# Contents

# Certifying the Thurston Norm via SL(2, $\mathbb{C}$)-twisted Homology

IAN AGOL AND NATHAN M. DUNFIELD

In memory of Bill Thurston: his amazing mathematics will live on, but as a collaborator, mentor, and friend he is sorely missed.

## 1 INTRODUCTION

For a compact orientable 3-manifold $M$, the Thurston norm on $H_2(M, \partial M; \mathbb{Z}) \cong H^1(M; \mathbb{Z})$ measures the minimal topological complexity of a surface representing a particular homology class. Twisted Alexander polynomials are a powerful tool for studying the Thurston norm; such a polynomial $\tau(M, \phi, \alpha)$ depends on a class $\phi \in H^1(M; \mathbb{Z})$ and a representation $\alpha \colon \pi_1(M) \to \mathrm{GL}(V)$, where $V$ is a finite-dimensional vector space over a field $K$. The polynomial $\tau(M, \phi, \alpha)$ is constructed from the homology with coefficients twisted by $\alpha$ of the cyclic cover of $M$ associated to $\phi$. The degree of any such $\tau(M, \phi, \alpha) \in K[t^{\pm 1}]$ gives a lower bound on the Thurston norm of $\phi$ [FK1]. Remarkably, Friedl and Vidussi [FV2] showed that given $M$ and $\phi$ one can always choose $\alpha$ so that this lower bound is sharp, with the possible exception of when $M$ is a closed graph manifold; their results rely on the fact that most Haken 3-manifold groups are full of cubulated goodness [Wis, Liu, PW1, PW2] so that [A] applies.

Here, we explore whether one can get sharp bounds from just representations to $\mathrm{SL}_2\mathbb{C}$, especially those that originate in a hyperbolic structure on $M$. When $M$ is the exterior of a hyperbolic knot $K$ in $S^3$, there is a well-defined *hyperbolic torsion polynomial* $\mathscr{T}_K \in \mathbb{C}[t^{\pm 1}]$ which is (a refinement of) the twisted Alexander polynomial associated to a lift to $\mathrm{SL}_2\mathbb{C}$ of the holonomy representation $\pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2\mathbb{C}$. The experimental evidence in [DFJ] forcefully led to

**Conjecture 1.1** (DFJ). *For a hyperbolic knot in $S^3$, the hyperbolic torsion polynomial determines the Seifert genus $g(K)$; precisely,* $\deg \mathscr{T}_K = 4g(K) - 2$.

Here, we prove this conjecture for a large class of knots, which includes infinitely many knots whose ordinary Alexander polynomial is trivial. We call a knot $K \subset S^3$ *libroid* if there is a collection $\Sigma$ of disjointly embedded minimal genus Seifert surfaces in its exterior $X = S^3 \setminus N(K)$ so that $X \setminus \Sigma$ is a union of books of $I$-bundles in a way that respects the structure of $X \setminus \Sigma$ as a sutured manifold; see Section 6.3 for the precise definitions. We show

**Theorem 6.2.** *Conjecture 1.1 holds for libroid hyperbolic knots in $S^3$.*

Libroid knots generalize the notion of a fibroid surface introduced in [CS], and includes all fibered knots. The class of libroid knots is closed under Murasugi sum (Lemma 6.6) and contains all *special* arborescent knots obtained from plumbing oriented bands (this includes 2-bridge knots), as well as many knots whose ordinary Alexander polynomial is trivial (Theorem 6.1). Previous to Theorem 6.2, Conjecture 1.1 was known only in the case of 2-bridge knots, by work of Morifuji and Tran [Mor, MT].

## 1.2 Motivation

While twisted Alexander polynomials give sharp bounds on the Thurston norm if one allows arbitrary representations to $\mathrm{GL}_n\mathbb{C}$ by [FV2], there are still compelling reasons to consider questions such as Conjecture 1.1. First, if the Thurston norm is detected by representations of uniformly bounded degree, then one should be able to use ideas from [Kup] to show that the KNOT GENUS problem of [AHT] is in $\mathbf{NP} \cap \mathbf{co\text{-}NP}$ for knots in $S^3$ using a finite-field version of $\tau(M,\phi,\alpha)$ as the $\mathbf{co\text{-}NP}$ certificate. (As with the results in [Kup], this would be conditional on the Generalized Riemann Hypothesis. Subsequent to our work here, Lackenby has shown that KNOT GENUS is in $\mathbf{NP} \cap \mathbf{co\text{-}NP}$ by different methods [Lac].) Second, since $\mathscr{T}_K$ is easily computable in practice, a proof of Conjecture 1.1 should lead to an effectively polynomial-time algorithm for computing $g(K)$ for knots in $S^3$. Finally, Conjecture 1.1 would be another beautiful Thurstonian connection between the topology and geometry of 3-manifolds.

## 1.3 Sutured manifolds

The Thurston norm bounds associated to twisted Alexander polynomials can be understood in the following framework of [FK2]. Throughout, see Section 2 for precise definitions. Let $M = (M, R_-, R_+, \gamma)$ be a sutured manifold. Given a representation $\alpha\colon \pi_1(M) \to \mathrm{GL}(V)$, we say that $M$ is an $\alpha$-homology product if the inclusion-induced maps

$$H_*(R_+; E_\alpha) \to H_*(M; E_\alpha) \quad \text{and} \quad H_*(R_-; E_\alpha) \to H_*(M, E_\alpha)$$

are all isomorphisms; here $E_\alpha$ is the system of local coefficients associated to $\alpha$. An $\alpha$-homology product is necessarily a taut sutured manifold (see Theorem 3.2 for the precise statement). Conversely, every taut sutured manifold is an $\alpha$-homology product for *some* representation $\alpha$ by [FK2]. A weaker, less geometric, parallel to Conjecture 1.1 is

**Conjecture 1.4.** *For a taut sutured manifold $M$, there exists $\alpha\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ for which $M$ is a homology product.*

Theorem 6.2 will follow easily from the next result, establishing a strong version of Conjecture 1.4 for books of $I$-bundles (see Section 4.3 for the definitions).

**Theorem 4.1.** *Let $M$ be a taut sutured manifold which is a book of $I$-bundles. Suppose $\alpha\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ has $\mathrm{tr}\left(\alpha(\gamma)\right) \neq 2$ for every curve $\gamma$ which is the core of a gluing annulus for an $I$-bundle page. Then $M$ is an $\alpha$-homology product.*

In trying to attack Conjecture 1.4, an intriguing aspect of Theorem 4.1 is the very weak hypothesis on the representation $\alpha$. Unfortunately, for more complicated taut sutured manifolds one must put additional restrictions on $\alpha$ to get a homology product, as the next result shows.

**Theorem 5.7.** *There exists a taut sutured manifold $M$ with a faithful discrete and purely hyperbolic representation $\alpha\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ where $M$ is not an $\alpha$-homology product. The manifold $M$ is acylindrical with respect to the pared locus consisting of the sutures.*

Another instance where we can prove Conjecture 1.4 is

**Theorem 4.2.** *Suppose $M$ is a sutured manifold which is a genus 2 handlebody with suture set $\gamma$ a single curve separating $\partial M$ into two once-punctured tori. If the pared manifold $(M, \gamma)$ is acylindrical and $M \setminus \gamma$ is incompressible, then $M$ is a homology product with respect to some $\alpha\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$.*

With both Theorems 4.1 and 4.2, it is easy to construct sutured manifolds satisfying their hypotheses which are *not* homology products with respect to $H_*(\,\cdot\,;\mathbb{Q})$.

## 1.5 Outline of contents

After reviewing the needed definitions in Section 2, we establish the basic properties of homology products in Section 3 and so relate Conjectures 1.1 and 1.4. Section 4 is devoted to proving Conjecture 1.4 in the two cases mentioned above. Section 5 studies one sutured manifold in detail, characterizing which $\mathrm{SL}_2\mathbb{C}$-representations make it a homology product (Theorem 5.5); Theorem 5.7 is an easy consequence of this. Finally, Section 6 is devoted to studying libroid knots, both showing that this is a large class of knots and also proving Theorem 6.2 follows from Theorem 4.1.

## 1.6 Acknowledgments

## 2  BACKGROUND

We begin with the precise definitions of the basic objects we will be working with. Throughout, all manifolds will be assumed orientable and moreover oriented.

