

Contents

<i>Acknowledgments</i>	vii
<i>Introduction</i>	1
CHAPTER ONE	
Mathematics as a Philosophical Challenge	4
CHAPTER TWO	
Frege's Logicism	21
CHAPTER THREE	
Formalism and Deductivism	38
CHAPTER FOUR	
Hilbert's Program	56
CHAPTER FIVE	
Intuitionism	73
CHAPTER SIX	
Empiricism about Mathematics	88
CHAPTER SEVEN	
Nominalism	101
CHAPTER EIGHT	
Mathematical Intuition	116
CHAPTER NINE	
Abstraction Reconsidered	126
CHAPTER TEN	
The Iterative Conception of Sets	139
CHAPTER ELEVEN	
Structuralism	154
CHAPTER TWELVE	
The Quest for New Axioms	170

© Copyright, Princeton University Press. No part of this book may be distributed, posted, or reproduced in any form by digital or mechanical means without prior written permission of the publisher.

Contents

<i>Concluding Remarks</i>	183
<i>Bibliography</i>	189
<i>Index</i>	199

Introduction

MATHEMATICS RAISES A WEALTH of philosophical questions, which have occupied some of the greatest thinkers in history. So when writing this book, some hard choices had to be made.

Let me begin with the aim of the book. Its target audience are advanced undergraduates and graduate students in philosophy, but also mathematicians and others interested in the foundations of one of the most successful, but also most puzzling, human endeavors. For the most part, the book does not presuppose much mathematics. Knowledge of elementary logic, the number systems from the natural numbers up through the reals, and some basic ideas from the calculus will be plenty for all except two late chapters devoted to set theory. While some familiarity with the philosophical mode of thinking will be a clear advantage, I have attempted to explain all relevant philosophical concepts.

I make no attempt to hide my own views concerning what is important and what works. Accordingly, my discussion has some general themes that serve to distinguish it from other introductions to the subject. First, Frege figures prominently in the book, both through his own views and his criticism of other thinkers. While my views often differ from Frege's, I share his fundamental conviction that mathematics is an *autonomous science*. Like other sciences, mathematics uses a meaningful language to express truths, ever more of which are discovered. Yet mathematics differs profoundly from the paradigmatic empirical sciences concerning the nature of its subject matter and the methods it employs. Following Frege, I am critical of any kind of formalism or fictionalism that deprives mathematics of its status as a body of truths, and of any attempt to assimilate mathematics to the empirical sciences. Frege famously defended the objectivity of mathematics. Just as geographers discover continents and oceans, so mathematicians explore numbers and sets. The two

Introduction

kinds of object are equally “real” and are described by equally objective truths.

A second theme of the book is how to understand the objects (such as numbers and sets) that mathematics explores. I pay more attention than is customary to the question of whether mathematical objects can be accepted without fully embracing a so-called platonistic conception of them. So I discuss some less demanding conceptions of mathematical objects. Might these objects be explicable in terms of a network of objective mathematical truths? Or might they be constructed by us? Or might they exist only potentially, not actually?

A final theme concerns mathematical knowledge. This knowledge must be explained in a way that links up with the subject matter of mathematics. It is not just an accident that our mathematical beliefs tend to be true. We would like to know why. What is it about our ways of forming mathematical beliefs which ensures that most of the beliefs correctly represent their subject matter? The answer must draw on an account of mathematical evidence. So what evidence do we have for our mathematical beliefs? A variety of answers have been proposed. Perhaps the evidence is logical or conceptual, or broadly perceptual in character, or of some indirect form that flows from mathematical principles’ ability to explain and systematize knowledge already established. My approach to the question of mathematical evidence will be *pluralist* and *gradualist*. That is, one form of evidence need not exclude another. And evidence may come in degrees, such that the elementary parts of mathematics enjoy a higher degree of evidence than the more advanced parts, especially those of a highly set-theoretic character.

Space considerations have forced me to downplay some issues to make room for a proper discussion of the themes just described. There is no systematic discussion of the philosophy of mathematics before Frege’s pioneering works of the 1880s and 1890s. I give only the briefest of introductions to Plato’s and Kant’s views on the subject. Traditional geometry receives little attention. Other important topics receive none. Examples include Wittgenstein on mathematics, explanation in mathematics, the philosophy of mathematical practice, the use of experimental

and other nontraditional methods in mathematics, and new developments such as homotopy type theory.¹

The first seven chapters cover topics that tend to be included in any good course in the philosophy of mathematics. The last five chapters discuss more recent developments. These chapters are more specialized and somewhat more demanding, both mathematically and philosophically, but are largely independent of one another (except for Chapter 12, which depends on Chapter 10).

