CONTENTS

Preface ix

1 Optimization and the Visual Arts? 1

2 Truchet Tiles 6

3 Linear Optimization and the Lego Problem 27

4 The Linear Assignment Problem and Cartoon Mosaics 54

5 Domino Mosaics 66

6 From the TSP to Continuous Line Drawings 83

7 TSP Art with Side Constraints 110

8 Knight’s Tours 126

9 Labyrinth Design with Tiling and Pattern Matching 135

10 Mosaics with Side Constraints 145

11 Game-of-Life Mosaics 166

Afterword: Beauty and Utility 181

Bibliography 183

Index 185
CHAPTER 1

Optimization and the Visual Arts?

Optimization is the branch of mathematics and computer science concerned with optimal performance, with finding the best way to complete a task. As such, it is extremely applicable, as everyone from time to time attempts to perform some task at the highest level possible. A UPS driver, for instance, may sequence their stops to minimize total distance traveled, time spent on the road, fuel costs, pollutant emissions, or even the number of left turns. Finding an optimal tour, or at least one that is close to optimal, will benefit not only the driver and UPS, but also their customers (through lower prices) and the rest of society (through reduced pollution).

Some optimization problems are easy, while others are extremely difficult. Which is the case depends in large part on the constraints—the rules, the restrictions, the limitations—that specify the underlying task. If every stop on the UPS driver’s list falls on the same thoroughfare, then finding the optimal route—and proving it to be optimal—is trivial. But if the city is filled with one-way streets, the stops are scattered throughout the city, and some stops must be made during specified time windows, then determining how to perform this task at a high level can require considerable algorithmic ingenuity and computing power.

Optimization has been put to good use in a great number of diverse disciplines: from advertising, agriculture, biology, business, economics, and engineering to manufacturing, medicine, telecommunications, and transportation (to name but a few). Numerous excellent books describe these important, practical applications, and if you turn to the bibliography, you will find my favorites.

The book you hold in your hands is quite different. It is a highly personal account of my more than sixteen-year-long obsession with using mathematical and computer-science-based optimization techniques to create visual artwork. As obsessions go, it is a harmless one, and not
nearly as strange as it sounds! Within these pages, I will provide evidence that supports a bold claim: that the mathematical optimizer and the artist have more similarities than differences.

The mathematical optimizer studies problems that involve optimizing—that is, maximizing or minimizing—some quantity of interest (profit or total cost, for example, in business applications). The optimizer’s goal is to come up with an optimal solution—perhaps a way of making the profit as large as possible or the total cost as low as possible. In some cases, the optimizer will be satisfied with a local optimum, a solution that is better than all neighboring solutions. If you find a local optimum, you can be confident that when you present it to the board, no one sitting there will be able to improve upon your solution by making minor tweaks to it. But in other cases, the optimizer will not rest until they find a global optimum, a solution that is provably better than every other solution. If you find a global optimum, you will be able to get a good night’s sleep before the board meeting, for you will be certain that no one there—or anywhere—will be able to find a solution that is better than yours.

The artist is also a problem solver and a seeker of high-quality solutions. The creation of a piece of artwork can be considered a problem to be solved. And isn’t it difficult to imagine an artist who, when creating a piece, does not try to do their best? For some small number of artists, the goal may be to maximize profit, but for most, the goal may be to make the piece as beautiful as possible, or to have as great an emotional impact on viewers as possible. Beauty and emotional impact are impossible to quantify, but haven’t we all been in the presence of the critic, the museum-goer, or the gallery-opening shmoozer who in a burst of enthusiasm blurts out something like, “Don’t you just love this piece? Don’t you think that if the artist had added anything more to it, or had left anything out, it would have failed to have the same impact?” (an assertion, to the mathematical optimizer, about local optimality).

Mathematical optimizers are mindful of the roles that constraints play. They know that in some cases, if they impose additional constraints on an optimization problem, the problem will become much more difficult, but in other cases it will become considerably easier. Some constraints seem to be structured in such a way that in their presence, algorithms have trouble working their way to the best part of the feasible region (the set of all feasible solutions—the solutions that satisfy all the constraints), whereas
other constraints provide the equivalent of handholds and toeholds that form an easily traversed path to optimality.

Artists are similarly mindful. Artists are well aware that they must deal with constraints. They must work within budgets. They must meet deadlines. If they enter competitions or juried shows, they must make sure that their pieces satisfy the rules of entry. If they take commissions, they must follow their clients’ instructions. And no matter what media they choose to work with, they must deal with the particular constraints—imposed by the laws of physics—that govern how those media work. Painting with watercolors is different from painting with oils, and painting on rice paper is different from painting on canvas.