### 2.1  Taut surfaces

For a connected surface, define $\chi_-(S) = \max\big(-\chi(S), 0\big)$; extend this to all surfaces via $\chi_-(S \sqcup S') = \chi_-(S) + \chi_-(S')$. For a 3-manifold $M$ and a (possibly empty) subsurface $A \subset \partial M$, the Thurston norm of $z \in H_2(M, A; \mathbb{Z})$ is defined by

$$\|z\| = \min \left\{ \chi_-(S) \mid S \text{ is a properly embedded surface representing } z \text{ with } \partial S \subset A \right\}.$$

A properly embedded compact surface $S$ in a 3-manifold $M$ is *taut* if $S$ is incompressible and realizes the Thurston norm for the class $[S, \partial S]$ in $H_2(M, N(\partial S); \mathbb{Z})$.

## 2.2   Sutured manifolds

A sutured manifold $(M, R_+, R_-, \gamma)$ is a compact 3-manifold with a partition of $\partial M$ into two subsurfaces $R_+$ and $R_-$ along their common boundary $\gamma$. The surface $R_+$ is oriented by the outward-pointing normal, and $R_-$ is oriented by the inward-pointing one. Note that the orientations of $R_\pm$ induce a common orientation on $\gamma$. A sutured manifold is *taut* if it is irreducible and the surfaces $R_\pm$ are both taut. A connected sutured manifold $M$ is *balanced* if it is irreducible, $\chi(R_-) = \chi(R_+)$, not a solid torus with $\gamma = \varnothing$, and if any component of $R_\pm$ has positive $\chi > 0$ then $M$ is $D^3$ with a single suture. A disconnected sutured manifold is balanced if each connected component is. Note that any taut sutured manifold is necessarily balanced.

## 2.3   Notes on conventions

We follow [Sch] in requiring taut surfaces to be incompressible; this is not universal, and the difference is just that the more restrictive definition excludes a solid torus with no sutures and a ball with more than one suture. Like [FK2] but unlike many sources, we do not allow torus sutures consisting of an entire torus component of $\partial M$. Our definition of balanced is slightly more restrictive than that of [FK2] and also differs from that of [Juh].

## 2.4   Twisted homology

Suppose $X$ is a connected CW complex with a representation $\alpha\colon \pi_1(X) \to \mathrm{GL}(V)$, where $V$ is a vector space over a field $K$. Let $E_\alpha$ be the system of local coefficients over $X$ corresponding to $\alpha$; precisely, $E_\alpha \to X$ is the induced vector bundle where we give each fiber the *discrete topology* so that $E_\alpha \to X$ is actually a covering map. (Alternatively, you can view $E_\alpha$ as an ordinary vector bundle equipped with a flat connection.) Throughout, we use the geometric definition of homology with local coefficients $H_*(X; E_\alpha)$ given in [Hat, pg. 330–336] which does not require a choice of basepoint; it is equivalent to the more algebraic definition of, e.g., [Hat, pg. 328–330]. More generally, if $X$ is not connected, we can consider a bundle $E \to X$ with fiber $V$ and the associated homology $H_*(X; E)$. We also use the geometrically defined cohomology $H^*(X; E)$ of [Hat, pg. 333]. Of course, both $H_*(X; E)$ and $H^*(X; E)$ satisfy all the usual properties: a relative version for $(X, A)$, long exact sequence of a pair, Mayer-Vietoris, etc.

If $X$ is a compact oriented $n$-manifold with $\partial X$ partitioned into two submanifolds with common boundary $A$ and $B$ then one has Poincaré duality:

$$D_M\colon\ H^k(X, A; E) \xrightarrow{\cong} H_{n-k}(X, B; E) \tag{2.5}$$

where $D_M$ is given by cap product with the ordinary relative fundamental class $[X, \partial X] \in H_n(X, \partial X; \mathbb{Z})$.

Let $E^* \to X$ denote the bundle where we have replaced each fiber with its dual vector space; for $E_\alpha$, this corresponds to using the dual or contragredient representation $\alpha^* \colon \pi_1(X) \to \mathrm{GL}(V^*)$ defined by $\alpha^*(g) = \left(\rho(g^{-1})\right)^*$. When $X$ has finitely many cells, the relevant version of universal coefficients is that $H_k(X; E) \cong H^k(X; E^*)$ as $K$-vector spaces.

When $E^* \cong E$ as bundles over $X$, we say that $E$ is self-dual. Examples include $E_\alpha$ where $\alpha \colon \pi_1(X) \to \mathrm{SL}_2 K$; specifically, $\alpha^*$ is conjugate to $\alpha$ via $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Seen another way, the action of $\mathrm{SL}_2 K$ on $K^2$ preserves the standard symplectic form $x_1 y_2 - x_2 y_1$ and hence $E_\alpha$ has a nondegenerate inner product on each fiber allowing us to identify $E_\alpha$ with $E_\alpha^*$. Representations that are unitary with respect to some involution on $K$ may not be self-dual, but still satisfy $H_*(X, A; E_\alpha) \cong H^*(X, A; E_\alpha)$, as $K$-vector spaces, for any $A \subset X$; we call such representations/bundles *homologically self-dual*.

## 3   BASICS OF TWISTED HOMOLOGY PRODUCTS

Throughout this section, $E$ will be a system of local coefficients over a sutured manifold $M$ with fiber a vector space of dimension $n \geq 1$. As in the introduction, we say that $M$ is an *E-homology product* if the inclusion induced maps $H_*(R_\pm; E) \to H_*(M; E)$ are both isomorphisms. This is equivalent to the notion of an *E-cohomology product* where $H^*(M; E) \to H^*(R_\pm; E)$ are isomorphisms: the former is the same as $H_*(M, R_\pm; E) = 0$, the latter is the same as $H^*(M, R_\pm; E) = 0$, and by Poincaré duality one has $H_k(M, R_\pm; E) \cong H^{3-k}(M, R_\mp; E)$. These concepts are parallel to [FK2], where they consider unitary representations of balanced sutured manifolds where $H_1(M, R_-; E_\alpha) = 0$ because of:

**Proposition 3.1.** *Suppose $M$ is a connected balanced sutured manifold with both $R_\pm$ nonempty. If $E$ is homologically self-dual, then $M$ is an $E$-homology product if and only if any one of the following eight groups vanish: $H_k(M, R_\pm; E)$ and $H^k(M, R_\pm; E)$ for $1 \leq k \leq 2$.*

*Proof.* Since both of $R_\pm$ are nonempty, it follows that $H_0(M, R_\pm; E) = H^0(M, R_\pm; E) = 0$, and so by Poincaré duality we have $H_3(M, R_\mp; E) = 0$. We focus on the case where $H_1(M, R_-; E) = 0$; the other cases are similar. Since $M$ is balanced, we have $\chi(R_-) = \chi(M)$ and hence $\chi\left(H_*(M, R_-; E)\right) = 0$. Since we know that $H_k(M, R_-; E) = 0$ for every $k \neq 2$, this forces $H_2(M, R_-; E) = 0$ as well. By Poincaré, we have $H^*(M, R_+; E) = 0$. Since $E$ is homologically self-dual, this gives $H_*(M, R_+; E) = 0$, and so $M$ is an $E$-homology product as claimed. $\square$

Our motivation for studying twisted homology products is the following two results:

**Theorem 3.2** (FK2, §3). *Suppose $M$ is an irreducible sutured manifold which is an E-homology product and where no component of $M$ is a solid torus without sutures. Then $M$ is taut.*

**Theorem 3.3** (FK2, §4). *Suppose $X$ is a compact irreducible 3-manifold with $\partial X$ a (possibly empty) union of tori. For $\phi \in H^1(X; \mathbb{Z})$ nontrivial and $\alpha \colon \pi_1(X) \to \mathrm{GL}(V)$, the torsion polynomial $\tau(X, \phi, \alpha)$ gives a sharp lower bound on the Thurston*

*norm* $\|\phi\|$ *if and only if when $S$ is a taut surface without nugatory tori dual to $\phi$ the sutured manifold $M$ which is $X$ cut along $S$ is an $\alpha$-homology product.*

Here, a set of torus components of a taut surface $S$ are *nugatory* if they collectively bound a submanifold of $X$ disjoint from $\partial X$. Theorem 3.2 is explicit and Theorem 3.3 is implicit in Sections 3 and 4 of [FK2] respectively; however, to make this paper more self-contained, we include proofs of both results.

