 56'$, and so forth.⁵⁸ We shall see identities used in this way again, in chapter 3.

Meanwhile, Rheticus had died in 1574, but the massive tables of the *Opus palatinum* were finally published in 1596 by Valentin Otho. We have already described these tables elsewhere.⁵⁹ The 700-page tables, the largest ever compiled up to that time, contain all six of the standard trigonometric functions. Computed for every $10''$ of arc to ten decimal places, they constitute one of the most intensive computational efforts in human history. However, the methods Rheticus used, although inventive, did not extend beyond the approximation methods we have seen in this section. In fact, in figure 1.10 we see that Rheticus encountered the same difficulties with numerically sensitive trigonometric values that plagued almost all of his colleagues. The errors in Rheticus's tables were noticed by Romanus⁶⁰ and repaired by Pitiscus in 1607. Six years later Pitiscus would release *Thesaurus mathematicus*, an even more precise set of tables based on some of Rheticus's unpublished calculations.⁶¹

⁵⁶ [Ursus 1588]. On sine tables, see especially the second of the seven chapters.

⁵⁷ See [Delambre 1821, vol. 1, 289–291, 299–301].

⁵⁸ See an account in [Delambre 1821, vol. 1, 306–307].

⁵⁹ See [Van Brummelen 2009, 273–282]. Since then a recomputation of the entire set of tables has appeared ([Roegel 2011e]).

⁶⁰ See [Bockstaele 1992] for a Latin edition of the passage and a modern account of Romanus's criticism.

⁶¹ See the description in [Van Brummelen 2009, 281–282]. Since then [Roegel 2011c] has given a recomputation.

■ Algebraic Gems by Viète

A tantalizing hint suggests that Rheticus was dissatisfied with existing methods for the construction of sine tables; he may have been aware that the $3m/2^{n^o}$ barrier could be broken by solving an appropriate cubic equation as al-Kāshī had done (unbeknownst to Rheticus) just over a century earlier. Rheticus visited Gerolamo Cardano in 1545, the year Cardano published his solution to the cubic in his *Ars Magna*, “hoping it would be of some use to me in grappling with the science of triangles.”⁶² But he was sent away empty handed, and the *Opus palatinum* contains no hint of the use of a cubic equation. Its accomplishment, then, owes as much to industry as it does to creativity.

On the other hand, François Viète managed to make the transition to the algebraic problem, showed how to solve the relevant equations, and described how they could be used to generate sine tables—but he seems never to have implemented the solution. His methods appear in *Ad angularium sectionum analyticen*, published by Alexander Anderson in 1615 more than a decade after Viète’s death.⁶³ The key to the solution comes early in this work where Viète determines recurrence relations for $\sin n\theta$ and $\cos n\theta$.

Text 1.3

Viète, Finding a Recurrence Relation for $\sin n\theta$

(from *Ad angularium sectionum analyticen*)

Theorem IIII: If beginning as a point on the circumference of a circle any number of equal segments are laid off and straight lines are drawn [from the beginning point] to the individual points marking the segments, as the shortest is to the one next to it, so any of the others above the shortest will be [to] the sum of the two nearest to it.

[A geometric proof follows.]

(After Theorem VII:) Cut the circumference of a circle into a number of equal parts beginning at any assumed point and from it draw straight lines to the ends of the equal arcs. Let the shortest of these lines be Z and the next shortest B . Hence, from Theorem IIII, the first is to the second as the second is to the sum of the first and the third. The third, therefore, will be $(B^2 - Z^2)/Z$. By the same method used in the preceding [theorem],

the fourth will be $\frac{B^3 - 2Z^2B}{Z^2}$

⁶² [Danielson 2006, 121].

⁶³ See [Viète 1615]; it also appears as “Theoremata ad sectiones angulares” in [Viète 1646]. See [Viète (Witmer) 1983, 418–450] for a translation.

the fifth will be $\frac{B^4 - 3Z^2B^2 + Z^4}{Z^3}$

...