So, given that artists are creative, we might think that if it were up to them, they would do away with constraints. After all, constraints constrain. They restrict. They limit our choices. It would seem that constraints inhibit creativity.

But actually there is much evidence to the contrary. Many artists embrace constraints. Some need deadlines to be able to finish their work, and some believe that when their choices are limited, they are much more focused and creative. Joseph Heller (while paraphrasing T. S. Eliot) wrote,

> When forced to work within a strict framework the imagination is taxed to its utmost—and will produce its richest ideas.

And the psychologist Rollo May wrote,

> Creativity arises out of the tension between spontaneity and limitations, the latter (like the river banks) forcing the spontaneity into the various forms which are essential to the work of art or poem.

In fact, many artists go so far as to create their own constraints. Consider George-Pierre Seurat. While viewing his painting *A Sunday on La Grande Jatte–1884* from up close, one sees a mass of colorful dots. While backing away from it, one’s eyes merge all of the dots into an image of a group of Parisians relaxing on an island on the Seine. To create this masterpiece, Seurat set himself the task of producing the best possible depiction of what he saw on the riverbank, subject to two highly restrictive, self-imposed constraints: he had to keep his colors separate,
and he could only apply paint to the canvas with tiny, precise, dot-like brush strokes. Seurat's self-imposed constraints gave rise to a spectacular piece of artwork, the most widely reproduced example of what we now call Pointillism.

In the mosaicking arena, self-imposed constraints abound. Every time a mosaicist states, “I will build a mosaic out of _____,” another self-imposed constraint is born (or at least conceived). In 400 BCE, the ancient Greeks were building mosaics out of differently colored pebbles, and around 200 BCE, they started building them out of specially manufactured tiles (tesserae) made out of ceramic, stone, or glass. Today’s mosaicists still use these traditional materials, but they also use whatever else they have on hand: dice, dominos, LEGO bricks, Rubik’s Cubes, toy cars, spools of thread, baseball cards, photographs, and even individual frames of films like Star Wars and It’s a Wonderful Life.

Some mosaicists like to go beyond the inherent materials constraints. The domino mosaics of Ken Knowlton, Donald Knuth, and myself are not only made out of dominos, they are made out of complete sets of dominos. Knowlton’s Joseph Scala (Domino Player) (from 1981) was made out of 24 complete sets of double-nine dominos, so it contains 24 dominos of each type: exactly 24 blank dominos, exactly 24 zero-one dominos, and so on. My domino portrait of President Obama, the 44th president of the United States, uses 44 complete sets. Knowlton’s portrait of Helen Keller is composed of the 64 characters of the Braille writing system, and each of these characters appears 16 times. Chris Jordan’s Denali/Denial mosaic arranges 24,000 (digitally altered) logos from the GMC Yukon Denali sports utility vehicle (six weeks of sales in 2004) into an image of Denali (also known as Mount McKinley). And a Robert Silvers photomosaic, commissioned by Newsweek for its 1997 pictures-of-the-year issue, portrays the late Princess Diana as a mosaic formed from thousands of photographs of flowers. All of these artists use computer software—usually computer programs that they have developed themselves—to design their mosaics.

In theory, you can design a photomosaic without software. You can take the senior portrait photos from your high-school yearbook, cut them out, and then arrange them in a rectangular grid so that from a distance, they will collectively resemble a photo of your favorite teacher. This is possible—some of the photos will be brighter and others will be darker. But you will need a good eye to assess the brightness of each
photo, and even then, you will have a tough time determining the best position for each photo. Likewise, you can make a domino mosaic without software—by printing the target image on a large piece of paper and then placing dominos on top of the print, saving the brightest dominos (the nine-nines) for the brightest sections and the darkest dominos (the zero-zeros or blanks) for the darkest sections. Here, though it is clear which dominos are brighter than others, it still will be difficult to determine where to place each domino.