Let $S$ be a properly embedded compact surface in a 3-manifold $N$; we do not assume either $S$ or $N$ is connected, and $N$ is allowed to be noncompact and have boundary. We say $S$ *separates* $N$ *into* $N_+$ *and* $N_-$ if $N = N_+ \cup_S N_-$, the positive side of $S$ is contained in $N_+$, the negative side of $S$ is contained in $N_-$, and every component of $N_\pm$ meets $S$. The linchpin for Theorems 3.2 and 3.3 is the following lemma, where all homology groups are with respect to some system $E$ of local coefficients on $N$, and all maps on homology are induced by inclusion:

**Lemma 3.4.** *Suppose $S$ separates $N$ into $N_\pm$. If both $H_*(N_\pm) \to H_*(N)$ are surjective then so are $H_*(S) \to H_*(N_\pm)$ and $H_*(S) \to H_*(N)$. Moreover, if for some $k$ both $H_k(N_\pm) \to H_k(N)$ are isomorphisms then so are $H_k(S) \to H_k(N_\pm)$ and $H_k(S) \to H_k(N)$.*

*Proof.* Since both $H_*(N_\pm) \to H_*(N)$ are surjective, the Mayer-Vietoris sequence for $N = N_+ \cup_S N_-$ splits into short exact sequences

$$0 \to H_k(S) \xrightarrow{i_+ \oplus i_-} H_k(N_+) \oplus H_k(N_-) \xrightarrow{j_+ - j_-} H_k(N) \to 0. \qquad (3.5)$$

To see that $H_k(S) \to H_k(N_+)$ is surjective, take $c_+ \in H_k(N_+)$ and choose $c_- \in H_k(N_-)$ which maps to the same element in $H_k(N)$ as $c_+$; then $(c_+, c_-) \mapsto 0$ under $j_+ - j_-$ and hence $c_+$ is the image of some element of $H_k(S)$ by exactness of (3.5). Symmetrically, $H_k(S) \to H_k(N_-)$ is also surjective, proving the first part of the lemma.

Suppose in addition that both $H_k(N_\pm) \cong H_k(N)$. Since $S$ is compact and $H_k(S)$ surjects $H_k(N_\pm)$ and $H_k(N)$, it follows that all four $K$-vector spaces are finite-dimensional. Since $H_k(N_\pm) \cong H_k(N)$, exactness of (3.5) forces $H_k(S) \cong H_k(N)$, and hence the surjections $H_k(S) \to H_k(N_\pm)$ must be isomorphisms as claimed.  $\square$

We now show that a sutured manifold which is a homology product must be taut.

*Proof of Theorem 3.2.* We may assume that $M$ is connected. All homology groups will have coefficients in $E$ unless otherwise indicated, and let $n$ be the dimension of the fiber of $E$. We first reduce to the case where every component of $R_\pm$ has $\chi \leq 0$. If a component of $R_\pm$ is a sphere, then $M$ must be $D^3$ by irreducibility with (say) $R_+ = \partial M$ and $R_- = \varnothing$. Since $E$ must be trivial over $D^3$, we get that $\dim(H_0(M)) = n$. However, $H_0(R_-) = 0$ contradicting that $M$ is an $E$-homology product. If some component of $R_\pm$ is a disc, say $D \subset R_+$, then $\dim(H_0(D)) = n$ and since a connected space will have $H_0$ of dimension at most $n$, we conclude

that $\dim(H_0(M)) = n$. However, then $E$ must be the trivial bundle, since nontrivial
monodromy around some loop would reduce $\dim(H_0(M))$ below $n$. It follows that
$M$ is also a homology product with respect to $H_*(\,\cdot\,; K)$, and hence $R_\pm$ are both
connected and thus discs; by irreducibility, $M$ is $D^3$ with one suture and hence
taut. So from now on we assume that every component of $R_\pm$ has $\chi \leq 0$.

Since we have excluded $M$ from being a solid torus with no sutures, all of the
torus components of $R_\pm$ are incompressible. Thus to prove that $M$ is taut it remains
to show that $R_\pm$ realize the Thurston norm of their common class in $H_2(M, N(\gamma); \mathbb{Z})$.
Note this is automatic if $R_+ = \varnothing$ since the homology product condition implies
$\chi(R_-) = 0$, so from now on we assume both $R_\pm$ are nonempty. Suppose $S$ is any
other surface in that homology class. Throwing away components of $S$ that bound
submanifolds of $M$ that are disjoint from $\partial M$, we can assume that $S$ separates $M$
into $M_\pm$, where each $M_\pm$ contains $R_\pm$ respectively. We next show that the theorem
follows from:

**Claim 3.6.** *The maps $H_k(R_\pm) \to H_k(M_\pm)$ are isomorphisms for $k \neq 1$ and injective
for $k = 1$. The maps $H_k(S) \to H_k(M_\pm)$ are isomorphisms for $k \neq 1$ and surjective
for $k = 1$.*

From the claim we get that $H_k(S) \cong H_k(R_\pm)$ for $k \neq 1$ and $\dim H_1(S) \geq \dim H_1(R_\pm)$;
hence

$$n \cdot \chi(S) = \chi\big(H_*(S)\big) \leq \chi\big(H_*(R_\pm)\big) = n \cdot \chi(R_\pm)$$

and so

$$\chi_-(S) \geq -\chi(S) \geq -\chi(R_\pm) = \chi_-(R_\pm).$$

Thus $R_\pm$ must realize the Thurston norm in its class, establishing the proposition
modulo Claim 3.6.

To prove the claim, first note that $S$, $R_\pm$, and $M_\pm$ are all homotopy equivalent to
2-complexes and so we need only consider $k \leq 2$. Since $R_\pm \hookrightarrow M$ gives isomorphisms
on $H_*$, we know $H_*(R_\pm) \to H_*(M_\pm)$ is injective and $H_*(M_\pm) \to H_*(M)$ is surjec-
tive. Since every component of $M_\pm$ meets $R_\pm$, it follows that $H_0(R_\pm) \to H_0(M_\pm)$
is onto and hence an isomorphism; consequently, so is $H_0(M_\pm) \to H_0(M)$. Since
$H_*(M, R_\pm) = 0$, the long exact sequence of the triple $(M, M_\pm, R_\pm)$ gives that
$H_2(M_\pm, R_\pm) \cong H_3(M, M_\pm)$; by excision and Poincaré duality, we have $H_3(M, M_\pm)$
$\cong H_3(M_\mp, S) \cong H^0(M_\mp, R_\mp)$ and the latter vanishes since each component of $M_\mp$
meets $R_\mp$. Thus we have shown $H_2(M_\pm, R_\pm) = 0$, and hence $H_2(R_\pm) \to H_2(M_\pm)$ is
an isomorphism.

By Lemma 3.4, we know that each $H_*(S) \to H_*(M_\pm)$ is surjective and more-
over is an isomorphism for $* = 0$. To see that $H_2(S) \to H_2(M_\pm)$ is an injection (and
hence an isomorphism), just note that $H_3(M_\pm, S) \cong H^0(M_\pm, R_\pm) = 0$. This proves
the claim and thus the theorem. $\square$

The last part of this section is devoted to proving the relationship between the
homology product condition and the Thurston norm bounds coming from twisted
torsion/Alexander polynomials.

*Proof of Theorem 3.3.* Let $n = \dim V$. All homology groups will have coefficients
in $E_\alpha$. Let $\widetilde{X}$ denote the infinite cyclic cover of $X$ corresponding to $\phi$; it has the

structure of a $\mathbb{Z}$'s worth of copies of $M$ stacked end to end so that the $R_+$ on one block is glued to the $R_-$ on the next. (Note that if $\phi$ is not primitive then $\widetilde{X}$ is disconnected; you can reduce to the case of $\phi$ primitive to avoid this issue if you prefer.) Let $\widetilde{S}$ be a lift of $S$ to $\widetilde{X}$ corresponding to the top of a preferred copy of $M$ in $\widetilde{X}$, and note that $\widetilde{S}$ separates $\widetilde{X}$ into $\widetilde{X}_+$ and $\widetilde{X}_-$ which consist of the blocks "above" and "below" $S$ respectively.