the tenth will be $\frac{B^9 - 8Z^2 + 21Z^4B^5 - 20Z^6B^3 + 5Z^8B}{Z^8}$.⁶⁴

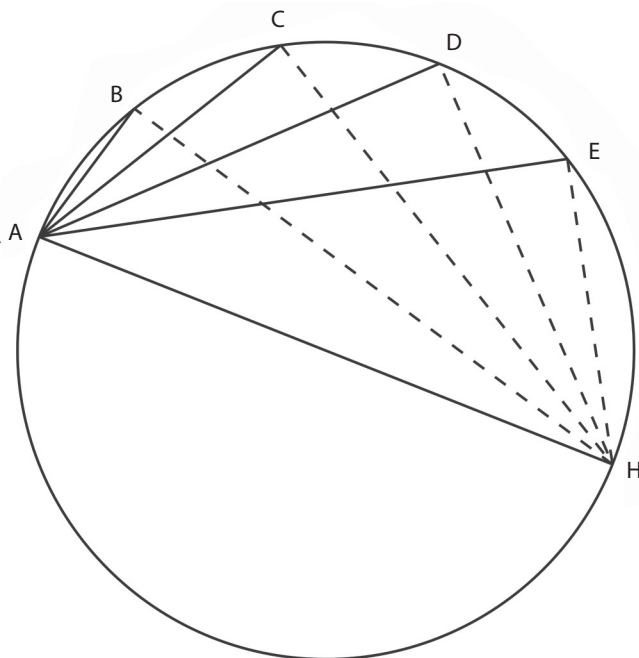


Figure 1.13

Viète's diagram for the sine multiple-angle recurrence relation (simplified). The dashed lines are used in our explanation but do not appear in Viète's figure.

Explanation: (See figure 1.13.) First, we must understand Viète's notation. Arcs $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$, and $\widehat{DE}$ are all equal; it is understood that AH has been divided into arbitrarily many arcs. AH is a diameter, which implies that the triangles ABH , ACH , and so forth are all right angled. Let θ be the angles $\angle AHB$, $\angle BHC$, $\angle CHD$, and $\angle DHE$; by *Elements* III.20, they are equal to half the posited arcs $\widehat{AB}$, $\widehat{BC}$, and so on. Then (assuming we are in a unit circle)

⁶⁴ [Viète (Witmer) 1983, 426, 435–436]. Viète's algebraic notation in the original differs somewhat

from Witmer's transcription; for instance, $(B^2 - Z^2)/Z$ is rendered as $\frac{Bq. - Zq.}{Z}$.

chord $Z=AB$ is equal to $2 \sin \theta$ while chord $B=AC$ is equal to $2 \sin 2\theta$. Viète asserts that

$$\frac{Z}{B} = \frac{D}{C+E}, \quad (1.12)$$

where D is the second-longest chord in the diagram, C is the third longest, and E is the longest. In modern notation, this turns out to be equivalent to the recurrence relation

$$\frac{\sin \theta}{\sin 2\theta} = \frac{\sin(n-1)\theta}{\sin(n-2)\theta + \sin n\theta}. \quad (1.13)$$

Viète also determines a recurrence relation for cosines:

$$\frac{1}{2 \cos \theta} = \frac{\cos(n-1)\theta}{\cos(n-2)\theta + \cos n\theta}. \quad (1.14)$$

By increasing n successively by one and solving for $\sin n\theta$ each time, Viète is able to generate formulas for $\sin n\theta$ for any n , including an equivalent to the sine triple-angle formula used by al-Kāshī.⁶⁵

Viète compiles a table of the coefficients in the formulas for $\cos n\theta$, going as far as $n=21$.⁶⁶ Clearly, this would have been virtually impossible without his symbolic notation.

Was Viète simply showing off by deriving higher and higher multiple-angle formulas in this way? Perhaps. Certainly, he could hardly have illustrated more effectively the power of combining symbolic algebra with trigonometry; higher-order formulas beyond the triple-angle formula had not been discovered anywhere else, even in the Islamic world. But there was more to it than demonstrating his prowess. He reveals at least part of his intent at the end of *Ad angularium sectionum analyticen*: to find a precise value for $\sin 1'$ in order to construct a table of sines. He begins with a value for $\sin 18^\circ$, which is a value that one can compute using geometric theorems. From it, Viète applies his sine quintuple-angle formula, generating $\sin 3^\circ 36'$. This requires solving a quintic equation, which Viète does not explain how to do; however,

⁶⁵ It came to light in the nineteenth century that Joost Bürgi had followed a similar algebraic path; see [Wolf 1872–1876, 7–28; 1890, vol. 1, 169–175] and [von Braunmühl 1900/1903, vol. 1, 205–208] for accounts and [Roegel 2010a, 5–7] for a discussion of his sine table. Unfortunately, Bürgi's failure to publish rendered his work a dead end.