With mathematical optimization it is quite easy to design photomosaics, and it isn’t all that difficult to design domino mosaics. With mathematical optimization, the artist/mathematician (or mathematician/artist) can explore all manner of constraints systems. This book is an account of my explorations of this world.
alternating knot, 111
Archimedean tiling, 21
Ariadne, 135
bend-detection variables, 136–137
Bezier curves, 124
binary variables, 56, 62, 69, 78, 84–85, 128, 136, 138, 142, 144, 145, 171, 177, 178
Bixby, Robert, 41
Blecher, Sharon, 80
blinker, Game-of-Life pattern, 165
block brightness values, 10, 20–22, 60, 68, 179–180
block constraints, 62, 79
Bosch, Charlotte Woebcke, 107
Bosch, Dima, 80, 106
Bosch, Robert Krauss, 107
Botticelli, Sandro, 107
bounding, 47
Braille, 4
branch-and-bound algorithm, 46–52
branch-and-bound tree, 46, 50
branch-and-cut algorithm, 90–97
branch-and-cut tree, 96, 109
branching, 46
Burton, Gail, 80
cartoon constraints, 62
cartoon-mosaic design problem, 65
cartoon mosaics, 60–65; example, 63
Ceci n’est pas une pomme, 171
cellular automaton, 166
Charles II, 6
checkerboard quadrilateral mosaics, 164
checkerboard quadrilateral tiles, 25–26, 165
Cheron le Hay, Elisabeth, 9
Concorde TSP Solver, 52, 98, 105, 106, 109, 117
Connecting the Dots, 104–105
constraints, 1–5; self-imposed, 3–5
continuous line drawing, 97–101, 105–109
continuous linear optimization problems, 45–46, 49, 52, 53
continuous relaxation, of a discrete linear optimization problem, 58, 65, 80, 148, 165, 178
contour lines, 30–31, 34
contour planes, 33, 34
contrast score, 75
convex polyhedron, 34
Conway, John Horton, 166
Conway’s rules, for the Game of Life, 166, 177, 178
Cook, William, 98, 107, 109
cookie cutter, 111
Courbet, Gustave, 107
The Creation of Adam, 98, 103–104
crossing number, 114–116
cutting planes, 93
Dantzig, George, 35
dart-throwing algorithm, 101, 102, 103
degree, of a vertex, 86
degree constraint, 86
density, of a still life, 166
Denali/Denial, 4
Diana, Princess of Wales, 4
discrete linear optimization problems, 46, 49, 50, 52, 53, 58, 72, 79, 132, 165, 173, 175, 177, 178, 179, 182, 183
domino constraints, 70
domino mosaics, 4, 5, 66–82, 182; examples, 74, 77, 79–82
domino-mosaic design problem, 67, 72, 79
domino-placement variables, 69, 78
downsampling, 10, 68, 69
Douât, Father Dominique, 7, 8, 9, 10
downsampling, 10, 68, 69
edge, of a graph, 83–84
downsampling, 10, 68, 69
downsampling, 10, 68, 69
edge, of a polyhedron, 33
downsampling, 10, 68, 69
edge-selection variables, 84–85, 136. See also edge-usage variables
downsampling, 10, 68, 69
downsampling, 10, 68, 69
downsampling, 10, 68, 69
Embrace, 111, 113
downsampling, 10, 68, 69
Escher, M. C., 105, 152, 153
downsampling, 10, 68, 69
Eliot, T. S., 3
downsampling, 10, 68, 69
extended Smith tiles, 158, 161
downsampling, 10, 68, 69
extended-Smith-tile mosaics, 161
feasible region, 2, 30, 32–33, 34, 37, 47, 48, 49, 58, 93
downsampling, 10, 68, 69
Fleron, Julian, 81
downsampling, 10, 68, 69
flexible star tiles, 20–25
downsampling, 10, 68, 69
flexible-star-tile mosaics, 22–24
downsampling, 10, 68, 69
flexible Truchet tiles, 11–15
downsampling, 10, 68, 69
flexible-Truchet-tile mosaics, 15, 16
downsampling, 10, 68, 69
fractional solution, 49
downsampling, 10, 68, 69
Frankenstein’s monster, 60, 61, 63, 64, 68, 72, 74, 77, 79, 80, 145, 146, 148, 149, 150–152, 157, 158, 160–164, 165
globa optimum, 2
Gosper, Bill, 166
graph theory, 86
grayscale values, 10–11
Gu, Zonghao, 41
guinea pig problem, 17–19
Gurobi Optimizer, 40–46, 49–53, 58, 59, 65, 73, 79, 86, 117, 140
Guy, Richard, 166
Gyre, James, 120
half-space, 29
Hamilton, Bethany, 106
Hamilton, Linda, 106
Hamilton, William Rowan, 106
Hamilton (musical), 106
Hamiltonian cycle, 105
Hands, 99, 100, 101
heat, of a Game-of-Life pattern, 178
Heller, Joseph, 3
high-dead constraints, 173
integer programming problems, 52.
See also discrete linear optimization problems
Interwoven, 123
Jenson, Sage, 133
Jordan, Chris, 4
Jordan curve theorem, 111, 117–119
The Jordan Curve Theorem, 117–119
Joseph Scala (Domino Player), 4
Karloff, Boris, 60, 61
Keller, Helen, 4
Kepler, Johannes, 22, 23, 24
King Jr., Martin Luther, 80, 81
knight’s tours, 126–134; twofold, 126, 127; nearly fourfold, 127–131; nearly mirror, 131, 133
The Knight Tours the Castle, 132, 133
Knot?, 110, 112
knot tiles, 158, 162
knot-tile mosaics, 162, 163
Knowlton, Ken, 4, 66–67, 74–75
Knowlton-Knuth method, 74–78, 79
Knuth, Donald, 4, 75