Unwinding the definitions, the precise form of the lower bound given in Theorem 14 of [FV1] (which is Theorem 6.6 in the arXiv version) is equivalent to

$$\|\phi\| \geq \frac{1}{n} \left( \dim H_1(\widetilde{X}) - \dim H_0(\widetilde{X}) - \dim H_2(\widetilde{X}) \right) \tag{3.7}$$

where if $H_1(\widetilde{X})$ is infinite-dimensional the convention is to declare the right-hand side as 0. (When $H_1(\widetilde{X})$ is finite-dimensional so is $H_2(\widetilde{X})$, see, e.g., [FV1].)

The only if direction is easy: if $M$ is an $\alpha$-homology product, the Mayer-Vietoris sequence and the fact that homology is compactly supported imply that $\widetilde{S} \hookrightarrow \widetilde{X}$ gives an isomorphism on $H_*$; thus one has

$$\|\phi\| = \chi_-(S) \geq -\chi(S) = -\frac{1}{n} \chi\big(H_*(S)\big) = -\frac{1}{n} \chi\big(H_*(\widetilde{X})\big) = \text{RHS of (3.7)} \tag{3.8}$$

where we have used that $H_3(\widetilde{X})$ must be 0 since $\widetilde{X}$ is noncompact.

Conversely, suppose that (3.7) is sharp. We will show:

**Claim 3.9.** *The maps $H_*(\widetilde{S}) \to H_*(\widetilde{X}_\pm) \to H_*(\widetilde{X})$ are all isomorphisms.*

The claim implies the theorem as follows: if we take $\widetilde{X}'_-$ to be $\widetilde{X}_-$ shifted down by one, we have $\widetilde{X} = \widetilde{X}'_- \cup_{\widetilde{S}'} M \cup_{\widetilde{S}} \widetilde{X}_+$. Applying the Mayer-Vietoris sequence to this decomposition, the claim gives that $H_*(M) \to H_*(\widetilde{X})$ is an isomorphism. Again by the claim, the inclusions of $\widetilde{S} = R_+$ and $\widetilde{S}' = R_-$ into $M$ induce isomorphisms on $H_*$, and so $M$ is a homology product.

To prove the claim, begin by noting that $H_*(\widetilde{X})$ is finitely generated, and the $\mathbb{Z}$-action on $\widetilde{X}$ can take any particular generating set to one which lies entirely in $\widetilde{X}_+$; hence $H_*(\widetilde{X}_+) \to H_*(\widetilde{X})$ is onto, as is $H_*(\widetilde{X}_-) \to H_*(\widetilde{X})$. By Lemma 3.4, we know $H_*(\widetilde{S}) \to H_*(\widetilde{X}_\pm)$ is onto and an isomorphism when $* = 2$ since $H_3(\widetilde{X}, \widetilde{X}_\pm) \cong H_3(\widetilde{X}_\mp, \widetilde{S}) \cong 0$ since (each component of) $\widetilde{X}_\mp$ is noncompact. For $* = 0$, we can build a compact subset $A$ of $\widetilde{X}_\pm$ so that $H_0(A) \to H_0(\widetilde{X}_\pm)$ is onto and $H_0(A) \to H_0(\widetilde{X})$ is an isomorphism; consequently, $H_0(\widetilde{X}_\pm) \to H_0(\widetilde{X})$ is an isomorphism and hence so is $H_0(S) \to H_0(\widetilde{X}_\pm)$ by Lemma 3.4. Finally, from (3.8), we see that $\chi(H_*(S)) = \chi(H_*(X))$ and hence the surjection $H_1(S) \to H_1(X)$ must be an isomorphism, proving the claim and thus the theorem. $\qquad\square$

## 4   SOME HOMOLOGY PRODUCTS

This section is devoted to the proof of Conjecture 1.4 in two nontrivial cases, both of which include many examples which are not $\mathbb{Q}$-homology products:

**Theorem 4.1.** *Let $M$ be a taut sutured manifold which is a book of $I$-bundles. Suppose $\alpha\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ has $\mathrm{tr}\big(\alpha(\gamma)\big) \neq 2$ for every curve $\gamma$ which is the core of a gluing annulus for an $I$-bundle page. Then $M$ is an $\alpha$-homology product.*

**Theorem 4.2.** *Suppose $M$ is a sutured manifold which is a genus 2 handlebody with suture set $\gamma$ a single curve separating $\partial M$ into two once-punctured tori. If the pared manifold $(M,\gamma)$ is acylindrical and $M \setminus \gamma$ is incompressible, then $M$ is a homology product with respect to some $\alpha\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$.*

## 4.3 Books of *I*-bundles

Recall that a *book of I-bundles* is a 3-manifold built from solid tori (the *bindings*) and $I$-bundles over possibly nonorientable compact surfaces (the *pages*) glued in the following way. For a page $P$ which is an $I$-bundle over a surface $S$, the *vertical annuli* are the components of the preimage of $\partial S$. One is allowed to glue such a vertical annulus to any homotopically essential annulus in the boundary of the binding. We do not require that all vertical annuli are glued; those that are not are called *free*. For a page $P$, the *vertical boundary* $\partial_v P$ is the union of all the vertical annuli; the *horizontal boundary* $\partial_h P$ is $\partial P \setminus \partial_v P$. We say a sutured manifold is a book of $I$-bundles if the underlying manifold has such a description where the sutures are exactly the cores of the free vertical annuli.

**Lemma 4.4.** *If $M$ is a taut sutured manifold which is a book of $I$-bundles, then it has such a structure where all the pages are product $I$-bundles. If the base surface of a page $P$ is not an annulus, then one component of the horizontal boundary is contained in $R_+$ and the other contained in $R_-$. The cores of the vertical annuli in the alternate description are homotopic to those in the original one.*

*Proof.* Suppose some page $P$ is a twisted $I$-bundle over a connected nonorientable surface $S$. Then the horizontal boundary $\partial_h P$ is connected and hence contained entirely in one of $R_{\pm}$, say $R_+$. Then $(R_+ \setminus \partial_h P) \cup \partial_v P$ is a surface homologous to $R_+$ with Euler characteristic $\chi(R_+) - 2\chi(S)$. Since $R_+$ is taut, we must have that $S$ is a Möbius band. The pair $(P, \partial_v P)$ is homeomorphic to a solid torus $B$ with an annulus that represents twice a generator of $\pi_1(B)$. Thus we can replace $P$ with a product bundle over the annulus to which we have attached a new component of the binding.

If a page $P$ is a product $I$-bundle over an orientable surface $S$, the same argument shows that if $\partial_h P$ is contained in just one of $R_+$ and $R_-$ then the base surface must be an annulus. This proves the lemma. $\square$

The proof of Theorem 4.1 rests on the following simple observation.

**Lemma 4.5.** *Suppose $\alpha\colon \pi_1(S^1) \to \mathrm{SL}_2\mathbb{C}$ is such that $\mathrm{tr}\big(\alpha(\gamma)\big) \neq 2$ where $\gamma$ is a generator of $\pi_1(S^1)$. Then $H_*(S^1; E_\alpha) = 0$.*

*Proof.* As with any space, $H_0(S^1; E_\alpha)$ is the set of co-invariants of $\alpha$, that is, the quotient of $\mathbb{C}^2$ by $\big\{ \alpha(g)v - v \;\big|\; g \in \pi_1(S^1), v \in \mathbb{C}^2 \big\}$. If $\alpha(\gamma)$ is diagonalizable, then this is 0 since neither eigenvalue of $\alpha(\gamma)$ can be 1 by the trace condition; alternatively, if $\alpha(\gamma)$ is parabolic then by the trace assumption it is conjugate to $\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$

and again the co-invariants vanish. Since $0 = 2\chi(S^1) = \chi\big(H_*(S^1; E_\alpha)\big)$ it follows that $H_1(S^1; E_\alpha) = 0$ as well, proving the lemma. $\qquad\square$

We next establish the first main result of this section.