⁶⁶ Viète also derives equivalents to multiple-angle sine and cosine formulas up to $n=5$ in Propositions 48–51 of his *Ad logisticem speciosam notae priores*, published in 1631 with notes by Jean de Beaugrand; it is the second treatise in [Viète (van Schooten) 1646]. For an English translation see [Viète (Witmer) 1983, 72–74]; for a French translation see [Ritter 1868, 245–276]. Witmer remarks (pp. 6–7) that Viète comes close to, but does not quite arrive at, general expressions for $\cos n\theta$ and $\sin n\theta$.

in another work he had shown how to approximate solutions to polynomial equations.⁶⁷ Likewise, using the sine triple-angle formula (and solving a cubic), we may move from $\sin 60^\circ$ to $\sin 20^\circ$. Trisect again to get $\sin 6^\circ 40'$; then bisect to get $\sin 3^\circ 20'$. Apply the sine difference law to $3^\circ 36'$ and $3^\circ 20'$ to get $\sin 16'$; finally, bisect four times, and we have $\sin 1'$.⁶⁸ Viète never did implement this method, but three decades later Henry Briggs would exploit it in the construction of massive trigonometric tables in his *Trigonometria Britannica*.

We are not yet finished with Viète's algebra. Before applying his multiple-angle formulas to sine tables in the *Ad angularium*, Viète shows how one may work sometimes in the other direction using trigonometry to solve problems in algebra. His most spectacular example is his 1595 *Ad problema quod omnibus mathematicis totius orbis construendum proposuit Adrianus Romanus*.⁶⁹ This dramatic story begins two years earlier. In 1593 Romanus had proposed to the world an apparently unsolvable problem, to find roots of the 45th-degree equation

$$\begin{aligned} 45x - 3795x^3 + 95634x^5 - 1138500x^7 + 7811375x^9 - 34512075x^{11} \\ + 105306075x^{13} - 232676280x^{15} + 384942375x^{17} - 488494125x^{19} \\ + 483841800x^{21} - 378658800x^{23} + 236030652x^{25} - 117679100x^{27} \\ + 46955700x^{29} - 14945040x^{31} + 3764565x^{33} - 740259x^{35} + 111150x^{37} \\ - 12300x^{39} + 945x^{41} - 41x^{43} + x^{45} = K. \end{aligned} \quad (1.15)$$

A quick examination reveals that this is no ordinary 45th-degree polynomial; for instance, all the powers of x are odd. However, at first glance it is a mystery how, when presented this problem by a Dutch ambassador through the king of France, Viète was able to come up with one solution almost immediately, and 22 others by the next day.

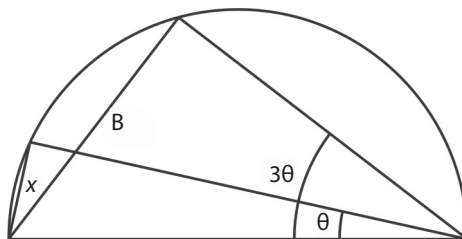
⁶⁷ *De numerosa potestatum purarum* [Viète 1600]; also available in [Viète 1646, 163–228]. The method for the extraction of roots is based on finding an initial approximation a to the solution x of the polynomial, substituting $a + b$ for x in the polynomial, and applying the binomial theorem to expand the result. See also [Goldstine 1977, 66–68].

⁶⁸ [Viète 1615, 47]; an English translation is in [Viète (Witmer) 1983, 450].

⁶⁹ [Viète 1595]; also available in [Viète 1646, 305–324]. Our account is based on [Viète (Witmer) 1983, 445n46], a translation of [Viète 1595, folio 12]. Viète deals with these issues in other treatises as well, including *De aequationum recognitione* and *Supplementum geometriae*, both available in [Viète 1646]. Viète's calculus of triangles, appearing also in *Ad logisticen speciosam notae priores* and *Zeteticorum*, has drawn attention; some of its calculations are isomorphic to the use of arithmetic with complex numbers, although [Glushkov 1977] is careful to point out the danger of such “unhistorical analysis”; see also [Itard 1968], [Bekken 2001], and [Reich 1973, chapter 3]. Also, [Bachmakova/Slavutin 1977] argue that Viète's calculations with triangles are dedicated to the solution of indeterminate equations.

Figure 1.14

Viète's solution of the irreducible cubic equation.