¹ Useful introductions to these topics can be found in Rodych (2011), Mancosu (2015, 2008), Baker (2015), and Awodey (2014), respectively.

Index

- abstraction, 125–30, 164, 183–84;
dynamic, 136–38; Fregean, 95,
165, 183; neo-Fregean, 131–36;
principles of, 30, 34–37, 135–36,
165
- abstract objects, 9–12, 38, 47, 76, 94,
97–98, 101–3, 105–6, 113, 121,
124–25, 128, 136, 150, 160, 183.
See also mathematical objects
- abstract structures, 49–56, 154–56,
158–59, 161, 164–66
- analytic, 15–16, 18–19, 23, 25–26, 33,
92–93; Frege's definition of,
24–25, 92
- ante rem* structuralism. *See*
structuralism: noneliminative
- antirealism, 76–77, 81–82, 84–86;
about set theory, 171
- a posteriori*, 15, 17–19, 25, 65, 92
- a priori*, 4, 6–7, 12, 15–19, 25, 76,
88–89, 134
- arithmetic, 16, 21, 25–27, 33–37,
40–43, 45–47, 66, 70, 89–91,
156–57, 178; Frege, 132–133;
Heyting, 84; primitive recursive,
65
- axioms, 13, 19, 21–24, 34–39, 48,
52–55, 95–96, 171–75; Axiom
of infinity, 131; Pairing axiom,
139–140; Powerset axiom, 146;
of first-order Dedekind-Peano
arithmetic, 157; of second-order
Dedekind-Peano arithmetic,
33–34, 36, 41, 157; of ZFC,
141–143
- Barrow, John, 155
- Basic Law V, 36–37, 129, 132–133,
135–137
- Benacerraf, Paul, 12n, 101–4
- Berkeley, George, 63, 76
- BHK-interpretation of intuitionistic
logic, 79–80. *See also* logic:
intuitionistic
- Bolzano, Bernard, 22–23, 25, 118
- Boolos, George, 140n, 145–48, 153
- bridge principles, 42–43, 105, 111
- Brouwer, L.E.J., 18–19, 68, 73–74,
76–81, 85–86, 116
- Burgess, John, 112
- Caesar problem, 36n15, 133
- Cantor, Georg, 30n8, 58–62, 68, 75,
152n, 170, 185
- calculus, 63–64, 66, 106
- cardinal numbers, 30, 59–61, 125–26,
130, 134, 151, 170; large, 173, 175
- cardinality, 26, 42, 90–91, 125
- Cartesian product, 167–68
- category theory, 167–169
- Cauchy, Augustin-Louis, 23
- Cauchy sequence, 85n. *See also* real
analysis
- causal theory of knowledge, 103–4
- CH. *See* continuum hypothesis
- choice sequence, 85–86
- criteria of identity, 124–125, 133; for
directions, 126; for numbers, 135
- computation, 45–46, 184
- confirmational holism, 92–93, 99, 103
- conservativeness, 105–6, 110–12
- consistency, 43–44, 48, 57, 69–72;
relative, 71, 177
- constructions, 54–55, 74, 168; mental,
73, 76–78
- context principle, 28–29
- continuum hypothesis, 61, 170–73,
176–81