*Proof of Theorem 4.1.* As usual, all homology will have coefficients in $E_\alpha$. Consider the decomposition of $M$ into $B \cup_A P$, where $B$ is the binding, $P$ is the union of all the pages, and $A$ is the union of attaching annuli. By Lemma 4.4, we can assume that $P = (S \times [-1, 1]) \cup Y$ where $S \times \{\pm 1\} \subset R_\pm$ and $Y$ is a union of (annulus) $\times I$. By our hypothesis on $\alpha$, Lemma 4.5 implies that $H_*(A) = 0$ and $H_*(Y) = 0$. Moreover, $H_*(B) = 0$ since the generator of $\pi_1$(component of $B$) has a power which has $\operatorname{tr}(\alpha) \neq 2$ and hence must have $\operatorname{tr}(\alpha) \neq 2$ as well. Set $B' = B \cup Y$ and let $A' \subset A$ be the interface between $B'$ and $S \times [-1, 1]$. Applying Mayer-Vietoris to the decomposition $M = B' \cup_{A'} (S \times [-1, 1])$ immediately gives that $H_*(S \times [-1, 1]) \to H_*(M)$ is an isomorphism. The same reasoning shows that $H_*(S \times \pm 1) \to H_*(R_\pm)$ are isomorphisms. Combining, we get that $H_*(R_\pm) \to H_*(M)$ are isomorphisms, and so $M$ is an $\alpha$-homology product as claimed. $\qquad\square$

## 4.6  Acylindrical sutured handlebodies

We turn now to the proof of Theorem 4.2. The following is an immediate consequence of the results in [MFP].

**Theorem 4.7.** *Suppose $M$ is a sutured manifold where each component of $R_\pm$ is a torus. If the interior of $M$ has a complete hyperbolic metric of finite volume, then there exists a lift $\alpha\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ of its holonomy representation so that $H_*(M; E_\alpha) = 0$ and $H_*(R_\pm; E_\alpha) = 0$. In particular, $M$ is an $\alpha$-homology product.*

*Proof.* By Lemma 3.9 of [MFP], there is a lift $\alpha$ of the holonomy representation so that for each component of $\partial M$ there is some curve $c$ with $\operatorname{tr}\big(\alpha(c)\big) = -2$. Corollary 3.6 of [MFP] now implies that $H^*(\partial M; E_\alpha) = 0$, and Theorem 0.1 of [MFP] then gives that $H^*(M; E_\alpha) = 0$ as well. Since $E_\alpha$ is self-dual, it follows that $H_*(\partial M; E_\alpha) = H_*(M; E_\alpha) = 0$; since $H_*(\partial M; E_\alpha) = H_*(R_-; E_\alpha) \oplus H_*(R_+; E_\alpha)$ we are done. $\qquad\square$

**Lemma 4.8.** *Let $M$ be a sutured manifold and $N$ be the sutured manifold resulting from attaching a 2-handle to $M$ along a component of the suture set $\gamma$. Let $E$ be a system of local coefficients on $N$. If $N$ is an $E$-homology product then $M$ is an $E|_M$-homology product.*

This is a natural result since if $N$ is taut then so is $M$, though the converse is not always true.

*Proof.* Throughout, all homology is with coefficients in $E$. Let $\bar{R}_+ \subset \partial N$ be the extension of $R_+$ to the new sutured manifold $N$. Note that $\bar{R}_+ = R_+ \cup D^2$ and $N = M \cup (D^2 \times I)$. Consider the associated Mayer-Vietoris sequences and natural maps:

$$
\begin{array}{ccccccc}
\longrightarrow & H_k(S^1) & \longrightarrow & H_k(R_+) \oplus H_k(D^2) & \longrightarrow & H_k(\bar{R}_+) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H_k(S^1 \times I) & \longrightarrow & H_k(M) \oplus H_k(D^2 \times I) & \longrightarrow & H_k(N) & \longrightarrow
\end{array}
$$

The leftmost vertical arrow is an isomorphism since it comes from a homotopy equivalence. The rightmost vertical arrow is an isomorphism by hypothesis. By the five lemma, the middle arrow must be an isomorphism; since it is the direct sum of the maps $H_k(R_+) \to H_k(M)$ and $H_k(D^2) \to H_k(D^2 \times I)$ we conclude that $H_k(R_+) \to H_k(M)$ is an isomorphism. The symmetric argument proves that $H_k(R_-) \to H_k(M)$ is an isomorphism for every $k$ and so $M$ is indeed an $E$-homology product. $\square$

**Theorem 4.9.** *Suppose that $M$ is a sutured manifold so that each component of $R_\pm$ is a (possibly) punctured torus. If adding 2-handles to $M$ along all the sutures results in a hyperbolic manifold, then there exists $\alpha \colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ so that $M$ is an $\alpha$-homology product.*

*Proof.* Let $N$ be the result of adding 2-handles to the sutures of $M$. Let $\alpha \colon \pi_1(N) \to \mathrm{SL}_2\mathbb{C}$ be the lift of the holonomy representation of the hyperbolic structure on $N$ given by Theorem 4.7. Applying Lemma 4.8 inductively shows that $M$ is a homology product with respect to the induced representation $\pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ as needed. $\square$

We can now prove the other main result of this section.

*Proof of Theorem 4.2.* By Theorem 4.9 it suffices to prove that the result $M_\gamma$ of attaching a 2-handle to $M$ along $\gamma$ is hyperbolic. Being a handlebody, $M$ is irreducible and atoroidal. Since $\partial M$ is compressible and $M \setminus \gamma$ is incompressible, Theorems A, 1, and 2 of [EM] together imply that $M_\gamma$ is irreducible, acylindrical, atoroidal, and has incompressible boundary (when applying Theorems 1 and 2, note that $\gamma$ is separating, which is one of the special cases mentioned in the final paragraph of the statements of these results). Thus $\mathrm{int}(M_\gamma)$ has a complete hyperbolic metric of finite volume as needed. $\square$

*Remark 4.10.* The representation $\alpha$ given in the proof of Theorem 4.2 may seem a bit unnatural since it is reducible on $\pi_1(R_\pm)$. However, it can be perturbed to $\beta$ for which $M$ is still a homology product and where $\beta$ is parabolic free on $\pi_1(M)$ and hence faithful. The point is just that the set of all such $\beta$ is the complement of a countable union of *proper* Zariski closed subsets in the character variety $X(M) \cong \mathbb{C}^3$, and hence is dense in $X(M)$. Specifically, as discussed in Section 5, the locus where $M$ is not a homology product is Zariski closed, as of course is the set where a fixed nontrivial $\gamma \in \pi_1(M)$ is parabolic.

## 5 AN EXAMPLE

Suppose $M$ is a balanced sutured manifold which is homeomorphic to a genus 2 handlebody. Assuming that each of $R_\pm$ is connected, then either $R_\pm$ are both tori with one boundary component or both pairs of pants. In this section, we compute

$H^1(M, R_+; E_\alpha)$ in a specific example as $\alpha$ varies over the $\mathrm{SL}_2\mathbb{C}$ character variety of $\pi_1(M)$, and so characterize the $\alpha$ for which $M$ is an $\alpha$-homology product. This leads to the proof of Theorem 5.7 which was discussed in the introduction.

## 5.1  Basic setup

Both $\pi_1(R_+)$ and $\pi_1(M)$ are free groups of rank two, say generated by $\langle x, y\rangle$ and $\langle a, b\rangle$ respectively; let $i_*\colon \pi_1(M)\to \pi_1(R_+)$ be the map induced by the inclusion $i\colon R_+ \hookrightarrow M$. For $w\in \langle x, y\rangle$ we denote its Fox derivatives in $\mathbb{Z}[\langle x, y\rangle]$ by $\partial_x w$ and $\partial_y w$, where

$$\partial_x x = 1, \quad \partial_x x^{-1} = -x^{-1}, \quad \partial_x y^{\pm 1} = 0, \quad \text{and} \quad \partial_x(w_1 \cdot w_2) = \partial_x w_1 + w_1 \cdot \partial_x w_2.$$

Now fix a representation $\alpha\colon \pi_1(M)\to \mathrm{GL}(V)$ where $\dim(V)=2$, and extend to a ring homomorphism $\alpha\colon \mathbb{Z}[\pi_1(M)]\to \mathrm{End}(V)$.

**Proposition 5.2.** *The sutured manifold $M$ is an $\alpha$-homology product precisely when the $4\times 4$ matrix*

$$\begin{pmatrix} \alpha\big(\partial_x i_*(a)\big) & \alpha\big(\partial_y i_*(a)\big) \\ \alpha\big(\partial_x i_*(b)\big) & \alpha\big(\partial_y i_*(b)\big) \end{pmatrix}$$

*has nonzero determinant.*

*Proof.* Consider the 2-complex $W$ with one vertex $v$, four edges $e_x, e_y, e_a, e_b$, and two faces $r_a, r_b$ with attaching maps specified by the words $i_*(a)\cdot a^{-1}$ and $i_*(b)\cdot b^{-1}$. For the subcomplex $B = e_a \cup e_b$, there is a map $j\colon (W, B)\to (M, R_+)$ which induces homotopy equivalences $W \to M$ and $B \to R_+$ corresponding to the natural maps on fundamental groups ($[e_x]\mapsto x$, $[e_a]\mapsto a$, etc.). By the long exact sequence of the pair and the five lemma, it follows that $j_*$ induces an isomorphism $H^*(M, R_+; E_\alpha)\to H^*(W, B; E_{\alpha\circ j_*})$.