We illustrate with a (thankfully) simpler case, an example of the first “Theoremation” of *Ad problema*: the equation $3x - x^3 = 1$, an example of the irreducible (sometimes called “depressed”) cubic $ax - x^3 = b$ that Scipione del Ferro, Tartaglia, and Gerolamo Cardano had solved several decades earlier. Viète recognizes that the form of this cubic equation is related to the sine triple-angle formula that he expresses as $3R^2x - x^3 = R^2B$, where R is the base circle radius, x is the chord subtending angle θ in figure 1.14, and B is the chord subtending 3θ . If we are in a unit circle, then we may verify that $B = 2 \sin 3\theta$ and $x = 2 \sin \theta$. For our example we have $B = 1$. This implies that $\sin 3\theta = \frac{1}{2}$. Thus $3\theta = 30^\circ$ or 150° , so $\theta = 10^\circ$ or 50° . Hence $x = 2 \sin 10^\circ = 0.347296$ or $x = 2 \sin 50^\circ = 1.53208$, and Viète has found two of the three roots of the cubic equation. (Since Viète can consider only angles between 0 and 180° he cannot find the third root, which is negative.)

This remarkable use of trigonometry to solve the irreducible cubic can be extended to certain polynomials of higher powers using higher multiple-angle formulas, thereby extending beyond Cardano’s solutions of the cubic and quartic equations. Of course, bringing in a sine table to solve a polynomial alters the problem by expanding the set of tools permitted to generate a solution. Nevertheless, it is ingenious and, within its parameters, successful. One can see now how Viète upheld the honor of French mathematics by solving the 45th-degree polynomial so quickly: he recognized that it is the result of two angle trisections and a quintisection ($3 \times 3 \times 5$). He was able to generate only 23 of the 45 solutions for the same reason that we generated only two of the three solutions in our cubic; the other solutions are negative.⁷⁰

Through this tour de force, Viète had clearly demonstrated the power of the new algebra. He ends the treatise, and we end our treatment of Viète’s contributions to trigonometry, as follows: “Embrace the new, lovers of knowledge; farewell, and consult the just and the good.”⁷¹

⁷⁰ [Hollingdale 1984, 135–136] contains an account of how Viète might have gone about solving Romanus’s equation.

⁷¹ [Viète 1595, unnumbered folio after folio 13].

■ New Theorems, Plane and Spherical

Complete solutions to all conceivable triangles, both plane and spherical, had existed in Europe since Regiomontanus's *De triangulis omnimodis*, which remained the dominant textbook for most of the sixteenth century. One might wonder, then, what there was left to do. But Regiomontanus's book was written before advances in the mid-sixteenth century made possible certain ways to streamline the theory. Primary among these was the advent of the new functions, especially the tangent and the secant. Regiomontanus, restricted to the sine, cosine (expressed as the sine of the complement of the angle), and the versed sine, naturally approached solutions of triangles with only these three functions in mind. As the tangent and secant (and their complements) gradually established themselves as members of an expanded set of primitive functions, new and more attractive options for solving triangles became readily available.

Today, the most well known of the new sixteenth-century formulas is the planar Law of Tangents,⁷²

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}. \quad (1.16)$$

Most modern sources assign the first European appearance of this formula to Thomas Fincke in proposition X.15 of his 1583 *Geometriae rotundi*.⁷³ He introduces the law to solve triangles where two sides and the included angle are known. His first example illustrates how it works. Let $a=21$, $b=13$, and $\angle C=67^\circ 22' 49''$; then $\frac{1}{2}(a+b)=17$ and $\frac{1}{2}(a-b)=4$.⁷⁴ We also know that $\frac{1}{2}(A+B)=\frac{1}{2}(180^\circ-C)=56^\circ 18' 35''$, so by the Law of Tangents, $\frac{1}{2}(A-B)=19^\circ 26' 24''$. Finally, A and B may be found as the sum and difference of $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$ respectively, namely, $75^\circ 45'$ and $36^\circ 52' 11''$.

Many other authors picked up the Law of Tangents shortly after its appearance in Fincke's book.⁷⁵ We find it used for the same purpose in, for instance, Christoph Clavius's 1586 *Triangula rectilinea*,⁷⁶ Philip van Lansberge's 1591 *Triangulorum geometriae*,⁷⁷ and Viète's 1593 *Variorum de*

⁷² The theorem was known in medieval Islam, but (as far we know) it was not transmitted to Europe.

⁷³ [Fincke 1583, 292–293].

⁷⁴ Fincke expresses the left side of the Law of Tangents as $\frac{1}{2}(a-b)/\frac{1}{2}(a+b)$, which simplifies the calculations slightly.

⁷⁵ See [Tropfke 1903, vol. 2, 238] for a short discussion.

⁷⁶ In an appendix to his edition of Theodosius's *Spherics* [Clavius 1586, 328–329].

⁷⁷ [Van Lansberge 1591, 162].

(continued...)

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