Index

- cumulative hierarchy, 139, 148, 150, 152, 154, 173, 177, 186. *See also* iterative conception of sets
- Curry, Haskell, 71
- Dedekind, Richard, 35, 62, 155–156, 158, 162, 164–65
- deductivism, 39, 48, 51–57, 71
- Descartes, René, 13, 106, 117
- Diophantine equations, 175
- Dummett, Michael, 27n, 81
- empiricism, 88–89, 116, 185; holistic form of, 92–96, 99
- epsilon-delta analysis, 22, 63
- equivalence relation, 125–126
- Euclid, 21, 24, 59
- evidence, 2, 116–17; intrinsic, 172–73; extrinsic, 173–76, 180–81; pluralism and gradualism about, 117, 187
- extrapolation, 116, 175, 184–85
- Feferman, Solomon, 53, 95, 171, 180–81
- fictionalism, 1, 19, 69
- Field, Hartry H., 102, 104–14
- finitary mathematics, 56–57, 65, 69–73, 120
- formalism, 1, 19, 39, 71–72, 185; game, 40–44, 48, 53, 57; term, 44–48, 56, 64, 184
- formal system, 24, 39–40, 48, 53, 55, 70–71, 185
- Fraenkel, Abraham, 141–142
- Frege, Gottlob, 1–2, 8, 10–11, 13, 16, 18–19, 21, 23–45, 47, 62, 77–79, 89–92, 94, 98, 116, 126–30, 132–33, 164–65, 185–86; his argument for the existence of abstract objects, 27, 101–2; his bootstrapping argument, 131, 137. *See also* abstraction: Fregean, arithmetic: Frege and theorems: Frege's theorem
- Friedman, Michael, 96
- Føllesdal, Dagfinn, 122–24
- Galois, Évariste, 170
- Gauss, Carl Friedrich, 4
- geometry, 2, 6–7, 8n, 16, 23, 50–52, 74, 111, 176–78, 181, 184; coordinate-free (synthetic) vs. coordinate-based (analytic), 106; Euclidean, 50–51, 76, 107, 176–77; non-Euclidean, 50, 76, 155, 176–77; projective, 69; Riemannian, 96
- Goldman, Alvin, 104
- Gödel, Kurt, 70–71, 116–18, 120, 140, 143–45, 147, 149–51, 170–75, 177–80, 187
- Gödel sentence, 111–12, 172–73
- group theory, 49, 52
- Hale, Bob, 129, 132–35
- Heine, H. E., 45
- Heyting, Arend, 74, 78–79. *See also* arithmetic: Heyting
- Hilbert, David, 50–52, 56–58, 62–74, 77, 83, 107, 116, 118, 120, 122–23, 155, 160, 170, 184; his program, 44, 56–57, 71, 73
- Hilbert space, 109, 115
- Hilbert strokes, 56, 64–65, 68, 77, 118, 120, 184
- (HP). *See* Hume's Principle
- Hume, David, 30, 88
- Hume's Principle, 30, 33–37, 125, 132–35, 137–38
- Husserl, Edmund, 78, 121–122
- ideal elements, 68–70
- idealization, 184
- if-then-ism. *See* deductivism
- imaginary number i , 45, 69
- indispensability argument, 89, 97–99, 115
- inference to the best explanation, 175, 181

- infinitary mathematics, 56–57, 68–69, 73, 123,
infinity, 184–185; actual, 58, 61, 65, 75, 85; Dedekind, 59n; potential, 58, 64–68, 75, 82–85, 151, 186
infinitesimals, 63
in re structuralism. *See* structuralism:
 eliminative
integration challenge, 12–15, 28, 102–4, 124, 186–87
intuition, 16–18, 21–23, 28, 65, 77, 88, 93, 95, 116–25, 172–73, 175, 187
intuitionism, 18, 73–87, 184
isomorphism, 49–50, 60, 107, 156, 158, 164–68, 176, 178–79, 181
iterative conception of sets, 139–54, 164, 172–73, 178–80, 185; modal explication of the, 151–53

Kant, Immanuel, 2, 15–19, 21, 24–26, 33, 56, 64–65, 72, 76–78, 88–89, 93, 95, 116, 118, 120, 122
Kolmogorov, A. N., 79
Kreisel, Georg, 32

law of excluded middle, 67–68, 75, 80
Leibniz's Law, 133, 163
Liouville, Joseph, 75
logic, 1, 18–19, 21, 24–25, 29, 31, 33–34, 37, 67–68, 74, 76, 131; first-order, 24, 90, 105, 110, 157; intuitionistic, 79–82, 84–85; modal, 67n, 152–153; plural, 89–90; second-order, 24, 49, 110, 132, 160
logical truth, 16, 21, 131, 133–34
logicism, 19, 21–37

Maddy, Penelope, 9n5, 102, 119–120, 123, 181
Malament, David, 109, 114
mathematical objects, 2, 9–11, 21, 27–28, 31–32, 50, 67, 75–76, 78, 97–98, 101–2, 110, 123, 127, 151, 160–61, 164, 165n, 175, 186;
 incompleteness claim about, 162–63; as representational aids, 112–113
Melia, Joseph, 112
metamathematics, 24, 53, 55–57, 74, 185
Mill, John Stuart, 18–19, 89–92, 94
model, 54, 178–179; of the constructible hierarchy, *L*, 177; problem of model existence, 54–57, 159–61

naturalism, 98, 109
necessity, 7–9, 67n, 160
neo-Fregeanism, 131–35
Newtonian theory of gravitation, 106–8, 175
nominalism, 98–101, 109, 112–114
nondistributive properties, 90

ontological dependence, 149
ordinal numbers, 59–60, 77, 139, 145, 147–48; finite von Neumann, 162

paradox, 58–62, 143–44, 150, 152; Galileo's, 58–60; Russell's, 36, 62, 129–31, 133, 137, 144
parallel postulate, 176–77
Parsons, Charles, 95, 117, 120–24, 148
patterns. *See* abstract structures
perception, 14–15, 21, 103, 118, 120–22, 124–25, 172, 174–75; quasi-, 120–22
Plato, 2, 6–7, 10–12, 17–19, 38, 88, 93, 102, 184
platonism, 2, 4–6, 7, 10–12, 21, 32, 38, 67–68, 70, 76, 97, 99, 104–6, 110–12, 127, 150, 156, 159, 161, 163, 186
platonistic conception of mathematics, *See* platonism
pluralist conception of mathematics, 87n13, 176–82
possible worlds, 8, 68n, 153