By Proposition 3.1, to show $M$ is an $\alpha$-homology product, it remains to show $H^1(W, B; E_{\alpha\circ j_*}) = 0$. As a left module over $\Lambda = \mathbb{Z}[\langle x, y\rangle]$, the chain complex of the universal cover $\widetilde{W}$ of $W$ has the form:

$$C_*\big(\widetilde{W}; \mathbb{Z}\big)\colon \quad 0 \to \Lambda r_a \oplus \Lambda r_b \xrightarrow{\partial_2} \Lambda e_x \oplus \Lambda e_y \oplus \Lambda e_a \oplus \Lambda e_b \xrightarrow{\partial_1} \Lambda v \to 0.$$

Since $\Lambda$ is noncommutative, it is most natural to write the matrices $[\partial_i]$ for the left module maps $\partial_i$ so that they act on row vectors to their left, that is, $\partial_i(v) = v\cdot [\partial_i]$. In this form, we have the following, where we have denoted $i_*(a)$ and $i_*(b)$ in $\langle x, y\rangle$ by just $a$ and $b$:

$$[\partial_1] = \begin{pmatrix} x - 1 \\ y - 1 \\ a - 1 \\ b - 1 \end{pmatrix} \qquad [\partial_2] = \begin{pmatrix} \partial_x a & \partial_y a & -1 & 0 \\ \partial_x b & \partial_y b & 0 & -1 \end{pmatrix}$$

Applying the functor $\mathrm{Hom}(\,\cdot\,, V_\alpha)$ to get $C^*(W; E_{\alpha\circ j_*})$ has the effect of replacing each copy of $\Lambda$ with $V$, where the matrices of the coboundary maps $d^i$ are the
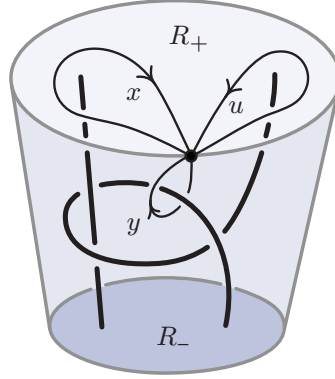
Figure 5.3: The sutured manifold $M$ sketched at left is $D^3$ with open neighborhoods of the two dark arcs removed, where $R_+$ and $R_-$ are the pairs of pants indicated. The manifold $M$ is homeomorphic to a handlebody, with $\pi_1(M)$ freely generated by the loops $x$ and $y$; the element $u$ in $\pi_1(M)$ is $yxyx^{-1}y^{-1}$. These claims can be checked by a straightforward calculation starting with a Reidemeister-like presentation for $\pi_1(M)$.

result of applying $\alpha\colon \Lambda \to \operatorname{End}(V)$ entrywise to the $[\partial_i]$; here the matrices $[d^i]$ act on column vectors to their right. Restricting to the subcomplex of cochains vanishing on $B$ gives:

$$C^*(W, B; E_{\alpha \circ j_*})\colon \quad 0 \leftarrow V^2 \xleftarrow{d^1} V^2 \leftarrow 0 \leftarrow 0$$

where $d^1$ is precisely the matrix in the statement of the proposition; the result follows. $\qquad\square$

## 5.4 Pants example

Let $M$ be the sutured manifold shown in Figure 5.3, where the free group $\pi_1(R_+)$ has generators

$$\pi_1(R_+) = \left\langle x,\, yxyx^{-1}y^{-1} \right\rangle.$$

Let $X(M)$ be the $\mathrm{SL}_2\mathbb{C}$ character variety of $\pi_1(M) = \langle x, y \rangle$. Now $X(M) \cong \mathbb{C}^3$ with coordinates $\{\bar{x}, \bar{y}, \bar{z}\}$ corresponding to the trace functions of $\{x, y, xy\}$. Despite the fact that $M$ is a product with respect to ordinary $\mathbb{Z}$ homology, we will show:

**Theorem 5.5.** *The locus $L$ of $[\alpha] \in X(M)$ where $M$ is not an $\alpha$-homology product is a (complex) 2-dimensional plane, namely $\left\{\bar{x} + \bar{y} - \bar{z} = 3\right\}$.*

*Remark 5.6.* Unlike for irreducible representations, characters $[\alpha] \in X(M)$ consisting of *reducible* representations may contain nonconjugate representations. For such classes, there is thus ambiguity in which local system $E$ to associate with $[\alpha]$. However, it turns out that whether $M$ is an $E$-homology product is independent of this choice. Similar to [DFJ, Lemma 7.1], the point is that reducible representations with the same character share the same diagonal part and one uses this with Proposition 5.2 to verify the claim; since our focus is on irreducible representations, we leave the details to the interested reader.

*Proof.* By Proposition 5.2, we are interested in when

$$\det \begin{pmatrix} \alpha(1) & 0 \\ \alpha(y - yxyx^{-1}) & \alpha(1 + yx - yxyx^{-1}y^{-1}) \end{pmatrix} = 0$$

or equivalently when $\det\big(\alpha(w)\big) = 0$ for $w = 1 + xy - xyx^{-1} \in \mathbb{Z}[\langle x, y\rangle]$. Any irreducible $\alpha$ can be conjugated so that

$$\alpha(x) = \begin{pmatrix} 0 & 1 \\ -1 & \bar{x} \end{pmatrix} \quad \text{and} \quad \alpha(y) = \begin{pmatrix} \bar{y} & -u \\ u^{-1} & 0 \end{pmatrix} \quad \text{where } u + u^{-1} = \bar{z}.$$

Applying this $\alpha$ to $w$ and eliminating variables yields that $\det\big(\alpha(w)\big) = 0$ if and only if $\bar{x} + \bar{y} - \bar{z} - 3 = 0$; thus $L$ is as claimed. $\qquad\square$

One representation in $L$ is $(\bar{x}, \bar{y}, \bar{z}) = (4, 4, 5)$ which can be realized by

$$\alpha(x) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad \alpha(y) = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}.$$

An easy calculation shows that the axes of these hyperbolic elements cross in $\mathbb{H}^2$; since $\alpha\big(xyx^{-1}y^{-1}\big)$ is also hyperbolic with negative trace, it follows that $\alpha\big(\langle x, y\rangle\big)$ is a Fuchsian Schottky group [Pur]. In particular, $\alpha$ is discrete, faithful, and purely hyperbolic. This proves:

**Theorem 5.7.** *There exists a taut sutured manifold $M$ with a faithful discrete and purely hyperbolic representation $\alpha\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$ where $M$ is not an $\alpha$-homology product. The manifold $M$ is acylindrical with respect to the pared locus consisting of the sutures.*

*Remark 5.8.* Representations that cover the same homomorphism $\pi_1(M) \to \mathrm{PSL}_2\mathbb{C}$ need not give rise to isomorphic cohomology. For example, the Schottky representation above covers the same $\mathrm{PSL}_2\mathbb{C}$ representation as $\beta$ where $(\bar{x}, \bar{y}, \bar{z}) = (-4, 4, -5)$, which is not in $L$, and hence $M$ is a $\beta$-homology product. In fact, in this example, every irreducible representation to $\mathrm{PSL}_2\mathbb{C}$ has *some* lift to $\mathrm{SL}_2\mathbb{C}$ for which $M$ is a homology product.