Girl with a Pearl Earring, 153, 154
Glider, Game-of-Life pattern, 166, 167, 171, 172

Game of Life, 166
Game of Life animations, 176, 180
Game-of-Life mosaics, 171, 172, 175
Gardner, Martin, 66

Knowlton-Knuth method, 74–78, 79
Knuth, Donald, 4, 75
labyrinths, 135–144
labyrinth tiles, 138
LEGO problem: two-product, 27–28; three-product, 31–32; SnS-product, 40
Leonardo da Vinci, 107, 149
linear assignment problem, 54–59, 65, 69, 72, 75–76, 80, 145
linear programming problems, 52. See also continuous linear optimization problems
linear optimization problems, 31
Lin-Kernighan TSP heuristic, 107
linking constraints, 129, 144
local optimum, 2
Louis XIV, 6
low-dead constraints, 172
MacQueen’s “tractor beam” algorithm, 102–103; symmetric version of, 119–120
Magritte, René, 171
map-colored mosaics, 148, 149, 150–152, 154
map coloring, 145–148
map-coloring constraints, 147
mathematical modeling, 15–20
maximum density still life problem, 173–174
May, Rollo 3
McKay, Brendan, 126
Michelangelo, 98, 104
Milky Way, 22, 23, 24
King Minos, 135–136, 138, 139
The Minotaur, 135, 136, 138, 139, 141
Miranda, Lin-Manuel, 106
Mona Lisa TSP Challenge, 107–109
Morris, Scot, 66
mountain climbing, 31
Muybridge, Eadweard, 176
Nagata, Yuichi, 109
node, 46; child, 49; leaf, 49; parent, 49; root, 47

Obama, President Barack, 4, 81, 82
objective function, 18, 28, 141–144
Olivieri, Julia, 169
One Fish, Two Fish, Red Fish, Black Fish, 117
operations research, 181
opt-emoji, 98, 99
Outside Ring, 117, 118
parquet deformation, 149, 153, 154
path-wall sequence, 139
pattern-matching constraints, 139–140, 153–156
Pendegraft, Norman, 28, 52
phoenix, Game-of-Life pattern, 166, 167, 175–176
Phoenix Mosaic of Buddha, 175
photomosaic, 4–5
Pointillism, 4
pond, Game-of-Life pattern, 166, 167, 168
PostScript, 60, 153
precisely-one-of-these-must-happen constraints, 173
product mix problem, 181. See also LEGO problem
professor-course assignment problem, 58. See also linear assignment problem
properly colored map, 145, 146
relaxation. See continuous relaxation
RGB values, 24–25
Roelofs, Rinus, 123
Rothberg, Edward, 41
Running Cat (After Muybridge), 176
Scala, Joseph, 4, 66
Seurat, George-Pierre, 3–4
shaded Smith tiles, 153, 155, 159
shaded-Smith-tile mosaics, 157, 160
Silvers, Robert, 4
simple closed curve, 100, 105, 111–112, 125
simplex algorithm, 34–42
slot, 68
slot constraints, 69–70
Smith, Cyril Stanley, 153
sofaborde, 31
King Solomon’s knot, 116, 117

For general queries, contact webmaster@press.princeton.edu
Square Knot, 124
Starry Eyed, 22
Starry Night, 24
Statue of Liberty, 80, 81
stayin’-alive constraints, 174
still life, Game-of-Life pattern, 166
still-life mosaics, 171, 172
Still Life with Glider, 171, 172
stippling, 101–104; symmetric, 119–120
subtour, 87
subtour-elimination constraint, 87
A Sunday on La Grande Jatte–1884, 3
The Surfer Is Connected to the Wave, 106, 107
symmetry constraints, 120, 169, 174
symmetry-detection variables, 128–129, 136, 142–144

The Terminator, 106
Theseus, 135, 138, 139, 141

Thomassin, Henri-Simon, 9
tile-bending problem, 19–20
tile brightness value, 11, 60, 153, 170
tile-placement variables, 138
tour, 83
traveling salesman problem, 52, 83–97
Truchet, Father Sébastien, 6–7, 9
Truchet tiles, 6–10
TSP Art, 97–101, 103–109; with side constraints, 110–125
two-region constraints, 115–116

van Gogh, Vincent, 24, 107
variables, 18
Velázquez, Diego, 107
Vermeer, Johannes, 107, 153, 154
vertex, of a graph, 86
vertex, of a polyhedron, 34–35
vertex, of a tiling, 22
Voldemort, 55, 56, 57

whack-a-mole strategy, for handling subtours, 88–90, 91, 141