Index

- powerset, 61. *See also* axioms:
 Powerset axiom
- proof, 5, 15, 22–25, 38–39, 79–81,
 185–86; nonconstructive
 existence, 74–75; by *reductio ad
 absurdum*, 80; theory, 71, 110–12
- psychologism, 77–78
- Putnam, Hilary, 52, 54, 97, 105
- quasi-concreteness, 47, 53, 56–57,
 64–65, 120–21, 123
- Quine, W. V., 12, 18–19, 89, 92–97,
 99, 116, 124; his attack on the
 analytic-synthetic distinction,
 92–93
- Ramsey, F. P., 130
- real analysis, 85–87; rigorization of,
 23, 118
- realism, 31–32, 79, 171; object, 9–10,
 11n, 27, 31–32, 67, 101, 104, 161,
 186; truth-value, 31–32, 161;
 working, 68
- realizability interpretation of
 intuitionistic logic, 84
- reconceptualization, 128, 134–35
- reflection principle, 147, 153
- reification, 124
- Resnik, Michael, 162
- rewrite rules, 45–46
- Russell, Bertrand, 36, 116, 119,
 129–31, 141, 145, 163, 174–75.
 See also paradox: Russell's
- schematic generalities, 66–67, 83
- semantics, 27, 38, 79–81, 98, 101, 161;
 proof-conditional vs.
 truth-conditional, 80–81
- set comprehension, 62, 141, 142
- set theory, 1, 55, 58–63, 75, 95,
 111–12, 118–20, 131, 135,
 139–45, 148–50, 154, 159–60,
 167–68, 170–72, 180–81;
 actualism and potentialism
 about, 150–153; monism and
 pluralism about, 176–79; naive,
 62, 141–53
 simple theory of types, 130–31, 141,
 145
 simply infinite system, 41, 158–62,
 165
 Skolem, Thoralf, 65, 141–42
 stage theory, 145–147. *See also* ZFC
 and iterative conception of sets
 strict potentialism, 83–84. *See also*
 infinity: potential
 structuralism, 154–56, 167–69, 184;
 eliminative, 156–61;
 methodological form of, 154–55;
 modal, 160–61; noneliminative,
 156, 161–64, 169, 186;
 set-theoretic, 55, 159–60, 184
 structural properties, 52, 158–59
 STT. *See* simple theory of types
 substantivalism, 108
 successor function, 46, 67–68, 82,
 157
 syntax, 38–39, 40, 44–46, 53, 56, 120,
 184
 synthetic, 15, 19, 25, 88, 92–93;
 a priori, 16–19, 25, 88
- Tait, William, 65
- Tarski, Alfred, 107
- theorems, 18–19, 22, 27, 51; Cantor's
 theorem, 61; Dedekind's
 categoricity theorem, 158, 165n;
 Euclid's theorem, 65–66; Frege's
 theorem, 33–36, 133; Gödel's First
 and Second Incompleteness,
 70–73, 75, 111; of Heyting
 arithmetic, 84; intermediate
 value theorem, 22, 118;
 Löwenheim-Skolem theorem,
 179n; representation, 107–8, 114,
 184; Zermelo's quasi-categoricity
 theorem, 179
- Thomae, Carl Johannes, 38, 40–41
- topology, 74, 123, 167
- transcendental numbers, 75

Index

- type theory, 143–144. *See also* simple theory of types
- type-token distinction, 39–40, 47, 118, 121–22, 165
- universals, 119, 156; structured, 166
- use-mention distinction, 39–40
- verificationist account of meaning, 92–93
- von Neumann, John, 71
- Weierstrass, Karl, 23, 63
- Weyl, Hermann, 87
- Whitehead, Alfred North, 129–31, 141
- Wittgenstein, Ludwig, 2, 14, 82
- Wright, Crispin, 37, 129, 132–35
- Yablo, Stephen, 109, 112
- Zermelo, Ernst, 141, 151–52, 174, 179
- ZFC, 139–43, 145–46, 148, 170–73, 177. *See also* set theory
- ZF2, 179–180