*Remark 5.9.* For each $N \geq 2$, the group $\mathrm{SL}_2\mathbb{C}$ has a unique irreducible $N$-dimensional complex representation, which we denote $\iota_N\colon \mathrm{SL}_2\mathbb{C} \to \mathrm{SL}_N\mathbb{C}$. Let $L_N$ be the locus of $[\alpha]$ in $X(M)$ where $M$ is not an $\iota_N \circ \alpha$ homology product. A straightforward calculation with Gröbner bases finds:

$$L_3 = \big\{ 2\bar{x}\bar{y}\bar{z} - \bar{x}^2 - \bar{y}^2 - 3\bar{z}^2 + 3 = 0 \big\}$$

$$\begin{aligned} L_4 = \big\{ &3\bar{x}^2\bar{y}^2\bar{z} - 3\bar{x}^2\bar{y}\bar{z}^2 - 3\bar{x}\bar{y}^2\bar{z}^2 + \bar{x}^4 - 2\bar{x}^3\bar{y} - 2\bar{x}\bar{y}^3 + \bar{y}^4 + 2\bar{x}^3\bar{z} \\ &+ 3\bar{x}^2\bar{y}\bar{z} + 3\bar{x}\bar{y}^2\bar{z} + 2\bar{y}^3\bar{z} - 3\bar{x}\bar{y}\bar{z}^2 + 2\bar{x}\bar{z}^3 + 2\bar{y}\bar{z}^3 + \bar{z}^4 - 3\bar{x}^3 - 3\bar{y}^3 \\ &+ 3\bar{z}^3 - 3\bar{x}^2 + 6\bar{x}\bar{y} - 3\bar{y}^2 - 6\bar{x}\bar{z} - 6\bar{y}\bar{z} - 3\bar{z}^2 + 6\bar{x} + 6\bar{y} - 6\bar{z} + 9 = 0 \big\}. \end{aligned}$$

The intersection $L_2 \cap L_3 \cap L_4$ is zero-dimensional, as one would expect from the intersection of three (complex) surfaces in $\mathbb{C}^3$. Computing out a bit farther, we found that $\bigcap_{N=2}^5 L_N = \bigcap_{N=2}^{10} L_N$ contains a single point $(\bar{x}, \bar{y}, \bar{z}) = (2, 2, 1)$ outside the reducible representations; in particular, there are no purely hyperbolic representations in this intersection.

## 6  LIBROID SEIFERT SURFACES

In this last section, we study libroid knots, a notion generalizing fibered knots and fibroid surfaces which is defined in Section 6.3 below. We will show that this is a large class of knots for which Conjecture 1.1 holds:

**Theorem 6.1.** *All special arborescent knots, except the $(2, n)$–torus knots, are hyperbolic libroid knots. Moreover, there are infinitely many hyperbolic libroid knots whose ordinary Alexander polynomial is trivial.*

**Theorem 6.2.** *Conjecture 1.1 holds for libroid hyperbolic knots in $S^3$.*

### 6.3  Library sutured manifolds

We call a taut sutured manifold $(M, R_\pm, \gamma)$ a *library* if there is a taut surface $(\Sigma, \partial\Sigma) \subset (M, N(\gamma))$ such that $[\Sigma] = n[R_+] \in H_2(M, N(\gamma); \mathbb{Z})$ for some $n \geq 0$, and the sutured manifold $M \setminus \Sigma$ is a book of $I$-bundles in the sense of Section 4.3. Note that $M \setminus \Sigma$ has at least $n + 1$ connected components, and thus is a collection of books of $I$-bundles, that is, a "library." We say that a taut surface $S \subset X^3$ is a *libroid surface* if $X \setminus S$ is a library sutured manifold. This generalizes the notion of a *fibroid surface* [CS], and in fact the surface $S \cup \Sigma$ is a fibroid surface. We say that a knot in $S^3$ is libroid if it has a minimal genus Seifert surface which is libroid. Definitions in hand, we now deduce Theorem 6.2 from Theorem 4.1.

*Proof of Theorem 6.2.* Let $K$ be a libroid knot with $X$ its exterior, and let $\alpha\colon \pi_1(X) \to \mathrm{SL}_2\mathbb{C}$ be a lift of the holonomy representation of the hyperbolic structure on $X$. Let $S$ be a minimal genus Seifert surface for $K$ which is libroid. By Theorem 3.3, we just need to show that the sutured manifold $M = X \setminus S$ is an $\alpha$-homology product. This is immediate if $M$ is a product, so we will assume from now on that $X$ is not fibered. Let $\{\Sigma_i\}$ be disjoint minimal genus Seifert surfaces cutting $M$ up into sutured manifolds that are each a book of $I$-bundles; for notational convenience, set $\Sigma_0 = S$. It is enough to show that each such book $B$ is an $\alpha$-homology product, since they are stacked one atop another to form $M$. To apply Theorem 4.1, we need to check that no core $\gamma$ of a gluing annulus has $\mathrm{tr}\big(\alpha(\gamma)\big) = 2$. Assume $\gamma$ is such a core, so in particular $\alpha(\gamma)$ is parabolic.

First note that $\gamma$ is isotopic to an essential curve in some $\Sigma_i$. Since $\Sigma_i$ is minimal genus and not a fiber, by Fenley [Fen] it is a quasi-Fuchsian surface in $X$ and in particular the only embedded curve in $\Sigma_i$ whose image under $\alpha$ is parabolic is $\partial\Sigma_i$, which is the homological longitude $\lambda \in \pi_1(\partial X)$. But by [Cal, Corollary 2.6] or [MFP, Corollary 3.11], one always has $\mathrm{tr}\big(\alpha(\lambda)\big) = -2$, which contradicts that $\alpha(\gamma)$ has trace $+2$. So we can apply Theorem 4.1 as desired, proving the theorem. $\square$

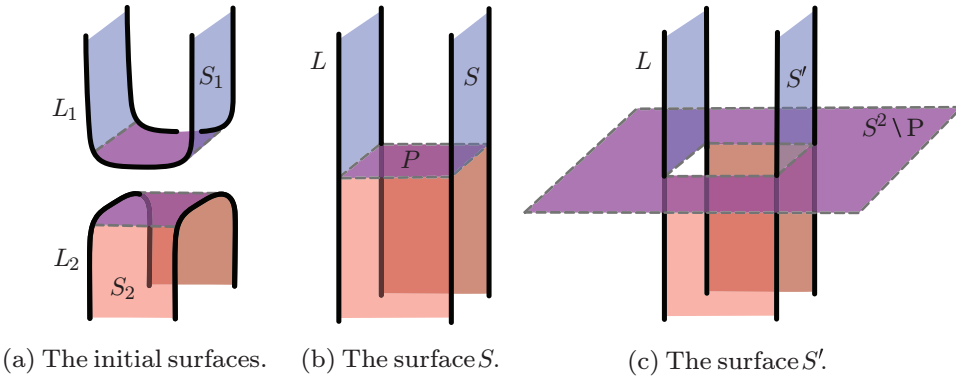(a) The initial surfaces.    (b) The surface $S$.    (c) The surface $S'$.

Figure 6.5: The Murasugi sum with $k = 2$.

## 6.4   A plethora of libroid knots

We now turn to showing that there are many hyperbolic libroid knots. A key tool for this will be the notion of Murasugi sum, which we quickly review. Consider two oriented surfaces with boundary $S_1$ and $S_2$ in $S^3$, and let $L_i = \partial S_i \subset S^3$ be the associated links. Suppose that $S_1$ and $S_2$ intersect so that there is a sphere $S^2 \subset S^3$ with $S^3 = B_1 \cup_{S^2} B_2$, so that $S_i \subset B_i$; see Figure 6.5(a, b), where the interface between $B_1$ and $B_2$ is a horizontal plane separating $S_1$ and $S_2$, which have been pulled apart slightly for clarity. Moreover, assume that $S_1 \cap S_2 = P \subset S^2$ is a $2k$-sided polygon, where the edges of $\partial P$ are cyclically numbered so that the odd edges lie in $L_1$, and the even edges lie in $L_2$. Also, assume that the orientations of $S_1$ and $S_2$ agree on $P$. Let $L = \partial(S_1 \cup S_2)$ be the link obtained as a boundary of the union of the two surfaces. Then $L$ is said to be obtained by *Murasugi sum* from $L_1$ and $L_2$. If $k = 1$, this is connected sum, and if $k = 2$, then this operation is known as *plumbing*. There are two natural Seifert surfaces for $L$ shown in Figure 6.5(b, c), given by $S = S_1 \cup S_2$, and $S' = \big((S_1 \cup S_2) - P\big) \cup \overline{(S^2 - P)}$. Note that $S'$ is also a Murasugi sum of the surfaces $(S_i - P) \cup \overline{S^2 - P}$, which are isotopic to $S_i$.

Gabai showed that if each $S_i$ is minimal genus, then so is $S$; similarly if each $S_i$ is a fiber, then so is $S$ [Gab1]. We generalize these results to:

**Lemma 6.6.** *If $S_1$ and $S_2$ are libroid surfaces, and $S$ is obtained from $S_1$ and $S_2$ by Murasugi sum, then $S$ is also a libroid surface.*

*Proof.* The Seifert surfaces $S$ and $S'$ for $L$ can be disjointly embedded as sketched in Figure 6.7. In detail, take a regular neighborhood $N(L)$, and form the exterior $E(L) = \overline{S^3 - N(L)}$. We'll use the notation above for Murasugi sum. Then $S^2 \cap E(L)$ is a $2k$-punctured sphere, dividing $E(L)$ into tangle complements $T_i = E(L) \cap B_i$. Take a regular neighborhood $R_{3-i}$ of $\overline{S_i - P} \cap T_i$ inside $T_i$; then the relative boundary of $R_{3-i}$ in $T_i$ is two parallel copies of $\overline{S_i - P}$. The union with $S^2 - (R_1 \cup R_2)$ gives our two disjointly embedded Seifert surfaces $S \cup S'$.

The complements $S^3 - S_i$, $S^3 - S$, and $S^3 - S'$ naturally admit sutured manifold structures as described in Section 4 of [Sak]. Moreover, the two complementary regions $S^3 - (S \cup S')$ may be identified with $(S^3 - S_i) \cup R_i$, where $R_i$ is the product sutured manifold described above, and $R_i$ is attached to $S^3 - S_i$ along $k$ product disks in the sutures corresponding to $R_i \cap S^2$ (recall $k$ is defined by $S_1 \cap S_2 = P$ is
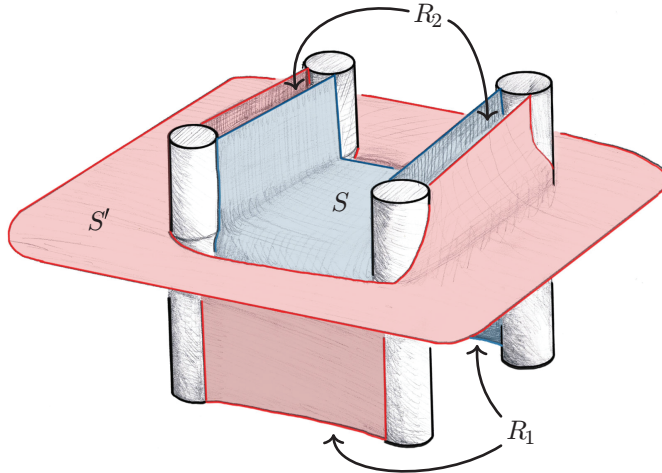
Figure 6.7: The surfaces $S$ and $S'$ made disjoint.

a $2k$-gon). But $S^3 - S_i$ is a library sutured manifold, which may be extended as products into $R_i$ to obtain a library decomposition of $S^3 - (S \cup S')$. Thus $S$ and $S'$ are libroid Seifert surfaces for $L$. □

*Remark 6.8.* The sutured manifold decomposition in the above proof is the same as that in [Sak, Condition 4.2]; while we first decompose along $S \cup S'$ and then along the $2k$ product disks and remove the product sutured manifolds $R_i$, Sakuma first decomposes along $S$ and then along the disk $S^2 - P$, resulting in the union of the sutured manifolds $S^3 - S_i$.

The class of *arborescent links* are those obtained by plumbing together twisted bands in a tree-like pattern (see, e.g., [Gab2, BS] for a definition). It is important to note that the bands are allowed to have an odd number of twists. With a few known exceptions, these links are hyperbolic (see [BS] or [FG, Theorem 1.5]). The subclass of *special arborescent links* studied by Sakuma [Sak] are those obtained by plumbing bands with even numbers of twists, and hence the plumbed surface is a Seifert surface for the link. Inductively applying Lemma 6.6 shows that all *special* arborescent knots are libroid. A famous family of non-special arborescent knots are the Kinoshita-Terasaka knots; to complete the proof of Theorem 6.1, it suffices to show:

**Theorem 6.10.** *The Kinoshita-Terasaka knots $KT_{2,n}$ shown in Figure 6.9(a) are libroid hyperbolic knots with trivial ordinary Alexander polynomial.*

*Proof.* These knots are hyperbolic since they are arborescent and not one of the exceptional cases, and their Alexander polynomials were calculated in [KT]. Minimal genus Seifert surfaces were found by Gabai [Gab2, §5]; we review his construction to verify that these knots are libroid.

Let $L$ be the $(3, -2, 2, -3)$–pretzel link shown in Figure 6.9(b); a Seifert surface $S$ for one orientation of $L$ is shown in Figure 6.9(c). The surface $S$ is a twice-punctured torus, and hence taut since $S^3 \setminus L$ is hyperbolic. The $KT_{2,n}$ knot can be obtained

(a) The knot $KT_{2,n}$.      (b) The pretzel link $L$.

(c) The surface $S$.

(d) $M$ viewed from inside.      (e) $T$ viewed from inside.
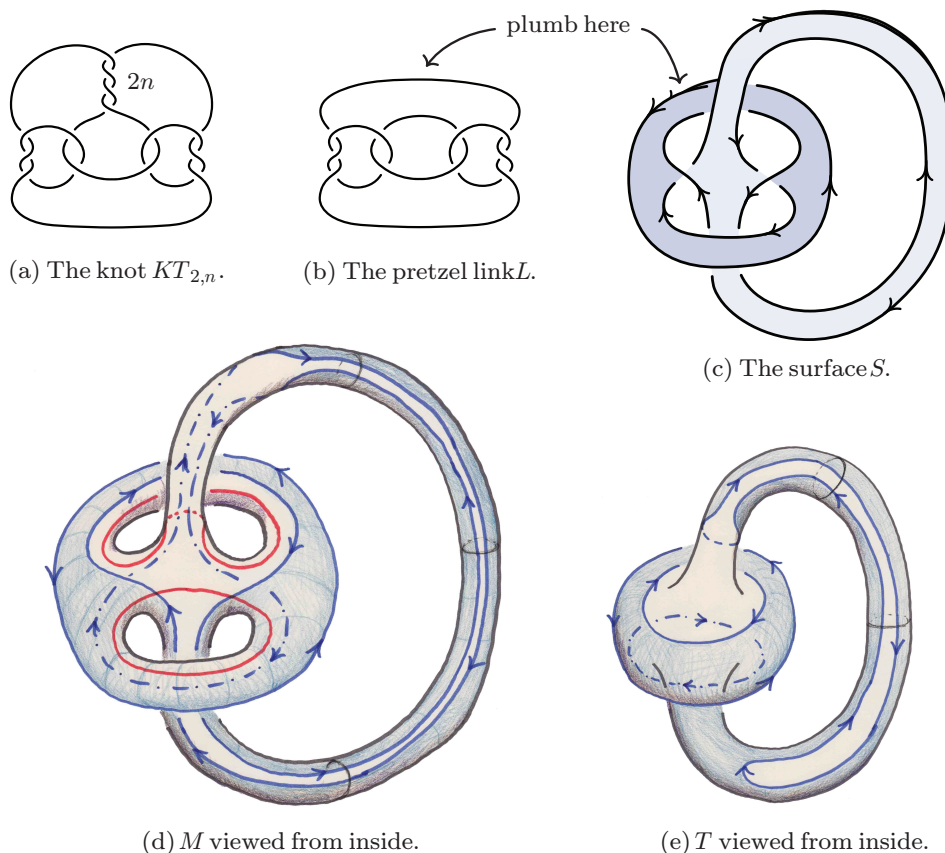
Figure 6.9: Kinoshita-Terasaka knots and the proof of Theorem 6.10.

by plumbing a band with $2n$-twists onto $S$ in the location shown, so by Lemma 6.6
it suffices to prove that the complement of $S$ is a book of $I$-bundles.

Thickening $S$ to a handlebody, we get the picture in Figure 6.9(d); the outside
of this handlebody is the sutured manifold $M$ we seek to understand. Each short
red curve meets the long blue oriented sutures in two points and bounds an obvious
disk in $M$. These are product discs in the sense of [Gab2], so we decompose along
them to get the sutured manifold $T$ which is the *exterior* of the solid torus shown
in Figure 6.9(e). Note that $T$ is a solid torus with four sutures that each wind once
around in the core direction. In particular $T$ is taut and hence so is $M$; moreover,
thinking backwards to build $M$ from $T$ by reattaching the product discs shows that
$M$ is a book of $I$-bundles with a single binding which is basically $T$.                □

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